# Sources of Entropy in Representative Agent Models

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# Questions

Q1. What does the pricing kernel look like?

- **Dispersion**: **entropy**
- **Dynamics**: multiperiod entropy and **horizon dependence**
- **Disasters**: entropy and **high-order cumulants**
- Illustration: Vasicek model

Q2. How do these pricing kernels compare?

- Power utility
- Recursive preferences
- Habits
- Jumps and disasters

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# Questions

Q1. What does the pricing kernel look like?

- **Dispersion**: **entropy ("big")**
- **Dynamics**: multiperiod entropy and **horizon dependence ("small")**
- **Disasters**: entropy and **high-order cumulants**
- Illustration: Vasicek model

Q2. How do these pricing kernels compare?

- Power utility
- Recursive preferences
- Habits
- Jumps and disasters

# Facts about excess returns (monthly)

| Asset | Mean | Standard Deviation | Skewness | Excess Kurtosis |
|---|---|---|---|---|
| S&P 500 | 0.0040 | 0.0556 | −0.40 | 7.90 |
| Fama-French (small, low) | −0.0030 | 0.1140 | 0.28 | 9.40 |
| Fama-French (small, high) | 0.0090 | 0.0894 | 1.00 | 12.80 |
| S&P 500 6% OTM puts | −0.0184 | 0.0538 | 2.77 | 16.64 |
| Pound Sterling | 0.0035 | 0.0316 | −0.50 | 1.50 |
| 5-year bond | 0.0015 | 0.0190 | 0.10 | 4.87 |

Also: nominal yield spread at 60 months is about 0.001 a month

# Facts: summary

Facts

- "Big" excess returns $\gg$ equity premium
- "Small" bond premiums
- Skewness and kurtosis evident

Each tells us something about the pricing kernel

# Facts: summary

Facts

- "Big" excess returns $\gg$ equity premium **(dispersion)**
- "Small" bond premiums **(dynamics)**
- Skewness and kurtosis evident **(disasters)**

Each tells us something about the pricing kernel

# Dispersion: entropy

"Entropy" defined: for $m_{t+1} > 0$, (conditional) entropy is

$$L_t(m_{t+1}) \;=\; \log E_t m_{t+1} - E_t \log m_{t+1}$$

Cumulant expansion

$$L_t(m_{t+1}) \;=\; \underbrace{\kappa_{2t}(\log m_{t+1})/2!}_{\text{normal term}} + \underbrace{\kappa_{3t}(\log m_{t+1})/3! + \kappa_{4t}(\log m_{t+1})/4! + \cdots}_{\text{high-order cumulants}}$$

# Dispersion: entropy bound

Asset pricing theory: there exists $m_{t+1} > 0$ satisfying

$$E_t\left(m_{t+1} r_{t+1}\right) = 1$$

Entropy bound

$$\begin{aligned} L_t(m_{t+1}) &= \log \underbrace{E_t m_{t+1}}_{q_t^1 = 1/r_{t+1}^1} - E_t \log \underbrace{m_{t+1}}_{=1/r_{t+1}^*} \\ &\geq E_t(\log r_{t+1} - \log r_{t+1}^1) \\ \Rightarrow \quad E L_t(m_{t+1}) &\geq E(\log r_{t+1} - \log r_t^1) \end{aligned}$$

Conditional and unconditional entropy

$$L(m_{t+1}) = E L_t(m_{t+1}) + L(E_t m_{t+1})$$

# Dynamics: horizon dependence

Multiperiod entropy

$$
\begin{aligned}
m_{t,t+n} &= m_{t+1} m_{t+2} \cdots m_{t+n} \\
L_t(m_{t,t+n}) &= \underbrace{\log E_t m_{t,t+n}}_{\log q_t^n = -n y_t^n} - E_t \log m_{t,t+n} \\
n^{-1} E L_t(m_{t,t+n}) &= -E y_t^n - E \log m_{t+1}
\end{aligned}
$$

Horizon dependence

$$
H(n) \;=\; \underbrace{n^{-1} E L_t(m_{t,t+n})}_{\text{avg over n periods}} - \underbrace{E L_t(m_{t+1})}_{\text{one period}} \;=\; -E(y_t^n - y_t^1)
$$

# What the pricing kernel looks like: summary

Dispersion

- Entropy $\geq 0.01 = 1\%$ a month

Dynamics

- $|$Horizon dependence$| \leq 0.001 = 0.1\%$ a month (@ 60 months)

