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Notes on Quantitative OG Models*

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The idea is to work through some overlapping generations models with realistic quantitative inputs. We describe the model and explore properties of steady-state closed and small-open-economy versions. There's a literature review at the end of both open economy work with OG models and a selection of demography papers. The former includes notable work by Attanasio, Kitao, and Violante (JME, 2007), Auerbach and Kotlikoff (book, 1987), Domeij and Floden (IER, 2006), Henriksen (ms, 2002), and Krueger and Ludwig (JME, 2007).

Work in progress. Comments and questions marked by ?? in square brackets [ ].

1 Demographic environment

The usual setting: discrete time with dates $t = 0, 1, \ldots$. We use a time interval of one year, which is more natural (to us anyway) than the five years (or more) typically used in the literature.

Stationary survival rates. The economy is populated by overlapping generations (or cohorts) who live up to H periods. Newborns are age zero. For each individual, survival is stochastic and s_h is the probability an individual of age h survives to age $h+1$. This could, in principle, depend on the date t as well, but will not for now. The probability of surviving for n periods is $s_h^{(n)} = \Pi_{j=1}^{n} s_{h+j-1}$. By convention, $s_h^{(1)} = s_h$ and $s_h^{(0)} = 1$.

[?? There's a question here about the convention: should s_h be the prob of surviving to age h or $h + 1$? Not clear which is cleaner. AKV use the former, Victor and Espen the latter. We'll stick with the latter for now.]

More notation. Let x_{ht} be the size of a cohort of age h at date t. Evidently they are members of generation $g = t - h$, the generation born at date g. In our settings, there's uncertainty about mortality for individuals, but not in the aggregate, so cohort sizes evolve like this: $x_{h+1,t+1} = s_h x_{ht}$.

This has a convenient matrix form. Our starting point is an economy with one agent born at each date t. In matrix form we have

$$x_{t+1} = Dx_t + m_{t+1}$$

*Working notes, no guarantee of accuracy or sense.

where

$$x_t = \begin{bmatrix} x_{1t} \\ x_{2t} \\ \vdots \\ x_{It} \end{bmatrix}, \quad D = \begin{bmatrix} \varphi_1 & \varphi_2 & \varphi_3 & \cdots & \varphi_I \\ s_1 & 0 & 0 & \cdots & 0 \\ 0 & s_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & s_{I-1} & 0 \end{bmatrix}, \quad m_t = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

The φ_j's are fertility rates, which are all zero here, but we put them in on the off-chance we'll want them later. The additive component m_t is births here, but could include immigration. [?? Start x with age zero?]

Changing survival rates. Survival rates have been increasing. A simple one-dimensional version of this might be viewed as a poor man's Lee-Carter adjustment. In the demography lit, the focus is on mortality rates: $m_{ht} = 1 - s_{ht}$. Lee and Carter (JASA, 1992) suggest this one-dimensional process:

$$\begin{aligned} \log m_{ht} &= a_h + b_h k_t \\ E_t k_{t+1} &= \mu + k_t. \end{aligned}$$

[?? need another letter, k already in use.] We use a simpler version, with $b_h = 1$. The result is that mortality rates decline proportionately at rate $-\mu$. Lee and Miller (Demography, 2001, esp fig 6) show that this isn't a bad approximation. The most obvious departures are at young ages, which aren't the focus of our work anyway.

2 Economic environment

We go back and forth between sequence and date-0 versions, not sure which is more helpful yet.

Commodity space. There is one good plus labor at every date t. The good can be used for consumption or investment and labor can be used for production or (in future versions) leisure.

Preferences and endowments. Agents are endowed with labor and consume the good. They also receive bequests, but we'll hold off on that.

Denote consumption at date t by agents of age h by c_{ht}. Individuals have the time-additive utility function

$$U_{ht} = \sum_{j=0}^{I-h} \beta^j s_h^{(j)} u(d_{h+j} c_{h+j,t+j}) = u(d_h c_{ht}) + \beta s_h U_{h+1,t+1}, \tag{1}$$

where the instantaneous utility function is $u(x) = (x^{1-\sigma} - 1)/(1-\sigma)$ and d_h is an age adjustment. Total consumption by all agents at date t is $c_t = \sum_h c_{ht} x_{ht}$, where the sum is over all allowable ages h.

