COMMUNICATIONS ON PURE AND APPLIED MATHEMATICS, VOL. XI, 573–586 (1958)

On Positive Eigenvectors of Positive Infinite Matrices*

TOSIO KATO

University of Tokyo

Introduction

According to a classical theorem of Perron and Frobenius [1, 8], a finite, positive square matrix A has one and only one eigenvalue with a positive[1] eigenvector, and this eigenvalue is larger than the absolute value of any other eigenvalue of A. This theorem is not true for a general infinite positive matrix. On the one hand such a matrix need not have any eigenvalue at all, as is seen from the example of the matrix with all its elements equal to unity. On the other hand such a matrix can have a continuum of positive eigenvalues with positive eigenvectors. This is seen from a recent result of Rosenblum [9], who has shown that the (generalized) Hilbert matrix[2]

$$H_\theta = \left((i+k+2\theta)^{-1}\right), \qquad i, k = 0, 1, 2, \cdots, \quad \theta > 0, \tag{0.1}$$

has all complex numbers λ with $\mathscr{Re}\,\lambda > 0$ as eigenvalues and, in particular, that every real eigenvalue $\lambda \geqq \pi$ has a positive eigenvector.

Rosenblum has obtained these results by solving the eigenvalue problem for the Hilbert matrix explicitly in terms of certain integrals involving Whittaker functions. Thus his method is not easily applicable to other matrices even those differing slightly from the Hilbert matrix.

The object of the present paper is to show that there is a certain class of positive matrices, including the Hilbert matrix as a special case, having a continuum of positive eigenvalues with positive eigenvectors.[3]

It should be remarked that the analogues of these results have long been

*This paper was written while the author was at the Institute of Mathematical Sciences. The work was supported in part by the U. S. Air Force, through the Air Force Office of Scientific Research of the Air Research and Development Command under Contract No. AF 18(600)-367.

[1]A vector or a matrix is here said to be positive if all its elements are positive. This is to be distinguished from the notion "positive-definite" applied to Hermitian matrices.

[2]The notation is slightly different from that of the author's previous paper [5].

[3]In the previous paper [5] we proved that $\lambda = \pi$ is an eigenvalue with a positive eigenvector of the Hilbert matrix H_θ if $\theta \geqq 1/4$. The present paper contains an independent proof that every $\lambda \geqq \pi$ has this property, see Examples 2 and 5 below.

known for integral equations with a positive kernel. The Perron-Frobenius theorem holds for a certain class of completely continuous kernels (Jentsch [4])[4] whereas the existence of a continuum of positive eigenfunctions has been proved by Hopf [3] for certain types of singular kernels. The method of Hopf is based exclusively on a positivity argument, and as such can be applied to matrix problems without modification. It appears, however, that there are some practical inconveniences in applying his method to matrices, owing to the simple fact that infinite series are often harder to handle than integrals. In particular it is usually difficult to find the two "trial vectors" which are needed to start the iteration procedure used in constructing the eigenvector for a given eigenvalue.

In the method employed below we also resort to an iteration procedure, but we need only one trial vector, the convergence of the iteration being proved by an elementary theorem on linear operators of Banach space. We do not even have to assume that the matrix or the eigenvalue is positive. However, this simplicity is gained at the cost of having to estimate the bound of the matrix regarded as a linear operator in the Banach space under consideration; this estimate is practicable only under certain additional assumptions.[5]

For convenience we give here a brief account of the definitions and conventions to be used. We consider infinite vectors $x = (x(i))$ and infinite matrices $A = (a(i, k))$ with complex elements. *The indices i and k always take the values* $0, 1, 2, \cdots$. The transposed matrix of A is denoted by A'. The linear operations $\alpha x+\beta y$ on vectors and $\alpha A+\beta B$ on matrices, where α, β are complex numbers, are defined in the usual way. The product Ax of a matrix A and a vector x is defined if and only if the series

$$y(i) = \sum_{k=0}^{\infty} a(i, k)x(k) \tag{0.2}$$

is convergent for each index i; then we set $Ax = y = (y(i))$ by definition. An iterated product such as $ABCx$ is to be defined as equal to $A[B(Cx)]$ provided the latter is meaningful. In particular the $A^n x$, $n = 1, 2, 3, \cdots$, are defined successively by $A^n x = A(A^{n-1}x)$ as long as this is possible. We do not make use of the product AB of two matrices. A vector $x \neq 0$ is an *eigenvector* of a matrix A for the *eigenvalue* λ if and only if Ax exists and is equal to λx.

In general we do not require that the vectors x have a finite "norm" of any kind. But we shall often make use of the *p-norm*

---

[4]See also Krein and Rutman [6].

[5]It should be noted that our method is also applicable to integral equations; again the situation is much simpler there than in the case of matrices.

$$(0.3) \qquad ||x||_p = [\sum_i |x(i)|^p]^{1/p}, \qquad ||x||_\infty = \sup_i |x(i)|,$$

where $1 \leqq p \leqq \infty$. As is well-known,[6] $p < q$ implies $||x||_p \geqq ||x||_q$. The Banach space consisting of all vectors x with $||x||_p$ finite is as usual denoted by l_p. We have $l_p \subset l_q$ for $p < q$. For convenience we denote by l_{p+0} the intersection of all l_q with $q > p$; thus $x \in l_{p+0}$ means that $x \in l_q$ for all $q > p$. Obviously we have $l_p \subset l_{p+0}$. A sequence $\{x_n\}$ of vectors will be said to converge to x in the p-norm (p-convergence) if $||x_n - x||_p \to 0$ for $n \to \infty$; *here x_n and x need not belong to l_p separately.* Obviously the p-convergence implies the component-wise convergence.

