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Multiresolution approximation scale and time-shift subspaces

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Abstract Multiresolution Approximation subspaces are $\mathcal{L}^2(\mathbb{R})$-subspaces defined for each scale over all time shifts, i.e., "scale subspaces", while with respect to a given wavelet, the signal space $\mathcal{L}^2(\mathbb{R})$ not only admits orthogonal scale subspaces basis, but orthogonal "time shift subspaces" basis as well. It is therefore natural to expect both scale subspaces and time shift subspaces to play a role in Wavelet Theory and, in particular, in Multiresolution Approximation as well. This is what will be discussed in the paper.

Keywords Wavelet · Multiresolution approximation scale · Time-shift subspaces · Time-shift discrete wavelet transform · Time-shift multiresolution approximation

Mathematics Subject Classifications 42C05 · 47A15

1 Introduction

Multiresolution Approximation (or Analysis) (MRA) (Meyer, 1987, Mallat,, 1989a, Mallat, 1989b) plays a key role in Wavelet Theory. Heuristically speaking, a MRA is a family of nested $\mathcal{L}^2(\mathbb{R})$-subspaces—called *scaling approximation subspaces,* or simply scaling subspaces—with prescribed properties, defined for each scale over *all* time shifts. The MRA results in a method of constructing wavelets from scaling functions—generating the MRA, as well as decomposition of $\mathcal{L}^2(\mathbb{R})$-signals into scaling component and detail component.

It appears that time shift, for the most part, is "invisible" in Wavelet Theory. This, perhaps is due to the fact that with respect to a wavelet, the signal space $\mathcal{L}^2(\mathbb{R})$ is usually represented

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by scale orthogonal subspaces basis (o.s.b.) consisting of subspaces—defined for *each scale over all time shifts.* As a consequence any signal is the sum—over all scales—of its "layers of detail". However, the space $\mathcal{L}^2(\mathbb{R})$ can also be represented by time shift o.s.b. consisting of subspaces—defined for *each time shift over all scales* (Levan & Kubrusly, 2003). Hence any signal is also the sum —over all time shifts—of its layers of detail.

This paper is a sequel to Levan and Kubrusly (2004) in which we studied Time-Shifts Generalized MRA associated with a wavelet. Our goal here is to bring time shift to MRA. We show that each scaling MRA subspace, besides being expressed in terms of scaling function and its time shifts, as well as being represented by scale detail subspaces defined in terms of wavelet functions, can also be expressed in terms of time shift detail subspaces. This is obtained by means of $\mathcal{L}^2(\mathbb{R})$-time-shift o.s.b. which is the key tool of the paper.

Section 2 briefly recalls scale and time shift subspaces constructed from wavelet functions. These subspaces then serve as building blocks for representing Scaling MRA subspaces and Associated Scale Detail MRA subspaces, to be discussed in Sect. 3. Section 4 introduces the concept of Time Shift Detail MRA subspaces, while Sect. 5 discusses scale-time-shift approximation in $\mathcal{L}^2(\mathbb{R})$. Section 6 closes the paper with a formulation of the concept of Time Shift Multiresolution Approximation.

2 Scale and time shift detail subspaces

Let $\psi(\cdot) \in \mathcal{L}^2(\mathbb{R})$ be a wavelet (Lemarié & Meyer, 1986, Chui, 1992, Mallat, 1998) then the space $\mathcal{L}^2(\mathbb{R})$, by definition, inherits the double-indexed orthonormal basis (o.n.b.) $\{\psi_{m,n}(\cdot)\}_{(m,n)\in\mathbb{Z}^2}$ consisting of wavelet functions $\psi_{m,n}(\cdot)$ defined in terms of the wavelet $\psi(\cdot)$ as

$$\psi_{m,n}(\cdot) := \sqrt{2}^{\,m}\psi\big(2^m(\cdot) - n\big), \tag{2.1}$$

$$= D^m T^n \psi(\cdot), \quad (m,n) \in \mathbb{Z}^2, \tag{2.2}$$

where

$$Tf(\cdot) := f\big((\cdot) - 1\big) \quad \text{and} \quad Df(\cdot) := \sqrt{2} f\big(2(\cdot)\big) \tag{2.3}$$

are, respectively, time-shift-by-1 and dilation-by-2 bilateral shift operators of countably infinite multiplicity (Halmos, 1961, Sz-Nagy & Foias, 1970) on $\mathcal{L}^2(\mathbb{R})$.

In the following $m \in \mathbb{Z}$ always associates with scale 2^m and it is also referred to as "*level m*" (Keinert, 2004), while $n \in \mathbb{Z}$ is responsible for time shift. Moreover, since

$$\psi_{m,n}(t) = \sqrt{2}^{\,m}\psi\big(2^m\big[t - n\,\tfrac{1}{2^m}\big]\big), \quad (m,n) \in \mathbb{Z}^2$$

the wavelet function $\psi_{m,n}(\cdot)$ is shifted in step of $\frac{1}{2^m}$—as n varies. Also, the closure of the span (called closed spans) of a set A is denoted by $\bigvee A$ (i.e., $\bigvee A = \overline{\operatorname{span}}\,A$) and the orthogonal direct sum of a collection $\{\mathcal{M}_k\}$ of orthogonal subspaces is written as $\bigoplus_k \mathcal{M}_k$.

