1 Model economy

Time is discrete, $t = 0, 1, \ldots$.

1.1 Demographics

The economy is populated by overlapping generations (or cohorts) who live up to I periods. The size of a cohort that was born at $t = g$ and that has reached age h is denoted by $x_{g,h}$. For each individual of a cohort, survival is stochastic and s_h denotes the survival probability for an individual of age h to be still alive at age $h+1$. Therefore, we have $x_{g,h+1} = s_h x_{g,h}$. We write the cumulative probability of an individual of age h to be still alive at age h' as $S_h^{h'} = \prod_{j=h}^{h'-1} s_j$. At each t, a new cohort $x_{t,0}$ of size x_0 is born.

1.2 Preferences

There is one good at every date t, used for consumption and investment. At time t, an individual from generation g with age $\bar{h}$ has time-separable preferences over consumption according to

$$U_{g,\bar{h}} = \sum_{j=\bar{h}}^{I} \beta^{j-\bar{h}} S_h^j u(c_{g,j})$$

where the instantaneous utility function is $u(c_{g,h}) = \frac{c_{g,h}^{1-\sigma}-1}{1-\sigma}$. At time t total consumption in the economy is $c_t = \sum_h c_{t-h,h} x_{t-h,h}$.

Labor supply is fixed, but age-dependend. An individual of age h supplies e_h efficiency units of labor. Total labor supply in the economy at time t is $n_t^s = \sum_h e_h x_{t-h,h}$.

Individuals can hold assets to save or borrow. We denote by $a_{g,h}$ the asset holdings of an individual from generation g of age h. Each individual starts life with zero assets, i.e. ($a_{g,0} = 0$), and cannot have negative assets at the certain death age $I+1$, i.e. ($a_{g,I+1} \geq 0$). The gross return from holding one unit of the asset from period t to $t+1$ is r_t. An individual receives wage w_t per efficiency unit of labor supplied. Remember that in our formulation we have $t = g+h$. Therefore, the budget constraint writes as

$$a_{g,h+1} = (1 + r_{t-1}) a_{g,h} + e_h w_t + b_{g,h} - c_{g,h},$$

where $b_{g,h}$ denotes accidental bequests (to be specified). Aggregate asset holdings in the economy are denoted by $a_t = \sum_h a_{t-h,h} x_{t-h,h}$. With q_t denoting the time zero price of the time t good, we can establish a present value budget constraint:

$$\sum_{j=h}^{I} q_j \left[(1 + r_{g+j-1}) a_{g,j} + e_j w_{g+j} + b_{g,j} - a_{g,j+1} - c_{g,j}\right] \geq 0 \qquad \text{(PV-BC)}$$

1.3 Technology

Output is produced by final good producers with inputs capital and labor according to $y_t = f(k_t, z_t n_t)$. They rent capital from households at interest rate r_t and also bear the depreciation cost δk_t.

1.4 Equilibrium

Definition 1 (Competitive equilibrium)**.** *Define* $\mathscr{M} = \{(g,h)|g \in \{-I, 1, \dots, \infty\}, h \in \{\max[0,-g], 1, \dots, I\}\}$. *Given* k_0 *and* $\{x_{g,h}\}_{(g,h)\in\mathscr{M}}$*, a competitive equilibrium consists of an allocation* $\{c_{g,h}, a_{g,h}\}_{(g,h)\in\mathscr{M}}$*,* $\{k_t\}_{t=1}^{\infty}$*,* $\{n_t\}_{t=0}^{\infty}$*, and prices* $\{r_t, w_t\}_{t=0}^{\infty}$ *such that*

1. *For each individual in cohort* g *and is now of age* $\bar{h}$*,* $\{c_{g,h}, a_{g,h}\}_{\bar{h}=\max[0,-g]}^{I}$ *solves*

$$\max_{\{\tilde{c}_{g,h},\tilde{a}_{g,h}\}_{\bar{h}=\max[0,-g]}^{I}} \sum_{j=\bar{h}}^{I} \beta^{j-\bar{h}} S_{\bar{h}}^{j} u(\tilde{c}_{g,j})$$

s.t. $\forall h \in \{\max[0,-g], 1, \dots, I\}$

$$\sum_{j=\bar{h}}^{I} q_j \left[(1+r_{g+j-1})\tilde{a}_{g,j} + e_j w_{g+j} + b_{g,j} - \tilde{a}_{g,j+1} - \tilde{c}_{g,j}\right] \geq 0 \qquad (\lambda_g)$$

$$\tilde{a}_{g,0} = 0,\ \tilde{a}_{g,I+1} \geq 0$$

where λ_g *is the Langrange multiplier on the budget constraint for cohort* g*. Notice that we use* $g + h = t$ *in the individual's problem.*

Aggregation over individuals and cohorts gives

$$n_t^s = \sum_h e_h x_{t-h,h}$$
$$a_t = \sum_h a_{t-h,h} x_{t-h,h}$$
$$c_t = \sum_h c_{t-h,h} x_{t-h,h}$$

2. *For the final goods producers* $\{n_t\}_{t=0}^{\infty}$ *and* $\{k_t\}_{t=0}^{\infty}$ *solve at any* t

$$\max_{\tilde{n}_t,\tilde{k}_t} f(\tilde{k}_t, z_t\tilde{n}_t) - (r_t + \delta)\tilde{k}_t - w_t\tilde{n}_t \qquad (1)$$

