Backus, Cooley, & Henriksen

Notes on Misallocation of Capital*

(Started: May 27, 2011; Revised: June 1, 2011)

The idea is to look at aggregate productivity and returns to capital when productivity is firm-specific and capital is allocated arbitrarily. It has some of the flavor of Hsieh-Klenow (QJE, 2007) and Olley-Pakes (Econometrica, 1996).

A lognormal example

Here's an example, just to see how things work. We start with an arbitrary distribution of firm-level productivity, allocate capital arbitrarily, and allocate labor to equate marginal products. Question 1: What is aggregate productivity? Question 2: How does the aggregate return on capital compare to the marginal product based on the aggregate production function? Question 3: Can we think of countries with misallocated capital as being unattractive to outside investors relative to what the aggregate production function would suggest?

1. Production. Firms produce according to

$$y \quad = \quad a k^{\alpha} n^{1-\alpha}.$$

We'll use lower-case letters for firm quantities and upper-case letters for aggregate quantities.

2. Misallocation. We specify an arbitrary allocation of (a, k) across firms. We'll use a bivariate lognormal distribution:

$$\log(a, k) \quad \sim \quad \mathcal{N}(\mu, \Sigma).$$

Subscripts (a, k) denote elements. Average ("aggregate") capital is $K = Ek = \exp(\mu_k + \sigma_{kk}/2)$. If we consider changes in σ_{kk}, we'll adjust μ_k to keep K constant. Similarly, average productivity is $A = Ea = \exp(\mu_a + \sigma_{aa}/2)$. Productivity implied by an aggregate version of the production function can be completely different (more on this coming up).

3. Labor demand. Allocate labor to equate marginal products to the wage. Each firm chooses n to maximize $y - wn$ given the production function, the wage w, and (a, k). In this problem, the combination $z = ak^{\alpha}$ works like one term, which is also lognormal. For future reference, note that $\mu_z = \mu_a + \alpha\mu_k$ and $\sigma_{zz} = \sigma_{aa} + \alpha^2\sigma_{kk} + 2\alpha\sigma_{ak}$. This leads to the labor demand schedule

$$n \quad = \quad \left[\frac{(1-\alpha)z}{w}\right]^{1/\alpha}$$

or

$$\log n \quad = \quad (1/\alpha)\left[\log(1-\alpha) + \log z - \log w\right].$$

*Working notes, no guarantee of accuracy or sense.

Since w is constant, $\log n \sim \mathcal{N}(\mu_n, \sigma_{nn})$ with

$$\begin{aligned}\mu_n &= (1/\alpha)\,[\log(1-\alpha) + \mu_z - \log w] \\ \sigma_{nn} &= (1/\alpha)^2 \sigma_{zz}.\end{aligned}$$

Aggregate labor demand is $En = \exp(\mu_n + \sigma_{nn}/2)$, a function of the wage through μ_n.

4. Labor market equilibrium. If labor supply is N, then $N = En$ gives us the wage:

$$(1/\alpha)\log w \;=\; (1/\alpha)\,[\log(1-\alpha) + \mu_z] - \log N + \sigma_{nn}/2.$$

[For later: show how misallocation of capital affects the wage.] A firm's labor usage is therefore

$$\begin{aligned}\log n &= (1/\alpha)\,[\log(1-\alpha) + \log z - \log w] \\ &= \log N + (1/\alpha)\log z - [(1/\alpha)\mu_z + (1/\alpha)^2\sigma_{zz}/2].\end{aligned}$$

[Better arrangement of terms?]