A matrix A may (or may not) define a bounded linear operator on l_p into itself. This is the case if Ax is defined for every $x \in l_p$ and satisfies the inequality

$$(0.4) \qquad ||Ax||_p \leqq M\,||x||_p$$

with a finite constant M. In such a case we shall say that the matrix A is *p-bounded*. The smallest number M with this property is called the *p-norm* of A and denoted by $||A||_p$. As is well known A is p-bounded, $1 \leqq p < \infty$, if (0.4) is true for every x with only a finite number of non-vanishing components. We also have (see footnote 6)

$$(0.5) \qquad ||A||_p = ||A'||_{p'} \quad \text{for } p^{-1} + p'^{-1} = 1, \qquad 1 < p < \infty.$$

A vector or a matrix is said to be *positive* (respectively *non-negative*) if all its elements are positive (respectively non-negative). We shall write $x \leqq y$ or $A \leqq B$ if $y - x$ or $B - A$ is non-negative.

1. Theorems on Existence of Extra-Bound Eigenvalues

The following is our basic theorem.[7]

THEOREM 1. *Let A be a q-bounded matrix for some q, $1 \leqq q \leqq \infty$. Let there exist a vector $x^0 \notin l_q$ and a complex number λ such that $|\lambda| > ||A||_q$, Ax^0 exists and $Ax^0 - \lambda x^0 \in l_q$. Then λ is an eigenvalue of A. An associated eigenvector x can be constructed by iteration in the following way: the vectors*

$$(1.1) \qquad x^n = \lambda^{-n} A^n x^0, \qquad n = 1, 2, 3, \cdots,$$

exist and we have

$$(1.2) \qquad \lim_{n\to\infty} x^n = x \notin l_q, \qquad x - x^0 \in l_q, \qquad Ax = \lambda x,$$

where the limit exists in the sense of q-convergence. If, in particular, A, λ and x^0

[6] See Hardy, Littlewood and Polya [2].

[7] Actually Theorem 1 could be given a more abstract form. It is obvious from the proof given below how such a generalization should be formulated.

are positive (respectively non-negative), the eigenvector x is also positive (respectively non-negative).

Remark. For convenience λ will be called an *extra-bound eigenvalue* of A because $|\lambda| > ||A||_q$. It should be noticed that such an eigenvalue does not exist for a finite matrix A. We note also that, when A, λ and x^0 are positive, we can employ a different normalization for the approximating vectors x^n. For instance, on setting

$$y^n = [A^n x^0(0)]^{-1} A^n x^0 = x^n(0)^{-1} x^n, \qquad y = x(0)^{-1} x, \tag{1.3}$$

we have

$$y^n(0) = y(0) = 1, \qquad \lim y^n = y, \qquad Ay = \lambda y, \tag{1.4}$$

where the limit exists also in the sense of q-convergence. This follows directly from the fact that $\lim x^n(0) = x(0) > 0$.

Proof of Theorem 1: Since A is q-bounded and $z = Ax^0 - \lambda x^0$ belongs to l_q, Az exists and belongs to l_q. Hence $A^2x^0 = Az + \lambda Ax^0$ exists and $Az = A^2x^0 - \lambda Ax^0$. Repeating the same argument, we see that all $A^n x^0$, $n = 1, 2, 3, \cdots$, exist. We now have

$$x^n - x^0 = \lambda^{-n} A^n x^0 - x^0 = \lambda^{-1} \sum_{k=0}^{n-1} \lambda^{-k} A^k z.$$

Since $z \in l_q$ and $|\lambda| > ||A||_q$ by hypothesis, the Neumann series $u = \sum_{k=0}^{\infty} \lambda^{-k} A^k z$ is convergent in the q-norm and the sum $u \in l_q$ satisfies the equation $u - \lambda^{-1} Au = z$. Consequently $\lim x^n = x$ exists in the sense of q-convergence and $x - x^0 = \lambda^{-1} u$. Then Ax exists and

$$Ax = Ax^0 + \lambda^{-1} Au = Ax^0 - z + u = \lambda x^0 + u = \lambda x.$$

Since $x^0 \notin l_q$ and $x - x^0 = \lambda^{-1} u \in l_q$, x does not belong to l_q; in particular $x \neq 0$ and x is an eigenvector of A for the eigenvalue λ.

If A, λ and x^0 are non-negative (actually this implies that λ is positive because $\lambda > ||A||_q$), x^n is non-negative for every n. Hence the limit x is likewise non-negative. If in addition A is positive, $x = \lambda^{-1} Ax$ is positive. This completes the proof.

Theorem 1 is concerned with extra-bound eigenvalues of the matrix A. The following theorem and its corollary show that essentially all positive eigenvectors must have extra-bound eigenvalues when the matrix A is *symmetric* and non-negative.