3. *All markets clear, i.e.,* $\forall t$

   (i) Goods: $y_t = c_t + i_t$

   (ii) Assets: $a_t = k_t$

   (iii) Labor: $n_t^s = n_t$

1.5 Solving the individual's problem

The Lagrangian for an individual from cohort g who is now of age $\bar{h}$ writes as

$$\mathscr{L} = \sum_{j=\bar{h}}^{I} \beta^{j-\bar{h}} S_{\bar{h}}^{j} u(c_{g,j}) + \lambda_g \sum_{j=\bar{h}}^{I} q_{g+j} \left[(1+r_{g+j-1}) a_{g,j} + e_j w_{g+j} + b_{g,j} - a_{g,j+1} - c_{g,j} \right] \tag{2}$$

The first order condition for cohort g's consumption at age h, $c_{g,h}$ is

$$\frac{\beta^{h-\bar{h}} S_{\bar{h}}^{h} c_{g,h}^{-\sigma}}{\lambda_g} = q_{g+h} \tag{3}$$

The first order condition with respect to asset holdings $a_{g,h+1}$ is

$$-q_{g+h} + (1+r_{g+h}) q_{g+h+1} = 0$$

Combining the two FOCs gives the consumption Euler equation

$$\beta s_h (1+r_{g+h}) \left(\frac{c_{g,h+1}}{c_{g,h}} \right)^{-\sigma} = 1 \tag{EE}$$

More general, we have a relation between consumption at age h relative to consumption at base age $\bar{h}$:

$$c_{g,h} = c_{g,\bar{h}} \left(\beta^{h-\bar{h}} S_{\bar{h}}^{h} \prod_{j=\bar{h}}^{h-1} (1+r_{g+j}) \right)^{\frac{1}{\sigma}} \tag{4}$$

Now we plug (3) and (4) into the present value budget constraint (PV-BC) which will be binding. Asset holdings cancel out except for $a_{g,\bar{h}}$ and $a_{g,I}$, where the latter will be zero. We obtain

$$\frac{c_{g,\bar{h}}^{-\sigma}}{\lambda_g} \left\{ a_{g,\bar{h}} + \sum_{l=\bar{h}}^{I} \frac{e_l w_{g+l} + b_{g,l} - c_{g,\bar{h}} \left(\beta^{l-\bar{h}} S_{\bar{h}}^{l} \prod_{j=\bar{h}}^{l-1} (1+r_{g+j}) \right)^{\frac{1}{\sigma}}}{\prod_{j=\bar{h}}^{l-1} (1+r_{g+j})} \right\} = 0 \tag{5}$$

Because of our choice of the utility function $c_{g,h} > 0 \ \forall h$, and with (PV-BC) binding, $\lambda_g > 0$ usually holds. Then by solving for $c_{g,\bar{h}}$ and by virtue of (4) we have a closed form expression for the entire consumption path (for a given path of prices):

$$c_{g,h} = \left\{ \sum_{l=\bar{h}}^{I} \frac{\left(\beta^{l-h} \frac{S_{\bar{h}}^{l}}{S_{\bar{h}}^{h}} \frac{\prod_{j=\bar{h}}^{l-1} (1+r_{g+j})}{\prod_{j=\bar{h}}^{h-1} (1+r_{g+j})} \right)^{\frac{1}{\sigma}}}{\prod_{j=\bar{h}}^{l-1} (1+r_{g+j})} \right\}^{-1} \left\{ a_{g,\bar{h}} + \sum_{l=\bar{h}}^{I} \frac{e_l w_{g+l} + b_{g,l}}{\prod_{j=\bar{h}}^{l-1} (1+r_{g+j})} \right\}, \ \forall h \in \{\bar{h},, \ldots, I\} \tag{6}$$

The second curly bracket is the present value of the individual's lifetime income. The first curly bracket is a function of prices and survival probabilities (and the discount factor) and shows which fraction of the lifetime income will be consumed in a given period.

1.6 Solving the firms problems

For the final goods producers the problem is static. The first order conditions are $f_1(.) \equiv \frac{\partial f(.)}{\partial k_t} = r_t + \delta$ and $f_2(.) \equiv \frac{\partial f(.)}{\partial n_t} = w_t$.

1.7 Steady state

Definition 2 (Steady state)**.** *A steady state is a competitive equilibrium in which*

1. *the demographic structure is constant/ time-invariant, i.e.* $x_{g,h} = \bar{x}_h$, $\forall g$*,*
2. *labor productivity is constant, i.e.,* $z_t = \bar{z}$, $\forall t$*,*
3. *the capital-output ratio is constant, i.e.,* $\frac{k_t}{y_t} = \frac{\bar{k}}{\bar{y}}$, $\forall t$*.* *Or should rather capital be constant? I guess the two are synon*

The first order conditions for the final goods producers imply - taking into account the CRS property of the production function - that the interest rate and the wage are constant in steady state. Consequently, if we let each cohort start with zero assets, i.e., $a_{g,0} = 0$ $\forall g$, the paths for consumption and asset holdings are identical for each cohort. Since the population shares $\bar{x}_h$ are time-invariant, it follows that aggregate asset holdings (and aggregate consumption) are constant for all t. The market clearing condition for assets tells us subsequently that capital is constant in steady state.