The Elasticity of Substitution

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(A) Measuring Substitutability

Let us now turn to the issue of measuring the degree of substitutability between any pair of factors. One of the most famous ones is the elasticity of substitution, introduced independently by John Hicks (1932) and Joan Robinson (1933). Formally, the elasticity of substitution measures the percentage change in factor proportions due to a change in marginal rate of technical substitution. In other words, for our canonical production function, Y = � (K, L), the elasticity of substitution between capital and labor is given by:

$$\sigma = d \ln (L/K)/d \ln (�_K/�_L)$$

$$= [d(L/K)/d(�_K/�_L)]�[(�_K/�_L)/(L/K)]$$

The elasticity of substitution was designed as "a measure of the ease with which the varying factor can be substituted for others" (Hicks, 1932: p.117). [on the relationship between the Hicks and Robinson definitions, see R.F. Kahn (1933) and F. Machlup (1935).]

As Abba Lerner (1933) was quick to point out, the elasticity of substitution σ is effectively a measure of the curvature of an isoquant. Heuristically, this can be understood by referring to Figure 5.1. Suppose we move from point e to point e� on the isoquant. At point e, the MRTS is $�_K/�_L$, as represented by the slope of the line tangent to point e, while the labor-capital ratio is L/K, as represented by the slope of the chord connecting e to the origin. When we move to e� , the MRTS *increases* to $�_{K�}/�_{L�}$ while the labor-capital ratio increases to L� /K� . The elasticity of substitution, thus, compares the movement in the chord from L/K to L� /K� (denoted heuristically by Δ^R in Figure 5.1) to the movement in the MRTS from $�_K/�_L$ to $�_{K�}/�_{L�}$ (represented by Δ^M). The elasticity of substitution is thus, intuitively speaking, merely $\sigma = \Delta^R/\Delta^M$.

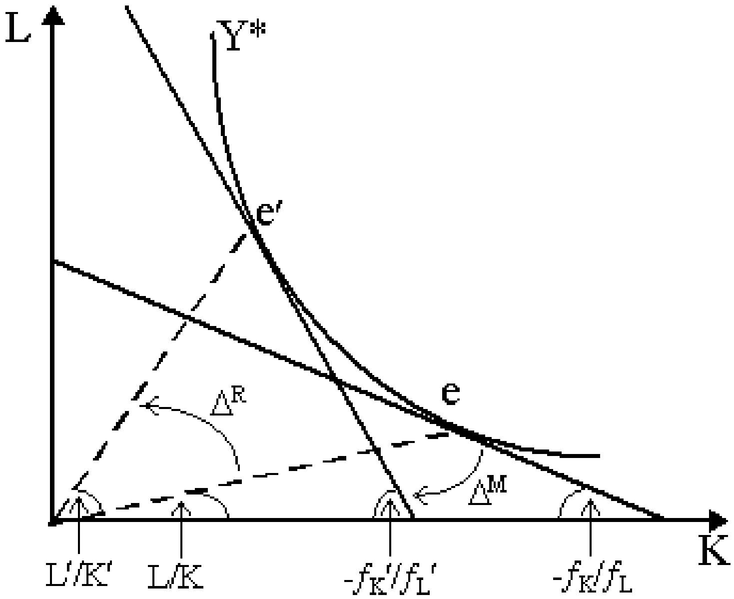

Figure 5.1 - Elasticity of Substitution

It is immediately deducible that, intuitively, the *more* curved or convex the isoquant is, the *less* the resulting change in the factor proportions will be (Δ^R is lower for the same Δ^M), thus the elasticity of substitution σ is *lower* for very curved isoquants. In the extreme case of Leontief (no-substitution) technology, where the L-shaped isoquants are as "curved" as can be (as shown in our earlier Figure 4.1), a change in MRTS will *not* lead to *any* change in the factor proportions, i.e. $\Delta^R = 0$ for any Δ^M. Thus, $\sigma = 0$ for Leontief isoquants.