Labor supply is fixed but age-dependent: an individual of age h supplies e_h efficiency units of labor. Total labor supply in the economy at time t is $n_t^s = \sum_h e_h x_{ht}$.

[?? how do we want to handle growth — within or across generations?]

One issue here is how to set e_h, in particular how it changes when we increase productivity growth. Is growth within or across generations? Suppose we have a zero-growth benchmark $\hat{e}_h$ [?? better notation?]. The wage/income profile is going to be $\hat{e}_h w_t \sim \hat{e}_h \gamma^h$, so if we increase growth we make it steeper. But what if growth occurs across generations? Then you'd like something like $(\hat{e}_h/\gamma^h) w_t \sim \hat{e}_h$. An intermediate version is

$$e_h \;\;=\;\; \eta(\hat{e}_h/\gamma^h) + (1-\eta)\hat{e}_h,$$

with η between zero and one. The simplest version sets $\hat{e}_h = 1$: a flat wage profile. If we set $\eta = 1$, we still have a flat profile when there's growth.

[?? work the following into the model — or are people born at age 21?]

In what follows, agents do nothing until age 21: for the first 20 years, consumption and labor supply are zero. The latter might be represented by $e_{ht} = 0$ for $h \leq 20$. The former we impose directly.

Technology. Output y_t is produced with inputs of capital and labor with the hd1 function $y_t = f(k_t, z_t n_t)$ Capital is the same good as consumption and follows the law of motion

$$k_{t+1} \;\;=\;\; (1-\delta)k_t + i_t,$$

where δ is the depreciation rate and i_t is investment.

Budget constraints. There is a single riskfree asset that can be used to borrow or lend. We denote by a_{ht} the holdings of this asset by an individual of age h at date t. Individuals start and end life with zero assets: $a_{1t} = a_{I+1,t} = 0$.

[?? bequests need work]

We distribute the wealth of agents who die young as bequests to living members of the same generation. The gross return from holding one unit of the asset from period t to $t+1$ is r_t. If the wage is w_t per efficiency unit of labor, the sequence budget constraint is

$$a_{h+1,t+1} \;\;=\;\; r_t a_{ht} + e_h w_t + b_{ht} - c_{ht},$$

where b_{ht} is receipt of bequests. We handle bequests with annuities as in Yaari (1965). Ditto many others, including especially Hansen and Imrohoroglu (RED, 2008). Since death is an idiosyncratic event, perfect insurance is possible. We get the perfect insurance allocation if we use the budget constraint

$$s_h a_{h+1,t+1} \;\;=\;\; r_t a_{ht} + e_h w_t - c_{ht}. \qquad (2)$$

They're the same if $b_{ht} = (1-s_h) r_t a_{h+1,t+1}$. [?? This is slightly different from Espen's version: we distribute — see the timing of a on the right. Related question: should the a's be shifted back one period? See also ACK's eq (6), which mirrors (2).]

In some settings, it's easier to think of the budget constraint in terms of present value prices. Let the price at t of one unit of the good at $t+n$ be $q_t^{(n)} = 1/\Pi_{j=1}^n r_{t+j-1}$. By convention $q_t^{(0)} = 1$.

We can use the sequence constraint (2) and the terminal conditions on net worth to get a present value budget constraint. If we flip the constraint around, we have

$$a_{ht} \quad = \quad q_t^{(1)} \left(s_h a_{h+1,t+1} - e_h w_t + c_{ht} \right).$$

...

$$\sum_{j=h}^{I} q_j \left[(1 + r_{g+j-1}) a_{g,j} + e_j w_{g+j} + b_{g,j} - a_{g,j+1} - c_{g,j} \right] \geq 0 \qquad \text{(PV-BC)}$$

[?? Need to check this out, check bequests.]

Aggregate asset holdings in the economy are denoted by $a_t = \sum_h a_{ht} x_{ht}$.

