# Option Pricing using Integral Transforms

Peter Carr

NYU Courant Institute

*joint work with H. Géman, D. Madan, L. Wu, and M. Yor*

# Introduction

Call
values
are often
obtained by integ-
rating their payoff against
a risk-neutral probability density
function. When the characteristic function
of the underlying asset is known in closed form,
call values can also be obtained by a single integration.

# A Brief History of Sines

- The history of integral transforms begins with d'Alembert in 1747.
- D'Alembert proposed using a superposition of sine functions to describe the oscillations of a violin string.
- The recipe for computing the coefficients, later associated with Fourier's name, was actually formulated by Euler in 1777.
- Fourier proposed using the same idea for the heat equation in 1807.
- Since the introduction of periodic functions, mathematics has never been the same...

# Fourier Frequency in Finance

- McKean (IMR 65) used Fourier transforms in his appendix to Samuelson's paper.
- Buser (JF 86) noticed that Laplace transforms with real arguments give present value rules.
- Shimko (92) championed the use of Laplace transforms in his book.
- Beaglehole (WP 92) used Fourier series to value double barrier options.
- Stein & Stein (RFS 91) and Heston (RFS 93) started the ball rolling with their use of Fourier transforms to analytically value European options on stocks with stochastic volatility.
- While not necessary, Fourier methods simplify the development of option pricing models which reflect empirical realities such as jumps (Ait-Sahalia JF 02), volatility clustering (Engle 81), and the leverage effect (Black 76).
- A bibliography at the end of this presentation lists 76 papers applying integral transforms to option pricing.

# My Fast Fourier Talk (FFT)

- To survey integral transforms for option pricing in one hour, I restrict the presentation to the use of Fourier transforms to value European options on a single stock.
- Here's an overview of my FFT:
  1. What is a Fourier Transform (FT)?
  2. What is a Characteristic Function (CF)?
  3. Relating FT's of Option Prices to CF's
  4. Pricing Options on Lévy Processes
  5. Pricing Options on Lévy Processes w. Stochastic Volatility

# Fourier Transformation and Inversion

- Let $f(x)$ be a suitably integrable function
- Letting $\delta(\cdot)$ be Dirac's delta function:

$$f(x) = \int_{-\infty}^{\infty} f(y)\delta(y - x)dy.$$

- The next page shows that $\delta(y - x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iu(y-x)} du$.
- Substituting in this fundamental result implies:

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} f(y) \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iu(y-x)} du dy \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iux} \int_{-\infty}^{\infty} f(y) e^{iuy} dy du. \end{aligned}$$

- Define the Fourier transform (FT) of $f(\cdot)$ as:

$$\mathcal{F}_f(u) \equiv \int_{-\infty}^{\infty} e^{iuy} f(y) dy.$$

- Thus given the FT of $f$, the function $f$ can be recovered by:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iux} \mathcal{F}_f(u) du.$$

- It is sometimes necessary to make $u$ complex. When $\text{Im}(u) \equiv u_i \neq 0$, the FT is referred to as a *generalized* Fourier transform.

$$f(x) = \frac{1}{2\pi} \int_{iu_i - \infty}^{iu_i + \infty} e^{-iux} \mathcal{F}_f(u) du.$$

# No Potato, One Potato, Two Potato, Three...

- Note that:

  1. the average of 1 and 1 is 1
  2. the average of 1 and $e^{i\pi} = -1$ is 0.

  1. The average of 1 and 1 and 1 is 1
  2. The average of 1 and $e^{\frac{2\pi}{3}i}$ and $e^{\frac{4\pi}{3}i}$ is 0.
  3. The average of 1 and $e^{\frac{4\pi}{3}i}$ and $e^{\frac{8\pi}{3}i}$ is 0.

  1. The average of 1 and 1 and 1 and 1 is 1
  2. The average of 1 and $e^{\frac{\pi}{2}i}$ and $e^{\pi i}$ and $e^{\frac{3\pi}{2}i}$ is 0.
  3. The average of 1 and $e^{\pi i}$ and $e^{2\pi i}$ and $e^{3\pi i}$ is 0.
  4. The average of 1 and $e^{\frac{3\pi}{2}i}$ and $e^{3\pi i}$ and $e^{\frac{9\pi}{2}i}$ is 0.

- As financial engineers, we conclude that for all $d = 2, 3, 4 \ldots$:

$$\frac{1}{d}\sum_{k=0}^{d-1} e^{\frac{2\pi i}{d}jk} = 1_{j=0},$$

for $j = 0, 1, \ldots, d-1$.

# Yes, but...

- Recall our engineering style proof that for $d = 2, 3, \ldots,$:
$$\frac{1}{d}\sum_{k=0}^{d-1} e^{\frac{2\pi i}{d}jk} = 1_{j=0}, \quad j = 0, 1, \ldots, d-1.$$
- A mathematician would note that if we define $r = e^{\frac{2\pi i}{d}j}$, then the LHS is $\frac{1}{d}\sum_{k=0}^{d-1} r^k$. If $j = 0$, then $r = 1$ and the LHS is clearly 1, while if $j \neq 0$, then the sum is a geometric series:
$$\frac{1}{d}\sum_{k=0}^{d-1} r^k = \frac{1}{d}\frac{r^d - 1}{r - 1} = 0, \text{ since } r^d = e^{2\pi i j} = 1.$$
- Multiplying the top equation by $d$ implies $d1_{j=0} = \sum_{k=0}^{d-1} e^{\frac{2\pi i}{d}jk}$.
- Putting our engineering cap back on on, letting $d \uparrow \infty$ and $j = y - x$:
$$\delta(y - x) = \int_{-\infty}^{\infty} e^{i2\pi\omega(y-x)} d\omega.$$
- Letting $u = 2\pi\omega$:
$$\delta(y - x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iu(y-x)} du.$$
- Fortunately, this can all be made precise.

