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Empirical assessment of an intertemporal option pricing model with latent variables

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Abstract

This paper assesses the empirical performance of an intertemporal option pricing model with latent variables which generalizes the Black–Scholes and the stochastic volatility formulas. We derive a closed-form formula for an equilibrium model with recursive preferences where the fundamentals follow a Markov switching process. In a simulation experiment based on the model, we show that option prices are more informative about preference parameters than stock returns. When we estimate the preference parameters implicit in S&P 500 call option prices given our model, we find quite reasonable values for the coefficient of relative risk aversion and the intertemporal elasticity of substitution.

JEL classification: C1; C5; G1

Keywords: Stochastic volatility; Black–Scholes implied volatility; Smile effect; Equilibrium option pricing; Recursive utility

1. Introduction

The empirical option pricing literature has revealed a considerable divergence between the risk-neutral distributions estimated from option prices after the 1987 crash and conditional distributions estimated from time series of returns on the underlying index. Three facts come out clearly. First, the implied volatility extracted from at-the-money options differs substantially from the realized volatility over the lifetime of the option. Second, risk-neutral distributions feature a substantial negative skewness

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which is revealed by the asymmetric implied volatility curves when plotted against moneyness. Third, the shape of these volatility curves changes over time, in other words the skewness is time varying. [1]

One possible explanation for the divergence between the objective and the risk-neutral distributions is the existence of time-varying risk premia. Pan (2002) estimates a jump-diffusion model proposed by Bates (2000) and investigates how volatility and jump risks are priced in S&P 500 index options. Based on a joint time series of the spot price and of one at-the-money option, Pan (2002) shows that the addition of both volatility and jump risk premia allows to fit well the joint time series of spot and option price data. The model can explain well the changing shapes of the implied volatility curves over time and the skewed patterns are largely due to investors' aversion to jump risks. However, it is not clear how this non-arbitrage continuous-time model relates to the preferences of a representative agent since in this approach investors may have different risk attitudes towards the diffusive return shocks, volatility shocks and jump risks. In a non-parametric framework, Aït-Sahalia and Lo (2000) and Jackwerth (2000) uncover the risk-aversion function implied by the comparison between the objective and the risk-neutral distributions, while Rosenberg and Engle (1999) investigate the empirical characteristics of investor risk aversion by estimating a daily semiparametric pricing kernel. Jackwerth (2000) finds that the preferences are oddly shaped, with marginal utilities increasing for some wealth levels. However, the implied-tree and the kernel methodologies used to recover the risk-neutral and the subjective probabilities are not likely to separate neatly the preferences from the probabilities, especially if the stochastic discount factor depends on state variables. These results underline the potential importance of investors' preferences for option prices but leave the question of knowing if option prices are compatible with reasonable preferences largely unanswered.

In this paper, we propose a utility-based option pricing model with stochastic volatility and jump features to better understand the relationships between the preferences embedded in risk premia and the aforementioned empirical facts. The model is cast within the recursive utility framework of Epstein and Zin (1989) in which the respective roles of discounting, risk aversion and intertemporal substitution are disentangled. This separation might be important for option pricing since an option contract will naturally be affected by the value of time as well as the price of risk associated with the underlying asset. We derive an option pricing formula which generalizes the Black and Scholes (1973) and the Hull and White (1987) and Heston (1993) stochastic volatility formulas, hereafter referred to as BS and SV formulas. [2]

An essential feature of this generalized option pricing formula is that it is not in general preference free. In so-called preference free formulas of which BS and SV are examples, it happens that these parameters are eliminated from the option pricing formula through the observation of the bond price and the stock price. In other

[1] These facts come out of a string of studies by Bakshi et al. (1997), Bates (1996, 2000), Chernov and Ghysels (2000), and Pan (2002), among others.

[2] Our formula can be seen as a discrete-time Heston-type formula. However, in contrast with Heston, the risk premia are explicitly linked to the preference parameters of a representative agent.

words, preference parameters are hidden in the observed stock and bond prices. In our case, the bond pricing formula and the stock pricing formula provide two dynamic restrictions relating the characteristics of the stochastic discount factor of the model (which include the preferences) to the bond and stock price processes. The key assumption underlying this result is the presence of an unobserved state variable driving the fundamentals (consumption and dividends) of the economy as in Cecchetti et al. (1990, 1993) and Bonomo and Garcia (1994a, b, 1996). This state variable captures the states of the economy which are typically represented by a low consumption growth associated with a high volatility of dividend growth or by a high consumption growth together with a low volatility of dividend growth. A contemporaneous correlation between the state variable and the fundamentals makes the preference parameters play an additional role over and above their impact on stock and bond prices. Therefore, it appears natural to investigate the informativeness of option prices about preference parameters and to confirm the dependence of option prices on preference parameters.

First, based on simulations, we show that option prices are more informative than stock returns about the structural parameters of the asset pricing model. More precisely, we show that a set moment conditions based on the mean, variance and autocovariance of order one of stock returns does not provide good estimates of the preference parameters in finite samples. Therefore, one can possibly question the empirical tests of intertemporal asset pricing models that have been based mostly on bond and stock returns. On the other hand, similar moment conditions with option prices recover with great accuracy the preference parameters. Part of the explanation lies probably in the better spanning of the stochastic discount factor (or the underlying risk neutral probability distribution) by a panel of option prices. The non-linear nature of the option payoffs could also help given the non-linearity in parameters of the model.

