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Real and Nominal Interest Rates: A Discrete-Time Model and Its Continuous-Time Limit

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I provide a general equilibrium theory of the term structure of real interest rates in a discrete-time economy. I derive the prices for one-period and two-period real bonds and a simple recursive formula for general k-period bonds, and prove that the price formula with appropriately specified parameters converges to that of the Cox, Ingersoll, and Ross model (1985). In addition, I consider the behavior of nominal bond prices in a partial equilibrium setting in which an exogenous price level process is correlated with the real economy. Finally, I provide an illustrative empirical investigation of the model. The results indicate a significant correlation between the price level and the growth rate of consumption, which does not support the "money neutrality" assumption underlying Cox, Ingersoll, and Ross's nominal bond prices and related empirical studies, such as Gibbons and Ramaswamy (1992), Heston (1991), and Pearson and Sun (1991).

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The term structure is the function that maps the time to maturity of a discount bond to its current price (or yield to maturity). The study of this functional relationship has long been of interest to economists. In this article, I provide a general equilibrium theory of the term structure of real interest rates in a sequence of discrete-time exchange economies. I present a discrete-time analog of the Cox, Ingersoll, and Ross model (1985; henceforth, CIR) and elucidate the essential ideas behind the theory of term structure and the mechanism underlying the CIR model. In addition, I also investigate the term structure of nominal interest rates in a partial equilibrium setting in which an exogenous price level process is affected by the real state of economy.

As emphasized by CIR (1985), modeling the term structure in an equilibrium setting can accommodate all the elements of the traditional theories, such as the expectations hypothesis and the liquidity preference hypothesis [Hicks (1946)], which imply little more than whether expected spot interest rates are equal to implied forward rates. In contrast to the arbitrage models, such as Brennan and Schwartz (1979) and Richard (1978), an equilibrium approach automatically ensures internal consistency and need not make assumptions on the risk premium required by investors for bearing interest rate risk.[1] In addition, an equilibrium model provides information about how underlying variables in the economy will affect the term structure.

There are three motivations for this work. First, the real discount bond price is derived from a self-evident and well-known first-order condition of the consumption and investment decision in an exchange economy: for any financial asset, the loss of marginal utility of current consumption must equal the gain of expected marginal utility of future consumption if one share of that financial asset is purchased today and sold in the future. In this respect, the article complements those of Campbell (1986) and Turnbull and Milne (1991), who assume that either the log endowment or the first difference of the log endowment follows a univariate, stationary, invertible process. Instead, the endowment process in this article generalizes the latter, and the derived bond price formula with appropriately specified parameters converges to that of the CIR model as the time interval between periods approaches zero.[2]

Second, given an exogenous price level, I derive the nominal dis-

[1] CIR (1985, p. 398) shows that inappropriate specification of the risk premium can lead to a model that guarantees arbitrage opportunities rather than precluding them.

[2] Note that the convergence here is different from the approximation results discussed in He (1990), Hull and White (1990), and Nelson and Ramaswamy (1990). These authors provide various discrete-time approximations to diffusion processes in financial models. Their results rely on the unsatisfactory economic assumption that the equilibrium discrete-time interest rate or price is exogenously given rather than endogenously determined.

count bond price from the same first-order condition and look into the relation between the real and the nominal interest rates. Since the price level is correlated with the growth rate of endowments, closed-form formulas are much more difficult to derive in a continuous-time model.

Third, when the price level is correlated with the real economy, the conditional density or the steady-state distribution is known only for extremely simple continuous-time processes. For example, Pennacchi (1991) assumes that both the state variable and the expected inflation rate follow Gaussian processes. This poses a formidable obstacle for econometricians and explains why the existing empirical work on CIR, such as Gibbons and Ramaswamy (1992), Heston (1991), and Pearson and Sun (1991), cannot do away with the "money-neutrality" assumption. Since this model contains the real bond price formula in CIR in the limit, it provides an empirical framework to investigate the validity of the "money-neutrality" assumption. It also illustrates that while it is conceptually and computationally straightforward to estimate a discrete-time model, it is infeasible to estimate the corresponding continuous-time model because neither the conditional density nor the steady-state distribution is known.

The rest of this article is organized as follows. In Section 1, I describe a discrete-time exchange economy similar to Lucas (1978) and specify the assumptions on the growth rate of endowments and the price level. I also show how to "support" the exchange economy in a production economy, which is a special case of Brock (1982). In Section 2, fixing the number of periods in a constant time interval, I derive explicit pricing formulas for one-period and two-period real discount bonds, and discuss the relations between the bond prices and parameters of the economy. In Section 3, I deal with the price for a one-period nominal bond and the relation between the real and the nominal interest rates. Even though the pricing formula is extremely complicated when there are more than three periods, a simple recursive formula is available for a general k-period bond.

Letting the time interval between periods go to zero, I derive, in Section 4, the limit pricing formula for the nominal discount bond. This limit pricing formula is equal to that in the corresponding continuous-time model and contains the CIR model's pricing formula as a special case. In Section 5, I provide an illustrative empirical investigation of the model. The method of maximum likelihood is applied to estimate the parameters of the model, based on the three-month and one-year Treasury-bill prices and the consumer price index. The results show that the price level is significantly correlated with the real economy, which does not support the "money neutrality" assumption. While the one-month real interest rate exhibits the mean-

reverting property, the expected inflation rate does not. A short conclusion is given in Section 6.

1. A Discrete-Time Exchange Economy

In this section, I first describe an exchange economy and show how it can be "supported" in a production economy. Then I specify the assumptions on the growth rate of endowments and the inflation rate underlying the exchange economy.

1.1 An exchange economy

Following Lucas (1978), consider a one-good discrete-time exchange economy with m financial assets in which a representative agent maximizes the expected discounted sum of a strictly increasing concave von Neumann–Morgenstern utility function, U:

$$\max \mathcal{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t U(C_t) \right\},$$

where $\mathcal{E}_t\{\cdot\}$ is the conditional expectation given all information up to time t, $\beta \in (0, 1)$ the time discount factor, and C_t the consumption at time t. Let C denote the consumption process $\{C_t, t = 0,1,2, \ldots\}$, with similar notation for other processes. In equilibrium, the consumption process C must be equal to the exogenous endowment process E. In addition, the total market value of the ith financial asset V_i satisfies the following first-order condition of the optimal consumption and investment decision at any time t for any holding period $s = 1,2, \ldots$:

$$U'(C_t)V_{i,t} = \mathcal{E}_t\{\beta^s U'(C_{t+s})V_{i,t+s}\}, \qquad i = 1,2, \ldots, m, \tag{1}$$

where $U'(\cdot)$ is the derivative of $U(\cdot)$.

Suppose that p_τ is the price level at time τ. Let $P_t(s)$ denote the *real* price at time t of a real discount bond maturing at time s, and $N_t(s)$ the *nominal* price of a corresponding nominal discount bond. A real (nominal) discount bond maturing at time τ is a financial asset that promises to pay one $(1/p_\tau)$ unit of consumption at time τ and nothing at any other time. Suppose that the utility function U has constant relative risk aversion equal to α (>0), that is

$$U(C) = \frac{C^{1-\alpha} - 1}{1 - \alpha}, \qquad \alpha > 0.$$

The above first-order conditions for real and nominal discount bonds are

$$P_t(s) = \mathcal{E}_t\left\{\beta^{s-t}\left(\frac{C_s}{C_t}\right)^{-\alpha}\right\}, \tag{2}$$

$$\frac{N_t(s)}{p_t} = \mathcal{E}_t\left\{\beta^{s-t}\left(\frac{C_s}{C_t}\right)^{-\alpha}\frac{1}{p_s}\right\}. \tag{3}$$

Let g_τ denote the ratio of the endowments (or the consumptions) at time τ and $\tau - 1$, and π_τ the ratio of the price levels. That is, $g_\tau \equiv C_\tau/C_{\tau-1}$ and $\pi_\tau \equiv p_\tau/p_{\tau-1}$. Then the prices of the discount bonds in the discrete-time exchange economy can be expressed as

$$P_t(s) = \beta^{s-t}\mathcal{E}_t\left\{\prod_{i=t}^{s-1} g_{i+1}^{-\alpha}\right\}, \tag{4}$$

$$N_t(s) = \beta^{s-t}\left\{\prod_{i=t}^{s-1} g_{i+1}^{-\alpha}\,\pi_{i+1}^{-1}\right\}. \tag{5}$$

Before specifying the assumptions on the growth rate of endowments and the inflation rate, I would like to show how to reinterpret these results in a production economy.

1.2 A production economy

The discrete-time production economy is a special case of Brock (1982) and analogous to that of CIR. Consider a one-good economy such that the production function is stochastic constant returns to scale. In addition to m financial assets, the representative agent can invest directly in the production process. The agent is assumed to maximize the expected discounted sum of a logarithmic utility function:

$$\max_{\{C_t, a_{i,t}\}} \mathcal{E}_0\left\{\sum_{t=0}^{\infty} \beta^t \ln(C_t)\right\},$$

subject to the budget constraint

$$W_{t+1} = (W_t - C_t)\left\{\sum_{i=1}^{m}\left[a_{i,t}\frac{V_{i,t+1}}{V_{i,t}}\right] + \left[1 - \sum_{i=1}^{m} a_{i,t}\right]\Lambda_{t,t+1}\right\},$$

where W_t is the total amount of produce at time t, $a_{i,t}$ is the proportion of produce at time t invested in the ith asset, and $\Lambda_{t,t+1}$ is the gross rate of return of the production opportunity from time t to time $t + 1$.