# Basic Properties of Fourier Transforms

- Recall that the (generalized) FT of $f(x)$ is defined as:

$$\mathcal{F}_f(u) \equiv \int_{-\infty}^{\infty} e^{iux} f(x)dx,$$

where $f(x)$ is suitably integrable.

- Three basic properties of FT's are:

1. **Parseval Relation:**
   Define the *inner product* of 2 complex-valued $L^2$ functions $f(\cdot)$ and $g(\cdot)$ as $\langle f, g \rangle \equiv \int_{-\infty}^{\infty} f(x)\overline{g}(x)dx$. Then:

$$\langle f, g \rangle = \langle \mathcal{F}_f(u), \mathcal{F}_g(u) \rangle.$$

2. **Differentiation**:

$$\mathcal{F}_{f'}(u) = -iu\mathcal{F}_f(u)$$

3. **Modulation:**

$$\mathcal{F}_{e^{\ell x} f}(u) = \mathcal{F}_f(u - i\ell)$$

# What is a Characteristic Function?

- A *characteristic function* (CF) is the FT of a PDF.
- If $X$ has PDF $q$, then:

$$\mathcal{F}_q(u) \equiv \int_{-\infty}^{\infty} e^{iux} q(x) dx = E e^{iuX}.$$

- For $u$ real and fixed, the CF is the expected value of the location of a random point on the unit circle. Hence the norm of the CF is never bigger than one:

$$|\mathcal{F}_q(u)| \leq 1.$$

- The bigger the absolute value of the real frequency $u$, the wider is the distribution of $uX$. Hence, if the PDF of $uX$ is wrapped around the unit circle, larger $|u|$ leads to more uniform distribution of probability mass on the circle, and hence smaller norms of the CF.
- Symmetric PDF's centered about zero have real CF's.
- When the argument $u$ is complex with non-zero imaginary part, the PDF is wrapped around a spiral rather than a circle. The larger is $\text{Im}(u)$, the faster we spiral into the origin.

# From Fourier to Finance

- Suppose we interpret the function $f$ as the final payoff to a derivative security maturing at $T$.
- Recall that $f(F_T) = \int_{-\infty}^{\infty} f(K)\delta(F_T - K)dK$.
- This is a spectral decomposition of the payoff $f$ into the payoffs $\delta(\cdot)$ from an infinite collection of Arrow Debreu securities.
- From Breeden & Litzenberger, $\frac{\partial^2}{\partial K^2}(F_T - K)^+ = \delta(F_T - K)$.
- Hence, static positions in calls can create any path-indep. payoff including $e^{r_1 x}\sin(r_2 x)$ & $e^{r_1 x}\cos(r_2 x), r_1, r_2$ real. The payoffs from these sine and cosine claims are created model-free.
- As we saw, $\delta(F_T - K) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iu(F_T - K)}du$.
- When $u$ is complex and $u = u_r + iu_i$:
$$e^{iux} = e^{u_r x}\cos(u_i x) + ie^{u_r x}\sin(u_i x).$$
- Hence, the payoff from each A/D security can in turn be replicated by a static position in sine claims and cosine claims.
- Just as the payoffs from A/D securities may be a more convenient basis to work with than option payoffs, the payoffs from sine and cosine claims may be an even more convenient basis.
- The use of complex numbers is even more convenient. After all, it is a lot easier to evaluate $i^2$ than $\sin(u_1 + u_2)$ or $\cos(u_1 + u_2)$.

# Parsevaluation

- Let $g(k)$ be the Green's function (a.k.a the pricing kernel, butterfly spread price, and discounted risk-neutral PDF).
- Letting $V_0$ be the initial value of a claim paying $f(X_T)$ at $T$, risk-neutral valuation implies:
$$V_0 = \int_{-\infty}^{\infty} f(k)g(k)dk = \langle f, g \rangle,$$
where for any functions $\phi_1(x)$ and $\phi_2(x)$, the inner product is:
$$\langle \phi_1, \phi_2 \rangle \equiv \int_{-\infty}^{\infty} \phi_1(x)\overline{\phi_2}(x)dx.$$
- By the Fourier Inversion Theorem $f(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iuk}\mathcal{F}_f(u)du$:
$$\begin{aligned} V_0 &= \int_{-\infty}^{\infty} \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iuk}\mathcal{F}_f(u)du g(k)dk \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{F}_f(u)\int_{-\infty}^{\infty} g(k)e^{-iuk}dkdu \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{F}_f(u)\overline{\mathcal{F}_g(u)}du. \end{aligned}$$
- Hence $V_0 = \langle f, g \rangle = \frac{1}{2\pi}\langle \mathcal{F}_f, \mathcal{F}_g \rangle$ by a change of basis.
- Note that $\mathcal{F}_g(u) = B_0(T)\mathcal{F}_q(-u)$, i.e. discount factor $\times$ CF.
- By restricting the payoff, more efficient Fourier methods can be developed.