5. Aggregate productivity. Consider an aggregate production function

$$Y \;=\; \widehat{A} K^{\alpha} N^{1-\alpha}.$$

What is Y? $\widehat{A}$? Output for a firm is

$$\begin{aligned}\log y &= \log z + (1-\alpha)\log n \\ &= \log z + (1-\alpha)\log N + [(1-\alpha)/\alpha]\log z - (1-\alpha)[(1/\alpha)\mu_z + (1/\alpha)^2\sigma_{zz}/2] \\ &= (1/\alpha)\log z + (1-\alpha)\log N - (1-\alpha)[(1/\alpha)\mu_z + (1/\alpha)^2\sigma_{zz}/2].\end{aligned}$$

Aggregate output is therefore $Y = Ey = \exp[\mu_z + (1/\alpha)\sigma_{zz}/2]N^{1-\alpha}$. The exponential term can be written

$$\begin{aligned}\mu_z + (1/\alpha)\sigma_{zz}/2 &= \mu_a + \alpha\mu_k + (1/\alpha)(\sigma_{aa} + \alpha^2\sigma_{kk} + 2\alpha\sigma_{ak})/2 \\ &= [\mu_a + (1/\alpha)\sigma_{aa}/2] + \alpha(\mu_k + \sigma_{kk}/2) + \sigma_{ak} \\ &= \log A + \alpha\log K + [(1-\alpha)/\alpha]\sigma_{aa}/2 + \sigma_{ak}.\end{aligned}$$

[Buried in algebra here, not sure about second to last term.] That gives us measured aggregate productivity of

$$\widehat{A} \;=\; Y/(K^{\alpha}N^{1-\alpha}) \;=\; Ae^{[(1-\alpha)/\alpha]\sigma_{aa}/2 + \sigma_{ak}}$$

or

$$\log\widehat{A} \;=\; \mu_a + (1/\alpha)\sigma_{aa}/2 + \sigma_{ak}.$$

That is: measured productivity depends on the dispersion of productivity and how capital is allocated.

The last term (the covariance): With constant returns to scale, what you'd like to do is allocate all the capital (and labor) to the firm with the highest productivity. With a lognormal distribution, that means you want the correlation between a and k to be one and the standard deviation of k as large as possible. The other term shows how α matters: as α approaches one, labor drops of out the model and the term disappears, leaving us with only the covariance (Olley-Pakes, so to speak). As α approaches zero, the model approaches one with perfect allocation and productivity grows with bound.

6. Returns to capital. Misallocation of capital means we're not equating marginal products across firms. Does it also lower returns on average? Conjecture: the average marginal product is less than the marginal product based on the aggregate production function, namely $\alpha Y/K = \alpha \widehat{A}(N/K)^{1-\alpha}$.

We'll look at the marginal product of a firm. Its marginal product is

$$\text{mpk} \;=\; \alpha y/k \;=\; \alpha a(n/k)^{1-\alpha}.$$

Output is (see above)

$$\begin{aligned}\log y &= (1/\alpha)\log z + (1-\alpha)\log N - (1-\alpha)[(1/\alpha)\mu_z + (1/\alpha)^2\sigma_{zz}/2] \\ &= (1/\alpha)\log a + \log k + (1-\alpha)\log N - (1-\alpha)[(1/\alpha)\mu_z + (1/\alpha)^2\sigma_{zz}/2].\end{aligned}$$

For the mpk, we knock out the k term and add $\log \alpha$:

$$\log \text{mpk} \;=\; \log\alpha + (1/\alpha)\log a + (1-\alpha)\log N - (1-\alpha)[(1/\alpha)\mu_z + (1/\alpha)^2\sigma_{zz}/2].$$

Aggregate MPK $= E(\text{mpk})$ is therefore

$$\begin{aligned}\log \text{MPK} &= \log\alpha + (1-\alpha)\log N + (1/\alpha)\mu_a + (1/\alpha)^2\sigma_{aa}/2 - (1-\alpha)[(1/\alpha)\mu_z + (1/\alpha)^2\sigma_{zz}/2] \\ &= \log\alpha + (1-\alpha)\log N + [\mu_a + (1/\alpha)\sigma_{aa}/2] - (1-\alpha)(\mu_k + \sigma_{kk}/2) - [(1-\alpha)/\alpha]\sigma_{ak} \\ &= \log\alpha + \log\widehat{A} + (1-\alpha)(\log N - \log K) - [(1-\alpha)/\alpha]\sigma_{ak}.\end{aligned}$$

The last term shows that the average marginal product is less than the marginal product constructed from the aggregate production function. All subject to getting the algebra right.