Inasmuch as the net supply of any financial asset is zero in equilibrium, that is, $a_{i,t} = 0$ for all i and t, we have

$$W_{t+1} = (W_t - C_t)\,\Lambda_{t,t+1}.$$

Furthermore, it is well known that the assumptions of a logarithmic utility function and a constant-returns-to-scale technology lead to proportional consumption with β being the constant saving rate [e.g., see Brock (1982, p. 30)]. Therefore, in equilibrium,

$$C_t = (1 - \beta) W_t = \beta C_{t-1} \Lambda_{t-1,t}.$$

Applying the above equation to the first-order conditions (2) and (3) gives the bond prices:

$$P_t(s) = \mathcal{E}_t \left\{ \prod_{i=t}^{s-1} \Lambda_{i,i+1}^{-1} \right\},$$

$$N_t(s) = \mathcal{E}_t \left\{ \prod_{i=t}^{s-1} \Lambda_{i,i+1}^{-1} \pi_{i+1}^{-1} \right\}.$$

Accordingly, Equations (4) and (5) with $\alpha = 1$ can be "supported" in a production economy if g is reinterpreted to be the process of the gross rate of return of the production opportunity.[3]

There is, however, an important difference in the pricing formulas between the exchange economy and the production economy: the time discount factor β does not appear in the production economy. In an exchange economy, stronger preference for present consumption (lower β) must decrease bond prices and increase returns on bonds to induce the agents not to consume more than their endowments. On the other hand, under the assumptions of a logarithmic utility function and a constant-returns-to-scale production opportunity, the discount factor β happens to be the saving rate in the production economy. Other things being equal, the lower the saving rate, the less future consumption will be. This implies that the certain future payoffs provided by bonds are more valuable and leads to increases in bond prices. Stronger preference for present consumption is, therefore, accompanied by a lower saving rate in such a way that the time-preference parameter has no net effect on bond prices [Gilles and LeRoy (1986)].

In the following, I proceed with the exchange economy to look into the relation between risk aversion and the term structure. Of course, a model with a production economy can be derived in a pure exchange economy if the endogenous process for consumption in

[3] Not every exchange economy can be "supported" in a production economy such that the exogenous consumption process in the exchange economy is the consumption process endogenously determined by the production process and the utility-maximizing rule in the production economy. On the other hand, a production economy always induces an exchange economy in which the exogenous consumption process is specified to be the one endogenously determined in the production economy.

the production economy is assumed to hold exogenously in the pure exchange economy.

1.3 Assumptions

I will apply the pricing formulas (4) and (5) to a fixed time interval $[0, \tau]$, and investigate the evolution of $P_0(\tau)$ and $N_0(\tau)$ when the number of periods between time 0 and time τ is increased, under the following assumptions on the growth rate of endowments and the inflation rate.[4]

Suppose that there are n periods in the fixed interval $[0, \tau]$, with the length of each period being τ/n. Let $\xi_1^{(y)}, \xi_2^{(y)}, \ldots, \xi_n^{(y)}$ represent a sequence of independent random variables such that each is normally distributed with zero mean and unit variance. Assume that the real state of economy at the (end of) ith period, denoted[5] as Y_i, satisfies the following.

Assumption 1.

$Y_0 \geq 0,$

$$Y_i = Y_{i-1} + \frac{\tau}{n}\eta(\delta - Y_{i-1}) + \sqrt{\frac{\tau}{n}}\sigma_y\sqrt{Y_{i-1}}\xi_i^{(y)}, \qquad 1 \leq i \leq n, \tag{6}$$

where δ, η, and σ_y are constants such that $\delta\eta > 0$ and $\sigma_y \geq 0$. Whenever the right-hand side of Equation (6) is not positive, Y_i is defined to be zero.

Since $\eta\delta$ is strictly positive, the sample path of Y will become so whenever it reaches zero. In other words, zero is a reflection boundary of Y. Moreover, if both η and δ are strictly positive, the process Y is mean-reverting in the sense that its movement is adjusted toward a strictly positive natural level δ (or the long-term mean) of the state of economy. The parameter η determines the speed of adjustment. The higher η is, the quicker the adjustment toward the natural level will be. Assumption 1 also specifies that Y is what is called a square-root process—that is, the relative variance of the change in Y, $\mathrm{var}(Y_i^{+1} - Y_i)/Y_i$, is $(\tau/n)\sigma_y^2$, constant through time. (Note that τ/n is just a normalizing factor since it is the length of one period.) These three properties characterize the process of the underlying uncertainty in CIR. However, the mean-reverting property is not presumed here.

Given the specification of the state of economy, it is reasonable to

[4] The same approach can be applied to analyze $P_i(\tau)$ and $N_i(\tau)$.

[5] Rigorously, it should be denoted $Y^{(n)} \equiv \{Y_0^{(n)}, Y_1^{(n)}, \ldots, Y_n^{(n)}\}$. Throughout the article, the dependence on n is suppressed to simplify notation.

assume that the process of the growth rate of endowments $\ln g \equiv \{\ln g_1, \ln g_2, \ldots, \ln g_n\}$ is described by the following.

Assumption 2.

$$\ln g_{i+1} = \frac{\tau}{n}\ln \epsilon + \frac{\tau}{n}\mu_c Y_i + \sqrt{\frac{\tau}{n}}\sigma_c\sqrt{Y_i}\xi_{i+1}^{(y)}, \qquad 0 \le i \le n-1,$$

where ϵ (> 0), μ_c, and σ_c are constants, and $\ln g_{i+1}$ is the growth rate of endowments from the ith period to the (i + 1)th period.

This assumption says that conditional on the state of the economy at the ith period, the growth rate of endowments from the ith period to the $(i + 1)$th period is normally distributed. In addition to a fixed autonomous growth $\ln \epsilon$, the conditional mean and variance of the growth rate of endowments are proportional to the state of the economy with constant coefficients $(\tau/n)\mu_c$ and $(\tau/n)\sigma_c^2$.

Even though the state of economy Y is a Markov process, the growth rate of endowments $\ln g$ is, in general, not Markovian. For the conditional distribution of $\ln g_i^{+1}$ to depend only on $\ln g_i$, it is necessary that Y be deterministic—that is, $\sigma_y = 0$. In this case, the growth rate of endowments can be represented as a first-order autocorrelated process as in Campbell (1986) and Turnbull and Milne (1991). When $\sigma_y \neq 0$, it can easily be shown that the sample path of $\ln g$ does depend on all of its past realizations.

So far, the state variable and the growth rate of endowments are assumed to be driven by the same sequence of random variables $\xi_1^{(y)}, \xi_2^{(y)}, \ldots, \xi_n^{(y)}$. This is equivalent to the formulation of two serially uncorrelated, but contemporaneously perfectly correlated, sequences of random variables. It is more reasonable to assume that the uncertainties underlying the state variable and the growth rate of endowments are correlated, but not necessarily perfectly so. This generalization will be employed in the price level process to be discussed. Here we can accommodate a possible negative correlation between the state variable and the growth rate of endowments by allowing a negative σ_c.

Finally, we turn to the evolution of the price level. Similar to CIR, I assume an exogenous price level process without explicitly modeling the role of money supply and demand that underlies the process. However, the price level is correlated with endowments. Let $\xi_1^{(z)}, \ldots, \xi_n^{(z)}$ be a sequence of independent random variables, which are uncorrelated with $\xi^{(y)}$ and normally distributed with zero means and unit variances. The expected inflation rate is assumed to be independent of the state variable Y and described by the following.

Assumption 3.

$$Z_0 \geq 0,$$

$$Z_i = Z_{i-1} + \frac{\tau}{n}\eta_z(\delta_z - Z_{i-1}) + \sqrt{\frac{\tau}{n}}\sigma_z\xi_i^{(z)}, \qquad 1 \leq i \leq n, \tag{7}$$

where η_z, δ_z, and σ_z (≥ 0) are constants.

Contrary to the real state of economy, $\eta_z\delta_z$ and the expected inflation rate can be negative. Of course, if the parameters are strictly positive, the expected inflation rate is mean-reverting with a constant conditional variance.

The process of the price level is assumed to satisfy the following.

Assumption 4.

$$p_0 > 0, \qquad \ln \pi_{i+1} = \frac{\tau}{n}Z_i + \sqrt{\frac{\tau}{n}}\sigma_p\sqrt{Y_i}\xi_{i+1}^{(p)}, \qquad 0 \leq i \leq n-1,$$

where σ_p (≥ 0) is constant, and $\xi_1^{(p)}, \xi_2^{(p)}, \ldots, \xi_n^{(p)}$ are independent random variables that are uncorrelated with $\xi^{(z)}$, but contemporaneously correlated with $\xi^{(y)}$ with coefficient ρ, and normally distributed with zero means and unit variances.

If we notice that $\ln \pi_{i+1}$ is the actual inflation rate from time i to time $i+1$, the economic meaning of Assumption 4 is evident. Conditional on Y_i and Z_i, the inflation rate is normally distributed with mean $(\tau/n)Z_i$ and variance $(\tau/n)\sigma_p^2 Y_i$. The parameter ρ measures the correlation between the random shocks underlying the price level and the real economy, and σ_p determines how the price level will be affected by the real economy. When σ_p equals zero, we have "money neutrality" and all the uncertainty about the price level is caused by the nominal shock $\xi^{(z)}$. For simplicity, I only allow the real state of economy to affect the inflation rate through the conditional variance. It is straightforward to extend the analysis to incorporate a real state-dependent conditional mean or possible effects in the opposite direction as in Pennacchi (1991). This generalization will not be adopted here because it does not add much insight to the model.