# Breeden Litzenberger in Logs

- Let $C(K,T)$ relate call value to strike $K$ and maturity $T$
- The Green's f'n $G(K,T)$ is $B_0(T)Q\{F_T \in d(K, K+dK)\}$.
- From Breeden & Litzenberger (JB 78), $G(K,T) = \frac{\partial^2}{\partial K^2}C(K,T)$.
- Let $k \equiv \ln(K/F_0)$ measure moneyness of the $T$ maturity call.
- Let $\gamma(k,T) \equiv C(K,T)$ relate call value to $k$ and $T$.
- Let $X_t \equiv \ln(F_t/F_0)$ be the log price relative.
- Let $g(k,T) \equiv G(K,T)$ be the Green's function of $X_T$.
- How are $g$ and $\gamma$ related?
- By no arbitrage, the call value is related to $g$ by:
$$\gamma(k,T) = F_0 \int_k^\infty (e^y - e^k) g(y,T)dy.$$
- To invert this relationship, differentiate w.r.t. $k$:
$$\frac{\partial}{\partial k}\gamma(k,T) = -F_0 \int_k^\infty e^k g(y,T)dy.$$
$$\text{Hence: } \frac{e^{-k}}{F_0}\frac{\partial}{\partial k}\gamma(k,T) = -\int_k^\infty g(y,T)dy.$$
- Differentiating w.r.t. $k$ again gives the desired result:
$$g(k,T) = \frac{\partial}{\partial k}\frac{e^{-k}}{F_0}\frac{\partial}{\partial k}\gamma(k,T).$$

# Relationship Between Fourier Transforms

- Recall that Green's function $g$ of $X_T = \ln(F_T/F_0)$ is related to the call price as a function $\gamma$ of moneyness $k \equiv \ln(K/F_0)$:

$$g(k,T) = \frac{\partial}{\partial k}\frac{e^{-k}}{F_0}\frac{\partial}{\partial k}\gamma(k,T).$$

- Multiply both sides by $e^{i\theta k}$ where $\theta \in \mathcal{C}$ and integrate out $k$:

$$\mathcal{F}[g](\theta,T) \equiv \int_{-\infty}^{\infty} e^{i\theta k} g(k,T)dk = \mathcal{F}\left[\frac{\partial}{\partial k}\frac{e^{-k}}{F_0}\frac{\partial}{\partial k}\gamma\right](\theta,T).$$

- The differentiation and modulation rules for FT's imply:

$$\begin{aligned}
\mathcal{F}[q](\theta,T) &= (-i\theta)\mathcal{F}\left[\frac{e^{-k}}{F_0}\frac{\partial}{\partial k}\gamma\right](\theta,T) \\
&= \frac{(-i\theta)}{F_0}\mathcal{F}\left[\frac{\partial}{\partial k}\gamma\right](\theta+i,T) \\
&= \frac{-\theta(\theta+i)}{F_0}\mathcal{F}[\gamma](\theta+i,T).
\end{aligned}$$

- Letting $u = \theta + i$, solving for $\mathcal{F}[\gamma](\theta+i,T)$:

$$\mathcal{F}[\gamma](u,T) = \frac{F_0\mathcal{F}[g](u-i,T)}{(i-u)u}.$$

- Carr Madan (JCF 98) compute this (generalized) FT via FFT.

## More on the FFT Method

- Since $\gamma(k)$ is not integrable, its generalized FT is only defined on a subset of the complex plane that excludes the real line.
- If we invert an FT for call value at $n$ strikes, the work is $O(n^2)$ since each inversion is a numerical integration.
- Using the FFT to invert the FT of the call value reduces the work to $O(n \ln n)$, a considerable improvement.
- The formula is only for European options. However, Lee (03) extends it to a bigger class of path-independent payoffs and Dempster & Hong (WP 02) extend it to spread options.
- Lee (03) develops error bounds for the FFT method.
- Inversion returns option prices as a function of (log) strike, which is useful for calibrating to market option prices.
- Alternatively, one can calibrate directly in Fourier space relying on Parseval to ensure that errors do not magnify on inversion.
- If call values are homogeneous in spot and strike, one also gets option prices in terms of spot (hedging/risk management).
- When the CF is available in closed form, both Parsevaluation and the FFT method give closed form expressions for the generalized FT of a claim value. But where do we get closed form expressions for CF's?

# Options on Lévy Processes

- Fortunately, an important class of stochastic processes called Lévy processes are specified directly in terms of the CF of a random variable.
- Lévy processes are right continuous left limits processes with stationary independent increments.
- Important examples include arithmetic brownian motion (ABM) and compound Poisson processes.
- The only continuous Lévy process is ABM:

$$dA_t = bdt + \sigma dW_t, \quad t \geq 0.$$

- Note that the Black Scholes model assumes that the log price is ABM. Thus, if we are going to go beyond Black Scholes and we want to price options on assets whose log price is a Lévy process, then we are going to have to make our peace with pricing options in the presence of jumps.
- Work on this issue is also motivated by Hakansson's catch 22 and Sandy Grossman's crack on ketchup economics.

# Complete Markets

- The Black Scholes model prices all stock price contingent claims uniquely by no arbitrage using just a bond and the underlying stock in the replicating portfolio.
- The basic intuition comes from seeing Black Scholes as a continuous time limit of the binomial model, where at each discrete time the increment in the stock price is Bernoulli distributed.
- Some SV models and jump models have all of the above features of Black Scholes, eg. CEV model or Cox Ross (JFE 76) fixed jump model (also see Rogers & Hobson (MF 98) and Dritschel & Protter (MF 99)).
- However, these models are still limits of discrete time models in which the local movement in the stock price process is still conditionally binary.