THEOREM 2. *Let A be a non-negative, symmetric matrix and let there exist a positive vector x such that $Ax \leqq \alpha x$ for some real number $\alpha < \infty$. Then A is 2-bounded with $||A||_2 \leqq \alpha$.*

Proof: It is sufficient to show that $||z||_2 \leqq \alpha ||y||_2$, where $z = Ay$ and y

is a vector with only a finite number of non-vanishing components. We have by the Cauchy inequality

$$
\begin{aligned}
|z(i)|^2 &\leqq [\sum_k a(i,k)\,|y(k)|]^2 \\
&\leqq [\sum_k a(i,k)x(k)][\sum_k a(i,k)\,|x(k)|^{-1}|y(k)|^2] \\
&\leqq \alpha x(i) \sum_k a(i,k)|x(k)|^{-1}|y(k)|^2, \\
\|z\|_2^2 = \sum_i |z(i)|^2 &\leqq \alpha \sum_k |x(k)|^{-1}|y(k)|^2 \sum_i a(i,k)x(i) \\
&\leqq \alpha^2 \sum_k |y(k)|^2 = \alpha^2 \|y\|_2^2 .
\end{aligned}
$$

COROLLARY. *A non-negative, symmetric matrix A with a positive eigenvector is necessarily 2-bounded. Any eigenvalue λ of A with a positive eigenvector x cannot be smaller than $\|A\|_2$, and x does not belong to l_2 unless $\lambda = \|A\|_2$.*

2. Matrices With a Homogeneous Principal Part

Applying Theorem 1 we shall now establish the existence of eigenvalues of a certain class of matrices including the Hilbert matrix (0.1) as a special case. We consider matrices of the form

$$A = B - C, \tag{2.1}$$

where B is the principal part of A and C is a small "perturbation" in a sense to be described below.

The elements $b(i, k)$ of the principal part B are assumed to be given by

$$b(i,k) = f(i+\theta, k+\theta), \quad \theta > 0, \quad i, k = 0, 1, 2, \cdots, \tag{2.2}$$

where $f(u, v)$ is a complex-valued function of two real variables defined for $u, v > 0$ and satisfying the following conditions:[8]

(i) $f(u, v)$ is positive-homogeneous of degree -1, that is,

$$f(tu, tv) = t^{-1} f(u, v), \qquad t > 0,$$

(ii) $f(t, 1)$ and $f(1, t)$ are of bounded variation[9] over the closed interval $[0, \infty]$ of t.

A direct consequence of these two conditions is that $f(t, 1)$, $f(1, t)$, $tf(1, t) = f(t^{-1}, 1)$, $tf(t, 1) = f(1, t^{-1})$ are all of bounded variation and hence, in particular, bounded.

The assumptions on the perturbing term C will be introduced later.

---

[8]Similar matrices are considered in Hardy, Littlewood and Polya [2], p. 227, for different purposes.

[9]This condition could be weakened, but we assume (ii) to avoid unnecessary complications.

The main result to be proved in this section is that under some additional conditions the matrix A has the eigenvalue

$$\lambda_\gamma = \int_0^\infty f(1, t)t^{-\gamma}dt = \int_0^\infty f(t, 1)t^{\gamma-1}dt, \qquad 0 < \mathcal{R}e\,\gamma < 1. \tag{2.3}$$

Here the two integrals are absolutely convergent by the remark above. Their equality is a consequence of condition (i).

THEOREM 3. *Let $A = B-C$ be a matrix, where $B = (b(i,k))$ is given by* (2.2) *with f satisfying the conditions* (i) *and* (ii) *and where $C = (c(i,k))$ is a matrix which transforms every $x \in l_{p+0}$ into $Cx \in l_q$ for some constants p, q such that $1 < q \leqq p < \infty$. Let γ be a complex number with $\mathcal{R}e\,\gamma = p^{-1}$ and $\|A\|_q < |\lambda_\gamma|$, where λ_γ is given by* (2.3). *Then $\lambda = \lambda_\gamma$ is an eigenvalue of A with an eigenvector $x = x_\lambda$ belonging to l_{p+0} but not to l_p. If in addition A and λ_γ are positive* (*respectively non-negative*) *and γ is real, the eigenvector x_λ can be chosen positive* (*respectively non-negative*).

Remark. We could give several sufficiency conditions for C to map l_{p+0} into l_q. Here we will mention only the following condition (iii), the sufficiency of which can easily be proved by Hölder's inequality:

(iii) $\sum_i (\sum_k |c(i, k)|^r)^{q/r} < \infty$ for some $r < p' = p(p-1)^{-1}$.

Proof of Theorem 3: Set

$$x^0 = (x^0(i)), \qquad x^0(i) = (i+\theta)^{-\gamma}, \qquad i = 0, 1, 2, \cdots; \tag{2.4}$$

x^0 does not belong to l_q because $q\,\mathcal{R}e\,\gamma = qp^{-1} \leqq 1$. Thus Theorem 3 follows from Theorem 1 if one can show that $Ax^0-\lambda_\gamma x^0$ exists and belongs to l_q. It should be recalled that the eigenvector x constructed in Theorem 1 has the property that $x-x^0 \in l_q$. Since $x^0 \in l_{p+0}$ and $x^0 \notin l_q$ because $p\,\mathcal{R}e\,\gamma = 1$, and since $p \geqq q$, this implies that $x \in l_{p+0}$ and $x \notin l_p$.