Disasters

- Log pricing kernel unlikely to be normal

# Vasicek model: a loglinear example

Pricing kernel has loglinear dynamics

$$\begin{aligned} \log m_t &= \log m + a(B) w_t \\ &= \log m + \underbrace{a_0 w_t}_{\text{entropy}} + \underbrace{a_1 w_{t-1} + a_2 w_{t-2} + \cdots}_{\text{horizon dependence}} \\ \{w_t\} &\sim \text{NID}(0,1) \end{aligned}$$

Examples

- White noise (iid): $a_0$ arbitrary, $a_j = 0$ for $j \geq 1$
- AR(1): $a_0$, $a_{j+1} = \varphi a_j$ for $j \geq 0$
- ARMA(1,1): $(a_0, a_1)$ arbitrary, $a_{j+1} = \varphi a_j$ for $j \geq 1$

# Vasicek model: entropy and horizon dependence

Partial sums

$$A_n = a_0 + a_1 + a_2 + \cdots + a_n$$

Entropy

$$\begin{aligned} EL_t(m_{t+1}) &= a_0^2/2 = A_0^2/2 \Rightarrow a_0 \quad \textbf{(“big”)} \\ EL_t(m_{t,t+n}) &= \sum_{j=1}^{n} A_{j-1}^2/2 \end{aligned}$$

Horizon dependence

$$H(n) = n^{-1} \sum_{j=1}^{n} (A_{j-1}^2 - A_0^2)/2 \Rightarrow a_j \quad \textbf{(“small”)}$$

# Vasicek model: interest rates

Short rate

$$\begin{aligned} \log r^1_{t+1} &= y^1_t \\ &= -\log m - A_0^2/2 - [a(B)/B]_+ w_t. \end{aligned}$$

Forward rates

$$f^n_t = -\log m - A_n^2/2 - [a(B)/B^n]_+ w_t$$

Typical term in mean spread $f^n - f^0$ and horizon dependence

$$\begin{aligned} A_0^2 - A_n^2 &= A_0^2 - [A_0 + (A_n - A_0)]^2 \\ &= -2A_0(A_n - A_0) - (A_n - A_0)^2 \end{aligned}$$

# Vasicek model: estimation strategy

Short rate governed by $(a_1, a_2, \ldots)$

- ARMA(1,1) pricing kernel $\Rightarrow$ AR(1) short rate
- Choose $(a_1, \varphi)$ to match variance and autocorrelation

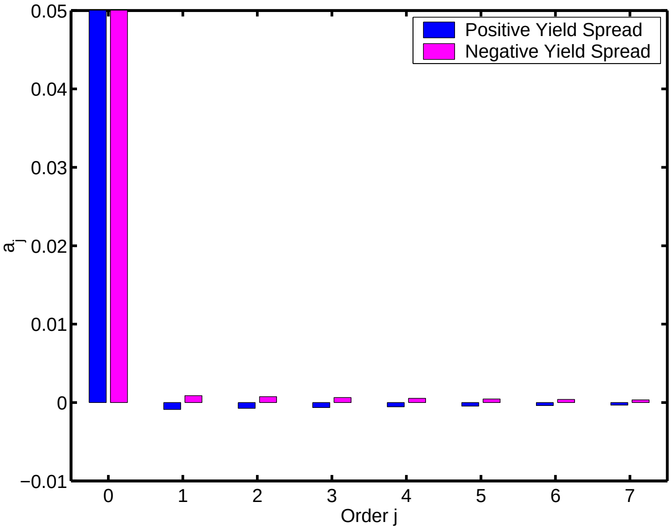
Mean yield spread governed by $a_0$ ("$\lambda$" in affine models)