The other extreme case of *perfect substitution* or *linear* production technology is shown in Figure 5.2. This represents the case when machines are perfectly substitutable for laborers. In other words, adding a laborer and taking out a machine will not lead to any change in the marginal products of either of them as one is perfectly substitutable for another. A production function which exhibits this can be written as a linear function:

$$Y = � (K, L) = \alpha K + \beta L$$

where α, β are constants. Notice that $dY/dK = \alpha$ and $dY/dL = \beta$, thus the marginal products of capital and labor are constant and MRTS = α/β, which is also constant. Thus, as shown in Figure 5.2, the isoquants are straight lines, indicating a constant marginal rate of technical substitution.

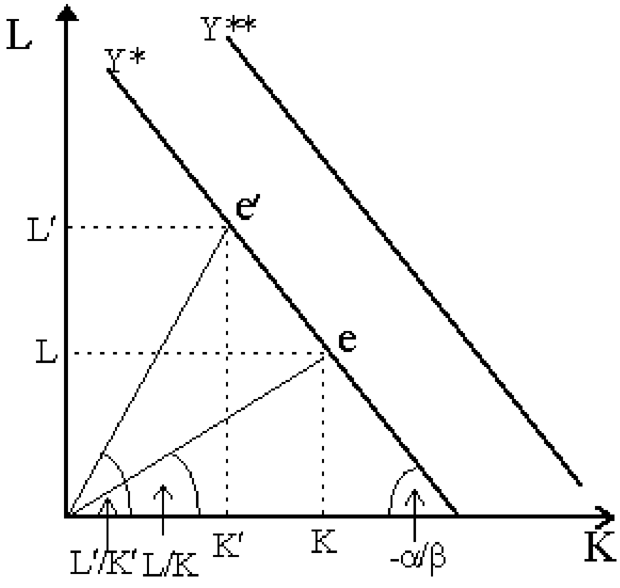

Figure 5.2 - Perfect Substitute Isoquants

Notice that as the MRTS does not change at all along the isoquant, then $\Delta^{M} = 0$. Consequently, the elasticity of substitution of perfect substitute production functions is infinite, i.e. σ = � .

In sum, then, we see that in general, for any production technology, as σ � � , we approach perfect substitutability between factors, while as σ � 0, we approach no substitution between factors. Intuitively, it is clear why. If σ is very high, then a small percentage change in the MRTS will engender a very large percentage change in the labor-capital ratio. In order for the input mix to react so violently, they must be very good substitutes. Conversely, if σ is very low, a large percentage shift in MRTS barely budges the factor input mix. Thus, if factor proportions are held on to so tightly, they must be needed in relatively fixed proportions.

As such, we can see that the assumption of diminishing marginal productivity, which the early economists struggled with, gains a more interesting and straightforward meaning when viewed in terms of the elasticity of substitution. As we see, diminishing marginal productivity necessarily implies that σ < � . Thus, as Joan Robinson points out, what the assumption of diminishing marginal productivity "really states is that there is a limit to the extent to which one factor of production can be substituted for another, or, in other words, the elasticity of substitution between factors is not infinite" (J. Robinson, 1933: p.330).

The elasticity of substitution can be expressed in various forms. Let Y = � (K, L) be our production function. Now, we know:

$$\sigma = [d(L/K)/d(�_K/�_L)]�(�_K/�_L)/(L/K)]$$

Now, totally differentiating the expression $�_K/�_L$ with respect to K and L, we obtain:

$$d(�_K/�_L) = [� (�_K/�_L)/� K]�dK + [� (�_K/�_L)/� L]�dL$$

and, by the definition of the isoquant, $�_K/�_L$ = - dL/dK, or dK = -($�_L/�_K$)dL, so:

$$d(�_K/�_L) = [� (�_K/�_L)/� K]�(-�_L/�_K)dL + [� (�_K/�_L)/� L]dL$$

or simply:

d(�$_K$/�$_L$) = {�$_K$[� (�$_K$/�$_L$)/� L] -�$_L$[� (�$_K$/�$_L$)/� K]}dL/�$_K$

Now, totally differentiating the expression L/K, we obtain:

d(L/K) = (KdL -LdK)/K^2

or, again as dK = -(�$_L$/�$_K$)dL by the isoquant, this becomes:

d(L/K) = [K + L�(�$_L$/�$_K$)]dL/K^2

= [�$_K$K + �$_L$L]dL/�$_K$K^2

thus dividing this through by d(�$_K$/�$_L$):

d(L/K)/d(�$_K$/�$_L$) = [�$_K$K + �$_L$L]/{K^2(�$_K$[� (�$_K$/�$_L$)/� L] -�$_L$[� (�$_K$/�$_L$)/� K])}

Now, dividing through by L/K and multiplying by �$_K$/�$_L$, we obtain the expression for the elasticity of substitution

σ = [d(L/K)/d(�$_K$/�$_L$)]�[(�$_K$/�$_L$)/(L/K)] =

{�$_K$[�$_K$K + �$_L$L]}/{�$_L$KL(�$_K$[� (�$_K$/�$_L$)/� L] -�$_L$[� (�$_K$/�$_L$)/� K])}

All that remains is to evaluate the terms � (�$_K$/�$_L$)/� L and � (�$_K$/�$_L$)/� K. Now,

� (�$_K$/�$_L$)/� K = [�$_{KK}$�$_L$ - �$_{LK}$�$_K$]/�$_L^2$

� (�$_K$/�$_L$)/� L = [�$_{KL}$�$_L$ - �$_{LL}$�$_K$]/�$_L^2$

thus, combining, we see that:

�$_K$[� (�$_K$/�$_L$)/� L] -�$_L$[� (�$_K$/�$_L$)/� K] = �$_K$[�$_{KL}$�$_L$ - �$_{LL}$�$_K$]/�$_L^2$ - �$_L$[�$_{KK}$�$_L$ - �$_{LK}$�$_K$]/�$_L^2$

= (2�$_{KL}$�$_L$�$_K$ - �$_{LL}$�$_K^2$ - �$_{KK}$�$_L^2$) /�$_L^2$

as, by Young's Theorem, �$_{KL}$ = �$_{LK}$. Thus, we see that plugging back into our expression, we now have:

σ ={�$_L$�$_K$[�$_K$K + �$_L$L]}/{ KL(2�$_{KL}$�$_L$�$_K$ - �$_{LL}$�$_K^2$ - �$_{KK}$�$_L^2$)}

which is our alternative expression for σ . This expression is notable for the fact that the term within the brackets in the denominator is merely the determinant of the bordered Hessian formed by the production function. Recall that for our particular case this is:

$$|B| = \begin{array}{ccc} 0 & \text{�}_{L} & \text{�}_{K} \\ \text{�}_{L} & \text{�}_{LL} & \text{�}_{KL} \\ \text{�}_{K} & \text{�}_{LK} & \text{�}_{KK} \end{array}$$

or:

$$|B| = 2\text{�}_{KL}\text{�}_{L}\text{�}_{K} - \text{�}_{LL}\text{�}_{K}^{2} - \text{�}_{KK}\text{�}_{L}^{2}$$

Also note that the term �$_{L}$�$_{K}$ is actually the cofactor of the LKth term in the Hessian matrix, i.e. �$_{L}$�$_{K}$ = $|B_{LK}|$. Thus, the elasticity of substitution can be written as:

$$\sigma = ((\text{�}_{K}K + \text{�}_{L}L)/KL)\text{�}(|B_{LK}|/|B|)$$

Now, recall that quasi-concavity implies that $|B| > 0$, thus automatically we obtain the result that $\sigma > 0$ for quasi-concave production functions with two factors. This, of course, is as is should be expected. Namely, recall that quasi-concavity of the production function implies convexity of the isoquants and that, in turn, implies a diminishing MRTS. Now, a diminishing MRTS, as is obvious from the earlier diagrammatic exposition, implies that K/L and �$_{K}$/�$_{L}$ move in opposite directions as we go along an isoquant, or, equivalently, that L/K and �$_{K}$/�$_{L}$ move in the same direction. But this last is precisely what σ measures, thus its positivity.