We further show that a simple method of moments with a panel of simulated option prices provides good estimates of all the parameters of the model, that is, parameters associated with the fundamentals in the economy along with the preference parameters. This lays the ground for an empirical assessment of the model with S&P 500 option prices in terms of out-of-sample pricing errors and a comparison with usual stochastic volatility and expected utility models which appear as particular cases of our general framework. Our results indicate clearly that the explicit incorporation of preferences improves the performance of the option pricing model and that time non-separable preferences improve the results further. Preference parameter estimates appear reasonable and stable over a 5-year period (1991–1995).

Apart from the economic interest of recovering preference parameters from this new option pricing formula, there is always the question of its practical use say for forecasting the price of other options. Taking options of all moneyness and all maturities at once, we confirm that the absolute and relative errors of the non-expected utility model are lower than the errors produced by the expected utility model and a stochastic volatility model. However, the magnitude of the errors remains very large with respect to the errors associated with practitioners' ad hoc approaches such as plugging in the BS formula implied volatilities of the day or the week before. To put the model on a level playing field with ad hoc approaches, we separate the options according to maturity for estimation, we reduce the period over which empirical moments are computed to the last 5 days and finally we introduce conditioning information in the estimation. The volatility of the dividend growth is made a function of the implied volatility of the same class of moneyness the day before. This has the effect of reducing the errors to levels more in line with the practitioners' ad hoc approaches, but given the complex structure of our model, it does not appear as a practical substitute to the simple practitioners' Black–Scholes. Moreover, this shorter-term calibration blurs the distinctions between the expected utility and the non-expected utility models since they perform quite similarly in terms of predictive ability.

The interplay between preferences and latent factors that affect the stochastic discount factor has been explored to a certain extent in the literature. Amin and Ng (1993) provide an extension of the equilibrium model of Rubinstein (1976) and Brennan (1979) with a systematic stochastic volatility in stock returns. Garcia et al. (2001) show that the option pricing model we estimate in this paper can reproduce the various patterns observed in implied volatility curves as well as changing skewness over time. David and Veronesi (1999) show that option prices are affected by investors' beliefs about the drift of a firm's fundamentals. In particular, they emphasize how investors' beliefs and their degree of risk aversion affect stock returns and hence option prices. Guidolin and Timmermann (1999) explain the empirical biases of the Black–Scholes option pricing model by Bayesian learning effects. The importance of preference parameters in explaining fluctuations in equity prices has also been explored by Mehra and Sah (1998) who show that small changes in investors' subjective discount factors and attitudes towards risk can induce volatility in equity prices. The main thesis of the paper is that some instantaneous causality effects between state variables and asset prices can capture the stylized facts of interest without having to introduce any fluctuation in beliefs or preferences or learning.

The rest of the paper is organized as follows. Section 2 develops a generalized option pricing formula with latent variables based on a recursive utility consumption-based asset pricing model. Section 3 explores, in a simulation experiment, the information about preference parameters contained in option prices compared with that in stock returns. Preference parameters are also estimated using S&P 500 option and stock prices. Section 4 calibrates the model for practical option pricing. Section 5 concludes.

2. An intertemporal option pricing model with latent variables

We adopt the recursive utility framework proposed by Epstein and Zin (1989). Many identical infinitely lived agents maximize their lifetime utility and receive each period an endowment of a single non-storable good. Their recursive utility function is of the form

$$V_t = W(C_t, \mu_t), \tag{2.1}$$

where W is an aggregator function that combines current consumption C_t with $\mu_t = \mu(\tilde{V}_{t+1} \mid J_t)$, a certainty equivalent of random future utility $\tilde{V}_{t+1}$, given J_t the information available to the agents at time t, to obtain the current-period lifetime utility

V_t. Following Kreps and Porteus (1978), Epstein and Zin (1989) propose the CES function as the aggregator function, i.e.,

$$V_t = [C_t^\rho + \beta \mu_t^\rho]^{\frac{1}{\rho}}. \tag{2.2}$$

The way the agents form the certainty equivalent of random future utility is based on their risk preferences, which are assumed to be isoelastic, i.e., $\mu_t^\alpha = \mathrm{E}[\tilde{V}_{t+1}^\alpha \mid J_t]$, where $\alpha \leqslant 1$ is the risk aversion parameter (1-α is the Arrow–Pratt measure of relative risk aversion). Given these preferences, the following Euler condition must be valid for any asset j if an agent maximizes his lifetime utility (see Epstein and Zin, 1989):

$$\mathrm{E}\left[\beta^\gamma \left(\frac{C_{t+1}}{C_t}\right)^{\gamma(\rho-1)} M_{t+1}^{\gamma-1} R_{j,t+1} | J_t\right] = 1, \tag{2.3}$$

where M_{t+1} represents the return on the market portfolio, $R_{j,t+1}$ the return on any asset j, and $\gamma = \alpha/\rho$. The parameter ρ is associated with intertemporal substitution, since the elasticity of intertemporal substitution is $1/(1-\rho)$. The position of α with respect to ρ determines whether the agent has a preference towards early resolution of uncertainty ($\alpha < \rho$) or late resolution of uncertainty ($\alpha > \rho$).