In all, the information structure in this economy is (vector) Markovian and resolved jointly by Y, Z, and π, and $\mathcal{E}_i\{\cdot\} \equiv \mathcal{E}\{\cdot \mid Y_i, Z_i, \pi_i\}$. Note that the state variable Y is just a convenient notation. It is clear from the assumptions that all future uncertainty in this economy is embedded in the three sequences of random variables $\xi^{(y)}$, $\xi^{(z)}$, and $\xi^{(p)}$. For example, each Y_i can be expressed as a function of Y_0, and $\xi_1^{(y)}, \ldots, \xi_{i-1}^{(y)}$. This, in turn, implies that the economy at the ith period

is completely determined by the realizations of p_0, Y_0, Z_0, and $\xi_t^{(y)}$, $\xi_t^{(z)}$, $\xi_t^{(p)}$, $t \leq i - 1$.

2. Pricing Real Discount Bonds

In this section, I derive the prices for one-period and two-period real discount bonds. Even though the pricing formula is formidable for a general k-period bond, there is a nice recursive expression. Since a real bond price equals the corresponding nominal bond price given a constant price level, I will present the general pricing formula in the next section when discussing the nominal bond.

2.1 One-period bonds

Applying Assumption 2 to Equation (4) gives the price for the one-period real bond:

$$P_0\left(\frac{\tau}{n}\right) = \beta^{\tau/n}\mathcal{E}_0\{g_1^{-\alpha}\}$$
$$= \left(\frac{\beta}{\epsilon^\alpha}\right)^{\tau/n} \exp\left\{-\alpha\left(\mu_c - \frac{\alpha\sigma_c^2}{2}\right)\frac{\tau}{n}Y_0\right\}.$$

The one-period (continuously compounded) interest rate implied by this pricing formula is

$$R_0\left(\frac{\tau}{n}\right) \equiv \frac{-\ln P_0(\tau/n)}{\tau/n} = \alpha\left(\mu_c - \frac{\alpha\sigma_c^2}{2}\right)Y_0 - \ln\left(\frac{\beta}{\epsilon^\alpha}\right). \tag{8}$$

As expected, the interest rate is a decreasing function of σ_c and β, and an increasing function of μ_c and ϵ. Stronger preference for current consumption (lower β) must be accompanied with a higher rate of return on a bond to induce the agent not to consume more than the total current endowment.

Breeden (1986) has provided intuitive explanations for the positive relation between the interest rate and the mean of the growth rates of endowments, and the negative relation between the interest rate and the variance of the growth rate of endowments. The price of a discount bond is equal to the expected marginal utility of endowment (consumption) at maturity divided by the marginal utility of endowment (consumption) today. Because of decreasing marginal utility, a better prospect (higher μ_c or higher ϵ) of future endowment, holding current endowment constant, decreases the expected marginal utility, which in turn decreases the bond price and increases the interest rate. On the other hand, since the marginal utility function is convex $[U'''(\cdot) > 0]$, a mean-preserving spread of the distribution for the future endowment (higher σ_c) increases the expected marginal utility

and decreases the bond price. In summary, a certain future payoff provided by a bond is more valuable when the future endowment is more uncertain, while a certain future payoff is not as important when the future endowment is expected to be higher.

As for the state variable, a better initial state of economy Y_0 increases both the conditional mean and the conditional variance of the growth rate of endowments. Its net effect on the interest rate depends on the relative magnitude of the positive mean effect and the negative variance effect, which is closely related to the degree of risk aversion. If the agent is more risk-averse, so that $\alpha > 2\mu_c/\sigma_c^2$, the variance effect dominates ($\mu_c < \alpha\sigma_c^2/2$) and the interest rate is negatively correlated with the initial state of the economy. On the other hand, if the agent is less risk-averse, so that $\alpha < 2\mu_c/\sigma_c^2$, the mean effect dominates ($\mu_c > \alpha\sigma_c^2/2$) and the interest rate is positively correlated with the initial state of the economy.

2.2 Two-period bonds

Let us turn to the two-period discount bond, which matures at time $2\tau/n$. First, I derive the current price and the future price at time τ/n for the bond, and the corresponding current two-period interest rate, the future spot interest rate at time τ/n, and the implied forward rate between time τ/n and $2\tau/n$. Then, I discuss how the parameters of the real state of economy and the growth rate of endowments affect the interest rates.

2.2.1 Prices and interest rates. Clearly, the future price at time τ/n of the two-period bond is

$$P_{\tau/n}\left(\frac{2\tau}{n}\right) = \left(\frac{\beta}{\epsilon^\alpha}\right)^{\tau/n} \exp\left\{-\alpha\left(\mu_c - \frac{\alpha\sigma_c^2}{2}\right)\frac{\tau}{n}Y_1\right\}.$$

The corresponding future spot rate is therefore

$$R_{\tau/n}\left(\frac{2\tau}{n}\right) \equiv \frac{-\ln P_{\tau/n}(2\tau/n)}{\tau/n} = \alpha\left(\mu_c - \frac{\alpha\sigma_c^2}{2}\right)Y_1 - \ln\left(\frac{\beta}{\epsilon^\alpha}\right).$$

Let $\Psi \equiv \alpha(\mu_c - \alpha\sigma_c^2/2)$. Applying Assumptions 1 and 2 to Equation (4) gives the current price of the two-period bond:

$$\begin{aligned} P_0\left(\frac{2\tau}{n}\right) &= \beta^{2\tau/n}\mathcal{E}_0\{g_1^{-\alpha}g_2^{-\alpha}\} \\ &= \left(\frac{\beta}{\epsilon^\alpha}\right)^{2\tau/n} \exp\left\{-\frac{\tau}{n}\Psi\left[\frac{\tau}{n}\eta(\delta - Y_0)\right.\right. \end{aligned}$$

$$+\left(2-\frac{\tau}{n}\alpha\sigma_c\sigma_y - \frac{[(\tau/n)\sigma_y]^2}{2}\Psi\right)Y_0\Bigg]\Bigg\}N(\Upsilon_1)$$

$$+\left(\frac{\beta}{\epsilon^\alpha}\right)^{2\tau/n}\exp\left\{-\frac{\tau}{n}\Psi Y_0\right\}N(\Upsilon_2), \tag{9}$$

where $\Phi(\cdot)$ is the cumulative standard normal distribution, and

$$\Upsilon_1 \equiv \frac{Y_0 + (\tau/n)\eta(\delta - Y_0)}{\sqrt{(\tau/n)}\,\sigma_y\sqrt{Y_0}} - \alpha\sqrt{\frac{\tau}{n}}\sigma_c\sqrt{Y_0} - \left(\frac{\tau}{n}\right)^{3/2}\Psi\sigma_y\sqrt{Y_0},$$

$$\Upsilon_2 \equiv \left(\frac{\tau}{n}\right)^{3/2}\Psi\sigma_y\sqrt{Y_0} - \Upsilon_1.$$

In Equation (9), $\Phi(\Upsilon_1)$ reflects the event that Y_1 is greater than zero, and $\Phi(\Upsilon_2)$, the event that Y_1 is equal to zero. When Y is deterministic, $\Phi(\Upsilon_1) = 1$ and $\Phi(\Upsilon_2) = 0$. While this is in general not true, $\Phi(\Upsilon_1) \approx 1$ and $\Phi(\Upsilon_2) \approx 0$ when n is large. To avoid unnecessary technical complexity, in the rest of the article the number of periods is assumed to be large enough so the probability that Y reaches zero is negligible.

Therefore, the two-period bond price is

$$P_0\left(\frac{2\tau}{n}\right) = \left(\frac{\beta}{\epsilon^\alpha}\right)^{2\tau/n}\exp\left\{-\frac{\tau}{n}\Psi\left[\frac{\tau}{n}\eta(\delta - Y_0) + \left(2-\frac{\tau}{n}\alpha\sigma_c\sigma_y - \frac{[(\tau/n)\sigma_y]^2}{2}\Psi\right)Y_0\right]\right\}.$$

The forward rate (between time τ/n and $2\tau/n$) implied by the one-period and the two-period bond prices is

$$R_{\tau/n,2\tau/n} \equiv \frac{\ln P_0(\tau/n) - \ln P_0(2\tau/n)}{\tau/n}$$

$$= \Psi\left[\frac{\tau}{n}\eta(\delta - Y_0) + \left(1-\frac{\tau}{n}\alpha\sigma_c\sigma_y - \frac{[(\tau/n)\sigma_y]^2}{2}\Psi\right)Y_0\right] - \ln\left(\frac{\beta}{\epsilon^\alpha}\right),$$

and the two-period interest rate is

$$R_0\left(\frac{2\tau}{n}\right) \equiv \frac{-\ln P_0(2\tau/n)}{2\tau/n} = \frac{1}{2}\left[R_{\tau/n,2\tau/n} + R_0\left(\frac{\tau}{n}\right)\right].$$

2.2.2 Comparative-static analysis. The pricing formula for the two-period bond is much more complicated because of the interaction between the parameters of the state of economy and the growth rate

of endowments. In the following, I discuss the case in which the state of economy is mean-reverting. The arguments can be easily adapted to other cases.