Remark 1 A procedure involving closed spans and orthogonal direct sum which will be used throughout is, as usual, *closed span of orthogonal* (*closed*) *subspaces of a Hilbert space* (i.e., *the topological sum of orthogonal* (*closed*) *subspaces*) *is identified with their orthogonal sum*—these are in fact unitarily equivalent (see e.g., [(Kubrusly, 2003), p37]). Also, subspaces spanned by wavelet functions are, by tradition, characterized as "detail" subspaces (Mallat, 1989b; Mallat, 1998).

A wavelet functions o.n.b. $\{\psi_{m,n}(\cdot)\}_{(m,n)\in\mathbb{Z}^2}$ can be converted into a scale detail orthogonal subspaces basis, denoted by $\{\mathcal{W}_m(\psi)\}_{m\in\mathbb{Z}}$, and defined as closed spans of $\psi_{m,n}(\cdot)$ over all n:

$$\mathcal{W}_m(\psi) := \overline{\text{span}}\left\{\psi_{m,n}(\cdot),\ \forall\, n \in \mathbb{Z}\right\}, \tag{2.4}$$

$$= \bigvee_{n=-\infty}^{\infty} D^m\psi\big((\cdot)-n\big) = D^m\mathcal{W}_0(\psi), \quad m \in \mathbb{Z} \tag{2.5}$$

and is called "scale detail subspace (or layer) for scale m". Moreover, since D has a bounded inverse, D^m can be taken out of the closed span symbol (cf. (Levan & Kubrusly, 2003, Kubrusly & Levan, 2004)), and the subspace

$$\mathcal{W}_0(\psi) := \bigvee_{n=-\infty}^{\infty} \psi\big((\cdot)-n\big) \tag{2.6}$$

is a generating wandering subspace for the bilateral shift D (Sz-Nagy & Foias, 1970) (see also (Kubrusly & Levan, 2004); wandering because

$$D^m\mathcal{W}_0(\psi) \perp D^{m'}\mathcal{W}_0(\psi), \quad \text{whenever} \quad m \neq m' \tag{2.7}$$

and generating since

$$\mathcal{L}^2(\mathbb{R}) = \bigoplus_{m\in\mathbb{Z}} \mathcal{W}_m(\psi) = \bigoplus_{m\in\mathbb{Z}} D^m\mathcal{W}_0(\psi). \tag{2.8}$$

This decomposition is classic, not only in Wavelet Theory (Goodman, Lee, & Tang, 1993, Antoniou & Gustafson, 1999), but in Hilbert space bilateral shift theory as well, since such a decomposition actually defines a bilateral shift whose multiplicity is the dimension of the generating wandering subspace $\mathcal{W}_0(\psi)$ (Sz-Nagy & Foias, 1970). What is not classic, perhaps, is the fact that the space $\mathcal{L}^2(\mathbb{R})$ can also be represented by o.s.b.$\{\mathcal{H}_n\}_{n\in\mathbb{Z}}$ consisting of closed spans of $\psi_{m,n}(\cdot)$ over all m (Levan & Kubrusly, 2003),

$$\mathcal{L}^2(\mathbb{R}) = \bigoplus_{n\in\mathbb{Z}} \mathcal{H}_n(\psi), \tag{2.9}$$

where

$$\mathcal{H}_n(\psi) := \overline{\text{span}}\left\{\psi_{m,n}(\cdot) := D^mT^n\psi(\cdot),\ \forall\, m \in \mathbb{Z}\right\}, \tag{2.10}$$

$$= \bigvee_{m=-\infty}^{\infty} D^m\psi\big((\cdot)-n\big), \quad n \in \mathbb{Z} \tag{2.11}$$

called "the time shift detail subspace (or layer) for time shift n".

The subspaces $\mathcal{W}_m(\psi)$ are neither D-invariant nor D^*-invariant, while the subspaces $\mathcal{H}_n(\psi)$ are D-invariant and D^*-invariant, which means that they are D-reducing. Similarly, the subspaces $\mathcal{H}_n(\psi)$ are neither $T_{\frac{1}{2^m}}$-invariant nor $T^*_{\frac{1}{2^m}}$-invariant, while $\mathcal{W}_m(\psi)$ are $T_{\frac{1}{2^m}}$-invariant and $T^*_{\frac{1}{2^m}}$-invariant (Keinert, 2004). These, in some sense, show "opposite" characteristics of scale subspaces and time shift subspaces. With respect to $T_{\frac{1}{2^m}}$-invariance and $T^*_{\frac{1}{2^m}}$-invariance we also refer to (Kubrusly & Levan, to appear) where these aspects are treated in detail.