Since the market portfolio price, say P_t^M at time t, is determined in equilibrium, it should also verify the first-order condition:

$$\mathrm{E}\left[\beta^\gamma \left(\frac{C_{t+1}}{C_t}\right)^{\gamma(\rho-1)} M_{t+1}^{\gamma} | J_t\right] = 1. \tag{2.4}$$

In this model, the payoff of the market portfolio at time t is the total endowment of the economy C_t. Therefore the return on the market portfolio M_{t+1} can be written as follows:

$$M_{t+1} = \frac{P_{t+1}^M + C_{t+1}}{P_t^M}.$$

Replacing M_{t+1} by this expression, we obtain

$$\lambda_t^\gamma = \mathrm{E}\left[\beta^\gamma \left(\frac{C_{t+1}}{C_t}\right)^{\gamma\rho} (\lambda_{t+1}+1)^\gamma | J_t\right], \tag{2.5}$$

where $\lambda_t = P_t^M/C_t$. Under some regularity and stationarity assumptions, there exists a unique solution λ_t to (2.5) of the form $\lambda_t = \lambda(J_t)$ with $\lambda(\cdot)$ solution of

$$\lambda(J)^\gamma = \mathrm{E}\left[\beta^\gamma \left(\frac{C_{t+1}}{C_t}\right)^{\gamma\rho} (\lambda(J_{t+1})+1)^\gamma | J_t = J\right]. \tag{2.6}$$

Similarly, we will be looking for a solution $\varphi_t = \varphi(J_t) = S_t/D_t$ to the stock pricing equation:

$$\varphi(J) = \mathrm{E}\left[\beta^\gamma \left(\frac{C_{t+1}}{C_t}\right)^{\gamma\rho-1} \left(\frac{\lambda_{t+1}+1}{\lambda_t}\right)^{\gamma-1} (\varphi(J_{t+1})+1)\, \frac{D_{t+1}}{D_t}\right| J_t = J\Bigg]. \tag{2.7}$$

It is then possible, for given λ and φ functions, to compute the market portfolio price and the stock price as $P_t^M = \lambda(J_t)C_t$ and $S_t = \varphi(J_t)D_t$. The dynamic behavior of these prices, or equivalently of the associated rates of return:

$$\log M_{t+1} = \log \frac{\lambda(J_{t+1}) + 1}{\lambda(J_t)} + \log \frac{C_{t+1}}{C_t} \tag{2.8}$$

and

$$\log R_{t+1} = \log \frac{S_{t+1} + D_{t+1}}{S_t} = \log \frac{\varphi(I_{t+1}) + 1}{\varphi(I_t)} + \log \frac{D_{t+1}}{D_t} \tag{2.9}$$

is determined by the joint probability distribution of the stochastic process (X_t, Y_t, J_t) where $X_t = \log C_t/C_{t-1}$ and $Y_t = \log D_t/D_{t-1}$.

2.1. A pricing model conditional on latent state variables

We shall define these dynamics through a stationary vector process of state variables U_t such that

$$J_t = \bigvee_{\tau \leqslant t} [X_\tau, Y_\tau, U_\tau]. \tag{2.10}$$

We want these state variables to be exogenous and stationary and to subsume all temporal links between the variables of interest (X_t, Y_t). This framework has already been used in asset pricing models (see Cecchetti et al., 1990, 1993; Bonomo and Garcia, 1994a, b, 1996; Amin and Ng, 1993). We achieve this through Assumptions 1, 2 and 3 below:

Assumption 1. The pairs $(X_t, Y_t)_{1 \leqslant t \leqslant T}$, $t = 1, \ldots, T$, are mutually independent given $U_1^T = (U_t)_{1 \leqslant t \leqslant T}$.

Assumption 2. The fundamentals (X, Y) do not cause the state variables U in the Granger sense or equivalently, given Assumption 1, the conditional probability distribution of (X_t, Y_t) given $U_1^T = (U_t)_{1 \leqslant t \leqslant T}$ coincides, for any $t = 1, \ldots, T$, with the conditional probability distribution given $U_1^t = (U_\tau)_{1 \leqslant \tau \leqslant t}$.

Assumption 3. The conditional probability distribution of $(X_{t+1}, Y_{t+1}, U_{t+1})$ given U_1^t only depends upon U_t.

Under Assumptions 1–3 we have

$$P_t^M = \lambda(U_t)C_t, \quad S_t = \varphi(U_t)D_t,$$

where $\lambda(U_t)$ and $\varphi(U_t)$ are, respectively, defined by

$$\lambda(U_t)^\gamma = \mathrm{E}\left[\beta^\gamma \left(\frac{C_{t+1}}{C_t}\right)^{\gamma\rho} (\lambda(U_{t+1}) + 1)^\gamma | U_t\right]$$

and

$$\varphi(U_t) = \mathrm{E}\left[\beta^\gamma \left(\frac{C_{t+1}}{C_t}\right)^{\gamma\rho - 1} \left(\frac{\lambda(U_{t+1}) + 1}{\lambda(U_t)}\right)^{\gamma - 1} (\varphi(U_{t+1}) + 1)\frac{D_{t+1}}{D_t} | U_t\right]. \tag{2.11}$$