The effects of β and ϵ on the forward rate and the interest rates are clear. The parameters of the mean of the state of economy (η and δ) have the following effects on the forward rate and the two-period interest rate. Since a better state of economy increases both the conditional mean and the conditional variance of the growth rate of endowments, we should expect the effects to depend on the relative strength between the positive mean effect and the negative variance effect. Suppose the agent is less risk-averse, so that the mean effect dominates $\mu_c > \alpha\sigma_c^2/2$. The interest rate is an increasing function of the long-term mean δ as expected. The interest rate is also an increasing function of the parameter of the speed of adjustment η if $Y_0 < \delta$, and a decreasing function if $Y_0 > \delta$. This is clear from the mean-reverting property of the state of economy. When it is below the long-term mean, the economy tends to improve. The speedier (higher η) the adjustment, the higher (even though more volatile) the future endowment and the lower the expected marginal utility of future endowment (since the mean effect dominates). Consequently, the interest rate will be positively related to η if $Y_0 < \delta$. The interpretation for the case that $Y_0 > \delta$ is similar. Clearly, the relations will be reversed when the variance effect dominates.[6]

The above arguments do not hold for σ_y. The two-period interest rate is a decreasing function of $|\sigma_y|$ when $\mathcal{V} \equiv \sigma_c\sigma_y(\mu_c - \alpha\sigma_c^2/2) > 0$, but the relation is ambiguous when $\mathcal{V} < 0$. A higher $|\sigma_y|$ increases the uncertainty of the endowment at time $2\tau/n$, and hence the value of the certain payoff provided by the two-period bond. This has a negative effect on the two-period interest rate. There is, however, another effect of $|\sigma_y|$ on $R_0(2\tau/n)$ through the conditional covariance between the current growth rate of endowment and the future one-period interest rate, which is

$$\operatorname{cov}\left(\ln g_1,\ R_{\tau/n}\left(\frac{\tau}{n}\right)\middle|\ Y_0\right) = \frac{\tau}{n}\alpha\mathcal{V}Y_0 \equiv \alpha\sigma_c\sigma_y\left(\mu_c - \frac{\alpha\sigma_c^2}{2}\right)\frac{\tau}{n}Y_0. \quad (10)$$

This covariance measures the ability of the two-period bond to

[6] CIR (1985, p. 393) incorrectly claimed that their bond price is an increasing (decreasing) function of the speed of adjustment parameter if the current interest rate is greater (less) than the long-term mean. A counterexample is reported in Nelson and Ramaswamy (1990, Table 2, p. 421). The CIR bond price decreases as the speed of adjustment parameter increases *when the volatility parameter is large* ($\sigma = 0.5$), though the current interest rate is specified to be greater than the long-term mean. (I thank Chester Spatt for raising this point and providing the counterexample.) The example is consistent with the discrete-time result here: given that the current interest rate is greater than the long-term mean, the bond price increases (decreases) as the speed of adjustment parameter increases if the mean (variance) effect dominates.

hedge against unfavorable states [Benninga and Protopapadakis (1986)]. The two-period bond is a poor hedge against unfavorable states if the covariance is negative, and a good hedge if the covariance is positive. To see this, suppose the covariance is negative. A low growth rate of endowments leads to a low level of endowment available at time τ/n and a low future price at time τ/n of the two-period bond. This implies that the market value at time τ/n of the two-period bond is low when the marginal utility of endowment is high and wealth is greatly desired, and vice versa. The two-period bond is, therefore, a bad hedge against unfavorable states. Conversely, the two-period bond is a good hedge against unfavorable states if the covariance is positive.

According to the consumption capital asset pricing model [see LeRoy (1982) and the articles cited there], a financial asset will be priced to yield a negative (positive) risk premium if it is a good (bad) hedge against unfavorable states. When $\mathcal{V} > 0$, the two-period bond is a good hedge against unfavorable states. A higher $|\sigma_y|$ enlarges the positive correlation and increases the value of the two-period bond. This reinforces the negative "uncertainty" effect of $|\sigma_y|$ on the two-period bond price. When $\mathcal{V} < 0$, the two-period bond is a bad hedge against unfavorable states. A higher $|\sigma_y|$ enlarges the negative correlation and decreases the value of the two-period bond. The net effect of $|\sigma_y|$ on R_0 $(2\tau/n)$ is therefore indeterminate because of two offsetting effects.

It comes as no surprise that the effects of Y_0, μ_c, and σ_c on the two-period interest rate and the forward rate are ambiguous. Because of the complexity of the interplay between the state of economy and the growth rate of endowments, exact conditions for definite relations can be derived, but no intuitive explanation is available.

This completes the analysis of the one-period and two-period real discount bonds. We will turn to the study of nominal bonds.

3. Pricing Nominal Discount Bonds

In this section, I derive the price for the one-period nominal discount bond and a nice recursive formula for a general k-period bond.

3.1 One-period bonds

Applying Assumptions 2 and 4 to Equation (5) gives the price for the one-period nominal bond:

$$\begin{aligned} N_0\left(\frac{\tau}{n}\right) &= \beta^{\tau/n}\mathcal{E}_0\{g_1^{-\alpha}\pi_1^{-1}\} \\ &= P_0\left(\frac{\tau}{n}\right)\exp\left\{-\frac{\tau}{n}Z_0 + (\sigma_p^2 + 2\alpha\sigma_c\sigma_p\rho)\frac{\tau}{n}\frac{Y_0}{2}\right\}. \end{aligned}$$

The one-period nominal interest rate implied by this pricing formula is

$$R_0^*\left(\frac{\tau}{n}\right) \equiv \frac{-\ln N_0(\tau/n)}{\tau/n} = R_0\left(\frac{\tau}{n}\right) + Z_0 - (\sigma_p^2 + 2\alpha\sigma_c\sigma_p\rho)\frac{Y_0}{2}. \quad (11)$$

First, the nominal interst rate could be negative. This seemingly unreasonable result occurs because of the lack of a nominally riskless numeraire (e.g., fiat money) in the economy. Without such a numeraire, a nominal bond is the only means to provide a certain nominal future payoff. Accordingly, it may be worthwhile to pay more than one dollar today to secure one certain dollar in the future, especially when the future prospect is too bleak (extremely low μ_c and ϵ) or too uncertain (extremely high σ_c and σ_p).

Let m denote the expected one-period depreciation rate of purchasing power, namely,

$$m \equiv \frac{-\ln \mathcal{E}_0\{p_0/p_1\}}{\tau/n} = Z_0 - \sigma_p^2\frac{Y_0}{2}.$$

Hence, Equation (11) is equivalent to

$$R_0\left(\frac{\tau}{n}\right) = R_0^*\left(\frac{\tau}{n}\right) - m + \alpha\sigma_p\sigma_c\rho Y_0. \quad (12)$$

The last term in the above equation is the covariance between the price level p_1 and the growth rate of endowments $\ln g_1$. When the random shocks underlying the real state and the price level are uncorrelated ($\rho = 0$) or the "money neutrality" holds ($\sigma_p = 0$), the covariance is zero and the one-period real interest rate equals the one-period nominal interest rate adjusted by the expected depreciation rate of purchasing power. On the other hand, the real interest rate will be higher than the nominal interest rate adjusted by the depreciation rate of purchasing power if the covariance is positive, while it will be lower if the covariance is negative.

To see this, suppose that the covariance is positive. The real payoff of a real bond is always fixed and independent of the marginal utility of endowment. On the other hand, the real payoff of a nominal bond and the marginal utility of endowment tend to move together since a lower endowment will accompany a lower price level and a higher real payoff of a nominal bond, and vice versa. A real bond is therefore less appealing than a nominal bond and yields a higher interest rate.

The effects on the nominal interest rate of the parameters of the nominal variables are as follows. An increase in the expected inflation rate increases the nominal interest. The interpretation is obvious: a higher expected inflation rate decreases the real payoff provided by

a nominal bond and increases the nominal interest rate. When the covariance between the price level and the growth rate of endowments is positive, a higher variance of the inflation rate (σ_p) always increases the value of the certain payoff provided by a nominal bond and decreases the nominal interest rate. When the covariance is negative, the effect of an increase in σ_p is indeterminate because the real payoff of a nominal bond and the marginal utility of endowments tend to move in opposite directions. Finally, similar arguments show that an increase in ρ always decreases the nominal interest if σ_c is positive, while it decreases the nominal interest rate if σ_c is negative.

3.2 General *k*-period bonds

Similarly, an explicit expression can be derived for the two-period nominal bond price. However, the similarity between the formulation of the real structure and the nominal structure allows us to skip the intermediate steps and go to the general pricing formula for a *k*-period nominal bond. The following theorem is proved in the Appendix and provides a nice recursive representation of the general pricing formula.

Theorem 1. *For any given $k = 1,2,\ldots,n$, we have*

$$N_0\left(\frac{k\tau}{n}\right)=\left(\frac{\beta}{\epsilon^{\alpha}}\right)^{k\tau/n}\exp\left\{-\hat{A}\left(\frac{k\tau}{n}\right)-\hat{B}\left(\frac{k\tau}{n}\right)Y_0-\hat{C}\left(\frac{k\tau}{n}\right)-\hat{D}\left(\frac{k\tau}{n}\right)Z_0\right\}, \quad (13)$$

where $\hat{A}(\cdot)$, $\hat{B}(\cdot)$, $\hat{C}(\cdot)$, and $\hat{D}(\cdot)$ are functions defined on $\{0, \tau/n, \ldots, [(n-1)/n]\tau, \tau\}$, with $\hat{A}(0) = \hat{B}(0) = \hat{C}(0) = \hat{D}(0) = 0$, and satisfy

$$\begin{aligned}\hat{B}\left(t+\frac{\tau}{n}\right) &= \hat{B}(t) - \frac{\tau}{2n}\sigma_y^2\hat{B}^2(t) \\ &\quad + \frac{\tau}{n}\left[\alpha\mu_c - \frac{\alpha^2\sigma_c^2}{2} - \frac{\sigma_p^2}{2} - \alpha\sigma_c\sigma_p\rho \right. \\ &\quad \left. - (\eta + \alpha\sigma_c\sigma_y + \rho\sigma_p\sigma_y)\hat{B}(t)\right], \end{aligned} \quad (14)$$

$$\hat{A}\left(t+\frac{\tau}{n}\right) = \hat{A}(t) + \frac{\tau}{n}\eta\delta\hat{B}(t), \quad (15)$$