- Choose $a_0$ to match mean yield spread for maturity 60 months

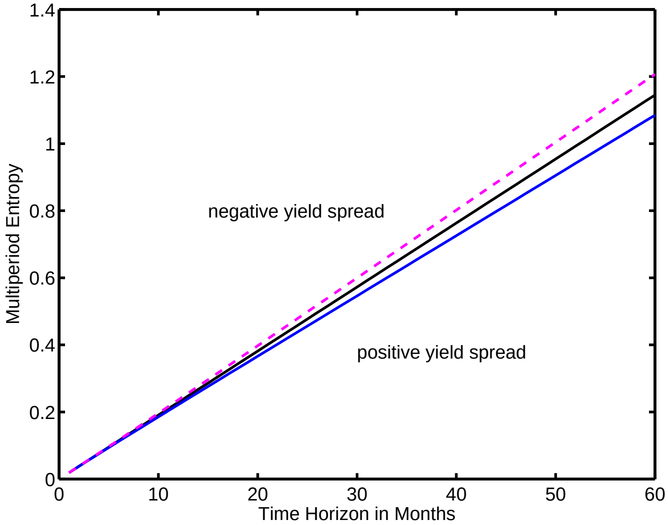
# Vasicek model: ARMA(1,1) review

show AR1, ARMA11

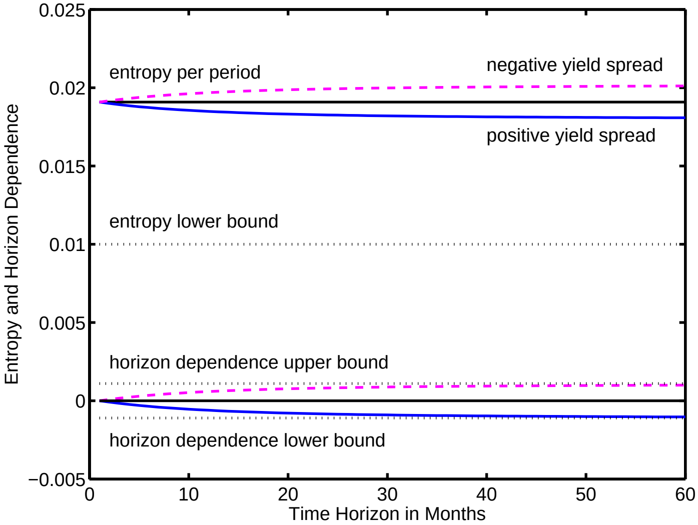
# Vasicek model: moving average coefficients



# Vasicek model: multiperiod entropy



# Vasicek model: horizon dependence



# Representative-agent models

Power utility

Recursive preferences

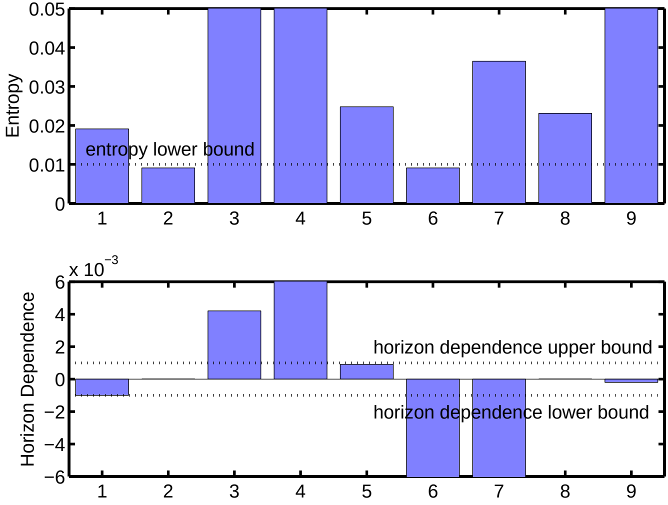
- Bansal-Yaron with persistent consumption growth
- ... and stochastic volatility

Habits

- Ratio habits
- Difference habits
- Campbell-Cochrane

Jumps and disasters

# Model summary



# Additive power utility

Consumption growth process ($g_t = c_t/c_{t-1}$)

$$\begin{aligned} \log g_t &= \log g + \gamma(B) v^{1/2} w_t \\ \{w_t\} &\sim \text{NID}(0,1) \end{aligned}$$

Pricing kernel

$$\begin{aligned} \log m_t &= \log \beta + (\rho - 1) \log g_t \\ &= \text{constants} + (\rho - 1)\gamma(B) v^{1/2} w_t \end{aligned}$$

# Recursive preferences

Preferences

$$\begin{aligned} U_t &= \left[(1-\beta)c_t^\rho + \beta\mu_t(U_{t+1})^\rho\right]^{1/\rho} \\ \mu_t(U_{t+1}) &= \left(E_t U_{t+1}^\alpha\right)^{1/\alpha} \\ & \alpha, \rho \leq 1 \end{aligned}$$