We note that since $\mathcal{H}_n(\psi)$ is D-reducing, it can be represented as

$$\mathcal{H}_n(\psi) = \bigoplus_{m\in\mathbb{Z}} [D_n]^m\,\mathcal{W}_{0,n}(\psi), \quad n \in \mathbb{Z}, \tag{2.12}$$

where $D_n := D|\mathcal{H}_n(\psi)$—the part of D on $\mathcal{H}_n(\psi)$—and

$$\mathcal{W}_{0,n}(\psi) := \bigvee \psi\big((\cdot) - n\big), \quad n \in \mathbb{Z} \tag{2.13}$$

with the indices $(0, n)$ indicating that $\mathcal{W}_{0,n}(\psi)$ is for scale 2^0 and for time shift n.

It is plain that (2.13) is simply a "multiplicity-1" analog of the orthogonal decomposition (2.9). It then follows that the D-reducing subspaces $\mathcal{H}_n(\psi)$ should, somehow, inherit some wavelet properties associated with the bilateral shift D. These will become clear later.

3 Scaling MRA scale subspaces and scale detail subspaces

We now recall definition of (Discrete) MRA—with scaling function $\phi(\cdot)$ (Meyer, 1987, Mallat,, 1989a, Mallat, 1989b).

Definition 1 A set of closed subspaces $\{\mathcal{V}_m\}_{m\in\mathbb{Z}}$, called scaling subspaces, is a MRA—with scaling function $\phi(\cdot)$ — over $\mathcal{L}^2(\mathbb{R})$ if the following properties are satisfied:

(0) $\exists\ \phi(\cdot):\ \mathcal{V}_0 := \bigoplus_{n\in\mathbb{Z}} \bigvee \phi\big((\cdot) - n\big)$ and $\mathcal{V}_{m+1} = D\mathcal{V}_m,\ \ (m, n) \in \mathbb{Z}^2$,
(1) $\mathcal{V}_m \subset \mathcal{V}_{m+1},\ \ m \in \mathbb{Z}$,
(2) $\bigcap_{m\in\mathbb{Z}} \mathcal{V}_m = \{0\}$,
(3) $\overline{\bigcup_{m\in\mathbb{Z}}} \mathcal{V}_m = \mathcal{L}^2(\mathbb{R})$.

Let $\{\mathcal{V}_m\}_{m\in\mathbb{Z}}$ be a MRA with scaling function $\phi(\cdot)$ then (Mallat, 1989a)

$$\mathcal{V}_{m+1} = \mathcal{V}_m \oplus \mathcal{W}_m(\psi), \quad m \in \mathbb{Z}. \tag{3.1}$$

We therefore have, on the one hand by definition,

$$\mathcal{V}_m := \bigvee_{n=-\infty}^{\infty} D^m \phi\big((\cdot) - n\big), \tag{3.2}$$

$$= D^m \bigvee_{n=-\infty}^{\infty} \phi\big((\cdot) - n\big) = D^m \mathcal{V}_0, \quad m \in \mathbb{Z} \tag{3.3}$$

and on the other hand by (3.1) and by MRA properties,

$$\mathcal{V}_m = \bigoplus_{m'=-\infty}^{m-1} \mathcal{W}_{m'}(\psi), \quad m \in \mathbb{Z}. \tag{3.4}$$

This shows that $\mathcal{V}_m$ can also be represented by the subspaces $\mathcal{W}_{m'}(\psi)$—for scales not greater than 2^{m-1}. We therefore denote the right-hand side of (3.4) by $\mathcal{V}_m(\psi)$

$$\mathcal{V}_m \ = \bigoplus_{m'=-\infty}^{m-1} \mathcal{W}_{m'}(\psi) \ := \ \mathcal{V}_m(\psi), \quad m \in \mathbb{Z} \tag{3.5}$$

and refer to $\mathcal{V}_m(\psi)$ as scale detail subspace "representing" $\mathcal{V}_m$.

It is easy to see that $\mathcal{V}_m(\psi)$, $m \in \mathbb{Z}$, also satisfy properties (1), (2) and (3) of Definition 1, as well as property (0') of Definition 2 below.

Definition 2 A MRA $\{\mathcal{V}_m\}_{m\in\mathbb{Z}}$—with scaling function $\phi(\cdot)$—is now called Scaling MRA, while the set of subspaces $\{\mathcal{V}_m(\psi)\}_{m\in\mathbb{Z}}$—representing $\{\mathcal{V}_m\}_{m\in\mathbb{Z}}$—satisfies Definition 1 with property (0) replaced by

$$(0')\ \ \mathcal{V}_0(\psi) := \bigoplus_{m'=-\infty}^{-1} \mathcal{W}_{m'}(\psi), \quad \text{and} \quad \mathcal{V}_{m+1}(\psi) = D\mathcal{V}_m(\psi), \quad (m,n)\in\mathbb{Z}^2$$

is called Associated Scale Detail MRA with wavelet $\psi(\cdot)$ derived from the scaling function $\phi(\cdot)$.