Firms. Firms buy capital and labor at prices one and w_t per unit. They produce output today and depreciated capital, which is effectively output tomorrow. At every date t, their profit is

$$f(k_t, z_t n_t) + q_t^{(1)} (1 - \delta) k_t - w_t n_t - k_t.$$

3 Equilibrium: definition

[?? fix up bequests]

[?? kill r's, use q's]

Definition 1 (competitive equilibrium)**.** *Define the set of feasible generations g and ages h: $\mathscr{M} = \{(g,h) | g \in \{-I, 1, \ldots, \infty\}, h \in \{\max[0, -g], 1, \ldots, I\}\}$. Given an initial capital stock k_0 and age distribution $\{x_{g,h}\}_{(g,h) \in \mathscr{M}}$, a competitive equilibrium consists of an allocation $\{c_{g,h}, a_{g,h}\}_{(g,h) \in \mathscr{M}}$, $\{k_t\}_{t=1}^\infty$, $\{n_t\}_{t=0}^\infty$, and prices $\{r_t, w_t\}_{t=0}^\infty$ such that*

1. *For each individual in cohort g and is now of age $\bar{h}$, $\{c_{g,h}, a_{g,h}\}_{\bar{h}=\max[0,-g]}^I$ solves*

$$\max_{\{\tilde{c}_{g,h}, \tilde{a}_{g,h}\}_{\bar{h}=\max[0,-g]}^I} \sum_{j=\bar{h}}^{I} \beta^{j-\bar{h}} S_{\bar{h}}^j u(\tilde{c}_{g,j})$$

s.t. $\forall h \in \{\max[0, -g], 1, \ldots, I\}$

$$\sum_{j=\bar{h}}^{I} q_j \left[(1 + r_{g+j-1}) \tilde{a}_{g,j} + e_j w_{g+j} + b_{g,j} - \tilde{a}_{g,j+1} - \tilde{c}_{g,j} \right] \geq 0 \qquad (\lambda_g)$$

$$\tilde{a}_{g,0} = 0, \; \tilde{a}_{g,I+1} \geq 0$$

where λ_g is the Langrange multiplier on the budget constraint for cohort g. Notice that we use $g+h=t$ in the individual's problem.

Aggregation over individuals and cohorts gives

$$n_t^s = \sum_h e_h x_{t-h,h}$$
$$a_t = \sum_h a_{t-h,h} x_{t-h,h}$$
$$c_t = \sum_h c_{t-h,h} x_{t-h,h}$$

2. *For the final goods producers $\{n_t\}_{t=0}^{\infty}$ and $\{k_t\}_{t=0}^{\infty}$ solve at any t*

$$\max_{\tilde{n}_t, \tilde{k}_t} f(\tilde{k}_t, z_t \tilde{n}_t) - (r_t + \delta)\tilde{k}_t - w_t \tilde{n}_t$$

3. *All markets clear, i.e., $\forall t$*
   - *(i) Goods: $y_t = c_t + i_t$*
   - *(ii) Assets: $a_t = k_t$*
   - *(iii) Labor: $n_t^s = n_t$*

4 Equilibrium: first-order conditions

[?? do we want sequence or date-0 versions? also: clean up annuities/bequests]

Household problem. The Lagrangian for an individual from cohort g who is now of age $\bar{h}$ is

$$\mathscr{L} = \sum_{j=\bar{h}}^{I} \beta^{j-\bar{h}} S_{\bar{h}}^{j} u(c_{g,j}) + \lambda_g \sum_{j=\bar{h}}^{I} q_{g+j} \left[(1 + r_{g+j-1}) a_{g,j} + e_j w_{g+j} + b_{g,j} - a_{g,j+1} - c_{g,j}\right] \quad (3)$$

The first-order condition for cohort g's consumption at age h is

$$\frac{\beta^{h-\bar{h}} S_{\bar{h}}^{h} c_{g,h}^{-\sigma}}{\lambda_g} = q_{g+h} \quad (4)$$

The first-order condition with respect to asset holdings $a_{g,h+1}$ is

$$-q_{g+h} + (1 + r_{g+h}) q_{g+h+1} = 0$$

Combining the two gives the consumption Euler equation

$$\beta s_h (1 + r_{g+h}) \left(\frac{c_{g,h+1}}{c_{g,h}}\right)^{-\sigma} = 1 \quad \text{(EE)}$$

With annuities, the s_h drops out. [?? elaborate on this, show how it works]

More general, we have a relation between consumption at age h relative to consumption at base age $\bar{h}$:

$$c_{g,h} = c_{g,\bar{h}} \left(\beta^{h-\bar{h}} S_{\bar{h}}^{h} \prod_{j=\bar{h}}^{h-1} (1 + r_{g+j}) \right)^{\frac{1}{\sigma}} \tag{5}$$