Interpretation

$$\begin{aligned} IES &= 1/(1-\rho) \\ CRRA &= 1-\alpha \\ \alpha &= \rho \quad \Rightarrow \quad \text{additive power utility} \end{aligned}$$

# Recursive preferences: basic analytics

Scale everything by $c_t$ ($u_t = U_t/c_t$, $g_{t+1} = c_{t+1}/c_t$)

$$u_t = [(1-\beta) + \beta\mu_t(g_{t+1}u_{t+1})^\rho]^{1/\rho}$$

Loglinear approximation (exact if $\rho = 0$: $b_0 = 0$, $b_1 = \beta$)

$$\log u_t \approx b_0 + b_1 \log \mu_t(g_{t+1}u_{t+1})$$

Pricing kernel

$$\begin{aligned} m_{t+1} &= \beta \underbrace{(c_{t+1}/c_t)^{\rho-1}}_{\text{short-run risk}} \underbrace{[U_{t+1}/\mu_t(U_{t+1})]^{\alpha-\rho}}_{\text{long-run risk}} \\ &= \beta\, g_{t+1}^{\rho-1} [g_{t+1}u_{t+1}/\mu_t(g_{t+1}u_{t+1})]^{\alpha-\rho} \end{aligned}$$

# Recursive preferences: consumption dynamics

Consumption growth

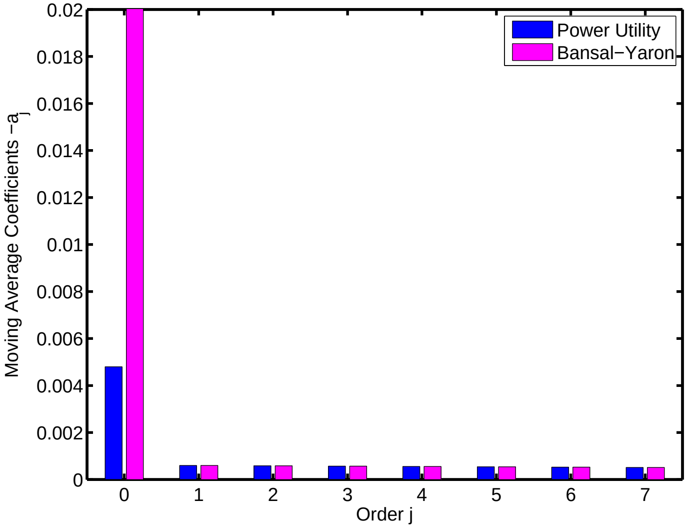
$$\begin{aligned} \log g_t &= g + \gamma(B) v^{1/2} w_t \\ \{w_t\} &\sim \text{NID}(0,1) \end{aligned}$$

Pricing kernel

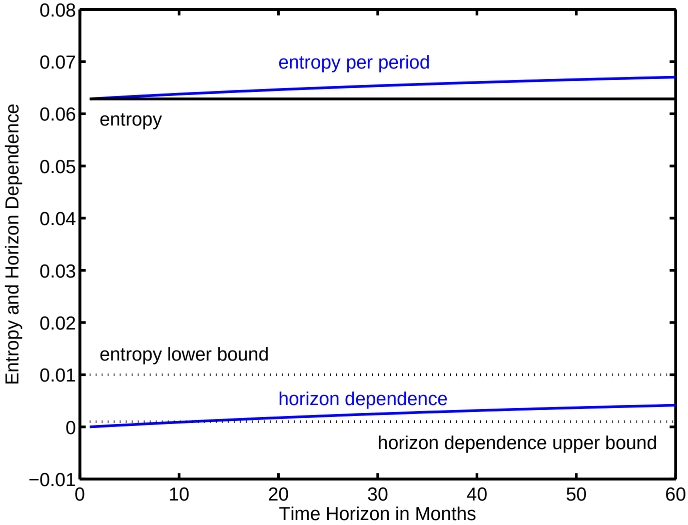
$$\begin{aligned} \log m_{t+1} = & \text{ constants} \\ & + \underbrace{[(\rho-1)\gamma_0 + (\alpha-\rho)\gamma(b_1)] v^{1/2}}_{a_0} w_{t+1} \\ & + \underbrace{(\rho-1)\gamma_1 v^{1/2}}_{a_1} w_t + \underbrace{(\rho-1)\gamma_1 v^{1/2}}_{a_2} w_{t-1} + \cdots \end{aligned}$$