Remark 2 In general, if a wavelet $\psi(\cdot)$ is not derived from a scaling function $\phi(\cdot)$, and if a set $\{\mathcal{V}_m(\psi)\}_{m\in\mathbb{Z}}$ satisfies properties (1), (2), and (3) of Definition 1, and if the "core subspace" $\mathcal{V}_0(\psi)$ is T-invariant, then $\{\mathcal{V}_m(\psi)\}_{m\in\mathbb{Z}}$ is defined as a Generalized MRA (Baggett, Medina, & Merill, 1999).

We now connect a Scaling MRA $\{\mathcal{V}_m\}_{m\in\mathbb{Z}}$ and its Associated Scale Detail MRA $\{\mathcal{V}_m(\psi)\}_{m\in\mathbb{Z}}$ with the time shift subspaces $\{\mathcal{H}_n(\psi)\}_{n\in\mathbb{Z}}$ which yields the following result.

Lemma 1 *Let $\psi(\cdot)$ be a wavelet derived from scaling function $\phi(\cdot)$. Then the Scaling MRA $\{\mathcal{V}_m\}_{m\in\mathbb{Z}}$ and its Associated Scale Detail MRA $\{\mathcal{V}_m(\psi)\}_{m\in\mathbb{Z}}$ admit the time shift detail decomposition*

$$\mathcal{V}_m = \mathcal{V}_m(\psi) = \bigoplus_{n'\in\mathbb{Z}} \mathcal{H}_{n'}^{(m)}(\psi), \quad m\in\mathbb{Z}, \tag{3.6}$$

where

$$\mathcal{H}_{n'}^{(m)}(\psi) := \bigvee_{m'=-\infty}^{m-1} D^{m'}\psi\big((\cdot)-n'\big), \quad n'\in\mathbb{Z}. \tag{3.7}$$

Moreover, for each $n\in\mathbb{Z}$, the set $\left\{\mathcal{H}_n^{(m)}(\psi)\right\}_{m\in\mathbb{Z}}$ is a Scale Detail MRA over the subspace $\mathcal{H}_n(\psi)$.

Proof We have, from (3.5) and (2.6),

$$\mathcal{V}_m(\psi) \ = \bigoplus_{m'=-\infty}^{m-1} \ \bigvee_{n'=-\infty}^{\infty} D^{m'}\psi\big((\cdot)-n'\big).$$

Therefore it is easy to see that

$$\mathcal{V}_m(\psi) \ = \bigvee_{n'=-\infty}^{\infty} \ \bigvee_{m'=-\infty}^{m-1} D^{m'}\psi\big((\cdot)-n'\big), \quad m\in\mathbb{Z} \tag{3.8}$$

or

$$\mathcal{V}_m(\psi) \ = \bigvee_{n'=-\infty}^{\infty} \mathcal{H}_{n'}^{(m)}(\psi), \quad m\in\mathbb{Z}. \tag{3.9}$$

Moreover, since the subspaces $\mathcal{H}_n(\psi)$ are pairwise orthogonal, and since (for each m,n) $\mathcal{H}_n^{(m)}(\psi)\subset\mathcal{H}_n(\psi)$,

$$\mathcal{H}_n^{(m)}(\psi)\perp\mathcal{H}_{n'}^{(m')}(\psi), \quad \text{whenever} \quad n\neq n', \ \forall\,(m,m')\in\mathbb{Z}^2. \tag{3.10}$$

Hence (3.9) can be rewritten as

$$\mathcal{V}_m(\psi) = \bigoplus_{n'\in\mathbb{Z}} \mathcal{H}_{n'}^{(m)}(\psi), \quad m \in \mathbb{Z}$$

as claimed. The rest of the Lemma is self-evident. This finishes the proof. □

4 MRA Time shift detail subspaces

We now turn to MRA time shift detail subspaces. Let $\psi(\cdot)$ be the wavelet derived from a scaling function $\phi(\cdot)$. Let the closed subspaces $\mathcal{G}_n(\psi),\ n \in \mathbb{Z}$—called "detail subspaces for time shifts not greater than $n-1$"—be defined as

$$\mathcal{G}_n(\psi) := \bigoplus_{n'=-\infty}^{n-1} \mathcal{H}_{n'}(\psi), \quad n \in \mathbb{Z}. \tag{4.1}$$

This is simply a time shift analog of the scale subspace $\mathcal{V}_m(\psi)$. Moreover, since each $\mathcal{H}_{n'}(\psi)$ is D-reducing, hence so is $\mathcal{G}_n(\psi)$. We have

$$\mathcal{G}_n(\psi)= \bigvee_{n'=-\infty}^{n-1} \ \bigvee_{m'=-\infty}^{\infty} D^{m'}\psi\big((\cdot)-n'\big), \tag{4.2}$$

$$= \bigvee_{m'=-\infty}^{\infty} \ \bigvee_{n'=-\infty}^{n-1} D^{m'}\psi\big((\cdot)-n'\big). \tag{4.3}$$