Now we plug (4) and (5) into the present value budget constraint (PV-BC) which will be binding. Asset holdings cancel out except for $a_{g,\bar{h}}$ and $a_{g,I}$, where the latter will be zero. We obtain

$$\frac{c_{g,\bar{h}}^{-\sigma}}{\lambda_g} \left\{ a_{g,\bar{h}} + \sum_{l=\bar{h}}^{I} \frac{e_l w_{g+l} + b_{g,l} - c_{g,\bar{h}} \left(\beta^{l-\bar{h}} S_{\bar{h}}^{l} \prod_{j=\bar{h}}^{l-1} (1 + r_{g+j}) \right)^{\frac{1}{\sigma}}}{\prod_{j=\bar{h}}^{l-1} (1 + r_{g+j})} \right\} = 0 \tag{6}$$

Because of our choice of the utility function $c_{g,h} > 0 \; \forall h$, and with (PV-BC) binding, $\lambda_g > 0$ usually holds. Then by solving for $c_{g,\bar{h}}$ and by virtue of (5) we have a closed form expression for the entire consumption path (for a given path of prices):

$$c_{g,h} = \left\{ \sum_{l=\bar{h}}^{I} \frac{\left(\beta^{l-h} \frac{S_{\bar{h}}^{l}}{S_{\bar{h}}^{h}} \frac{\prod_{j=\bar{h}}^{l-1} (1+r_{g+j})}{\prod_{j=\bar{h}}^{h-1} (1+r_{g+j})} \right)^{\frac{1}{\sigma}}}{\prod_{j=\bar{h}}^{l-1} (1 + r_{g+j})} \right\}^{-1} \left\{ a_{g,\bar{h}} + \sum_{l=\bar{h}}^{I} \frac{e_l w_{g+l} + b_{g,l}}{\prod_{j=\bar{h}}^{l-1} (1 + r_{g+j})} \right\}, \; \forall h \in \{\bar{h},, \dots, I\} \tag{7}$$

The second curly bracket is the present value of the individual's lifetime income. The first curly bracket is a function of prices and survival probabilities (and the discount factor) and shows which fraction of the lifetime income will be consumed in a given period.

[?? Derive Espen's second-order diff eq for a]

Firm's problem. For the final goods producers the problem is static. The first order conditions are $f_1(.) \equiv \frac{\partial f(.)}{\partial k_t} = r_t + \delta$ and $f_2(.) \equiv \frac{\partial f(.)}{\partial n_t} = w_t$.

5 Balanced growth path

[?? change from steady state to balanced growth path]

Definition 2 (Steady state)**.** *A steady state is a competitive equilibrium in which*

1. *the demographic structure is constant:* $x_{g,h}$ *same for all* g.

2. *labor productivity grows at a constant rate:* $z_{t+1}/z_t = \gamma$ *for all* t.

3. *the capital-output ratio is constant:* k_t/y_t *the same for all* t. $\forall t$.

The first order conditions for the final goods producers imply - taking into account the CRS property of the production function - that the interest rate and the wage are constant in steady state. Consequently, if we let each cohort start with zero assets, i.e., $a_{g,0} = 0$ $\forall g$, the paths for consumption and asset holdings are identical for each cohort. Since the population shares $\bar{x}_h$ are time-invariant, it follows that aggregate asset holdings (and aggregate consumption) are constant for all t. The market clearing condition for assets tells us subsequently that capital is constant in steady state.

6 Sequence budget constraints

A Literature review for quantitative OG models

Idiosyncratic list, governed by our own interests.

Yaari, REStud, 1965.

- Classic uncertain lifetime problem
- Annuities: with fairly priced annuities, risk of death drops out of first-order condition for consumption

Auerbach and Kotlikoff, Dynamic Fiscal Policy, 1987, chs 3, 4.

- Annual model, utility depends on consumption and leisure. CES period utility, power utility over time.
- Labor has age-dependent efficiency. There's a constraint due to retirement that I don't follow, maybe just that leisure has an upper bound of one.
- Taxes: proportional taxes on labor and capital income, consumption.
- Production CES with constant productivity.
- Capital and investment: no lom for capital, set MPK = r, add (quadratic) adjustment costs.
- Parameters: IES = 4, elasticity of consumption and leisure = 0.8, discount factor = 1/(1+0.015), wage profile from Welch (not reported?), production Cobb-Douglas with capital share of 0.25.
- Demographics buried pretty deep, not clear how they work.