Critical term: $\gamma(b_1) = \gamma_0 + b_1\gamma_1 + b_1^2\gamma_2 + \cdots$

# Recursive preferences: moving average coefficients



# Recursive preferences: entropy and horizon dependence



# Recursive preferences: and volatility dynamics

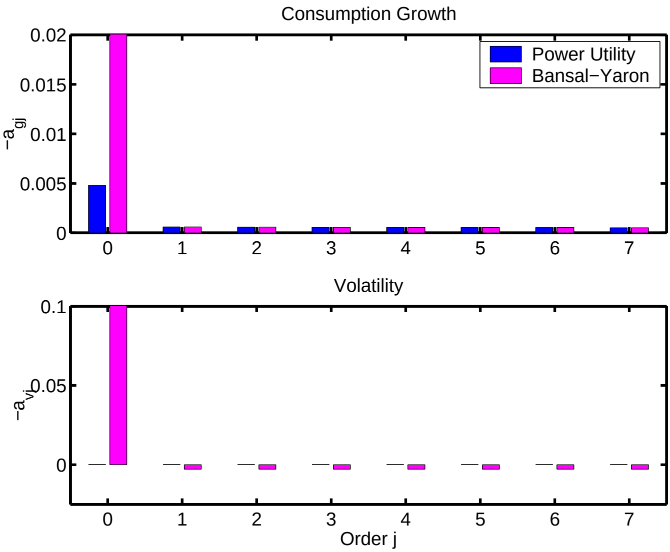
Consumption growth

$$
\begin{aligned}
\log g_t &= g + \gamma(B) v_{t-1}^{1/2} w_{gt} \\
v_t &= v + \nu(B) w_{vt} \\
\{w_{gt}, w_{vt}\} &\sim \text{NID}(0, I)
\end{aligned}
$$

Pricing kernel

$$
\begin{aligned}
\log m_{t+1} = \;& \text{constants} \\
& + [(\rho - 1)\gamma_0 + (\alpha - \rho)\gamma(b_1)] v_t^{1/2} w_{gt+1} \\
& + (\alpha - \rho)(\alpha/2)\gamma(b_1)^2 b_1 \nu(b_1) w_{vt+1} \\
& + (\rho - 1)[\gamma(B)/B]_+ v_{t-1}^{1/2} w_{gt} \\
& - (\alpha - \rho)(\alpha/2)\gamma(b_1)^2 \nu(B) w_{vt}
\end{aligned}
$$

# Recursive preferences: moving average coefficients



# Recursive preferences: numerical examples

| Parameter | Power Utility (1) | Bansal-Yaron Version 1 (2) | Bansal-Yaron Version 2 (3) | Bansal-Yaron Version 3 (4) |
|---|---|---|---|---|
| *Preferences* | | | | |
| $\alpha$ | $-9$ | $-9$ | $-9$ | $-9$ |
| $\rho$ | $-9$ | $1/3$ | $1/3$ | $1/3$ |
| *Consumption growth* | | | | |
| $\gamma_1$ | 0.1246 | 0.1246 | 0.1246 | 0.0250 |
| $\varphi_g$ | 0.9750 | 0.9750 | 0.9750 | |
| $\nu_0$ | 0 | 0 | $0.28 \times 10^{-5}$ | $0.28 \times 10^{-5}$ |
| $\varphi_v$ | | | 0.9990 | 0.9990 |
| *Entropy and horizon dependence* | | | | |
| $EL_t(m_{t+1})$ | 0.0026 | 0.0631 | 0.1319 | 0.0248 |
| $H(60)$ | 0.0305 | 0.0042 | 0.0077 | 0.0009 |

# Habits

Preferences

$$U_t = f(c_t, x_t) + \beta E_t U_{t+1}$$
$$x_t = \text{“external habit”}$$

Examples

- Ratio habit: $c_t / x_t$
- Difference habit: $c_t - x_t$
- Campbell-Cochrane: P2C2E

Standard inputs

$$\log g_t = \log g + \gamma(B) v^{1/2} w_t \quad [\gamma(B) = 1?]$$
$$\log x_t = \log x + \chi(B) \log c_{t-1} \quad [\chi(1) = 1?]$$