$$\hat{D}\left(t+\frac{\tau}{n}\right) = \hat{D}(t) + \frac{\tau}{n}[1 - \eta_z\hat{D}(t)], \quad (16)$$

$$\hat{C}\left(t+\frac{\tau}{n}\right) = \hat{C}(t) + \frac{\tau}{n}\eta_z\delta_z\hat{D}(t) - \frac{\tau}{2n}\sigma_z^2\hat{D}^2(t), \quad (17)$$

for

$$t = 0, \frac{\tau}{n}, \ldots, \frac{n-1}{n}\tau.$$

Hence, the k-period real discount bond price is

$$P_0\left(\frac{k\tau}{n}\right) = \left(\frac{\beta}{\epsilon^\alpha}\right)^{k\tau/n} \exp\left\{-\hat{A}_1\left(\frac{k\tau}{n}\right) - \hat{B}_1\left(\frac{k\tau}{n}\right)Y_0\right\},$$

where $\hat{A}_1(k\tau/n)$ and $\hat{B}_1(k\tau/n)$ are defined recursively in (14) and (15) with $\sigma_p = 0$. Recall that the one-period real interest rate equals the one-period nominal interest rate adjusted by the expected one-period depreciation rate of purchasing power if $\rho = 0$. This is no longer true for multiperiod bonds as long as $\sigma_p \neq 0$ because the correlation between the price level and growth rate of endowments is not zero except one period ahead.

Before proceeding, I must mention the key idea in the CIR model and the arbitrage models: the spot interest rate serves as the only instrumental variable for the underlying uncertainty in determining the term structure. This point can be clearly seen in this discrete-time economy by representing the state variable Y_0 in terms of the current one-period real interest rate $R_0(\tau/n)$ through Equation (8). Hence, the k-period real discount bond price can be expressed as a function of the one-period real interest rate:

$$P_0\left(\frac{k\tau}{n}\right) = \left(\frac{\beta}{\epsilon^\alpha}\right)^{k\tau/n} \exp\left\{-\hat{A}_0\left(\frac{k\tau}{n}\right) - \hat{B}_0\left(\frac{k\tau}{n}\right)\frac{R_0(\tau/n) + \ln(\beta/\epsilon^\alpha)}{\alpha(\mu_c - \alpha\sigma_c^2/2)}\right\}.$$

Similarly, the k-period nominal bond price can be expressed as a function of the one-period and two-period nominal interest rates. This has an important empirical implication. Despite the fact that the state variable Y and the expected inflation rate Z are unobservable, two observable interest rates can replace their roles in estimation and testing. I will discuss briefly the empirical implications of the model after deriving the pricing formula in the limit.

4. The Pricing Formula in the Limit

In this section, I first derive the limit pricing formula for the nominal discount bond by letting the number of periods between time 0 and time τ approach infinity. Then I show that the limit pricing formula is equal to the pricing formula in the corresponding continuous-time model.

4.1 The limit pricing formula

The following theorem, which is proved in the Appendix, gives the limit pricing formulas.

Theorem 2. *Suppose that*

$$(\eta + \alpha\sigma_c\sigma_y + \rho\sigma_p\sigma_y)^2 + 2\sigma_y^2(\alpha\mu_c - \alpha^2\sigma_c^2/2 - \sigma_p^2/2 - \alpha\sigma_c\sigma_p\rho) > 0. \quad (18)$$

When $n \to \infty$, we have

$$N_0(\tau) = (\beta/\epsilon^\alpha)^\tau A(\tau)C(\tau)\exp\{-B(\tau)Y_0 - D(\tau)Z_0\},$$

where

$$A(\tau) \equiv \begin{cases} \exp\left\{-\delta\left(\alpha\mu_c - \dfrac{\alpha^2\sigma_c^2}{2} - \dfrac{\sigma_p^2}{2} - \alpha\sigma_c\sigma_p\rho\right)\left[\tau + \dfrac{(e^{-\eta\tau} - 1)}{\eta}\right]\right\}, \\ \quad \text{if } \sigma_y = 0, \\ \left[\dfrac{2\gamma\exp\{\tau(\gamma + \eta + \alpha\sigma_c\sigma_y + \rho\sigma_p\sigma_y)/2\}}{(\gamma + \eta + \alpha\sigma_c\sigma_y + \rho\sigma_p\sigma_y)(\exp\{\gamma\tau\} - 1) + 2\gamma}\right]^{2\eta\delta/\sigma_y^2}, \\ \quad \text{otherwise,} \end{cases}$$

$$B(\tau) \equiv \frac{2(\exp\{\gamma\tau\} - 1)(\alpha\mu_c - \alpha^2\sigma_c^2/2 - \sigma_p^2/2 - \alpha\sigma_c\sigma_p\rho)}{(\gamma + \eta + \alpha\sigma_c\sigma_y + \rho\sigma_p\sigma_y)(\exp\{\gamma\tau\} - 1) + 2\gamma},$$

$$\gamma \equiv [(\eta + \alpha\sigma_c\sigma_y + \rho\sigma_p\sigma_y)^2 + 2\sigma_y^2(\alpha\mu_c - \alpha^2\sigma_c^2/2 - \sigma_p^2/2 - \alpha\sigma_c\sigma_p\rho)]^{1/2},$$

$$C(\tau) \equiv \exp\left\{\left(\frac{\sigma_z^2}{2\eta_z^2} - \delta_z\right)\tau + \left(\frac{\sigma_z^2}{\eta_z^3} - \frac{\delta_z}{\eta_z}\right)(e^{-\eta_z\tau} - 1) - \frac{\sigma_z^2}{4\eta_z^3}(e^{-2\eta_z\tau} - 1)\right\},$$

$$D(\tau) \equiv (1 - e^{-\eta_z\tau})/\eta_z.$$

Accordingly, the limit for the real bond price is $(\beta/\epsilon^\alpha)^\tau A(\tau) \cdot \exp\{-B(\tau)Y_0\}$ with $\sigma_p = 0$.

Let $r_0 \equiv -\ln P_0(\tau)/\tau$ and $r_0^* \equiv -\ln N_0(\tau)/\tau$ denote the current real and nominal instantaneous interest rates. Applying l'Hôpital's rule, we get

$$r_0 = \alpha\left(\mu_c - \frac{\alpha\sigma_c^2}{2}\right)Y_0 - \ln\left(\frac{\beta}{\epsilon^\alpha}\right),$$

$$r_0^* = r_0 + Z_0 - (\sigma_p^2 + 2\alpha\sigma_c\sigma_p\rho)Y_0/2.$$

Since an instant in time can be thought of as an infinitesimal time interval, it is no suprise that the roles of the instantaneous interest

rates in the continuous-time economy are exactly the same as the one-period rates in the discrete-time economy.

In the above, the limit pricing formula is derived by letting the time interval in the discrete-time economy go to zero. Is this limit the same as the nominal bond price in the continuous-time analogue? Not suprisingly, the answer is yes.

4.2 The continuous time analogue

The continuous-time analogues of Assumptions 1–4 are

$$dY = \eta(\delta - Y)\,dt + \sigma_y\sqrt{Y}\,dB^{(1)},$$

$$dC = [\ln \epsilon + (\mu_c + \sigma_c^2/2)Y]C\,dt + \sigma_c\sqrt{Y}C\,dB^{(1)},$$

$$dZ = \eta_z(\delta_z - Z)\,dt + \sigma_z\,dB^{(2)},$$

$$dp = (Z + \sigma_p^2 Y/2)p\,dt + \sigma_p\sqrt{Y}p\,dB^{(3)},$$

where $B^{(1)}$, $B^{(2)}$, and $B^{(3)}$ are three one-dimensional standard Brownian motions such that $B^{(2)}$ is independent of $B^{(1)}$ and $B^{(3)}$, while $B^{(1)}$ and $B^{(3)}$ are correlated with coefficient ρ. The following theorem shows that the pricing formula in the discrete-time model does converge to that in the continuous-time model.

Theorem 3. *The nominal bond price $N_0(\tau)$ in the continuous-time exchange economy is the same as that in Theorem 2.*

The proof of the theorem, given in the Appendix, is based on the first-order condition of the consumption and investment decision, specifically Equation (3). [Grossman and Shiller (1982), among others, have proved the necessity of this condition in a continuous-time economy if an equilibrium exists.] In contrast, the CIR model considers the term structure from the relationship between bond prices and equilibrium interest rates. The two approaches are equivalent because equilibrium interest rates are irrevocably linked to equilibrium consumption, a fact well recognized by financial economists and emphasized by Breeden (1986).

An immediate corollary of Theorem 3 is that the limit pricing formula for the real discount bond with appropriately specified parameters is the same as that given in the CIR model [1985, Equation (23), p.393].[7]

[7] Similar arguments as made in the proofs of Theorems 1 and 2 show that the limit distribution of $2cr_t$, conditional on r_0, is noncentral χ^2, a property characterizing the real interest rate in the CIR model.