Cx^0 belongs to l_q because $x^0 \in l_{p+0}$. In order to show that $Ax^0-\lambda_\gamma x^0 \in l_q$, it is therefore sufficient to prove that $Bx^0-\lambda_\gamma x^0 \in l_q$. To this end we construct the partial sums of the series which defines the components of $y^0 = Bx^0$ obtaining

$$y_n^0(i) = \sum_{k=0}^{n-1} b(i, k)x^0(k) = \sum_{k=0}^{n-1} f(i+\theta, k+\theta)(k+\theta)^{-\gamma}.$$

Since by (i), $f(u, v)$ is homogeneous of degree -1 we have

$$(i+\theta)^\gamma y_n^0(i) = \sum_{k=0}^{n-1} \frac{1}{i+\theta} f\left(1, \frac{k+\theta}{i+\theta}\right)\left(\frac{k+\theta}{i+\theta}\right)^{-\gamma}. \tag{2.5}$$

The right side is an approximating sum for the integral $\int_{t_0}^{t_n} f(1,t)t^{-\gamma}dt$ constructed with the mesh points $t_k = (k+\theta)(i+\theta)^{-1}$, $k = 0, 1, 2, \cdots, n$, having a constant mesh length $h = (i+\theta)^{-1}$. Such a sum can be estimated by means of the formula

$$\left| \sum_{k=0}^{n-1} hg(t_k) - \int_{t_0}^{t_n} g(t)dt \right| \leqq hV[g; t_0, t_n], \tag{2.6}$$

where $V[g; t_0, t_n]$ is the total variation of the function $g(t)$ over the interval $[t_0, t_n]$.

As is easily seen, the total variation of the function $g(t) = f(1, t)t^{-\gamma}$ over $[t_0, t_n]$ does not exceed $t_0^{-\alpha}(\alpha^{-1}|\gamma|M+V)$, where $\alpha = \mathcal{Re}\,\gamma = p^{-1}$, $M = \sup_t |f(1, t)|$ and $V = V[f; 0, \infty]$. Substituting $t_0 = \theta(i+\theta)^{-1}$ and $h = (i+\theta)^{-1}$, we thus obtain from (2.5) and (2.6)

$$\left| (i+\theta)^{\gamma} y_n^0(i) - \int_{t_0}^{t_n} f(1, t)t^{-\gamma}\, dt \right| \leqq M_1(i+\theta)^{\alpha-1}, \tag{2.7}$$

where M_1 is a constant which may depend on θ and γ but which is independent of n or i.

Letting $n \to \infty$ in (2.7) and noting that the integral (2.3) exists, we see that $\lim_{n\to\infty} y_n^0(i) = y^0(i) = Bx^0(i)$ exists and satisfies the inequality

$$|(i+\theta)^{\gamma} y^0(i) - \lambda_\gamma| \leqq \left| \int_0^{t_0} f(1, t)t^{-\gamma}\, dt \right| + M_1(i+\theta)^{\alpha-1} \leqq M_2(i+\theta)^{\alpha-1},$$

where M_2 is a constant independent of i. In virtue of (2.4), this implies that $|Bx^0(i) - \lambda_\gamma x^0(i)| \leqq M_2(i+\theta)^{-1}$ yielding the required result that $Bx^0 - \lambda_\gamma x^0 \epsilon\, l_q$.

3. Matrices With a Continuum of Positive Eigenvectors

In order to apply Theorem 3 to a given matrix, we need some estimate for $\|A\|_q$. In general such an estimate must be sought independently. In certain cases, however, it is possible to estimate $\|A\|_q$ in terms of the quantity λ_γ itself. The present section is devoted to the investigation of such a case.

Suppose that the function $f(u, v)$ introduced in the preceding section satisfies one of the following additional conditions:

(iv) $f(u, v) \geqq 0$ and both $f(t, 1)$ and $f(1, t)$ are non-increasing for $0 < t < \infty$,

(iv′) $f(u, v) \geqq 0$ and both $f(t, 1)$ and $f(1, t)$ are convex functions of t for $0 < t < \infty$.

It should be noticed that conditions (i) and (iv) imply condition (ii) and that $f(u, v)$ vanishes nowhere unless f is identically zero. Similarly, conditions (i), (ii) and (iv′) imply condition (iv) and hence again that $f(u, v) \neq 0$ unless $f \equiv 0$. In what follows we shall exclude the trivial case $f \equiv 0$.

We can now prove the following lemmas.

LEMMA 1. *Let B be a matrix with the elements* (2.2), *where $\theta \geqq 1$ and f is a function satisfying conditions* (i) *and* (iv). *Then $\|B\|_p \leqq \lambda_{1/p}$ for every p, $1 < p < \infty$, where $\lambda_{1/p}$ is given by* (2.3).