# Ratio habit

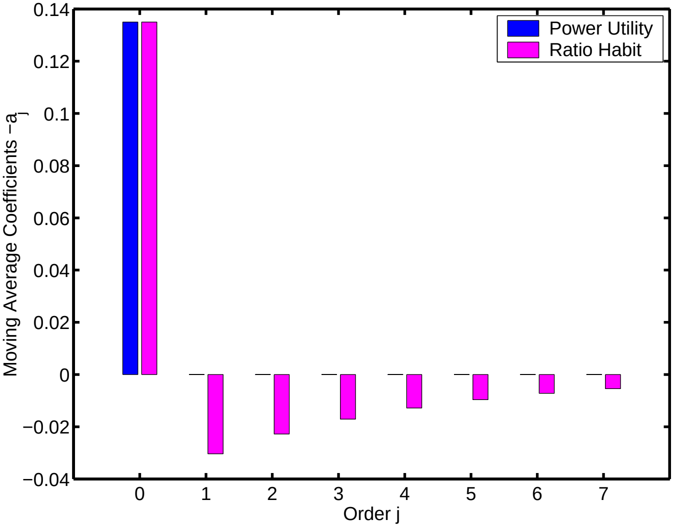
Preferences

$$f(c_t, x_t) \;=\; (c_t/x_t)^{\rho}/\rho, \quad \rho \leq 1$$

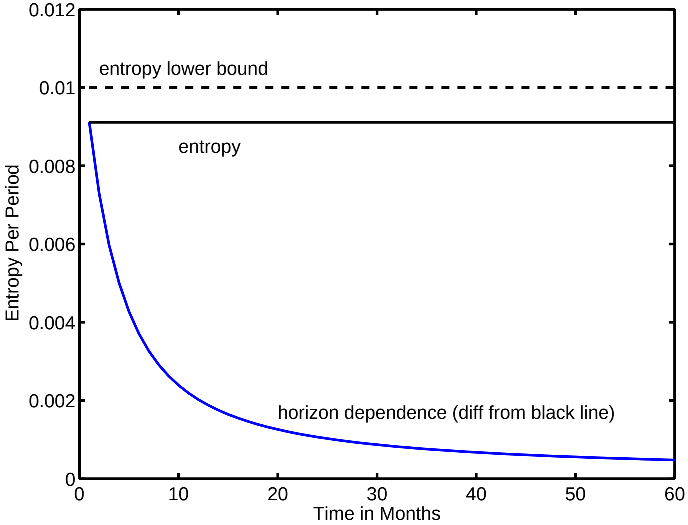
Pricing kernel

$$\begin{aligned} m_{t+1} &= \beta\,(c_{t+1}/c_t)^{\rho-1}\,(x_{t+1}/x_t)^{-\rho} \\ \log m_{t+1} &= \text{constants} \\ &\quad + [(\rho-1) - \rho B\chi(B)]\gamma(B) v^{1/2} w_{t+1} \end{aligned}$$