This can be rewritten as

$$\mathcal{G}_n(\psi) = \bigoplus_{m'\in\mathbb{Z}} \mathcal{W}_{m'}^{(n)}(\psi), \quad n \in \mathbb{Z}, \tag{4.4}$$

where $\mathcal{W}_{m'}^{(n)}(\psi) \subset \mathcal{W}_m(\psi)$ is defined as

$$\mathcal{W}_{m'}^{(n)}(\psi) := \bigvee_{n'=-\infty}^{n-1} D^{m'}\psi\big((\cdot)-n'\big). \tag{4.5}$$

Moreover, since the subspaces $\mathcal{W}_m(\psi),\ m \in \mathbb{Z}$, are pairwise orthogonal, hence so are $\mathcal{W}_{m'}^{(n)}(\psi),\ m' \in \mathbb{Z}$. It is plain that the subspaces $\{\mathcal{G}_n(\psi)\}_{n\in\mathbb{Z}}$ also satisfy properties (1), (2), and (3) of Definition 2.

We summarize the above in the next proposition.

Proposition 1 *Let $\{\mathcal{V}_m\}_{m\in\mathbb{Z}}$ be Scaling MRA with scaling function $\phi(\cdot)$ and wavelet $\psi(\cdot)$. Then, besides the Associated Scale Detail MRA $\{\mathcal{V}_m(\psi)\}_{m\in\mathbb{Z}}$, there also exists the Associated Time-Shift Detail MRA $\{\mathcal{G}_n(\psi)\}_{n\in\mathbb{Z}}$ defined by* (4.1) *and represented by the scale detail decomposition* (4.4). *Moreover, for each $m \in \mathbb{Z}$, the set $\{\mathcal{W}_m^{(n)}(\psi)\}_{n\in\mathbb{Z}}$ is a Time Shift Detail MRA over the subspace $\mathcal{W}_m(\psi)$.*

We must note that, unlike $\mathcal{V}_m(\psi)$, the subspace $\mathcal{G}_n(\psi)$ does not represent $\mathcal{V}_m$. This is due to the fact that $\mathcal{G}_n(\psi)$ is D-reducing while $\mathcal{V}_m$ is only D^*-invariant, since $\mathcal{V}_m(\psi)$ is. More can be said about the subspaces $\mathcal{V}_m(\psi)$ and $\mathcal{G}_n(\psi)$. First, $\mathcal{W}_m(\psi)$ can be rewritten as

$$\mathcal{W}_m(\psi) \ = \ \mathcal{W}_m^{(n)}(\psi) \oplus \mathcal{W}_{m,n}(\psi), \tag{4.6}$$

where

$$\mathcal{W}_{m,n}(\psi) := \bigvee_{n'=n}^{\infty} D^m \psi\big((\cdot) - n'\big). \tag{4.7}$$

Similarly, $\mathcal{H}_n(\psi)$ can be rewritten as

$$\mathcal{H}_n(\psi) = \mathcal{H}_n^{(m)}(\psi) \oplus \mathcal{H}_{n,m}(\psi), \tag{4.8}$$

where

$$\mathcal{H}_{n,m}(\psi) := \bigvee_{m'=m}^{\infty} D^{m'} \psi\big((\cdot) - n\big). \tag{4.9}$$

We now show the next Lemma.

Lemma 2 *The subspace $\mathcal{V}_m(\psi)$ admits the decomposition*

$$\mathcal{V}_m(\psi) \;=\; \bigoplus_{m'=-\infty}^{m-1} \mathcal{W}_{m'}^{(n)}(\psi) \;\oplus\; \bigoplus_{m'=-\infty}^{m-1} \mathcal{W}_{m',n}(\psi). \tag{4.10}$$

Similarly, the subspace $\mathcal{G}_n(\psi)$ admits the decomposition

$$\mathcal{G}_n(\psi) \;=\; \bigoplus_{n'=-\infty}^{n-1} \mathcal{H}_{n'}^{(m)}(\psi) \;\oplus\; \bigoplus_{n'=-\infty}^{n-1} \mathcal{H}_{n',m}(\psi). \tag{4.11}$$

Proof Equation (4.10) follows readily from (3.5) and (4.6), while (4.11) is a consequence of (4.1) and (4.8). □

To proceed we define

$$\mathcal{V}_{m,n}(\psi) := \bigvee_{m'=-\infty}^{m-1} \bigvee_{n'=-\infty}^{n-1} D^{m'} \psi\big((\cdot) - n'\big), \quad (m,n) \in \mathbb{Z}^2 \tag{4.12}$$

and

$$\mathcal{G}_{n,m}(\psi) := \bigvee_{n'=-\infty}^{n-1} \bigvee_{m'=-\infty}^{m-1} D^{m'} \psi\big((\cdot) - n'\big), \quad (m,n) \in \mathbb{Z}^2. \tag{4.13}$$

Then, since D has a bounded inverse, it is plain that

$$\mathcal{V}_{m,n}(\psi) = \mathcal{G}_{n,m}(\psi), \quad (m,n) \in \mathbb{Z}^2. \tag{4.14}$$