# Pricing Options on Assets which Jump

- Rejecting the conditionally binary assumption on empirical grounds, we can still price claims by using some subset of:
  1. Restricting the target claim space
  2. Restricting the underlying price process
  3. Increasing the basis asset space
  4. Assuming more than no arbitrage
  5. Giving up on unique pricing.
- For example, suppose the target claim space is restricted to arbitrary European options & the underlying price process is restricted to an arbitrary Lévy process. Suppose dynamic trading is restricted to stock & bond, but static positions are allowed in $N < \infty$ options with different strikes &/or maturities.
- Then we can still uniquely price options by assuming more than no arbitrage. For example, we can assume that asset markets are in equilibrium and hence is using this criterion to select one arbitrage-free pricing rule from the many that reprice the options and their underlying stock.
- To learn this pricing rule, we may specify *ex ante* a family of Lévy processes with $M < N$ parameters. Then the parameters are determined from market option prices by say a least squares fit. But how do we specify a Lévy process?

# Lévy Khintchine Theorem

- Once one specifies the distribution of an infinitely divisible random variable at time 1, the corresponding Lévy process is determined by $X_t \stackrel{d}{=} tX_1$.
- The Lévy Khintchine theorem characterizes all infinitely divisible random variables in terms of their CF.
- For simplicity, I will only present the theorem for Lévy processes started at 0 and whose jump component has sample paths of finite variation.
- By the Lévy Khintchine theorem, all such processes have a CF:

$$Ee^{iuX_t} = e^{t\left[ibu - \frac{\sigma^2 u^2}{2} + \int\limits_{\Re - \{0\}} \left(e^{iux} - 1\right)\ell(dx)\right]}, \quad t \geq 0.$$

- The Lévy process is specified by the drift rate $b$, the diffusion coefficient $\sigma$, and the so-called Lévy measure $\ell(dx)$.
- Loosely speaking, the Lévy measure $\ell(dx)$ specifies the arrival rate of jumps of size $(x, x+dx)$. Hence, it must be nonnegative and no measure is assigned to the origin. So that the process has well-defined quadratic variation, the Lévy measure must also integrate $x^2$ around the origin i.e.

$$\int\limits_{\Re - \{0\}} x^2 1(|x| < 1)\ell(dx) < \infty.$$

# Applying Lévy Khintchine

- Recall the Lévy Khintchine theorem:

$$Ee^{iuX_t} = e^{t\left[ibu - \frac{\sigma^2 u^2}{2} + \int\limits_{\Re - \{0\}} \left(e^{iux} - 1\right)\ell(dx)\right]}.$$

- If the Lévy process is ABM ($dA_t = bdt + \sigma dW_t, t \geq 0$), then:

$$Ee^{iuA_t} = e^{t\left[ibu - \frac{\sigma^2 u^2}{2}\right]}.$$

- For Black Scholes with constant interest rate $r$ and dividend yield $q$, the drift of the log stock price relative is $b = r - q - \frac{\sigma^2}{2}$ and we are done.

- Merton's jump diffusion model assumes that the log price relative is the sum of an ABM and an independent compound Poisson process. The conditional distribution of the jump size is normal with mean $\alpha$ and standard deviation $\sigma_j$. The Lévy measure is $\ell(dx) = \lambda \frac{e^{-\frac{1}{2}\left(\frac{x-\alpha}{\sigma_j}\right)^2}}{\sqrt{2\pi\sigma_j}} dx$. Hence, the CF is:

$$Ee^{iuX_t} = e^{t\left[ibu - \frac{\sigma^2 u^2}{2} + \frac{\lambda}{\sqrt{2\pi\sigma_j}} \int\limits_{\Re - \{0\}} \left(e^{iux} - 1\right) e^{-\frac{1}{2}\left(\frac{x-\alpha}{\sigma_j}\right)^2} dx\right]}.$$

- The last integral can be done in closed form. Once one relates the risk-neutral drift $b$ to the parameters $r$, $q$, $\sigma$, $\lambda$, $\alpha$, and $\sigma_j$, European option pricing is straightforward.

# An Interpretation of Lévy Khintchine

- All Lévy processes arise as limits of compound Poisson processes.
- To see why, recall the Lévy Khintchine theorem:

$$Ee^{iuX_t} = e^{t\left[ibu - \frac{\sigma^2 u^2}{2} + \int_{\Re - \{0\}} \left(e^{iux} - 1\right)\ell(dx)\right]}. \qquad (1)$$

- A Poisson process jumping by $k$ with arrival rate $\lambda$ has CF:

$$Ee^{iukN_t(\lambda)} = \sum_{n=0}^{\infty} e^{iukn} \frac{e^{-\lambda t}(\lambda t)^n}{n!} = e^{t\left(e^{iuk} - 1\right)\lambda}. \qquad (2)$$

- (1) and (2) are the same if $b = \sigma = 0$ and the Lévy measure is:

$$\ell(dx) = \lambda \delta(x - k) dx.$$

- The term $ibu$ in (1) is arising from cumulative drift $bt$. Now:

$$\lim_{k \downarrow 0} \frac{e^{iuk} - 1}{k} = iu.$$

- Hence, this term can come from (2) by letting $\lambda = \frac{b}{k}$ and letting $k \downarrow 0$.