# Ratio habit: moving average coefficients



# Ratio habit: entropy and horizon dependence



# Difference habit

Preferences

$$f(c_t, x_t) = (c_t - x_t)^\rho/\rho, \quad \rho \leq 1$$

Define surplus

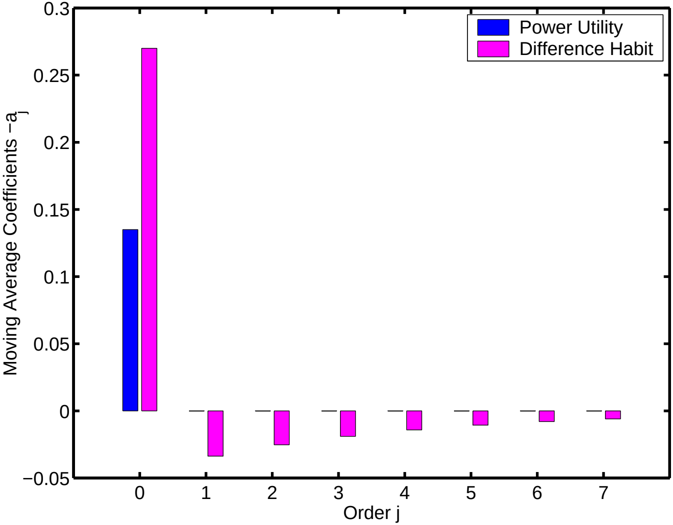
$$s_t = (c_t - x_t)/c_t = 1 - x_t/c_t$$

Pricing kernel

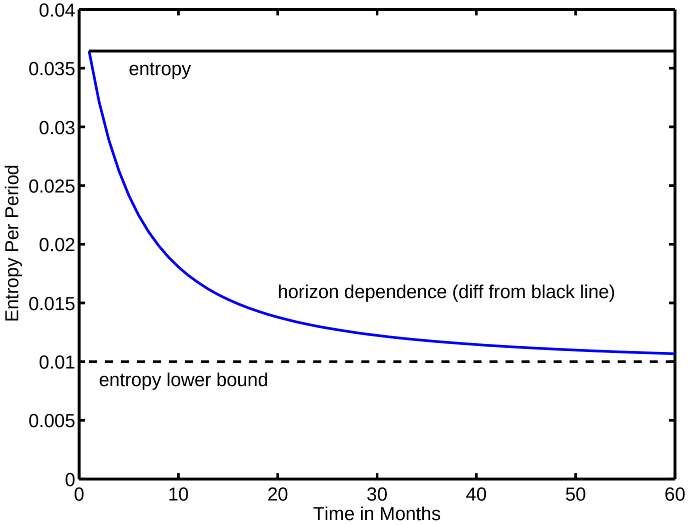
$$m_{t+1} = \beta g_{t+1}^{\rho-1} (s_{t+1}/s_t)^{\rho-1}$$

$$\log m_{t+1} = \text{constants} + (\rho - 1)(1/s)[1 - (1 - s)\chi(B)B]\gamma(B)v^{1/2}w_{t+1}$$

# Difference habit: moving average coefficients



# Difference habit: entropy and horizon dependence



# Campbell-Cochrane model

Surplus follows

$$\begin{aligned}
\log s_{t+1} - \log s_t &= (\varphi_s - 1)(\log s_t - \log s) + \lambda(\log s_t) v^{1/2} w_t \\
1 + \lambda(\log s_t) &= v^{-1/2} \left( \frac{(1-\rho)(1-\varphi_s) - b}{(1-\rho)^2} \right)^{1/2} \\
&\quad \times (1 - 2[\log s_t - \log s])^{1/2}
\end{aligned}$$

Designed to control horizon dependence

# Habits: numerical examples

| Parameter | Power Utility (1) | Ratio Habit (2) | Difference Habit (3) | Campbell-Cochrane (4) |
|---|---|---|---|---|
| *Preferences* | | | | |
| $\rho$ | $-9$ | $-9$ | $-9$ | $-1$ |
| *Consumption growth* | | | | |
| $v^{1/2}$ | 0.0135 | 0.0135 | 0.0135 | 0.0135 |
| *Habit* | | | | |
| $\chi_0$ | | 0.25 | 0.25 | |
| $\varphi_x$ or $\varphi_s$ | | 0.75 | 0.75 | 0.9885 |
| $s$ | | | 0.5 | |
| *Entropy and horizon dependence* | | | | |
| $EL_t(m_{t+1})$ | 0.0091 | 0.0091 | 0.0365 | 0.0231 |
| $H(60)$ | 0 | $-0.0086$ | $-0.0258$ | 0 |

# Recursive preferences: and jump risk

Consumption growth

$$\begin{aligned}
\log g_t &= g + v^{1/2} w_{gt} + z_t \\
h_t &= h + \eta(B) w_{ht} \\
(w_{gt}, w_{ht}) &\sim \text{NID}(0, I) \\
z_t | j &\sim \mathcal{N}(j\theta, j\delta^2) \\
j &\geq 0 \text{ has jump intensity } h_t
\end{aligned}$$