LEMMA 1′. *The conclusion of Lemma* 1 *is true already if $\theta \geqq 1/2$ provided f satisfies conditions* (i), (ii) *and* (iv′).

Proof: Set $x = (x(i))$ and $Bx = y = (y(i))$. We have by the Hölder inequality (writing $\alpha = p^{-1}$)

$$
\begin{aligned}
|y(i)| &\leqq \sum_k b(i,k)|x(k)| \\
&= \sum_k b(i,k)^{1-\alpha}\left(\frac{i+\theta}{k+\theta}\right)^{\alpha(1-\alpha)} b(i,k)^{\alpha}\left(\frac{k+\theta}{i+\theta}\right)^{\alpha(1-\alpha)}|x(k)| \qquad (3.1)\\
&\leqq \left[\sum_k b(i,k)\left(\frac{i+\theta}{k+\theta}\right)^{\alpha}\right]^{1-\alpha}\left[\sum_k b(i,k)\left(\frac{k+\theta}{i+\theta}\right)^{1-\alpha}|x(k)|^p\right]^{\alpha}.
\end{aligned}
$$

We shall show that the sum of the series in the first brackets on the right does not exceed λ_α. This series is the same approximating sum for the integral

$$\int_0^\infty g(t)dt = \lambda_\alpha \qquad \text{where } g(t) = f(1,t)t^{-\alpha} \tag{3.2}$$

which appeared in the proof of Theorem 3. This time we note that

$$hg(t_k) \leqq \int_{t_k-h}^{t_k} g(t)dt \tag{3.3}$$

because $g(t)$ is non-increasing (see the remark above). Thus the sum of the series in question does not exceed (3.2) since $t_0-h = (\theta-1)(i+\theta)^{-1} \geqq 0$ in virtue of $\theta \geqq 1$ (in the case of Lemma 1). If the stronger condition (iv′) is assumed (Lemma 1′), $g(t)$ is a convex, decreasing function so that (3.3) can be replaced by

$$hg(t_k) \leqq \int_{t_k-\frac{h}{2}}^{t_k+\frac{h}{2}} g(t)dt.$$

Hence the sum in question does not exceed (3.2) even for $\theta > 1/2$ and we have

$$|y(i)|^p \leqq \lambda_\alpha^{(1-\alpha)p}\sum_k b(i,k)\left(\frac{k+\theta}{i+\theta}\right)^{1-\alpha}|x(k)|^p,$$

$$\|y\|_p^p = \sum_i |y(i)|^p \leqq \lambda_\alpha^{p-1}\sum_k |x(k)|^p \sum_i b(i,k)\left(\frac{k+\theta}{i+\theta}\right)^{1-\alpha}.$$

The series with the index i on the right is the approximating sum for the integral $\int f(t,1)t^{\alpha-1}dt = \lambda_\alpha$ (see (2.3)) and again does not exceed λ_α for the same reason as above. Hence

$$||y||_p^p \leqq \lambda_\alpha^{p-1}\lambda_\alpha ||x||_p^p = \lambda_\alpha^p ||x||_p^p,$$

which gives the required result $||Bx||_p = ||y||_p \leqq \lambda_\alpha ||x||_p$.

It is obvious that these lemmas can be used in estimating the bound $||A||_q$ needed in the application of Theorem 3.

In what follows we shall deduce a theorem which is rather special but convenient for applications.

Before stating the theorem we note the following simple consequences of the assumption $f(u, v) > 0$. It follows from (2.3) that $\lambda_\alpha > 0$ for $0 < \alpha < 1$, and that λ_α is a strictly convex function of α tending to infinity for $\alpha \to 0$ as well as $\alpha \to 1$. Thus λ_α takes on its minimum value ω at a unique point $\alpha = \alpha_0$, $0 < \alpha_0 < 1$. We set $p_0 = \alpha_0^{-1}$.

We can now state

THEOREM 4. *Let $B = (b(i,k))$ be a matrix with the elements given by* (2.2) *where $\theta > 1$ and $f(u, v) > 0$ is a function satisfying conditions* (i) *and* (iv). *Let $C = (c(i,k))$ be a matrix such that $0 \leqq C \leqq 2B$ and $\sum_{i,k} |c(i,k)|^r < \infty$ for every $r > 1$. Let λ_γ be defined by* (2.3) *and let ω, α_0 be defined as above. Then each λ_γ such that $|\lambda_\gamma| > \omega$ is an eigenvalue of either $A = B-C$ or its transpose A' according as $\mathscr{Re}\,\gamma \leqq \alpha_0$ or $\mathscr{Re}\,\gamma \geqq \alpha_0$, and there exists an associated eigenvector belonging to l_{p+0} but not to l_p, where $p^{-1} = \mathscr{Re}\,\gamma$ or $p^{-1} = 1-\mathscr{Re}\,\gamma$, respectively. In particular, every real number*[10] *$\lambda > \omega$ is an eigenvalue both for A and A', and the associated eigenvectors can be chosen positive if A is positive.*

THEOREM 4′. *Let $f(u, v)$ of Theorem 4 satisfy the additional condition* (iv′), *while the constant θ is only required to satisfy $\theta > 1/2$. Then the conclusions of Theorem 4 remain true.*

Remark. The assumptions on C in these theorems are satisfied if $0 \leqq C \leqq 2B$ and if C has only a finite number of non-vanishing elements. In other words, the theorems are true for the matrix A which arises from B by replacing a finite number of elements $b(i, k)$ by elements $b_1(i, k)$ such that $|b_1(i, k)| \leqq b(i, k)$.