Corollary 1. *Suppose that the utility function is logarithmic ($\alpha = 1$), and*

$$\epsilon = \beta, \qquad \delta = \frac{\theta}{e}, \qquad \eta = \kappa, \qquad \sigma_y = \frac{\sigma}{\sqrt{e}},$$
$$\mu_c = \left(1 + \frac{\lambda^2}{2\sigma^2}\right)e, \qquad \sigma_c = \frac{\lambda\sqrt{e}}{\sigma}, \tag{19}$$

where θ (> 0), κ (> 0), e (> 0), and σ are constants. The real term structure in our economy converges to that in the CIR model:

$$\begin{aligned} P_0(\tau) &= A_1(\tau)\exp\{-B_1(\tau)eY_0\} \\ &= A_1(\tau)\exp\{-B_1(\tau)r_0\}, \end{aligned}$$

with

$$A_1(\tau) = \left[\frac{2\gamma_1 \exp\{\frac{1}{2}\tau(\kappa + \lambda + \gamma_1)\}}{(\gamma_1 + \kappa + \lambda)(\exp\{\gamma_1\tau\} - 1) + 2\gamma_1}\right]^{2\kappa\theta/\sigma^2},$$

$$B_1(\tau) = \frac{2(\exp\{\gamma_1\tau\} - 1)}{(\gamma_1 + \kappa + \lambda)(\exp\{\gamma_1\tau\} - 1) + 2\gamma_1},$$

$$\gamma_1 = [(\kappa + \lambda)^2 + 2\sigma^2]^{1/2}.$$

For comparison, the parameter θ in (19) is the long-term mean of the equilibrium (instantaneous) interest rate in the CIR model and the parameter κ determines the speed of adjustment of the interest rate moving toward the long-term mean. The parameter σ is the coefficient of the constant relative variance of the interest rate. In other words, the equilibrium interest rate in the CIR model satisfies

$$dr_t = \kappa(\theta - r_t)\,dt + \sigma\sqrt{r_t}\,dB_t,$$

where B is a one-dimensional standard Brownian motion. The factor risk premium λr is the covariance of changes in the instantaneous interest rate with the percentage changes in optimally invested wealth, and the expected rate of return on the bond is $r + (\lambda r\, P_r/P)$. This is consistent with our interpretation of the conditional covariance (10), equal to $(\tau/n)\lambda r_0$ under (19), as a measure of hedging ability of a discount bond against unfavorable states. Last, the parameter e is equal to the difference of α and Ω, which are the parameters determining the dynamics of the production opportunity process X:

$$dX_t = \alpha Y_t X_t\,dt + \sqrt{\Omega Y_t}X_t\,dB_t.$$

5. Empirical Implications

Since the real discount bond price in the discrete-time economy contains that of the CIR model in the limit, this model provides an empirical framework to investigate the validity of the "money-neutrality" assumption made in CIR's nominal term structure models.[8] When the price level is correlated with the real state of economy, the conditional density or the steady-state distribution of the continuous-time processes is, in general, unknown. This poses a formidable obstacle for econometricians and explains why the existing empirical work on CIR cannot do away with the neutrality assumption.

In this section, I apply the method of maximum likelihood to estimate the parameters of the model. The intention is not to provide a thorough empirical investigation of the model. Instead, the focuses are (i) to provide preliminary evidence on the validity of the "money-neutrality" assumption, which underlies CIR's nominal bond prices and related empirical studies such as Gibbons and Ramaswamy (1992), Heston (1991), and Pearson and Sun (1991); (ii) to demonstrate how to conduct an empirical study of the model by using observable interest rates as the instrumental variables for the unobservable state variables; and (iii) to show that it is feasible to estimate a discrete-time model, but not the corresponding continuous-time model because neither the conditional density nor the steady-state distribution is known.

5.1 Econometric methods

Under Assumptions 1–4, $(Y_{i+1}, Z_{i+1}, \ln p_{i+1})$ conditional on $(Y_i, Z_i, \ln p_i)$ has a multivariate normal distribution and the density is

$$f(Y_{i+1}, Z_{i+1}, \ln p_{i+1} \mid Y_i, Z_i, \ln p_i) \propto \frac{\exp\{-\mathbf{M}_i^T \mathbf{V}_i^{-1} \mathbf{M}_i/2\}}{\sigma_y \sigma_z \sigma_p Y_i \sqrt{1-\rho^2}},$$

where "$\propto$" and "^{T}" mean "proportional to" and "transpose of a vector," respectively, and

$$\mathbf{M}_i \equiv \begin{pmatrix} Y_{i+1} - Y_i - \frac{\tau}{n}\eta(\delta - Y_i) \\ Z_{i+1} - Z_i - \frac{\tau}{n}\eta_z(\delta_z - Z_i) \\ \ln \frac{p_{i+1}}{p_i} - \frac{\tau}{n} Z_i \end{pmatrix},$$

[8] In addition to the general equilibrium model of the real term structure, Section 7 of CIR extends the theory to a world of inflation in which the exogenous price level has no effect on the real equilibrium. Under this neutrality assumption and different hypotheses on the price level, various pricing formulas for a nominal discount bond are derived.

$$\mathbf{V}_i^{-1} \equiv \frac{n}{\tau}\begin{pmatrix} \sigma_y^2 Y_i & 0 & \rho\sigma_y\sigma_p Y_i \\ 0 & \sigma_z^2 & 0 \\ \rho\sigma_y\sigma_p Y_i & 0 & \sigma_p^2 Y_i \end{pmatrix}^{-1}$$

$$= \frac{n}{\tau}\begin{pmatrix} \dfrac{1}{\sigma_y^2(1-\rho^2)Y_i} & 0 & \dfrac{-\rho}{\sigma_y\sigma_p(1-\rho^2)Y_i} \\ 0 & \dfrac{1}{\sigma_z^2} & 0 \\ \dfrac{-\rho}{\sigma_y\sigma_p(1-\rho^2)Y_i} & 0 & \dfrac{1}{\sigma_p^2(1-\rho^2)Y_i} \end{pmatrix}.$$

The conditional density determines the evolution of the term structure through time and enables us to apply the method of maximum likelihood. If observations of Y, Z, and p were available, the method of maximum likelihood could be directly applied. However, the state variable Y and the expected inflation rate Z are not observable. As in Pearson and Sun (1991), two nominal bond prices need to serve as the instrumental variables for Y and Z in estimation and testing.[9] More specifically, given two particular nominal discount bonds with times to maturity j_1 and j_2, let $N_i(j_k)$, $k = 1,2$, denote the observable nominal prices at time i for the two bonds. From (13), we can recover the two state variables from the two bond prices:

$$Y_i = \frac{\hat{D}(j_1)[\ln N_i(j_2) + \hat{E}(j_2)] - \hat{D}(j_2)[\ln N_i(j_1) + \hat{E}(j_1)]}{\hat{B}(j_1)\hat{D}(j_2) - \hat{D}(j_1)\hat{B}(j_2)}, \quad (20)$$

$$Z_i = \frac{\hat{B}(j_2)[\ln N_i(j_1) + \hat{E}(j_1)] - \hat{B}(j_1)[\ln N_i(j_2) + \hat{E}(j_2)]}{\hat{B}(j_1)\hat{D}(j_2) - \hat{D}(j_1)\hat{B}(j_2)}, \quad (21)$$

where

$$\hat{E}(j_k) \equiv \hat{A}(j_k) + \hat{C}(j_k) - j_k \ln(\beta/\epsilon^\alpha), \qquad k = 1,2.$$

Let $\mathbf{O}_i$ be the observable vector $(\ln N_i(j_1), \ln N_i(j_2), \ln p_i)$. The density for $\mathbf{O}_{i+1}$ conditional on $\mathbf{O}_i$ is therefore

$$f(\mathbf{O}_{i+1} \mid \mathbf{O}_i) \propto \frac{\exp\{-\mathbf{M}_i^T\mathbf{V}_i^{-1}\mathbf{M}_i/2\}}{\sigma_y\sigma_z\sigma_p Y_i\sqrt{1-\rho^2}\,|\hat{B}(j_1)\hat{D}(j_2) - \hat{D}(j_1)\hat{B}(j_2)|},$$

which is a function of the two bond prices and the price level through (20) and (21).

[9] Since there are only three state variables in the economy, the distribution of any three nominal bonds and the price level must be "degenerated" such that the conditional density does not exist.

Suppose that there are K observations of the two bond prices and the price level. The logarithm of the likelihood function is

$$\mathcal{L}(\beta, \epsilon, \alpha, \mu_c, \sigma_c, \eta, \delta, \sigma_y, \eta_z, \delta_z, \sigma_z, \sigma_p, \rho) \equiv \ln f(\mathbf{O}_2, \mathbf{O}_3, \ldots, \mathbf{O}_K \mid \mathbf{O}_1)$$
$$= \sum_{i=2}^{K} \ln f(\mathbf{O}_i \mid \mathbf{O}_{i-1}),$$

where the second equality follows from the fact that Y, Z, and p form a vector Markov process. Now the method of maximum likelihood can be applied to estimate the parameters of the model, based on the time series of the two bond prices and the price level.

Several remarks about the econometric method are in order. First, as pointed out in Pearson and Sun (1991), the above method can be applied to any two portfolios of bonds, although the calculation is much more complicated if coupon bonds are included. The method takes full advantage of the probability distribution of the state variables. However, this procedure fails to incorporate efficiently the cross-sectional information since it employs only two points on the yield curve at any instant of time. The information contained in the time series of two points is limited and may not be enough for us to have parameter estimates consistent with the entire yield curve.

Second, the estimation procedure does not really need to use the data of price levels. The above method can be based only on the conditional density of Y and Z and the two bond prices. The price level is included to demonstrate that it is conceptually and computationally straightforward to estimate the discrete-time model. On the other hand, it is impossible to utilize the data of price levels in estimating the corresponding continuous-time model because the conditional density of Y, Z, and p is unknown without the "money-neutrality" assumption.

Last, not all of the parameters can be estimated from a sample of bond prices and price levels. Only β/ϵ^{α}, $\alpha\mu_c$, and $\alpha\sigma_c$ are identifiable in the bond price formula (13). If real consumption data were available, we could separate α from other parameters by replacing the price level with the growth rate of consumptions in the estimation procedure. Moreover, the state variable Y needs to be normalized because its unit is not specified. For the empirical results, I normalize Y by imposing $\alpha\mu_c - \alpha\sigma_c^2/2 = 1$ so that the one-month real interest rate equals the state variable plus $\ln(\beta/\epsilon^{\alpha})$ [Equation (8)].