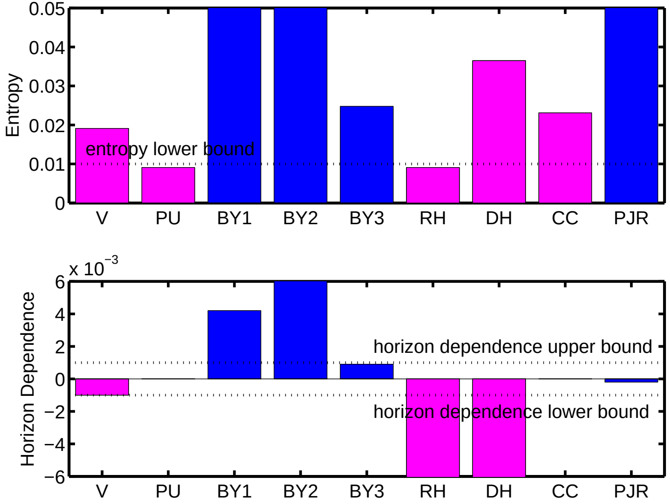
Pricing kernel

$$\begin{aligned}
\log m_{t+1} = & \text{ constants} \\
& + (\alpha - 1)(v^{1/2} w_{gt+1} + z_{gt+1}) \\
& + (\alpha - \rho)[(e^{\alpha\theta + (\alpha\delta)^2/2} - 1)/\alpha][b_1 \eta(b_1) w_{ht+1} - \eta(B) w_{ht}]
\end{aligned}$$

# Jump risk: numerical examples

| Parameter | Power Utility No Jump (1) | Power Utility Jump (2) | Persistent Jump Risk (3) |
|---|---|---|---|
| *Preferences* | | | |
| $\alpha$ | $-4$ | $-4$ | $-4$ |
| $\rho$ | $-4$ | $-4$ | 0 |
| *Consumption growth* | | | |
| $h \times 12$ | | 0.0100 | 0.0100 |
| $\theta$ | | $-0.3000$ | $-0.3000$ |
| $\delta$ | | 0.1500 | 0.1500 |
| $\eta_0$ | | | 0.0033 |
| $\varphi_h$ | | | 0.9934 |
| *Entropy and horizon dependence* | | | |
| $EL_t(m_{t+1})$ | 0.0023 | 0.0051 | 0.5193 |
| $H(60)$ | 0 | 0 | $-0.0002$ |

# Model summary



# Answers — and more questions

Q1. What does the pricing kernel look like?

- Excess returns suggest **entropy** $\geq 1\%$ monthly
- Yield spreads suggest **horizon dependence** $\leq 0.1\%$ monthly
- Probably not normal
- Useful diagnostics for any asset pricing model

Q2. How do representative-agent models compare?

- Most generate lots of entropy
- Several generate too much horizon dependence as a result
- Persistent jump risk helps
- All this conditional on parameter values

Q3. What's next?

- New parameter values?
- Heterogeneous agents?
- Imbed in business cycle models?

# Related work (some of it)

Bounds

- Alvarez-Jermann, Bansal-Lehmann, Hansen-Jagannathan

Recursive preferences

- Preferences: Epstein-Zin, Kreps-Porteus, Weil
- Asset pricing: Bansal-Yaron, Campbell, Hansen-Heaton-Li

Habits

- Abel, Campbell-Cochrane, Chan-Kogan, Constantinides, Heaton, Sundaresan

Jumps and disasters

- Barro, Barro-Nakamura-Steinsson-Ursua, Bekaert-Engstrom, Benzoni-Collin-Dufresne-Goldstein, Branger-Rodrigues-Schlag, Drechsler-Yaron, Eraker-Shaliastovich, Gabaix, Garcia-Luger-Renault, Longstaff-Piazzesi, Wachter

# What is entropy?

Humpy Dumpty (in "Through the Looking Glass")

> *"When I use a word," Humpty Dumpty said, "it means just what I choose it to mean — neither more nor less."*

Hans-Otto Georgii (quoted by Hansen and Sargent):

> *When Shannon had invented his quantity and consulted von Neumann on what to call it, von Neumann replied: "Call it entropy. It is already in use under that name and, besides, nobody knows what entropy is anyway."*

Paul Samuelson ("Gibbs in economics," 1989):

> *I have limited tolerance for the perpetual attempts to fabricate for economics concepts of "entropy."*

# What is entropy?

Reminder

$$L_t(m_{t+1}) \;=\; \log E_t m_{t+1} - E_t \log m_{t+1}$$

Note (with apologies for the notation)

$$\begin{aligned} L_t(k_t m_{t+1}) &= L_t(m_{t+1}) \text{ for } k_t > 0 \\ m_{t+1} &= q_t^1 p^*_{t+1}/p_{t+1} \end{aligned}$$

Therefore: entropy = relative entropy = Kullback-Leibler divergence

$$L_t(m_{t+1}) \;=\; L_t(p^*_{t+1}/p_{t+1}) \;=\; -E_t \log(p^*_{t+1}/p_{t+1})$$