# Diffusions as Jumps

- Recall the Lévy Khintchine theorem:

$$Ee^{iuX_t} = e^{t\left[ibu - \frac{\sigma^2 u^2}{2} + \int\limits_{\Re - \{0\}} \left(e^{iux} - 1\right)\ell(dx)\right]}. \quad (3)$$

and the CF for a Poisson process jumping by $k$ with arrival rate $\lambda$:

$$Ee^{iukN_t(\lambda)} = e^{t\left(e^{iuk} - 1\right)\lambda}. \quad (4)$$

- The difference of 2 IID Poissons jumping by $k$ has CF:

$$Ee^{iuk[N_{1t}(\lambda) - N_{2t}(\lambda)]} = e^{t\left[\left(e^{iuk} - 1\right) + \left(e^{-iuk} - 1\right)\right]\lambda}. \quad (5)$$

- The term $\frac{\sigma^2 u^2}{2}$ in (3) is arising from the CF of $\sigma W_t$, where $W$ is SBM. Now:

$$\lim_{k \downarrow 0} \frac{(e^{iuk} - 1) + (e^{-iuk} - 1)}{k^2} = -u^2.$$

- Hence, this term can come from (4) by letting $\lambda = \frac{\sigma^2}{2k^2}$ and letting $k \downarrow 0$.
- Where does the term $\int\limits_{\Re - \{0\}} \left(e^{iux} - 1\right) \ell(dx)$ in (3) come from?

# Poisson Processes as Building Blocks

- Recall the Lévy Khintchine theorem:

$$Ee^{iuX_t} = e^{t\left[ibu - \frac{\sigma^2 u^2}{2} + \int\limits_{\Re - \{0\}} \left(e^{iux} - 1\right)\ell(dx)\right]}. \qquad (6)$$

- Suppose that a Poisson process $P_t$ jumps by the fixed size $x$ and has an infinitessimally small arrival rate $\ell(dx)$:

$$P_t = xN_t(\ell(dx)).$$

- Its CF is:

$$Ee^{iuP_t} = e^{t\left(e^{iux} - 1\right)\ell(dx)}. \qquad (7)$$

- Now consider a continuum of such processes where each process is independent of every other.
- The integral in (6) can be thought of as arising from a calculation of the CF of a superposition of these processes:

$$I_t \equiv \int\limits_{\Re - \{0\}} xN_t(\ell(dx)).$$

  The rareness of jumps ensures that at every time, there are either no jumps or only one.
- Since drift and diffusion also come from limiting linear combinations of standard Poisson processes, we see that these processes are the building blocks of Lévy processes.

## You Say Tomato

- Many Lévy processes can go negative while futures prices of limited liability assets must be nonnegative.
- Suppose we assume that the log of the futures price relative is a Lévy process started at zero:

$$X_t = \ln(F_t/F_0).$$

- No arbitrage implies that the futures price $F_t = F_0 e^{X_t}$ is a positive martingale under a risk-neutral measure $Q$.
- Since the exponential function is convex, Jensen's inequality forces the process $X$ to have negative drift.
- For example in the Black model, the log futures price relative is ABM and the drift of the log futures price relative is $-\sigma^2/2$.
- This negative drift is often termed a convexity correction, but it should be called a concavity correction if the log is concave.

# Convexity Correction

- To determine the convexity correction when the log futures price relative is a Lévy process, let $L_t$ be a Lévy process with zero drift whose jump component has sample paths of finite variation. We term $L$ the driver of the futures price process.
- By the Lévy Khintchine theorem, we have:

$$Ee^{iuL_t} = e^{t[-\frac{u^2\sigma^2}{2} + \int_{\Re-\{0\}} (e^{iux}-1)\ell(dx)]} = e^{-t\Psi(u)},$$

where $\Psi(u) \equiv -\ln Ee^{iuL_1} = u^2\sigma^2/2 - \int_{\Re-\{0\}} (e^{iux} - 1)\ell(dx)$ is called the *characteristic exponent* of the driver.
- Assuming that expectations are finite, evaluating the top equation at $u = -i$:

$$Ee^{L_t} = e^{-t\Psi(-i)} \text{ and hence } Ee^{t\Psi(-i)+L_t} = 1.$$

Let:

$$b \equiv \Psi(-i) = -\sigma^2/2 - \int_{\Re-\{0\}} (e^x - 1)\ell(dx).$$

- Then $X_t \equiv bt + L_t$ is a Lévy process with the property that $e^{X_t}$ is a positive martingale started at 1.
- Hence, $S_t \equiv S_0 e^{(r-q)t+X_t}$ has the desired risk-neutral dynamics, since:

$$ES_t = S_0 e^{(r-q)t}, \quad t \geq 0.$$

## The thigh bone's connected to the..

- Recall from way back when that $\gamma(k)$ is a function relating the call price to the moneyness $k \equiv \ln(K/F_0)$.
- Carr Madan relate the FT of $\gamma$ to the CF of $X_T \equiv \ln\left(\frac{F_T}{F_0}\right)$:

$$\mathcal{F}_\gamma(u,T) = \frac{F_0 B_0(T)\mathcal{F}_q(u-i,T)}{(i-u)u},$$

  where $q(k,T) \equiv Q\left\{X_T \in (k, k+dk)\right\}$ is the risk-neutral PDF of $X_T$ and $B_0(T)$ is the price of a bond paying \$1 at $T$.
- The CF of $X_t = \ln\left(\frac{F_t}{F_0}\right)$ is det'd by the char. exponent $\Psi(u)$:

$$\mathcal{F}[q](u,T) = Ee^{iuX_T} = e^{iubT - T\Psi(u)}, \text{ where } b = \Psi(-i).$$

- The characteristic exponent is det'd by the Lévy measure $\ell(dx)$:

$$\Psi(u) = u^2\sigma^2/2 - \int_{\Re-\{0\}} (e^{iux}-1)\ell(dx).$$

- If $\ell(dx)$ is chosen so that $\int_{\Re-\{0\}} (e^{iux}-1)\ell(dx)$ can be evaluated in closed form, then the CF and FT of $\gamma$ are also closed form.
- The table on the next page gives several popular Lévy measures and closed form expressions for the corresponding characteristic exponents.