Moreover,

$$\mathcal{V}_{m,n}(\psi) \;=\; \mathcal{V}_m(\psi) \cap \mathcal{G}_n(\psi) \;=\; \bigoplus_{m'=-\infty}^{m-1} \mathcal{W}_{m'}^{(n)}(\psi) \tag{4.15}$$

by (4.10) and (4.4), and

$$\mathcal{G}_{n,m}(\psi) \;=\; \mathcal{G}_n(\psi) \cap \mathcal{V}_m(\psi) \;=\; \bigoplus_{n'=-\infty}^{n-1} \mathcal{H}_{n'}^{(m)}(\psi) \tag{4.16}$$

by (4.11) and (3.6). It then follows from Lemma 2, (4.15) and (4.16), that the subspaces $\mathcal{V}_m(\psi)$ and $\mathcal{G}_n(\psi)$ now admit the decompositions

$$\mathcal{V}_m(\psi) = \mathcal{V}_{m,n}(\psi) \oplus \bigoplus_{m'=-\infty}^{m-1} \mathcal{W}_{m',n}(\psi), \tag{4.17}$$

$$= \mathcal{G}_{n,m}(\psi) \oplus \bigoplus_{n'=n}^{\infty} \mathcal{H}_{n'}^{(m)}(\psi) \tag{4.18}$$

and

$$\mathcal{G}_n(\psi) = \mathcal{G}_{n,m}(\psi) \oplus \bigoplus_{n'=-\infty}^{n-1} \mathcal{H}_{n',m}(\psi), \tag{4.19}$$

$$= \mathcal{V}_{m,n}(\psi) \oplus \bigoplus_{m'=m}^{\infty} \mathcal{W}_{m'}^{(n)}(\psi). \tag{4.20}$$

Suppose now that the wavelet $\psi(\cdot)$ is derived from a scaling function $\phi(\cdot)$. Then

$$\mathcal{V}_{m,n}(\psi) = \bigvee_{m'=-\infty}^{m-1} \bigvee_{n'=-\infty}^{n-1} D^{m'}\psi\big((\cdot)-n'\big), \tag{4.21}$$

$$= \bigvee_{n'=-\infty}^{n-1} D^m\phi\big((\cdot)-n'\big) := \mathcal{V}_{m,n}, \quad (m,n) \in \mathbb{Z}^2. \tag{4.22}$$

In other words, the subspace $\mathcal{V}_{m,n}$ is also represented by the subspace $\mathcal{V}_{m,n}(\psi)$. We conclude from the above, from (4.14) and Lemma 2, the next result.

Proposition 2 *Let $\{\mathcal{V}_m\}_{m\in\mathbb{Z}}$ be a Scaling MRA with scaling function $\phi(\cdot)$ and wavelet $\psi(\cdot)$. Then*

$$\mathcal{V}_{m,n} = \mathcal{V}_{m,n}(\psi) = \mathcal{G}_{n,m}(\psi), \quad (m,n) \in \mathbb{Z}^2. \tag{4.23}$$

5 Time shift approximation

We now apply the above results to approximations over $\mathcal{L}^2(\mathbb{R})$ — containing a wavelet $\psi(\cdot)$. It is plain from (2.8) that any $f(\cdot) \in \mathcal{L}^2(\mathbb{R})$ can be represented in terms of detail over all levels m as

$$f(\cdot) = \sum_{m\in\mathbb{Z}} P_{\mathcal{W}_m} f(\cdot) \tag{5.1}$$

and from (2.9), in terms of detail over all time shifts n as

$$f(\cdot) = \sum_{n\in\mathbb{Z}} P_{\mathcal{H}_n} f(\cdot), \tag{5.2}$$

where $P_{\mathcal{M}}$ denotes orthogonal projection onto the closed subspace $\mathcal{M}$.

Suppose now that the wavelet $\psi(\cdot)$ is derived from a scaling function $\phi(\cdot)$ associated with the Scaling MRA $\{\mathcal{V}_m\}_{m\in\mathbb{Z}}$. Then it is plain from (2.8) and (3.5) that the space $\mathcal{L}^2(\mathbb{R})$ admits the “cascaded scaling-detail decomposition” for any level m:

$$\mathcal{L}^2(\mathbb{R}) = \mathcal{V}_m \oplus \bigoplus_{m'=m}^{\infty} \mathcal{W}_{m'}(\psi), \quad m \in \mathbb{Z}. \tag{5.3}$$

Therefore any $f(\cdot) \in \mathcal{L}^2(\mathbb{R})$ can now be represented as

$$f(\cdot) = P_{\mathcal{V}_m} f(\cdot) + \sum_{m'=m}^{\infty} P_{\mathcal{W}_{m'}} f(\cdot). \tag{5.4}$$

Now let m be any fixed level and $\mu > m$ be another level. Then, by Definition 1(i),

$$\mathcal{V}_m \subset \mathcal{V}_\mu.$$

Therefore

$$\mathcal{V}_\mu = \mathcal{V}_m \oplus \mathcal{V}_\mu \cap \mathcal{V}_m^{\perp}.$$

From which and from (3.5) it is evident that

$$\mathcal{V}_\mu = \mathcal{V}_m \oplus \bigoplus_{m'=m}^{\mu-1} \mathcal{W}_{m'}(\psi). \tag{5.5}$$

Suppose now that $f(\cdot) \in \mathcal{V}_\mu$. Then

$$f(\cdot) = P_{\mathcal{V}_\mu} f(\cdot) = P_{\mathcal{V}_m} f(\cdot) + \sum_{m'=m}^{\mu-1} P_{\mathcal{W}_{m'}} f(\cdot). \tag{5.6}$$

This results in a Discrete Wavelet Transform (DWT) (Keinert, 2004).