Finally, notice that as, by Young's Theorem, �$_{LK}$ = �$_{KL}$, we have the immediate implication that $|B_{LK}| = |B_{KL}|$ and thus that:

$$\sigma = d \ln (L/K)/d \ln (\text{�}_{K}/\text{�}_{L}) = d \ln (K/L)/d \ln (\text{�}_{L}/\text{�}_{K})$$

so that the elasticity of substitution is symmetric.

(B) Elasticity of Substitution under Constant Returns to Scale

The elasticity of substitution has interesting expressions when the two-input production function exhibits constant returns to scale. Firstly, under constant returns to scale, Euler's Theorem implies that Y = �$_{K}$K + �$_{L}$L, thus our expression becomes immediately:

$$\sigma = \{\text{�}_{L}\text{�}_{K}Y\}/\{ KL(2\text{�}_{KL}\text{�}_{L}\text{�}_{K} - \text{�}_{LL}\text{�}_{K}^{2} - \text{�}_{KK}\text{�}_{L}^{2})\}$$

Now, recall once again, that if the production function � is homogeneous of degree one in the factors (constant returns), then the marginal product �$_{i}$ is homogeneous of degree zero in teh factors. This implies, again by Euler's Theorem, that:

$$\text{�}_{KK}K + \text{�}_{KL}L = 0$$

$$�_{LK}K + �_{LL}L = 0$$

so $�_{KK} = -�_{KL}(L/K)$ and $�_{LL} = -�_{LK}(K/L)$. Thus substituting in:

$$\sigma = \{�_L �_K Y\}/\{ KL(2�_{KL}�_L �_K + �_{LK}(K/L)�_K^2 + �_{KL}(L/K)�_L^2)\}$$

$$= \{�_L �_K Y\}/\{�_{KL}(2�_L �_K KL + K^2 �_K^2 + L^2 �_L^2)\}$$

$$= \{�_L �_K Y\}/\{�_{KL}(�_K K + �_L L)^2\}$$

so, by Euler's Theorem again:

$$\sigma = �_L �_K Y/�_{KL} Y^2$$

or simply:

$$\sigma = �_L �_K/�_{KL} Y$$

which is considerably more simple. This expression for the elasticity of substitution in the constant returns to scale case was precisely the form in which it was first introduced by John Hicks (1932: p.117, 245).

Notice that this form implies that as $�_{KL}$ increases, σ declines. This has a very intuitive interpretation. The *easier* it is to substitute labor for capital, then the *less* the marginal rate of technical substitution rises during the process. This is precisely what the relation between $�_{KL}$ and σ expresses: namely, the less an increase in the amount of labor L raises the marginal product of capital $�_K$, the more is the ease and thus the elasticity of substitution.

Recall that when we have constant returns to scale, then we can express a production function $Y = �(K, L)$ in intensive form as $y = \phi(k)$, where $y = Y/L$ and $k = K/L$. We also know that $�_K = \phi_k$ and $�_L = y - \phi_k k$. Thus, $�_L/�_K = (y-\phi_k k)/\phi_k$. As a result, the elasticity of substitution can be written in intensive form as:

$$\sigma = d \ln (K/L)/ d \ln (�_L/�_K) = d \ln k/ d \ln ((y-\phi_k k)/\phi_k)$$

or:

$$\sigma = [d \ln ((y-\phi_k k)/\phi_k)/d \ln k]^{-1}$$

$$= \{[d((y-\phi_k k)/\phi_k)/dk]�[k\phi_k/(y-\phi_k k)]\}^{-1}$$

Now, since $d[(y-\phi_k k)/\phi_k]/dk = -\phi_{kk\phi k}k - \phi_{kk}[y - \phi_k k]\}/\phi_k^2$ thus:

$$\sigma = \{[-\phi_{kk\phi k}k - \phi_{kk}[y - \phi_k k]\}/\phi_k]�[k/(y-\phi_k k)]\}^{-1}$$

which simplifies to:

$$\sigma = \{- y\phi_{kk}k/\phi_{k}(y-\phi_{k}k)]\}^{-1}$$

or simply:

$$\sigma = - \phi_{k}(y-\phi_{k}k)/y\phi_{kk}k$$

which holds for any two-input constant returns to scale production function.

An alternative convenient expression of σ in the constant returns case is the following. Recall that since � $_L$ = y - ϕ_kk, then d� $_L$/dk = -ϕ_{kk}k. Thus, since (dy/d� $_L$)�(d� $_L$/dk) = ϕ_k, then dy/d� $_L$ = -ϕ_k/ϕ_{kk}k. As a result:

d ln y/d ln � $_L$ = (dy/d� $_L$)�(� $_L$/y)

= -[ϕ_k/ϕ_{kk}k]�[(y - ϕ_kk)/y]

= -ϕ_k(y - ϕ_kk)/yϕ_{kk}k

= σ

by the expression given earlier. Thus the elasticity of substitution of a constant returns to scale production function can be expressed as the elasticity of output per capita with respect to the marginal product of labor.

(C) Cobb-Douglas Production Functions

If we have $\sigma = 1$, then a 10% change in MRTS will yield a 10% change in the input mix. This unit-elasticity curve will give our isoquants their traditional, very nice, gently convex shape. A famous case is the well-known Cobb-Douglas production function introduced by Charles W. Cobb and Paul H. Douglas (1928), although anticipated by Knut Wicksell (1901: p.128, 1923) and, some have argued, J.H. von Th�nen (1863). [for a review of theoretical and empirical literature on the Cobb-Douglas production function, see Douglas (1934, 1967), Nerlove (1965) and Samuelson (1979)]

The Cobb-Douglas production function normally has the form akin to the following for our canonical case:

$$Y = AK^{\alpha} L^{\beta}$$

where A, α and β are constants. Let us derive the elasticity of substitution from this. As we know from before, in Cobb-Douglas production functions, � $_K$ = αY/K and � $_L$ = β Y/L, thus:

�$_K$/�$_L$ = (α/β)�(L/K)

Consequently, it follows that:

σ = (β /α)�[(α /β)�(L/K)]/(L/K) = 1

as we announced.

We can now turn to an interesting exercise: namely, that if we have a constant returns to scale production function *and* the elasticity of substitution is 1, then the form of the production function is *necessarily* Cobb-Douglas. To see this, recall that when we have constant returns to scale and $\sigma = 1$, then we can write it as:

$\sigma = d \ln y/d \ln$ � $_L = 1$

Integrating:

$\ln y = \ln$ � $_L + a$

where a is a constant of integration. Consequently, taking the antilog:

$y =$ � $_L e^a$

as � $_L = y - \phi_k k$ by constant returns, then:

$y = (y - \phi_k k)b$

where $b = e^a$. Then:

$(b-1)y = b\phi_k k$

Now, as $\phi_k = dy/dk$, then this can be rewritten as $(b-1)y = bk(dy/dk)$, or:

$(1/y)$�$dy = ((b-1)/bk)$�dk

integrating:

� $1/y\ dy =$ � $(b-1)/bk\ dk$

which yields:

$\ln y = [(b-1)/b]$�$\ln k + c$

$= \ln [k^{(b-1)/b}] + c$

where c is a constant of integration. Taking the anti-log:

$y = e^c k^{(b-1)/b}$

Letting $e^c = A$ and $(b-1)/b = \alpha$, then this becomes:

$y = Ak^{\alpha}$

Consequently, as $k^{\alpha} = (K/L)^{\alpha} = K^{\alpha} L^{-\alpha}$, then multiplying the expression through by L, we obtain:

$Y = AK^{\alpha} L^{1-\alpha}$

which is the Cobb-Douglas form. Thus,Cobb-Douglas is the *only* form which a constant returns to scale production function with $\sigma = 1$ can take.