# Lévy Measures & Characteristic Exponents

Table 1:

| Driver's Name | Lévy Measure $\ell(dx)/dx$ | Characteristic Exponent $\Psi(u) \equiv -\ln E e^{iuL_1}$ |
|---|---|---|
| | *Purely Continuous Lévy Driver* | |
| ABM $\mu t + \sigma W_t$ | $0$ | $-i\mu u + \frac{1}{2}\sigma^2 u^2$ |
| | *Finite Activity Pure Jump Lévy components* | |
| Merton Jump Part | $\lambda \frac{1}{\sqrt{2\pi\sigma_j^2}} \exp\left(-\frac{(x-\alpha)^2}{2\sigma_j^2}\right)$ | $\lambda\left(1 - e^{iu\alpha - \frac{1}{2}\sigma_j^2 u^2}\right)$ |
| Kou's Double Exp'l | $\lambda \frac{1}{2\eta} \exp\left(-\frac{\lvert x-k\rvert}{\eta}\right)$ | $\lambda\left(1 - e^{iuk}\frac{1-\eta^2}{1+u^2\eta^2}\right)$ |
| Eraker (2001) | $\lambda \frac{1}{\eta} \exp\left(-\frac{x}{\eta}\right)$ | $\lambda\left(1 - \frac{1}{1-iu\eta}\right)$ |
| | *Infinite Activity Pure Jump Lévy Driver* | |
| Normal Inv. Gauss | $e^{\beta x}\frac{\delta\alpha}{\pi\lvert x\rvert} K_1(\alpha\lvert x\rvert)$ | $-\delta\left[\sqrt{\alpha^2-\beta^2} - \sqrt{\alpha^2-(\beta+iu)^2}\right]$ |
| Hyperbolic | $\frac{e^{\beta x}}{\lvert x\rvert}\left[\int_0^\infty \frac{e^{-\sqrt{2y+\alpha^2}\lvert x\rvert}}{\pi^2 y\left(J_{\lvert\lambda\rvert}^2(\delta\sqrt{2y}) + Y_{\lvert\lambda\rvert}^2(\delta\sqrt{2y})\right)} dy \right.$ $\left. + 1_{\lambda \geq 0}\lambda e^{-\alpha\lvert x\rvert}\right)$ | $-\ln\left[\frac{\sqrt{\alpha^2-\beta^2}}{\sqrt{\alpha^2-(\beta+iu)^2}}\right]^\lambda \left[\frac{K_\lambda\left(\delta\sqrt{\alpha^2-(\beta+iu)^2}\right)}{K_\lambda\left(\delta\sqrt{\alpha^2-\beta^2}\right)}\right]$ |
| CGMY | $\begin{cases} Ce^{-G\lvert x\rvert}\lvert x\rvert^{-Y-1}, & x < 0, \\ Ce^{-M\lvert x\rvert}\lvert x\rvert^{-Y-1}, & x > 0 \end{cases}$ | $C\Gamma(-Y)\left[M^Y - (M-iu)^Y + G - (G+iu)^Y\right]$ |
| Variance Gamma | $\frac{\mu_\pm^2}{v_\pm}\frac{\exp\left(-\frac{\mu_\pm}{v_\pm}\lvert x\rvert\right)}{\lvert x\rvert}$ $(\mu_\pm = \sqrt{\frac{\alpha^2}{4\lambda^2} + \frac{\sigma_j^2}{2}} \pm \frac{\alpha}{2\lambda}, v_\pm = \mu_\pm^2/\lambda)$ | $\lambda \ln\left(1 - iu\alpha + \frac{1}{2}\sigma_j^2 u^2\right)$ |
| FiniteMomLogStbl | $c\lvert x\rvert^{-\alpha-1}, \quad x < 0$ | $-c\Gamma(-\alpha)\,(iu)^\alpha$ |

# Stochastic Volatility and Timex

- By definition, Lévy processes have stationary independent increments.
- As a consequence, squared returns are independent i.e. volatility does not cluster.
- However, there is much empirical evidence to the contrary.
- Fortunately, one can capture volatility clustering by *time-changing* a Lévy process. If a Lévy process is run on a stochastic clock whose increments are correlated, then the increments in the Lévy process inherit this correlation.
- Mathematically, a stochastic clock (technically a subordinator) is a right continuous increasing stochastic process started at 0.
- Intuitively, think of it as a $5 Rolex.
- If the Lévy process is standard Brownian motion and the stochastic clock is $\tau(t) \equiv \int_0^t \sigma_t^2 dt$, then:

$$dB_{\tau(t)} \stackrel{d}{=} \sigma_t dB_t,$$

  by the Brownian scaling property. If $\sigma_t$ is a random continuous process, then the increments of $B_{\tau(t)}$ become correlated.