In this setting, a contraction mapping argument may be applied as in Lucas (1978) to ensure existence and unicity of the functions $\lambda(\cdot)$ and $\varphi(\cdot)$.[3] Using the definitions of returns on the market portfolio and asset S_t, we can write

$$\log M_{t+1} = \log \frac{\lambda(U_{t+1}) + 1}{\lambda(U_t)} + X_{t+1} \tag{2.12}$$

and

$$\log R_{t+1} = \log \frac{\varphi(U_{t+1}) + 1}{\varphi(U_t)} + Y_{t+1}.$$

Hence, the return processes (M_{t+1}, R_{t+1}) are stationary as U, X and Y, but, contrary to the stochastic setting in the Lucas (1978) economy, are not Markovian due to the presence of unobservable state variables U.

Given this intertemporal model with latent variables, we will show how standard asset pricing models will appear as particular cases under some specific configurations of the stochastic framework. In particular, we will analyze the pricing of bonds, stocks and options and show under which conditions the usual models, such as the CAPM or the Black–Scholes model, are obtained. To achieve this, we introduce an additional assumption on the probability distribution of the fundamentals X and Y given the state variables U.

Assumption 4.

$$\begin{pmatrix} X_{t+1} \\ Y_{t+1} \end{pmatrix} | U_t^{t+1} \sim \aleph \left[\begin{pmatrix} m_{Xt+1} \\ m_{Yt+1} \end{pmatrix}, \begin{bmatrix} \sigma^2_{Xt+1} & \sigma_{XYt+1} \\ \sigma_{XYt+1} & \sigma^2_{Yt+1} \end{bmatrix} \right],$$

where $m_{Xt+1} = m_X(U_1^{t+1})$, $m_{Yt+1} = m_Y(U_1^{t+1})$, $\sigma^2_{Xt+1} = \sigma^2_X(U_1^{t+1})$, $\sigma_{XYt+1} = \sigma_{XY}(U_1^{t+1})$, $\sigma^2_{Yt+1} = \sigma^2_X(U_1^{t+1})$. In other words, these means and variance–covariance functions are time-invariant and measurable functions with respect to U_t^{t+1}, which includes both U_t and U_{t+1}.

This conditional normality assumption allows for skewness and excess kurtosis in unconditional returns.[4] It is also useful for recovering as a particular case the Black–Scholes formula.[5]

2.2. Pricing formulas for bonds, stocks and options

In all three following subsections we will price the respective assets using the Euler conditions and use our Assumptions 1–4 to derive a pricing formula. In each case, we

[3] It should be emphasized that this framework is more general than the Lucas one because the state variables U_t are given by a general multivariate Markovian process (while a Markovian dividend process is the only state variable in Lucas, 1978).

[4] Actually, it even allows for skewness and excess kurtosis in conditional returns, given the information available to the agents at the beginning of the period.

[5] It can also be argued that, if one considers that the discrete-time interval is somewhat arbitrary and can be infinitely split, log-normality (conditional on state variables U) is obtained as a consequence of a standard central limit argument given the independence between consecutive (X, Y) conditional on U.

will emphasize the prominent role of the latent variable in pricing the assets. We will insist especially on its contemporaneous correlation with the asset returns.

2.2.1. The pricing of bonds

Given the Euler condition (2.3) and Assumptions 1–3, the time t price of a bond delivering one unit of the good at time T, $B(t,T)$, is given by the following formula:

$$B(t,T)=\mathrm{E}_t\left[\beta^{\gamma(T-t)}\left(\frac{C_T}{C_t}\right)^{\alpha-1}\prod_{\tau=t}^{T-1}\left[\frac{(1+\lambda(U_1^{\tau+1})}{\lambda(U_1^{\tau})}\right]^{\gamma-1}\right],$$

which can be written as

$$B(t,T)=\mathrm{E}_t[\tilde{B}(t,T)], \tag{2.13}$$

with

$$\tilde{B}(t,T)=\beta^{\gamma(T-t)}a_t^T(\gamma)\exp\left((\alpha-1)\sum_{\tau=t}^{T-1}m_{X\tau+1}+\frac{1}{2}(\alpha-1)^2\sum_{\tau=t}^{T-1}\sigma^2_{X\tau+1}\right),$$

where $a_t^T(\gamma)=\prod_{\tau=t}^{T-1}[(1+\lambda(U_1^{\tau+1})/\lambda(U_1^{\tau})]^{\gamma-1}$.

This formula shows how the interest rate risk is compensated in equilibrium, and in particular how the term premium is related to preference parameters. Given the expression for $\tilde{B}(t,T)$ above, it can be seen that for von-Neuman preferences ($\gamma=1$) the term premium is proportional to the square of the coefficient of relative risk aversion (up to a conditional stochastic volatility effect). Another important observation is that even without any risk aversion ($\alpha=1$), preferences still affect the term premium through the non-indifference to the timing of uncertainty resolution ($\gamma\neq 1$).