Now let us convert (5.5) into a time shift representation. We have

$$\mathcal{V}_\mu = \mathcal{V}_m \oplus \bigvee_{m'=m}^{\mu-1} \bigvee_{n'=-\infty}^{\infty} D^{m'} \psi\big((\cdot) - n'\big). \tag{5.7}$$

Therefore

$$\mathcal{V}_\mu = \bigvee_{n'=-\infty}^{\infty} D^m \phi\big((\cdot) - n'\big) \oplus \bigoplus_{n'=-\infty}^{\infty} \bigvee_{m'=m}^{\mu-1} D^{m'} \psi\big((\cdot) - n'\big). \tag{5.8}$$

Thus, for any $f(\cdot) \in \mathcal{V}_\mu$,

$$f(\cdot) = P_{\mathcal{V}_\mu} f(\cdot) = \sum_{n'=-\infty}^{\infty} P_{\phi_{m,n'}} f(\cdot) + \sum_{n'=-\infty}^{\infty} \sum_{m'=m}^{\mu-1} P_{\psi_{m',n'}} f(\cdot), \tag{5.9}$$

where $P_{\phi_{m,n'}}$ is the orthogonal projection onto the vector $\phi_{m,n'}(\cdot) := D^m \phi\big((\cdot) - n'\big)$, and $P_{\psi_{m',n'}}$ is the orthogonal projection onto the vector $\psi_{m',n'}(\cdot) := D^{m'} \psi\big((\cdot) - n'\big)$. In addition, if the support of $f(\cdot)$ is compact, or if $f(\cdot)$ is a sample over a finite time window of a signal, then the infinite sums on the right hand side of (5.9) can be reduced to finite sums—by allowing the index n' to vary in a "suitable" finite range $[N', N]$ (say). This is expressed by the following equation:

$$f(\cdot)_{(N' \leq n' \leq N)} = \sum_{n'=N'}^{N} P_{\phi_{m,n'}} f(\cdot) + \sum_{n'=N'}^{N} \sum_{m'=m}^{\mu-1} P_{\psi_{m',n'}} f(\cdot), \tag{5.10}$$

which can be regarded as a "finite time" DWT.

To proceed, let us write (5.3) as

$$\mathcal{L}^2(\mathbb{R}) \;=\; \bigvee_{n'=-\infty}^{\infty} D^m \phi\big((\cdot) - n'\big) \;\oplus\; \bigvee_{m'=m}^{\infty} \; \bigvee_{n'=-\infty}^{\infty} D^{m'} \psi\big((\cdot) - n'\big) \tag{5.11}$$

or

$$\mathcal{L}^2(\mathbb{R}) \;=\; \bigvee_{n'=-\infty}^{\infty} D^m \phi\big((\cdot) - n'\big) \;\oplus\; \bigvee_{n'=-\infty}^{\infty} \; \bigvee_{m'=m}^{\infty} D^{m'} \psi\big((\cdot) - n'\big), \tag{5.12}$$

which is a "cascaded scaling-detail time shift decomposition" for level m. This can be further decomposed as

$$\mathcal{L}^2(\mathbb{R}) \;=\; \bigvee_{n'=-\infty}^{n-1} D^m \phi\big((\cdot) - n'\big) \;\oplus\; \bigvee_{n'=n}^{\infty} D^m \phi\big((\cdot) - n'\big) \;\oplus\; \bigoplus_{n'=-\infty}^{\infty} \mathcal{H}_{n',m}(\psi) \tag{5.13}$$

for each pair $(m, n) \in \mathbb{Z}^2$. It is clear that the first term on the right hand side is $\mathcal{V}_{m,n}$ by (4.22), while the second-term is the orthogonal complement of $\mathcal{V}_{m,n}$ in $\mathcal{V}_m$, which is equal to that of $\mathcal{V}_{m,n}(\psi)$ in $\mathcal{V}_m(\psi)$—since $\mathcal{V}_m(\psi)$ is equal to $\mathcal{V}_m$, and $\mathcal{V}_{m,n}$ is equal to $\mathcal{V}_{m,n}(\psi)$. Therefore (5.13) can be rewritten as

$$\mathcal{L}^2(\mathbb{R}) = \mathcal{V}_{m,n} \;\oplus\; \bigoplus_{n'=n}^{\infty} \mathcal{H}_{n'}^{(m)}(\psi) \;\oplus\; \bigoplus_{n'=-\infty}^{\infty} \mathcal{H}_{n',m}(\psi), \quad m \in \mathbb{Z}, \tag{5.14}$$

where we have made used of (4.16) and (4.18). This can be rearranged to yield

$$\mathcal{L}^2(\mathbb{R}) = \mathcal{V}_{m,n} \;\oplus\; \bigoplus_{n'=-\infty}^{n-1} \mathcal{H}_{n',m}(\psi) \;\oplus\; \bigoplus_{n'=n}^{\infty} \mathcal{H}_{n'}(\psi). \tag{5.15}$$