# FT of Time Changed Lévy Process

- Let $X$ be a Lévy process started at 0 and whose jump component has sample paths of finite variation. Then its CF is:

$$\mathcal{F}_{X_t}(u) \equiv E e^{iuX_t} = e^{-t\Psi_x(u)}, \quad t \geq 0,$$

where $\Psi_x(u) \equiv -ibu + \frac{\sigma^2 u^2}{2} - \int_{\Re-\{0\}} \left(e^{iux} - 1\right) \ell(dx)$ is the characteristic exponent of $X_t$.

- Let $\tau$ be a subordinator which is independent of $X$ and let $Y_t \equiv X_{\tau_t}, t \geq 0$. Then the CF of $Y$ involves expectations over 2 sources of randomness:

$$\mathcal{F}_{Y_t}(u) \equiv E e^{iuY_t} = E e^{iuX_{\tau_t}} = E\left[E\left[e^{iuX_{\tau_t}} | \tau_t = u\right]\right].$$

- If $\tau_t$ is independent of $X$, then the randomness due to the Lévy process can be integrated out using the top equation:

$$\mathcal{F}_{Y_t}(u) = E e^{-\tau_t \Psi_x(u)} \equiv \mathcal{L}_{\tau_t}(\Psi_x(u)),$$

where $\mathcal{L}_{\tau_t}(\lambda) \equiv E e^{-\lambda \tau_t}$ is the Laplace Transform (LT) of $\tau_t$, $\lambda \in \mathcal{C}, Re(\lambda) \geq 0$.

- Thus, the CF of $Y_t$ is just the LT of $\tau_t$ evaluated at the characteristic exponent of $X$.
- Clearly, if the LT of $\tau_t$ and the characteristic exponent of $X$ are both available in closed form, then so is the CF of $Y_t$.

# Laplace Transforms and Bond Prices

- Consider specifying the clock in terms of an *activity rate* $v_t$:
$$\tau_t = \int_0^t v_{s-} ds, \quad v_s \geq 0.$$
- Then the Laplace transform of the clock has form:
$$\mathcal{L}_{\tau_t}(\lambda) \equiv E\left[e^{-\lambda \tau_t}\right] = E\left[e^{-\lambda \int_0^t v_{s-} ds}\right].$$
- This formulation arises in the bond pricing literature if we regard $\lambda v_t$ as the instantaneous interest rate.
- If the Lévy process being time changed is Brownian motion, then $v$ is the variance rate.
- The instantaneous interest rate and the instantaneous activity rate are both required to be non-negative and are commonly thought to be mean reverting.
- Thus, one can adopt the vast literature on bond pricing to obtain Laplace transforms in closed form.
- In particular, one can apply 2 tractable bond pricing classes, namely affine and quadratic interest rate models.
- These classes are summarized in the table on the next page.
- See CGMY (MF 03) and Carr & Wu (JFE 03) for various pairings of Lévy processes and stochastic clocks.

# Activity Rate Processes & LT of Clock

Under each class of activity rate processes, the entries summarize the specification of the activity rate and the corresponding Laplace transform of the random time.

| Activity Rate Specification $v_t$ | Laplace Transform $\mathcal{L}_{T_t}(\lambda) \equiv E\left[e^{-\lambda T_t}\right]$ |
|---|---|
| *Affine: Duffie, Pan, Singleton (2000)* | |
| $v_t = \mathbf{b}_v^\top Z_t + c_v,$ <br> $\mu(Z_t) = a - \kappa Z_t,$ <br> $\left[\sigma(Z_t)\sigma(Z_t)^\top\right]_{ii} = \alpha_i + \beta_i^\top Z_t,$ <br> $\left[\sigma(Z_t)\sigma(Z_t)^\top\right]_{ij} = 0, \quad i \neq j,$ <br> $\gamma(Z_t) = a_\gamma + \mathbf{b}_\gamma^\top Z_t.$ | $\exp\left(-\mathbf{b}(t)^\top z_0 - c(t)\right),$ <br> $\mathbf{b}'(t) = \lambda \mathbf{b}_v - \kappa^\top \mathbf{b}(t) - \frac{1}{2}\beta \mathbf{b}(t)^2$ <br> $-\mathbf{b}_\gamma\left(Ł_q(\mathbf{b}(t)) - 1\right),$ <br> $c'(t) = \lambda c_v + \mathbf{b}(t)^\top a - \frac{1}{2}\mathbf{b}(t)^\top \alpha \mathbf{b}(t)$ <br> $-a_\gamma\left(Ł_q(\mathbf{b}(t)) - 1\right),$ <br> $\mathbf{b}(0) = 0,\ c(0) = 0.$ |
| *Generalized Affine: Filipovic (2001)* | |
| $\mathcal{A}f(x) = \frac{1}{2}\sigma^2 x f''(x) + (a' - \kappa x) f'(x)$ <br> $+ \int_{\mathbb{R}_0^+}\left(f(x+y) - f(x) + f'(x)(1 \wedge y)\right)$ <br> $\left(m(dy) + x\mu(dy)\right),$ <br> $a' = a + \int_{\mathbb{R}_0^+}(1 \wedge y)\, m(dy),$ <br> $\int_{\mathbb{R}_0^+}\left[(1 \wedge y)\, m(dy) + (1 \wedge y^2)\, \mu(dy)\right] < \infty.$ | $\exp\left(-b(t)v_0 - c(t)\right),$ <br> $b'(t) = \lambda - \kappa b(t) - \frac{1}{2}\sigma^2 b(t)^2$ <br> $+ \int_{\mathbb{R}_0^+}\left(1 - e^{-yb(t)} - b(t)(1 \wedge y)\right)\mu(dy),$ <br> $c'(t) = ab(t) + \int_{\mathbb{R}_0^+}\left(1 - e^{-yb(t)}\right) m(dy),$ <br> $b(0) = c(0) = 0.$ |
| *Quadratic: Leippold and Wu (2002)* | |
| $\mu(Z) = -\kappa Z, \quad \sigma(Z) = I,$ <br> $v_t = Z_t^\top A_v Z_t + \mathbf{b}_v^\top Z_t + c_v.$ | $\exp\left[-z_0^\top A(t) z_0 - \mathbf{b}(t)^\top z_0 - c(t)\right],$ <br> $A'(t) = \lambda A_v - A(t)\kappa - \kappa^\top A(\tau) - 2A(t)^2,$ <br> $\mathbf{b}'(t) = \lambda \mathbf{b}_v - \kappa \mathbf{b}(t) - 2A(t)^\top \mathbf{b}(t),$ <br> $c'(t) = \lambda c_v + trA(t) - \mathbf{b}(t)^\top \mathbf{b}(t)/2,$ <br> $A(0) = 0, \mathbf{b}(0) = 0, c(0) = 0.$ |