Taylor and Williamson, "Capital flows to the New World," JPE, 1994.

- Excess of young in New World kept saving low, so capital inflows made up the difference
- Their *dependency rate* is fraction of population age 0-15, *dependency rate gap* is difference from UK
- No formal model

Rios-Rull, "Baby boom," BEAM, 2001.

- Stationary demography with shocks: $x_{t+1} = \Gamma x_t + v_{t+1}$. Γ is constant, but v introduces fluctuations in age distribution. He makes v a one-dimensional AR(2), which is a little more complicated.
- Endogenous labor supply, retirement from setting efficiency equal to zero.
- Accidental bequests when people die with positive net worth, handled by distributing assets to other people in the same cohort/generation. At age i, sequence budget constraint is $a_{it+1} = (w_t e_i n_{it} + r_t a_{it}) / s_i$.

Domeij and Floden, IER, 2006.

- Multicountry OG model, 5-year time interval
- Households: age scale for consumption, ditto labor efficiency, fixed labor supply, utility from bequests
- Pension system: proportional tax on labor income, fixed payment after retirement (age 65) adjusts automatically to satisfy pay-as-you-go constraint
- Consumption scales tied to children, who count half
- Efficiency profile combines Hansen data on wage profiles with Fullerton data on participation rates
- Labor productivity differs across countries, but grows at the same rate (1% per year) to allow balanced growth path
- Demography: survival probabilities vary across time and countries, inferred from (projected) changes in population.

Hong and Rios-Rull, JME, 2007.

- Issues: annuities, life insurance
- Features: marriage and divorce, utility from bequests
- Parameters: IES = 1/3

Krueger and Ludwig, JME, 2007.

- Multicountry OG model
- Section 2: nice two-period model shows how this works.
- They add idiosyncratic income risk, but it doesn't have much effect
- Retirement age 65 to start, but they describe the impact of raising it
- Interest rate falls about 1%

Attanasio, Kitao, and Violante, JME, 2007.

- Two-country OG model: rich and poor countries (North and South)
- Focus on demographics differences between the two
- Parameters: time interval = 5 years, curvature = 2, $\beta = 1.036$, hits $K/Y = 1.5$
- Results: interest rate falls 1-2%, capital flows from North to South, flows depend on social security structure

Hansen and Imrohoroglu, "Annuities," RED, 2008.

- Issue is hump-shaped age profile of consumption
- They use partial annuity coverage to account for it
- Nice description of annuities and how they affect the budget constraint

B Literature review for demography research

Keyfitz, Applied Mathematical Demography, 1977. Chapter 2 on the life table includes these definitions:

$$
\begin{aligned}
l_x &= l(x) = \text{prob of surviving from birth to age } x \\
{}_n d_x &= l_x - l_{x+n} = n\text{-period number of deaths} \\
{}_n q_x &= {}_n d_x / l_x = (l_x - l_{x+n})/l_x = \text{probability of dying over next } n \text{ periods} \\
{}_n L_x &= \int_x^{x+n} l(a)da = \text{number of years lived from } x \text{ to } x+n \\
T_x &= {}_\infty L_x = \text{years remaining to the cohort} \\
e_x &= T_x / l_x = \text{life expectancy at age } x \\
{}_n m_x &= {}_n d_x / {}_n L_x < n \times {}_n q_x = \text{age-specific death/mortality rate} \\
{}_n M_x &= \text{observed death rate in a real population}
\end{aligned}
$$

Lee and Carter, JASA, 1992.

- One-dimensional change in mortality rates over time:

$$
\begin{aligned}
\log m_{xt} &= a_x + b_x k_t \\
k_{t+1} &= k_t + c + e_{t+1}
\end{aligned}
$$

- Table 1 shows values of (a_x, b_x)
- Prediction based on extrapolating k

Lee and Miller, Demography, 2001.

- Post-mortem on Lee-Carter
- Mortality rates have fallen faster than predicted
- Fig 6 shows roughly constant changes in $\log m_{xt}$ by age (constant b_x?)