There is however an important sub-case where the term premium will be preference free because the stochastic discount factor $\tilde{B}(t,T)$ coincides with the observed rolling-over discount factor (the product of short-term future bond prices, $B(\tau,\tau+1)$, $\tau=t,\dots,T-1$). Noticing that $\tilde{B}(t,T)=\prod_{\tau=t}^{T-1}\tilde{B}(\tau,\tau+1)$, this will occur as soon as $\tilde{B}(\tau,\tau+1)=B(\tau,\tau+1)$, that is, when $\tilde{B}(\tau,\tau+1)$ is known at time τ. From the expression of $\tilde{B}(t,T)$ above, it is easy to see that this last property holds if and only if the mean and variance parameters $m_{X\tau+1}$ and $\sigma_{X\tau+1}$ depend on $U_\tau^{\tau+1}$ only through U_τ. In this case, the conditional distribution of X_t given the whole past and future path of U is equal to the conditional distribution of X given only the past of U, that is $\ell(X_t|U_1^T)=\ell(X_t|U_1^{t-1})$. It is this property which ensures that short-term stochastic discount factors are predetermined, so the bond pricing formula becomes preference free:

$$B(t,T)=\mathrm{E}_t\prod_{\tau=t}^{T-1}B(\tau,\tau+1).$$

Of course, this does not necessarily cancel the term premia but it makes them preference free in the sense that the role of preference parameters is fully hidden in short-term bond prices. Moreover, when there is no interest rate risk because the consumption growth rates X_t are i.i.d., it is straightforward to check that constant m_{Xt+1} and σ^2_{Xt+1} imply constant $\lambda(\cdot)$ and in turn $\tilde{B}(t,T)=B(t,T)$, with zero term premia.

2.2.2. The pricing of stocks

By a recursive argument on the Euler condition (2.11), the stock price formula can be written as follows:

$$\mathrm{E}_t\left[\beta^{\gamma(T-t)} a_t^T(\gamma) b_t^T \left(\frac{C_T}{C_t}\right)^{\alpha-1} \frac{D_T}{D_t}\right] = 1, \tag{2.14}$$

with $b_t^T = \prod_{\tau=t}^{T-1} (1+\varphi(U_1^{\tau+1}))/\varphi(U_1^\tau)$. Using the conditional log-normality Assumption 3, we obtain

$$\mathrm{E}_t\left[\tilde{B}(t,T) b_t^T \exp\left(\sum_{\tau=t+1}^{T} m_{Y\tau} + \frac{1}{2}\sum_{\tau=t+1}^{T} \sigma_{Y\tau}^2 + (\alpha-1)\sum_{\tau=t+1}^{T} \sigma_{XY\tau}\right)\right] = 1. \tag{2.15}$$

With the definitional equation:

$$\mathrm{E}\left[\frac{S_T}{S_t} | U_1^T\right] = \frac{\varphi(U_1^T)}{\varphi(U_1^t)} \exp\left(\sum_{\tau=t+1}^{T} m_{Y\tau} + \frac{1}{2}\sum_{\tau=t+1}^{T} \sigma_{Y\tau}^2\right), \tag{2.16}$$

a useful way of writing the stock pricing formula is:

$$\mathrm{E}_t[Q_{XY}(t,T)] = 1, \tag{2.17}$$

where

$$Q_{XY}(t,T) = \tilde{B}(t,T) b_t^T \frac{\varphi(U_1^t)}{\varphi(U_1^T)} \exp\left((\alpha-1)\sum_{\tau=t+1}^{T} \sigma_{XY\tau}\right) \mathrm{E}\left[\frac{S_T}{S_t} | U_1^T\right]. \tag{2.18}$$

To understand the role of the factor $Q_{XY}(t,T)$, it is useful to notice that it can be factorized as

$$Q_{XY}(t,T) = \prod_{\tau=t}^{T-1} Q_{XY}(\tau,\tau+1),$$

and that there is an important particular case where $Q_{XY}(\tau,\tau+1)$ is known at time τ and therefore equal to one by (2.17). This is when $\ell(X_t, Y_t | U_1^T) = \ell(X_t, Y_t | U_1^{t-1})$. This means that neither the conditional means and variances of X_t or Y_t at time t nor the covariance σ_{XYt} depend on U_t. In this case, we have $Q_{XY}(t,T) = 1$. Since we also have $\tilde{B}(\tau,\tau+1) = B(\tau,\tau+1)$, we can express the conditional expected stock return as

$$\mathrm{E}\left[\frac{S_T}{S_t} | U_1^T\right] = \frac{1}{\prod_{\tau=t}^{T-1} B(\tau,\tau+1)} \frac{1}{b_t^T} \frac{\varphi(U_1^T)}{\varphi(U_1^t)} \exp\left((1-\alpha)\sum_{\tau=t+1}^{T} \sigma_{XY\tau}\right).$$

For pricing over one period (t to $t+1$), this formula provides the agent's expectation of the next period return (since in this case the only relevant information is U_1^t):

$$\mathrm{E}\left[\frac{S_{t+1}}{S_t} \frac{1+\varphi(U_1^{t+1})}{\varphi(U_1^{t+1})} | U_1^t\right] = \frac{1}{B(t,t+1)} \exp[(1-\alpha)\sigma_{XYt+1}],$$

that is

$$\mathrm{E}\left[\frac{S_{t+1}+D_{t+1}}{S_t} | U_1^t\right] = \frac{1}{B(t,t+1)} \exp[(1-\alpha)\sigma_{XYt+1}]. \tag{2.19}$$