5.2 Data description

The empirical results reported here are based on 182 monthly observations from November 1971 to December 1986. Data on Treasury bills were obtained from the Government Bond Files of the Center

for Research in Security Prices (CRSP) at the University of Chicago. Though the CRSP tape contains the month-end prices of all the Treasury securities starting from December 1925, this preliminary investigation employs only the "on-the-run" three-month, six-month, and one-year Treasury-bill prices. ("On-the-run" issues are those just auctioned and the most liquid in the Treasury markets. Currently, there are 10 "on-the-run" issues: the three bills, 2-, 3-, 4-, 5-, 7-, and 10-year notes, and the 30-year bond.) Since using different bill prices does not change the results significantly, I will report the results based on the three-month and one-year bills. There are two reasons for employing only bill prices: it simplifies the calculation considerably, and the results are more comparable to Gibbons and Ramaswamy (1992). Besides, I started the sample from November 1971 because all the maturities have been continuously available since then. Hence, the results here can be readily compared with future studies that utilize price data of other Treasury issues.

Price levels are measured by the consumer price index (CPI) series published by the Bureau of Labor Statistics. It should be noted that there are measurement errors in treating the CPI as the month-end price level because it is compiled with prices that are usually sampled during a month. There is an extensive literature dealing with this time-averaging problem. For example, Working's (1960) classic article points out how the use of averaging can introduce correlations not present in the original series. More recently, Grossman, Melino, and Shiller (1987) show that substantial biases may be introduced if time-averaged consumption data are used to estimate a continuous-time consumption-based asset pricing model. They also provide an estimation procedure to correct the effects of time-averaging. Though the problem is not tackled here, a thorough empirical investigation of this model should take into account the time-averaging problem or avoid using the data of price levels.

5.3 Empirical results

The parameter estimates and their asymptotic standard errors are given in Table 1.[10] All the estimates except $\alpha\sigma_c$ are different from zero at the 2.5 percent significance level, and their implications are briefly discussed below.

First, the empirical evidence does not support the "money neutrality" assumption made in the nominal CIR model and related empirical studies. The estimate for ρ, the correlation coefficient between uncertainties underlying the real economy and the price

[10] In the estimation, the time interval is measured in units of years, and one period in the discrete-time economy corresponds to one month in calendar time. More specifically, one period is equal to one-twelfth unit of time, and the one-year bill is the 12-period discount bond.

Table 1
MLE parameter estimates

Process	Parameter	Estimate (SE[1])
The state of the economy:	$\widehat{\eta\delta}$	0.0602 (0.0103)**
$Y_{t+1} = Y_t + \eta(\delta - Y_t) + \sigma_y\sqrt{Y_t}\xi_{t+1}^{(y)}$	$\hat{\eta}$	1.1570 (0.2304)**
	$\hat{\sigma}_y$	0.1223 (0.0088)**
The growth rate of endowments:[2]	$(\widehat{\beta/\epsilon^\alpha})$	1.0345 (0.0059)**
$\ln g_{t+1} = \ln \epsilon + \mu_c Y_t + \sigma_c\sqrt{Y_t}\xi_{t+1}^{(y)}$	$\widehat{\alpha\sigma_c}$	−2.4647 (1.4045)*
	$\hat{\delta}_z$	0.0583 (0.0033)**
The expected inflation rate:	$\hat{\eta}_z$	−1.9185 (0.2845)**
$Z_{t+1} = Z_t + \eta_z(\delta_z - Z_t) + \sigma_z\xi_{t+1}^{(z)}$	$\hat{\sigma}_z$	0.0090 (0.0017)**
The price level:	$\hat{\sigma}_p$	0.0557 (0.0042)**
$\ln \pi_{t+1} = Z_t + \sigma_p\sqrt{Y_t}\xi_{t+1}^{(p)}$	$\hat{\rho}$	0.1565 (0.0746)*

The parameter estimates are based on 182 monthly observations of the three-month and one-year on-the-run Treasury-bill prices and the consumer price index. The sample period is from November 1971 to December 1986.

[1] The numbers in parentheses are the corresponding asymptotic standard errors.

[2] The state variable Y is normalized to be the one-month real interest rate plus $\ln(\beta/\epsilon^\alpha)$ by the constraint that $\alpha\mu_c - \alpha\sigma_c^2/2 = 1$.

* Rejection at the 5% level of significance.

** Rejection at the 1% level of significance.

level, is positive. Furthermore, the estimate for σ_p is significantly different from zero so that the price level is indeed affected by the real state of economy. Without the neutrality assumption, for example, the generalized method of moments employed in Gibbons and Ramaswamy (1992) is no longer valid. To evaluate the first moment and the cross-moments—Equations (13) and (15) of Gibbons and Ramaswamy—they need to know the joint density of the price level and the real interest rate, which, in general, is unknown in a continuous-time model.

Next, a strictly positive estimate for $\eta\delta$ conforms to the assumption that zero is a reflection boundary of the real state variable Y. Further, a strictly positive η implies that Y is adjusted toward a long-term mean of 0.052. Since Y is normalized to be the one-month real interest rate plus $\ln(\beta/\epsilon^\alpha)$, the one-month real interest rate is also mean-reverting with a long-term mean of 1.81 percent. On the other hand, the expected inflation rate Z does not exhibit the mean-reverting property since the estimate for η_z is strictly negative.

Recall that the unobservable real state variable and the expected inflation rate can be recovered from the two bill prices through (20) and (21). The time series of the monthly actual inflation rate, expected inflation rate, and one-month real interest rate are presented in Figure 1. (All rates are annualized.) The sample mean of real interest rates is 1.91 percent, little different from the long-term mean, and the sample standard deviation is 2.68 percent. The time series of the

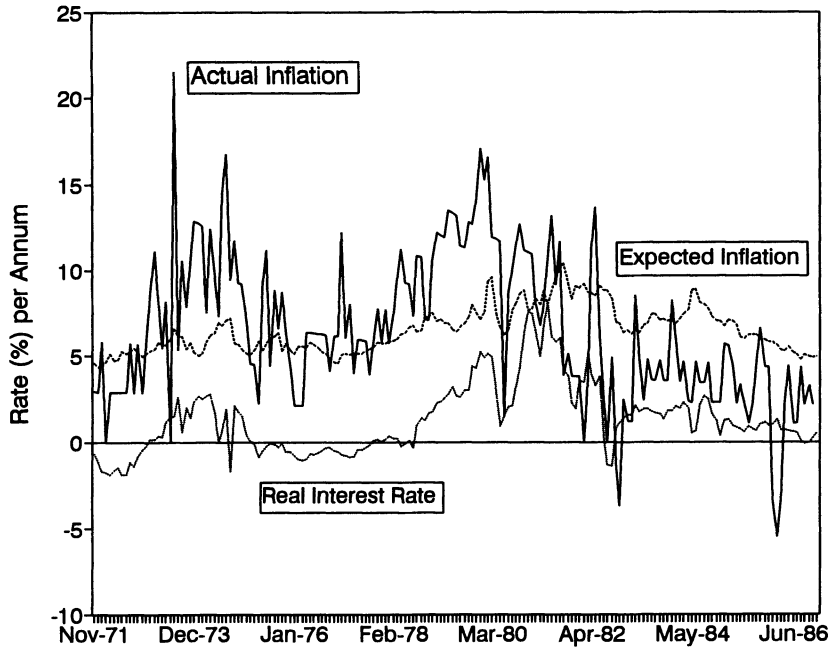

Figure 1
The monthly actual inflation rate, expected inflation rate, and one-month real interest rate from November 1971 to December 1986
The expected inflation rate and one-month real interest rate are recovered from Equations (20) and (21), where the estimates are maximum-likelihood estimates of the model based on 182 monthly observations of the three-month and one-year on-the-run Treasury-bill prices and the consumer price index.

implied real interest rate is very similar to that in Pennacchi (1991, Figure 7), where NBER–ASA survey forecasts of inflation, instead of the CPI, are used. Because $\alpha\sigma_c\sigma_p < 0$, Equation (12) indicates that the real interest rate is lower than the nominal interest rate adjusted by the depreciation rate of purchasing power. The expected inflation rate in Figure 1 is much smoother than the actual inflation rate. The sample mean and standard deviation are 6.59 percent and 4.39 percent for the actual inflation rate, and 5.92 percent and 0.48 percent for the expected inflation rate. This shows that most of the variability of the actual inflation rate is related to the real state of economy.

As for the parameter σ_c, a significantly negative estimate implies a positive factor premium. This finding is consistent with those in Gibbons and Ramaswamy (1992) and Pearson and Sun (1991).

There are two more empirical implications. The estimate for β/ϵ^α is not consistent with the real term structure model in CIR because it is significantly greater than 1. In addition, a positive estimate for σ_y means that the growth rate of endowments cannot be represented

as a first-order autocorrelated process as in Campbell (1986) and Turnbull and Milne (1991).

In summary, the preliminary empirical evidence indicates a significant correlation between the price level and the growth rate of endowments. Though the real state of economy is mean-reverting with a reflection boundary at zero as assumed in the CIR model, the parameter estimate for β/ϵ^{α} is not consistent with the model. Of course, all the claims are subject to more thorough future research.