Proof of Theorems 4 and 4′: From the hypothesis it follows that $-B \leqq A \leqq B$, and this implies that $||A||_q \leqq ||B||_q$ for any q. Lemma 1 (or 1′) shows that $||B||_{p_0} \leqq \lambda_{\alpha_0} = \omega$ where $p_0^{-1} = \alpha_0$, hence $||A||_{p_0} \leqq \omega$.

Suppose now that $|\lambda_\gamma| > \omega$ for some complex number γ, $0 < \mathscr{Re}\,\gamma < 1$. We see from what we just proved that $|\lambda_\gamma| > ||A||_{p_0}$. On the other hand, it follows from the hypothesis that C satisfies condition (iii) for any p and q such that $1 < p, q < \infty$. Thus Theorem 3 is applicable to our A with the constants $p = (\mathscr{Re}\,\gamma)^{-1}$ and $q = p_0 = \alpha_0^{-1}$ provided that $p \geqq p_0$, that is,

[10]It is not known whether in general $\lambda = \omega$ is an eigenvalue of A. But, under certain additional assumptions (see Theorem 5), this can be proved.

provided $\mathscr{Re}\,\gamma \leqq \alpha_0$, leading immediately to the results stated in the theorems. If on the other hand $\mathscr{Re}\,\gamma \geqq \alpha_0$, we have $\|A'\|_{p_0'} = \|A\|_{p_0} < |\lambda_\gamma|$ where $p_0^{-1}+p_0'^{-1} = 1$ (see (0.5)). Thus Theorem 3 can be applied to the matrix $A' = B'-C'$ with $p^{-1} = 1-\mathscr{Re}\,\gamma$ and $q = p_0'$; it is important to note here that λ_γ is changed into $\lambda_{1-\gamma}$ when B is replaced by B'. The last statement of Theorem 4 follows from the fact that any $\lambda > \omega$ can be put in the form $\lambda = \lambda_\alpha$ with two different real values α, one in the interval $(0, \alpha_0)$ and the other in $(\alpha_0, 1)$.

Let us now apply the foregoing theorems to some simple examples.

Example 1. Set

$$f(u, v) = [(au)^\rho + (bv)^\rho]^{-1/\rho}, \qquad a, b, \rho > 0. \tag{3.4}$$

It is easy to verify that f satisfies conditions (i), (ii) and (iv). The integral (2.3) gives

$$\lambda_\gamma = a^{-\gamma} b^{\gamma-1} \frac{\Gamma\left(\frac{\gamma}{\rho}\right)\Gamma\left(\frac{1-\gamma}{\rho}\right)}{\rho\,\Gamma\left(\frac{1}{\rho}\right)}. \tag{3.5}$$

Let ω be the minimum of λ_α for $0 < \alpha < 1$. Theorem 4 shows that any number $\lambda > \omega$ is an eigenvalue with a positive eigenvector both for the matrix B with the elements

$$b(i, k) = [a^\rho(i+\theta)^\rho + b^\rho(k+\theta)^\rho]^{-1/\rho}, \qquad \theta \geqq 1, \tag{3.6}$$

and its transpose B'. The same is true if, for example, a finite number of elements $b(i, k)$ are replaced by smaller non-negative numbers. If $\rho \leqq 1$, then f also satisfies condition (iv'). According to Theorem 4', the restriction $\theta \geqq 1$ can then be weakened to $\theta \geqq 1/2$.

Example 2. In the special case $a = b = \rho = 1$, the B of Example 1 reduces to the Hilbert matrix (0.1); (3.5) becomes $\lambda_\gamma = \pi/\sin \pi\gamma$ and its minimum for real γ is $\omega = \pi$. Hence any number $\lambda > \pi$ is an eigenvalue of the Hilbert matrix (0.1) for $\theta \geqq 1/2$ having a positive eigenvector. Actually this is true even for $\theta \geqq 1/4$, which is a direct consequence of Theorem 3 because it is known[11] that $\|B\|_2 = \pi$ for $\theta \geqq 1/4$. The matrix B may be modified without losing these properties in a manner described in Example 1.

On the other hand B cannot have any positive eigenvector with an eigenvalue $\lambda < \pi$; this follows immediately from the Corollary to Theorem 2 in virtue of the fact that $\|B\|_2 \geqq \pi$ for any $\theta > 0$. Actually it is known (see footnote 11) that $\|B\|_2 = \pi/\sin 2\pi\theta$ for $0 < \theta \leqq 1/4$. Thus in this case B has no positive eigenvector with the eigenvalue $\lambda < \pi/\sin 2\pi\theta$.

[11] See Schur [10] and Magnus [7].