This is a particularly interesting result since it is very close to a standard conditional CAPM equation (and unconditional in an i.i.d. world), which remains true for any value of the preference parameters α and ρ. While Epstein and Zin (1991) emphasize that the CAPM obtains for $\alpha = 0$ (logarithmic utility) or $\rho = 1$ (infinite elasticity of intertemporal substitution), we emphasize here that this relationship is obtained under a particular stochastic setting for any values of α and ρ. As we will see in the next section, the stochastic setting which produces this CAPM relationship will also produce most standard option pricing models (for example Black and Scholes, 1973; Hull and White, 1987), which are of course preference free. [6]

2.2.3. A generalized option pricing formula

The Euler condition for the price of a European option is given by

$$\pi_t = \mathrm{E}_t\left[\beta^{\gamma(T-t)}\left(\frac{C_T}{C_t}\right)^{\alpha-1}\prod_{\tau=t}^{T-1}\left[\frac{(1+\lambda(U_1^{\tau+1})}{\lambda(U_1^{\tau})}\right]^{\gamma-1}\max[0, S_T - K]\right]. \tag{2.20}$$

Under Assumptions 1–4, we arrive at a generalized Black–Scholes (GBS) formula (see proof in the appendix):

$$\frac{\pi_t}{S_t} = \mathrm{E}_t\left\{Q_{XY}(t,T)\Phi(d_1) - \frac{K\tilde{B}(t,T)}{S_t}\Phi(d_2)\right\}, \tag{2.21}$$

where

$$d_1 = \frac{\log[S_t Q_{XY}(t,T)/K\tilde{B}(t,T)]}{(\sum_{\tau=t+1}^{T}\sigma_{Y\tau}^2)^{1/2}} + \frac{1}{2}\left(\sum_{\tau=t+1}^{T}\sigma_{Y\tau}^2\right)^{1/2}$$

and

$$d_2 = d_1 - \left(\sum_{\tau=t+1}^{T}\sigma_{Y\tau}^2\right)^{1/2}.$$

It should be noticed that if $Q_{XY}(t,T) = 1$ and $\tilde{B}(t,T) = \prod_{\tau=t}^{T-1} B(\tau,\tau+1)$, the option price (2.21) is nothing but the conditional expectation of the Black–Scholes price, [7] where the expectation is computed with respect to the joint probability distribution of

[6] A similar parallel is drawn in an unconditional two-period framework in Breeden and Litzenberger (1978).

[7] We refer here to a BS option pricing formula where dividend flows arrive during the lifetime of the option and are accounted for in the definition of the risk neutral probability, while the option payoff does not include dividends. In other words, the BS option price is given by

$$\pi_t^{\mathrm{BS}} = \mathrm{e}^{-r(T-t)}\mathrm{E}_t[\max(0, S_T - K)] \tag{2.22}$$

$$= \mathrm{e}^{-\delta(T-t)}S_t\Phi(d_1) - K\mathrm{e}^{-r(T-t)}\Phi(d_2), \tag{2.23}$$

since in the risk neutral world:

$$\log\frac{S_T}{S_t} \rightsquigarrow \mathcal{N}((r-\delta)(T-t), \sigma^2(T-t)), \tag{2.24}$$

where δ is the intensity of the dividend flow.

the rolling-over interest rate $\bar{r}_{t,T} = -\sum_{\tau=t}^{T-1} \log B(\tau, \tau+1)$ and the cumulated volatility $\bar{\sigma}_{t,T} = \sqrt{\sum_{\tau=t+1}^{T} \sigma_{Y\tau}^2}$. This framework nests three well-known models. First, the most basic ones, the Black and Scholes (1973) and Merton (1973) formulas, when interest rates and volatility are deterministic. Second, the Hull and White (1987) stochastic volatility extension, since $\bar{\sigma}_{t,T}^2 = Var[\log(S_T/S_t)|U_1^T]$ corresponds to the cumulated volatility $\int_t^T \sigma_u^2 \,\mathrm{d}u$ in the Hull–White continuous-time setting. Third, the formula allows for stochastic interest rates as in Turnbull and Milne (1991) and Amin and Jarrow (1992). However, the usefulness of our general formula (2.21) comes above all from the fact that it offers an explicit characterization of instances where the preference-free paradigm cannot be maintained. Usually, preference-free option pricing is underpinned by the absence of arbitrage in a complete market setting. However, our equilibrium-based option pricing formula does not preclude incompleteness and points out in which cases this incompleteness will invalidate the preference-free paradigm, i.e., when the conditions $Q_{XY}(t,T) = 1$ and $\tilde{B}(t,T) = \prod_{\tau=t}^{T-1} B(\tau, \tau+1)$ are not fulfilled. In this case, preference parameters appear explicitly in the option pricing formula through $\tilde{B}(t,T)$ and $Q_{XY}(t,T)$. Amin and Ng (1993), who provide a similar framework by modeling directly stock returns and consumption growth, associate the preference-free property of the option pricing formula to the predictability of their respective mean, variance and covariance processes. In other words, these processes are known at the beginning of the period.