This allows us to represent any $f(\cdot) \in \mathcal{L}^2(\mathbb{R})$ as

$$f(\cdot) \;=\; \sum_{n'=-\infty}^{n-1} \langle f, \phi_{m,n'} \rangle \, \phi_{m,n'}(\cdot) \;+\; \sum_{n'=-\infty}^{n-1} P_{\mathcal{H}_{n',m}} f(\cdot) \;+\; \sum_{n'=n}^{\infty} P_{\mathcal{H}_{n'}} f(\cdot) \tag{5.16}$$

for each $n \in \mathbb{Z}$. Therefore, as in the case of (5.9), $f(\cdot)$ can be approximated over a "suitable" time shift range $[\nu', \nu]$ (say), for $-\infty < \nu' \leq \nu \leq n - 1$, as

$$f(\cdot)_{(\nu' \leq n' \leq \nu)} = \sum_{n'=\nu'}^{\nu} \langle f, \phi_{m,n'} \rangle \, \phi_{m,n'}(\cdot) \;+ \sum_{n'=\nu'}^{\nu} P_{\mathcal{H}_{n',m}} f(\cdot). \tag{5.17}$$

6 A formulation of time shift multiresolution approximation

We close the paper with a formulation of Time-Shift Multiresolution Approximation. We have shown that given a Scaling MRA $\{\mathcal{V}_m\}_{m\in\mathbb{Z}}$, with scaling function $\phi(\cdot)$ and wavelet $\psi(\cdot)$, there exists the Associated Time-Shifting Detail MRA $\{\mathcal{G}_n\}_{n\in\mathbb{Z}}$ defined by (4.1). Suppose now that, we are given a wavelet $\psi(\cdot)$, then it is natural to ask: "what Time Shift MRA—denoted by $\{\mathcal{T}_n\}_{n\in\mathbb{Z}}$—which admits $\{\mathcal{G}_n(\psi)\}_{n\in\mathbb{Z}}$, constructed from the given wavelet $\psi(\cdot)$, as its Associated Time Shift Detail MRA?" In other words, does there exist a function $\varphi(\cdot) \in \mathcal{L}^2(\mathbb{R})$ which generates the subspaces $\mathcal{T}_n$—with suitable MRA properties—so that

$$\mathcal{T}_n = \mathcal{G}_n(\psi), \quad \forall\, n \in \mathbb{Z}\,? \tag{6.1}$$

First, since $\mathcal{G}_n(\psi)$ are D-reducing, hence so must be $\mathcal{T}_n$. Suppose now that $\varphi(\cdot)$ in $\mathcal{L}^2(\mathbb{R})$ is a unit vector such that $\varphi\big((\cdot)-n\big),\ n \in \mathbb{Z}$, are wandering vectors for the dyadic-scaling operator D, that is,

$$\varphi\big((\cdot)-n\big) \perp D^m\varphi\big((\cdot)-n\big), \quad \forall\ (m,n) \in \mathbb{Z}^2.$$

Then let $\mathcal{T}_n,\ n \in \mathbb{Z}$, be the closed D-reducing subspaces defined by

$$\mathcal{T}_n := \bigvee_{m'\in\mathbb{Z}} D^{m'}\varphi\big((\cdot)-n\big), \quad n \in \mathbb{Z} \tag{6.2}$$

and satisfy the properties:

(1) $\mathcal{T}_n \subset \mathcal{T}_{n+1},\ \ n \in \mathbb{Z}$,
(2) $\bigcap_{n\in\mathbb{Z}} \mathcal{T}_m = \{0\}$,
(3) $\overline{\bigcup_{n\in\mathbb{Z}} \mathcal{T}_m} = \mathcal{L}^2(\mathbb{R})$.

Moreover,

$$\mathcal{T}_{n+1} = \mathcal{T}_n \oplus \mathcal{H}_n(\psi), \quad n \in \mathbb{Z}. \tag{6.3}$$

It then follows that, as in the case of $\mathcal{V}_m$, the subspace $\mathcal{T}_n$ is represented by the time shift detail subspace $\mathcal{G}_n(\psi)$

$$\mathcal{T}_n \ = \bigoplus_{n'=-\infty}^{n-1} \mathcal{H}_{n'}(\psi) \ := \ \mathcal{G}_n(\psi), \quad n \in \mathbb{Z}. \tag{6.4}$$

Thus the set $\{\mathcal{T}_n\}_{n\in\mathbb{Z}}$ forms a Time-Shift MRA.

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