6. Conclusion

In this article, I have provided a general equilibrium theory of the term structure of real interest rates in a sequence of discrete-time economies. I have presented a discrete-time analogue of the CIR model as a special case and have elucidated the essential ideas behind that model. Analysis based on a discrete-time framework can provide more intuition about the mechanism underlying the model. The key idea of the CIR model and the arbitrage models that the instantaneous interest rate is the only instrumental variable in determining the price of a discount bond can easily be seen in the discrete-time model by expressing the state variable in terms of the one-period spot interest rate. The characteristics of the state variable in the CIR model, a mean-reverting square-root process with a reflection boundary at zero, are also more easily understood in a discrete-time setting.

In addition, I have looked into the relation between the real and the nominal interest rates in a partial equilibrium setting in which an exogenous price level is correlated with the real economy. Since neither the conditional density nor the steady-state distribution is known in continuous time, it is infeasible to estimate the corresponding continuous-time model. However, it is straightforward in concept and computation to estimate the discrete-time model. Because the real bond price formula in CIR is nested in the model in the limit, I have provided an empirical framework to investigate the validity of the "money-neutrality" assumption.

Two possible directions for further research can be pursued along the line of this model. It can be used to investigate the relationship between bond and stock returns as in Campbell (1986), or to price the interest-rate contingent claims as in Turnbull and Milne (1991).

Appendix

Proof of Theorem 1

This theorem is an immediate corollary of the following proposition: for any given $k = 1,2,\ldots,n$, and $l = 1,2,\ldots,k$,

$$\mathcal{E}_{k-l}\left\{\prod_{i=k-l}^{k-1} g_{i+1}^{-\alpha}\,\pi_{i+1}^{-1}\right\} = \epsilon^{-\alpha l\tau/n}\exp\{-A_l - B_lY_{k-l} - C_l - D_lZ_{k-l}\},$$

where $A_0 = B_0 = C_0 = D_0 = 0$, and A_l, B_l, C_l, and D_l are determined by the recursion formulas

$$B_{i+1} = B_i + \frac{\tau}{n}\left[\alpha\mu_c - \frac{\alpha^2\sigma_c^2}{2} - \frac{\sigma_p^2}{2} - \alpha\sigma_c\sigma_p\rho - (\eta + \alpha\sigma_c\sigma_y + \rho\sigma_p\sigma_y)B_i\right] - \frac{\tau}{2n}\sigma_y^2B_i^2,$$

$$A_{i+1} = A_i + \frac{\tau}{n}\eta\delta B_i,$$

$$D_{i+1} = D_i + \frac{\tau}{n}(1 - \eta_zD_i),$$

$$C_{i+1} = C_i + \frac{\tau}{n}\eta_z\delta_zD_i - \frac{\tau}{2n}\sigma_z^2D_i^2, \qquad i = 0,1,\ldots,l-1.$$

The theorem follows from this by taking $l = k$, and defining $\hat{A}(i\tau/n) = A_i$ and so on.

We prove the proposition by induction on l. When $l = 1$, the proposition is clearly true: $B_1 = (\tau/n)(\alpha\mu_c - \alpha^2\sigma_c^2/2 - \sigma_p^2/2 - \alpha\sigma_c\sigma_p\rho)$, $D_1 = \tau/n$, and $A_1 = C_1 = 0$. A straightforward calculation shows that the proposition holds for l periods as long as it is true for $l - 1$ periods. This completes the induction arguments. ■

Proof of Theorem 2

For any $t = 0, \tau/n,\ldots,(n-1)\tau/n$, Equation (14) gives

$$\left(\frac{\tau}{n}\right)^{-1}\left[\hat{B}\left(t + \frac{\tau}{n}\right) - \hat{B}(t)\right] = \alpha\mu_c - \frac{\alpha^2\sigma_c^2}{2} - \frac{\sigma_p^2}{2} - \alpha\sigma_c\sigma_p\rho - (\eta + \alpha\sigma_c\sigma_y + \rho\sigma_p\sigma_y)\hat{B}(t) - \frac{\sigma_y^2}{2}\hat{B}^2(t).$$

Letting $n \to \infty$, we have the ordinary differential equation

$$\frac{d\hat{B}(t)}{dt} = \alpha\mu_c - \frac{\alpha^2\sigma_c^2}{2} - \frac{\sigma_p^2}{2} - \alpha\sigma_c\sigma_p\rho - (\eta + \alpha\sigma_c\sigma_y + \rho\sigma_p\sigma_y)\hat{B}(t) - \frac{\sigma_y^2}{2}\hat{B}^2(t),$$

with the initial condition $\hat{B}(0) = 0$. Under the restriction (18), the unique solution of this initial-valued ordinary differential equation is [see, e.g., Ford (1955, p. 35)]

$$\hat{B}(t) = \frac{2(\exp\{\gamma t\} - 1)(\alpha\mu_c - \alpha^2\sigma_c^2/2 - \sigma_p^2/2 - \alpha\sigma_c\sigma_p\rho)}{(\gamma + \eta + \alpha\sigma_c\sigma_y + \rho\sigma_p\sigma_y)(\exp\{\gamma t\} - 1) + 2\gamma}.$$

Similarly, taking $n \to \infty$ from Equation (15) gives the ordinary differential equation

$$\frac{d\hat{A}(t)}{dt} = \eta\delta\hat{B}(t),$$

with the initial condition $\hat{A}(0) = 0$. Integrating $\int_0^t \hat{B}(s)\, ds$, we have

$$\hat{A}(t) = \begin{cases} \delta\left(\alpha\mu_c - \dfrac{\alpha^2\sigma_c^2}{2} - \dfrac{\sigma_p^2}{2} - \alpha\sigma_c\sigma_p\rho\right)\left[t + \dfrac{e^{-\eta t} - 1}{\eta}\right], \\ \qquad \text{if } \sigma_y = 0, \\ \dfrac{-2\eta\delta}{\sigma_y^2}\ln\left[\dfrac{2\gamma\exp\{t(\gamma + \eta + \alpha\sigma_c\sigma_y + \rho\sigma_p\sigma_y)/2}{(\gamma + \eta + \alpha\sigma_c\sigma_y + \rho\sigma_p\sigma_y)(\exp\{\gamma t\} - 1) + 2\gamma}\right], \\ \qquad \text{otherwise.} \end{cases}$$

In exactly the same way we can show that as n approaches infinity, Equations (16) and (17) lead to

$$C(t) \equiv -\left(\frac{\sigma_z^2}{\eta_z^2} - \delta_z\right)t - \left(\frac{\sigma_z^2}{\eta_z^3} - \frac{\delta_z}{\eta_z}\right)(e^{-\eta_z t} - 1) + \frac{\sigma_z^2}{4\eta_z^3}(e^{-2\eta_z t} - 1),$$

$$D(t) \equiv (1 - e^{-\eta_z t})/\eta_z.$$

We complete the proof by defining $A(t) \equiv \exp\{-\hat{A}(t)\}$, $B(t) \equiv \hat{B}(t)$, $C(t) \equiv \exp\{-\hat{C}(t)\}$, and $D(t) \equiv \hat{D}(t)$. ■

Proof of Theorem 3

The unique solutions to the stochastic differential equations for the consumption and the price level are [see, e.g., Arnold (1974, Theorem 8.4.2)]

$$C_\tau = C_0 \exp\left\{\tau \ln \epsilon + \int_0^\tau \mu_c Y_t\, dt + \int_0^\tau \sigma_c\sqrt{Y_t}\, dB_t^{(1)}\right\},$$

$$p_\tau = p_0 \exp\left\{\int_0^\tau Z_t\, dt + \int_0^\tau \sigma_p\sqrt{Y_t}\, dB_t^{(3)}\right\}.$$

The nominal bond price from Equation (3) is therefore

$$N_0(\tau) = \left(\frac{\beta}{\epsilon}\right)^\tau \mathcal{E}_0\left\{\exp\left\{-\int_0^\tau Z_t\, dt - \int_0^\tau \alpha\mu_c Y_t\, dt \right.\right.$$
$$\left.\left. - \int_0^\tau \alpha\sigma_c\sqrt{Y_t}\, dB_t^{(1)} - \int_0^\tau \sigma_p\sqrt{Y_t}\, dB_t^{(3)}\right\}\right\}.$$

Let $F(\tau, Y_0, Z_0)$ denote the conditional expectation in the above equation. A multidimensional version of Lemma 6.1 in Sun (1987, p.65)

shows that $F(\tau, Y_0, Z_0)$ satisfies the following initial-valued partial differential equation:[11]

$$0 = \frac{1}{2}\sigma_y^2 Y \frac{\partial^2 F}{\partial Y^2} + \frac{1}{2}\sigma_z^2 \frac{\partial^2 F}{\partial Z^2} + [\eta_z(\delta_z - Z)]\frac{\partial F}{\partial Z}$$

$$+ [\eta(\delta - Y) - \alpha\sigma_c\sigma_y Y - \rho\sigma_p\sigma_y Y]\frac{\partial F}{\partial Y}$$

$$+ \left(\frac{\sigma_p^2 Y}{2} + \frac{\alpha^2\sigma_c^2 Y}{2} + \alpha\sigma_c\sigma_p Y - Z - \alpha\mu_c Y\right)F - \frac{\partial F}{\partial \tau},$$

$$1 = F(0, Y_0, Z_0).$$

The solution to the equation is $A(\tau)C(\tau)\exp\{-B(\tau)Y_0 - D(\tau)Z_0\}$, which proves that the nominal bond price $N_0(\tau)$ is exactly the same as in Theorem 2. ■

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[11] The lemma used here extends Theorem 8.4 in Durrett (1984, p. 234), which deals with only one-dimensional standard Brownian motion, to the case for multidimensional diffusion processes. The proof is omitted because it inevitably involves technical arguments in stochastic calculus. Please refer to Durrett (1984) or Sun (1987) for the ideas of the proof.

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