It is worth noting that our results of equivalence between preference-free option pricing and no instantaneous causality between state variables and asset returns are consistent with another strand of the option pricing literature, namely GARCH option pricing. Duan (1995) derived it first in an equilibrium framework, but Kallsen and Taqqu (1998) have shown that it could be obtained with an arbitrage argument. Their idea is to complete the markets by inserting the discrete-time model into a continuous time one, where conditional variance is constant between two integer dates. They show that such a continuous-time embedding makes possible arbitrage pricing which is per se preference free. It is then clear that preference-free option pricing is incompatible with the presence of an instantaneous causality effect, since it is such an effect that prevents the embedding used by Kallsen and Taqqu (1998).[8]

3. Estimation of the option pricing model

In this empirical section, we want to assess to what extent one can recover preference parameters from option prices and to establish if the parameters recovered from actual option price data are reasonable. Recent evidence brought forward by Jackwerth (2000) in a non-parametric framework tends to extract preferences that are not in accordance with theoretical properties such as decreasing marginal utility. Our theoretical equilibrium model suggests that, in general, option prices are not preference free in the sense that the information about preference parameters is not solely contained in

[8] Heston and Nandi (2000) point out that the GARCH option pricing model of Duan (1995) is valid if and only if BS is valid over one period.

bond and stock prices. Therefore, option prices might allow us to obtain more precise estimates of preference parameters in finite samples. We verify this point in a simulation experiment by comparing the preference estimates obtained with a simple method of moments applied first to stock returns and then to option prices while assuming that the other parameters of the model are known. We further devise an estimation framework that allows us to recover jointly all the parameters of the model, that is, the parameters characterizing the stochastic process of the state variables as well as the parameters of the mean, variance and covariance functions of the fundamentals of the economy, along with the price–consumption and price–dividend ratios and the preference parameters. We apply this estimation method first to simulated data to verify if parameters are well estimated and then to S&P 500 call option prices.

3.1. A Markov-chain process for the state variables

Until now, we have not made any specific assumption about the nature of the stochastic process governing the state variables U_t apart from its stationarity and its Markovianity. In order to estimate the model, we adopt a Markov-chain setup for these state variables as in Cecchetti et al. (1990, 1993) and Bonomo and Garcia (1994a, b, 1996), based on the regime-switching model introduced by Hamilton (1989). The process describing the joint evolution of X_t and Y_t is parameterized as follows:

$$X_t = m_X(U_t) + \sigma_X(U_t)\varepsilon_{Xt}, \tag{3.1}$$

$$Y_t = m_Y(U_t) + \sigma_Y(U_t)\varepsilon_{Yt}. \tag{3.2}$$

The time-varying means and variances are assumed to be a function of the state variable process $\{U_t\}$, which is assumed to be a two-state discrete first-order Markov chain. The transition probabilities between the two states are given by $p_{ij} = \Pr(U_t = j | U_{t-1} = i)$ for $i, j = 1, 2$. The unconditional probability of being in state 1 is denoted π_1 and is equal to $(1 - p_{22})/(2 - p_{11} - p_{22})$ and $\pi_2 = 1 - \pi_1$. We further assume that the dependence of the consumption mean and dividend variance parameters on the state can be written in a linear form, without loss of generality:

$$m_X(U_t) = m_{X_1} + m_{X_2} U_t,$$

$$\sigma_Y(U_t) = \sigma_{Y_1} + \sigma_{Y_2} U_t. \tag{3.3}$$

Moreover, for simplicity sake and based on empirical evidence, we consider that the consumption variance and dividend mean parameters are constant between regimes.

In order for Assumptions 1–3 to hold, the $(\varepsilon_{Xt}, \varepsilon_{Yt})$ are supposed to be serially independent, identically distributed and independent of the state variable process U_t. In accordance with Assumption 4, the vector $(\varepsilon_{Xt}, \varepsilon_{Yt})'$ follows a standard bivariate normal distribution with correlation coefficient ρ_{XY}.

3.2. Informational content of option prices about preference parameters

Our first goal is to compare the informational content of stock returns and option prices with respect to the preference parameters. That is, we wish to see from which series can one better infer the values of the preference parameters of the structural model. From a comparison of the option pricing formula (2.21) and the stock return in (2.12), we can see intuitively why option prices might be more informative than stock prices about the preference parameters. In (2.12) the preference parameters only appear indirectly through the stock price–earnings ratios, which in equilibrium are determined as solutions of the Euler conditions in (2.11). On the other hand, these ratios also appear in the option price through the term $Q_{XY}(t,T)$. This term in (2.21) along with $\tilde{B}(t,T)$ depend directly on the preference parameters in addition to the price–earnings ratios for the stock and the market portfolio.