# Correlation and the Leverage Effect

- To capture the well documented volatility clustering phenomenon, we time-changed a Lévy process using an independent subordinator.
- It is also well documented that percentage changes in the underlying's price and volatility are correlated, typically negatively.
- Whether or not this correlation is due to leverage, it is commonly referred to as the leverage effect.
- Carr and Wu (JFE 03) show how to calculate the CF of a Lévy process time-changed by a correlated subordinator.
- Monroe (1978) showed that any semi-martingale can be characterized as Brownian motion time-changed by a possibly correlated subordinator.
- Hence, the entire class of processes used in derivatives pricing can now be captured by Fourier methods.

# Motivating Complex Measure

- Recall that if the Lévy process $X$ and the clock $\tau_t$ are independent, then the CF of $Y_t \equiv X_{\tau_t}$ is given by:

$$\phi_{Y_t}(u) = \mathcal{L}_{\tau_t}(\Psi_x(u)) = E^Q\left[e^{-\Psi_x(u)\tau_t}\right],$$

  where $\Psi_x(u)$ is the characteristic exponent of $X$.

- Suppose for simplicity that the PDF of $X$ is symmetric (eg. SBM). Then the PDF of Y is also symmetric and the characteristic functions and exponents of both $X$ and $Y$ are real.

- One can introduce skewness into the distribution of $Y$ by introducing correlation between increments in $\tau$ and $X$. Then, the CF of $Y$ takes on a non-zero imaginary part.

- Suppose we still want to relate CF's to LT's and to characteristic exponents, as in the top equation.

- As the characteristic exponent of $X$ and the clock $\tau_t$ are both real, the only way to accomplish this is to allow the probability measure $Q$ used in the expectation to become complex.

# Changing to Complex Measure

- Let $Q$ be the usual risk-neutral measure under which a Lévy process $X$ has characteristic exponent $\Psi_x(u)$.
- Let $\tau_t$ be a stochastic clock and let $M_t(u) \equiv e^{iuX_{\tau_t}+\tau_t \Psi_x(u)}$.
- Carr and Wu use the Optional stopping theorem to show that $M_t(u)$ is a well-defined complex-valued $Q$-martingale and hence can be used to change the real valued probability measure $Q$ into a complex valued measure $Q(u)$:

$$\begin{aligned} E^Q\left[e^{iuY_t}\right] &= E^Q\left[e^{iuY_t+\tau_t\Psi_x(u)-\tau_t\Psi_x(u)}\right] = E^Q\left[M_t(u)e^{-\tau_t\Psi_x(u)}\right] \\ &= E^{Q(u)}\left[e^{-\tau_t\Psi_x(u)}\right] \equiv \mathcal{L}^u_{\tau_t}\left(\Psi_x(u)\right). \end{aligned}$$

- Thus, the generalized CF of the time-changed Lévy process $Y_t \equiv X_{T_t}$ under measure $Q$ is just the (modified) Laplace transform of $\tau$ under the complex-valued measure $Q(u)$, evaluated at the characteristic exponent $\Psi_x(u)$ of $X_t$.
- Just as Cox-Ross (JFE 76) show that correct valuation arises if a risk-neutral investor uses $Q$ in place of statistical measure, we show that correct valuation arises if an investor who believes in independence uses $Q(u)$ in place of $Q$. For this reason, we term $Q(u)$ the *leverage-neutral* measure. We illustrate many old models (eg. Heston (RFS 93)) and many new models from the perspective of this generalized framework.

# Summary and Future Research

- After reviewing the meanings of FT and CF, we showed how options can be priced by a single integration once one knows the CF of the underlying log price in closed form.
- To obtain this CF in closed form, we can time-change a Lévy process using a stochastic clock whose increments are in general correlated with returns.
- This allows us to rapidly develop and test a wide variety of option pricing models, which reflect empirical realities such as jumps, volatility clustering, and the leverage effect.
- A quick glance at my bibliography should convince you that Fourier methods are already being applied to many areas of finance besides option pricing.
- Judging from the enormity of the applications of harmonic analysis in mathematics and the hard sciences, it is not too hard to predict that much work remains to be done.
- Copies of these transparencies can be downloaded from: `www.petercarr.net` (and clicking on Papers) or `www.math.nyu.edu\research\carrp\papers\pdf`

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