(D) Constant Elasticity of Substitution (CES) Production Functions

Now, recall that the bordered Hessian, |B|, is evaluated at a particular point on the production function. Different points on the production function might yield different |B|. Consequently, as |B| enters directly into σ , it is not surprising that σ could be different at different places on the production function. Thus, in general, σ is not constant.

A special class of production functions, known as *Constant Elasticity of Substitution* (CES) production functions, were introduced by Arrow, Chenery, Minhas and Solow (1961) (thus it is also known as the ACMS function). It was generalized to the n-factor case by Hirofumi Uzawa (1963) and Daniel McFadden (1963). A CES function, as its name indicates, possesses a constant σ throughout. The CES production function takes the following famous form in the two-input case:

$$Y = \tau [\alpha K^{-\rho} + (1-\alpha)L^{-\rho}]^{-r/\rho}$$

where r denotes the degree of homogeneity of the function; $\tau > 0$ is the efficiency parameter which represents the "size" of the production function; α is the distribution parameter which will help us explain relative factor shares (so 0 � α � 1); and ρ is the substitution parameter, which will help us derive the elasticity of substitution. Notice that marginal products are:

�$_K = \alpha\tau^{\rho-1}(Y/K)^{\rho+1}$

�$_L = (1-\alpha)\tau^{\rho-1}(Y/L)^{\rho+1}$

thus, immediately we see that MRTS is:

�$_K$/�$_L = (\alpha/(1-\alpha))(L/K)^{\rho+1}$

Thus, in order for there to be decreasing MRTS (i.e. convex isoquants), we must assume that the substitution parameter takes on the value ρ � -1. It can be shown that for the constant returns to scale case (r = 1), the elasticity of substitution of a CES production function will be $\sigma = 1/(1+\rho)$, thus we can see immediately that it does not depend on *where* on the production function we are as ρ is given exogenously.

Notice also that if we have a Cobb-Douglas production function with constant returns to scale, then r = 1 and $\rho = 0$ so that $\sigma = 1$. It is not difficult to show that in this case, the CES production function takes the familiar Cobb-Douglas constant returns to scale form (apply l'H�pital's rule to obtain this). Other substitution parameter values are also rather straightforward: ρ � � implies σ � 0, i.e. Leontief (no-substitution); ρ � -1 implies σ � � (i.e. perfect substitutes).

(E) Elasticities of Substitution in Multi-Input Cases

It should be noted that the positivity of σ relies to a good extent on the fact that we are, so far, assuming that L and K are substitutes. Specifically, as noted, σ measures the degree of substitutability between two goods and thus the only allowance for complementarity we make is the Leontief case, when $\sigma = 0$. However, we are, so far, restricting ourselves to a two-input world, where the degree of complementarity is necessarily restricted. In a more general case, when there are many inputs available, the degree of complementarity may be such that the elasticity of substitution is negative, i.e. $\sigma < 0$.

Extending the concept of the elasticity of substitution from a two-input production function into one with three or more inputs invites complications. When measuring the elasticity of substitution between two factors when there are other factors in the production function, one must take care of controlling for possible cross effects. There are different schools of thought on the appropriate measure for the elasticity of substitution between inputs i and j in the context of a wider, multiple-input production function $y = \text{�} (x_1, x_2, .., x_m)$.