It should be noted that Theorem 3 also gives some information on non-real eigenvalues of B. In fact, the function $\gamma \to \lambda_\gamma = \pi/\sin \pi\gamma$ maps the strip $0 < \mathcal{R}e\, \gamma < 1$ onto the half-plane $\mathcal{R}e\, \lambda > 0$. Theorem 3 implies that any complex number λ with $\mathcal{R}e\, \lambda > 0$ and $|\lambda| > \pi = \| B \|_2$ is an eigenvalue of B for $\theta \geqq 1/4$.

Example 3. The limiting case $\rho \to \infty$ with $a = b = 1$ in Example 1 gives

$$b(i, k) = \frac{1}{\max\,(i+\theta, k+\theta)}, \qquad \theta \geqq 1, \tag{3.7}$$

$$\lambda_\gamma = \frac{1}{\gamma(1-\gamma)}, \qquad \omega = 4. \tag{3.8}$$

Thus any number $\lambda > 4$ is an eigenvalue of the matrix B given by (3.7) having a positive eigenvector. Again B may be modified to some extent preserving these properties. We can further obtain some information on non-real eigenvalues of B by considering the mapping (3.8) in the manner described in Example 2.

Example 4. Set

$$f(u, v) = \begin{cases} v^{-1} & \text{for } u \leqq v \\ 0 & \text{for } u > v. \end{cases} \tag{3.9}$$

This function f satisfies conditions (i) and (ii) but not (iv). Indeed we have

$$\lambda_\gamma = \gamma^{-1}, \tag{3.10}$$

and hence $\lim \lambda_\alpha = 1$ for $\alpha \to 1$, contrary to the situation in Theorem 4. In any case we can apply Theorem 3 to the matrix B with the elements

$$b(i, k) = \begin{cases} (k+\theta)^{-1} & \text{for } i \leqq k \\ 0 & \text{for } i > k, \end{cases}$$

where we assume that $\theta \geqq 1$. To this end we make use of the known result[12] that $\| B \|_p \leqq p$, $1 < p < \infty$, for $\theta = 1$ and hence for all $\theta \geqq 1$.

For any $\lambda > 1$ set $\gamma = \lambda^{-1}$ and take a $q > 1$ such that $q < \lambda$. Then $\lambda_\gamma = \gamma^{-1} = \lambda > q \geqq \| B \|_q$ and we can conclude, from Theorem 3, (set $p = \lambda$) that any number $\lambda > 1$ is an eigenvalue of B with a positive eigenvector belonging to $l_{\lambda+0}$ but not to l_λ. Again B may be modified to some extent as in Example 1. It should be pointed out that one can give an explicit solution to the eigenvalue problem for the unmodified B. But it would be of some interest to be able to deduce the main results from a general theorem. Notice also that Theorem 3 cannot give any information on the eigenvalue of B'; in fact B' has actually only one eigenvalue $\lambda = 1$.

[12]See Hardy, Littlewood and Polya [2], p. 240.

4. Special Types of Positive Matrices

For positive matrices A satisfying the assumptions of Theorem 4 every positive number $\lambda > \omega$ is an eigenvalue both for A and A' having positive eigenvectors. However, the relationship between different eigenvectors for different eigenvalues is not known in general. In order to give such a relationship it appears necessary to introduce further assumptions.

In conformity with the notation used in a previous paper [5], we write $A \ll B$ for two positive matrices $A = (a(i,k))$ and $B = (b(i,k))$ if the ratio $b(i,k)/a(i,k)$ is a non-decreasing function of i and k. Similarly, we write $x \ll y$ for two positive vectors $x = (x(i))$ and $y = (y(i))$ if $y(i)/x(i)$ is non-decreasing in i. The relation $\ll$ is not changed when both sides are multiplied with (not necessarily equal) positive numbers. Also $x \ll y$ implies $x \leqq y$ if x and y are normalized in such a way that $x(0) = y(0)$. A positive matrix A will be called a P-matrix if every minor determinant of A of order 2 consisting of four neighboring elements is non-negative. The Hilbert matrix (0.1) is an example of a P-matrix.

The following lemma was proved in [5] for finite matrices, but its extension to infinite matrices is obvious.

LEMMA 2. *Let A, B be positive matrices such that $A \ll B$ and, moreover, let B be a P-matrix. Let x, y be positive vectors such that $x \ll y$. Then $Ax \ll By$ whenever Ax and By exist.*

We can. now prove

THEOREM 5. *Let A be a P-matrix satisfying the conditions of Theorem 4 (or 4'). Then every number $\lambda \geqq \omega$ is an eigenvalue of A as well as of A' with positive eigenvectors x_λ and x'_λ, respectively, such that*

$$x_\lambda \ll x_\mu, \quad x'_\lambda \ll x'_\mu \quad \text{for } \omega \leqq \lambda \leqq \mu. \tag{4.1}$$

Proof: The existence of the eigenvectors x_λ and x'_λ for $\lambda > \omega$ has been proved in Theorem 4 (or 4'). We shall prove first (4.1) for the case $\lambda > \omega$. Since with A, A' is also a P-matrix, it is sufficient to prove the first inequality.