Three famous measures will be briefly mentioned. The simplest and most obvious measure is the *direct elasticity of substitution* between two factors x_i and x_j and is denoted:

$$\sigma_{ij}^D = ((\text{�}_i x_i + \text{�}_j x_j)/x_i x_j)\text{�}(|B_{ij}|/|B|)$$

Specifically, x_i and x_j are the quantities of the inputs, �_i and �_j are their marginal products, $|B|$ is the determinant of the bordered Hessian and $|B_{ij}|$ is the cofactor of �_{ij} (in our earlier case, this was $|B_{KL}| = \text{�}_{K\text{�}L}$). Thus, the direct elasticity is identical to our earlier two-input case, thus, effectively, it is assuming that the other factors in the production function are fixed and thus can be ignored.

Roy G.D. Allen (1938: p.503-5) proposed a different measure, the *Allen elasticity of substitution* (also known as the *partial elasticity of substitution*) and is defined as:

$$\sigma^A_{ij} = ((\text{�}_i \text{�}_i x_i)/x_i x_j)\text{�}(|B_{ij}|/|B|)$$

where, notice, the numerator holds a larger sum. Notice that if the total number of factors is two, this reduces to the direct elasticity of substitution, i.e. $\sigma_{ij}^D = \sigma_{ij}^A$. This is perhaps the most popular measure of the elasticity of substitution in general applications, although, intuitively, it seems somewhat amorphous.

We can obtain an interesting alternative expression for the Allen elasticity of substitution. As we shall see later, it turns out that from cost-minimization decision of the firm, we will obtain:

$$\sigma_{ij}^A = \varepsilon_{ij}/s_j$$

where $\varepsilon_{ij} = \text{�} \ln x_i / \text{�} \ln w_j$, i.e. the elasticity of the demand for the ith factor (x_i) with respect to the price of the jth factor (w_j). The term $s_j = w_j x_j / \text{�}_{i=1}^m w_i x_i$, where the numerator $w_j x_j$ is the expenditure by the producer on the jth factor and the denominator $\text{�}_i w_i x_i$ is total expenditures. Thus, s_j is the the jth factor's share of total expenditures by the producer. This will be useful later in determining the properties of the derived demand for factors.

An alternative measure of elasticity of substitution in the multi-factor case was proposed by Michio Morishima (1967) known as the *Morishima elasticity of substitution* and defined as:

$$\sigma_{ij}^M = (\text{�}_j/x_i)\text{�}(|B_{ij}|/|B|) - (\text{�}_j/x_j)\text{�}(|B_{ij}|/|B|)$$

which has the seemingly unusual property of being asymmetric, i.e. σ_{ij}^{M} � σ_{ji}^{M}. This, as Blackorby and Russell (1981, 1989) argue, *should* be natural for a multi-factor case. It is an algebraic matter to note that we re-express the Morishima measure in terms of the Allen measure as follows:

$$\sigma_{ij}^{M} = (\text{�}_j x_j / \text{�}_i x_i)(\sigma_{ij}^{A} - \sigma_{jj}^{A})$$

where σ_{ij}^{A} and σ_{jj}^{A} are Allen elasticities of substitution. One of the implications we should observe is that the Morishima measure also classifies factors somewhat differently from Allen's measure. More specifically, for any two inputs, x_i and x_j, it may be that $\sigma^{M}_{ij} > 0$ but that $\sigma_{ij}^{A} < 0$, so that by the Morishima measure, the inputs are substitutes, but by the Allen measure, the inputs are complements. In general, factors that are substitutes by the Allen measure, will be substitutes by the Morishima measure; but factors that are complements by the Allen measure may still be substitutes by the Morishima measure. Thus, the Morishima measure has a bias towards treating inputs as substitutes (or, alternatively, the Allen measure has a bias towards treating them as complements). This apparently paradoxical result in the Allen and Morishima measures is actually not too disturbing: it reflects the fluidity of the concept of elasticity of substitution in a multiple factor world. For a comparison between them (and defense of the Morishima elasticity), see Blackorby and Russell (1981, 1989).

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