According to Theorems 1, 3, 4, x_λ can be constructed by iteration starting with the vector $x^0_\lambda = ((i+\theta)^{-\alpha})$, where α is a number such that $\lambda = \lambda_\alpha$, $0 < \alpha < \alpha_0$. As is stated in the remark after Theorem 1, the approximating vectors x^n_λ, $n = 1, 2, 3, \cdots$, can be normalized in such a way that $x^n_\lambda(0) = 1$. We shall now show that

$$x^n_\lambda \ll x^n_\mu, \qquad \omega < \lambda \leqq \mu. \tag{4.2}$$

Going to the limit $n \to \infty$ we then get (4.1) for $\lambda > \omega$. Relation (4.2) is true for $n = 0$ since we have $x^0_\lambda = ((i+\theta)^{-\alpha})$ and $x^0_\mu = ((i+\theta)^{-\beta})$ where α, β are such that $\lambda = \lambda_\alpha$, $\mu = \lambda_\beta$ and $\alpha, \beta < \alpha_0$; but $\lambda \leqq \mu$ implies $\alpha \geqq \beta$ because λ_γ is a decreasing function of γ for $0 < \gamma < \alpha_0$. Suppose now that (4.2) has been proved for an n; we shall show that it is also true for n replaced by $n+1$, thus completing the induction. Since A is a P-matrix, it follows from Lemma 2 and (4.2) that $Ax_\lambda^n \ll Ax_\mu^n$, which after normalization gives $x_\lambda^{n+1} \ll x_\mu^{n+1}$.

We can now establish the existence of the eigenvector x_ω and the validity of (4.2) for $\lambda = \omega$. Since $x_\lambda \ll x_\mu$ implies $x_\lambda \leqq x_\mu$ in virtue of the normalization $x_\lambda(0) = x_\mu(0) = 1$, each component $x_\lambda(i)$ of x_λ is a non-decreasing function of λ for $\lambda > \omega$. Therefore, the limit $x_\omega(i) = \lim_{\lambda\to\omega} x_\lambda(i)$ exists for each i. The vector $x_\omega = (x_\omega(i))$ satisfies the normalization condition $x_\omega(0) = 1$ and the inequality $x_\omega \ll x_\lambda$ for $\lambda \geqq \omega$. Furthermore, it follows easily from the monotonicity of x_λ that x_ω is an eigenvalue of A with the eigenvalue ω.

Incidentally this shows that x_ω is actually positive. This completes the proof of Theorem 5.

The following theorem can be proved in quite the same way.

THEOREM 6. *$A = A_\theta$ of Theorem 5 depends on the parameter θ (see Theorems 4 or 4'). Assume that A_θ is always a P-matrix and that $\theta < \eta$ implies $A_\theta \ll A_\eta$. Denoting by $x_{\theta,\lambda}$ and $x'_{\theta,\lambda}$ the x_λ and x'_λ of Theorem 5, respectively, we then have*

$$x_{\theta,\lambda} \ll x_{\eta,\mu}, \qquad x'_{\theta,\lambda} \ll x'_{\eta,\mu} \qquad \text{for } \theta \leqq \eta,\ \omega \leqq \lambda \leqq \mu. \tag{4.3}$$

Example 5. The Hilbert matrix H_θ given by (0.1) satisfies the assumptions of Theorems 5 and 6 at least for $\theta \geqq 1/2$. Hence any number $\lambda \geqq \pi$ is an eigenvalue of H_θ with a positive eigenvector $x_{\theta,\lambda}$ (see also Example 2) for which (4.3) holds. This is true even if H_θ is modified by the term C of Theorem 4' as long as the modified matrix is a P-matrix satisfying the assumptions of Theorem 6.

Bibliography

[1] Frobenius, G., *Über Matrizen aus positiven Elementen*, Sitzungsber. Preuss. Akad. Wiss., 1908, pp. 471–476, 1909, pp. 514–518.

[2] Hardy, G. H., Littlewood, J. E., and Polya, G., *Inequalities*, Cambridge University Press, Cambridge, 1952.

[3] Hopf, E., *Über lineare Integralgleichungen mit positivem Kern*, Sitzungsber. Preuss. Akad. Wiss., 1928, pp. 233–245.

[4] Jentsch, R., *Über Integralgleichungen mit positivem Kern*, J. Reine Angew. Math., Vol. 141, 1912, pp. 235–244.

[5] Kato, T., *On the Hilbert matrix*, Proc. Amer. Math. Soc., Vol. 8, 1957, pp. 73–81.

[6] Krein, M. G., and Rutman, M. A., *Linear operators leaving invariant a cone in a Banach space*, Uspehi Mat. Nauk N. S. 3, No. 1(23), 1948, pp. 3–95. A. M. S. Trans. No 26.

[7] Magnus, W., *Über einige beschränkte Matrizen*, Arch. Math., Vol. 2, 1949–1950, pp. 405–412.
[8] Perron, O., *Grundlagen für eine Theorie des Jacobischen Kettenbruchalgorithmus*, Math. Ann., Vol. 64, 1907, pp. 1–76.
[9] Rosenblum, M., *On the Hilbert matrix, I*, Proc. Amer. Math. Soc. Vol. 9, 1958, pp. 137–140.
[10] Schur, I., *Bemerkungen zur Theorie der beschränkten Bilinearformen mit unendlich vielen Veränderlichen*, J. Reine Angew. Math., Vol. 140, 1911, pp. 1–28.

Received April 1958.