3.2.1. A Monte Carlo experiment

In this section we compare the empirical performances of the estimates based on option prices and on stock returns in the framework of a simulation experiment. We simulate asset prices in the economy described by our model. The experiment was carried out as follows. For given values (β,γ,α) characterizing preferences and $(p_{11},p_{22},m_{X1},m_{X2},\sigma_{X1},\sigma_{X2},m_{Y1},m_{Y2},\sigma_{Y1},\sigma_{Y2},\rho_{XY})$ describing the endowment and state variable processes, we first obtain the equilibrium values of the price–dividend ratios $(\lambda_1,\lambda_2,\varphi_1,\varphi_2)$ by numerically solving the following set of simultaneous equations:

$$\lambda_i^{\gamma} = \sum_{j=1}^{2} p_{ij}\left[\beta^{\gamma}\exp\left\{\alpha m_{Xj}+\frac{1}{2}(\alpha\sigma_{Xj})^2\right\}(\lambda_j+1)^{\gamma}\right],$$

$$\varphi_i = \sum_{j=1}^{2} p_{ij}\left[\beta^{\gamma}A_j\left(\frac{\lambda_1+1}{\lambda_1}\right)^{\gamma-1}(\varphi_j'+1)\right],$$

where

$$A_j = \exp\{(\alpha-1)m_{Xj}+m_{Yj}+\tfrac{1}{2}((\alpha-1)^2\sigma_{Xj}^2+\sigma_{Yj}^2+2(\alpha-1)\rho_{XY}\sigma_{Xj}\sigma_{Yj})\}.$$

The stock returns $\{r_t,\ t=1,\dots,N\}$ are obtained as

$$r_t = \log\frac{\varphi_t+1}{\varphi_{t-1}} + Y_t, \tag{3.4}$$

with $Y_t=\log D_t/D_{t-1}$ given by (3.2). Given the Markov chain process assumed for U_t, we generate paths of the state variable from time 1 through T, which we set at 100. For each path of the state variable U_t, we generate normalized option prices $C_t(U_t=i,\kappa,\tau)=\pi_t/K$,[9] where π_t is the price of a European call option as given by the generalized Black–Scholes pricing formula (2.21) when state i is operative at time t and the option's moneyness is equal to $\kappa=S_t/K$ and time to maturity is $\tau=(T-t)$. We therefore obtain series of stock returns and normalized option prices. We repeat the simulation 1000 times.

[9] Given the non-stationarity of S_t, option prices will also be non-stationary since S_t enters as an argument in the option pricing formula. However, the variable S_t/K will be stationary as strike prices are set at issuing time to bracket the underlying asset price. This suggests using $C_t(U_t=i,\kappa,\tau)$ to estimate the parameters of interest instead of π_t.

3.2.2. Estimating preference parameters with simulated prices

To start the estimation in the simplest way, we apply an exact method of moments to recover jointly the three preference parameters β, ρ and α while assuming that the other parameters of the model are known. In this first estimation, we focus our attention on preference parameters but later on, we will estimate all parameters at once to prepare the ground for estimation with actual data.

The moments for the stock returns that we consider are

$$\mathrm{E}[r_t] = \sum_{i=1}^{2}\sum_{j=1}^{2}\pi_i p_{ij}\left(\log\frac{\varphi_j+1}{\varphi_i} + m_{Yj}\right), \tag{3.5}$$

$$\begin{aligned} Var[r_t] = \sum_{i=1}^{2}\sum_{j=1}^{2}\pi_i p_{ij}\Bigg[\left(\log\frac{\varphi_j+1}{\varphi_i}\right)^2 &+ 2m_{Yj}\left(\log\frac{\varphi_j+1}{\varphi_i}\right) \\ &+ m_{Yj}^2 + \sigma_{Yj}^2\Bigg] - \mathrm{E}[r_t]^2 \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} &Cov[r_t, r_{t-1}] \\ &= \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2}\pi_i p_{ij} p_{jk}\left[\left(\log\frac{\varphi_j+1}{\varphi_i}\right)^2 + m_{Yj}^2\right] \\ &\quad \left[\left(\log\frac{\varphi_k+1}{\varphi_j}\right)^2 + m_{Yk}^2\right] - \mathrm{E}[r_t]^2. \end{aligned} \tag{3.7}$$

For the moments of option prices, it should be noticed that option prices allow for more flexibility than stock returns in the sense that we observe more than one option at each date, but only one price for the underlying stock. We can for example apply the method of moments to option prices of different moneynesses and maturities as follows:

$$\mathrm{E}\left[\frac{\pi_t}{K}\right] = \sum_{i=1}^{2}\pi_i C_t(U_t = i, \kappa, \tau). \tag{3.8}$$

It should be noticed that for a given set of values of the moneyness (possible values of S_t/K), option prices are deterministic functions of the current state variable. In our two-state setting, there are two values of the normalized option price, one for each state, as there are two price–dividend ratios. [10] Estimating parameters on the basis of this simulated series would have resulted in a perfect fit as the generalized option pricing model has more parameters that there are sources of randomness driving the transformed option price series. Therefore, we added noise to the ratio $\log(S_t/K)$ as $\log(S_t/K) + \sigma_Y(U_t)\varepsilon_t$ where ε_t is an i.i.d. $\mathrm{N}(0,1)$ process. Note that the added error term is proportional to the state-contingent standard error of the dividend process. The

[10] The division of the option prices by their strike price results in a binary process in the sense that for given values of κ and τ the transformed option prices take one of two values depending on which state is operative at time t.

Table 1
Descriptive statistics for the method-of-moments estimator of preference parameters based on simulated option prices

Options prices (*time series*)	ρ	α	β
Mean	−10.1585	−4.6162	0.9445
Median	−10.2131	−4.7979	0.9445
Std err	1.0524	1.8975	0.0093
RMSE	1.0638	1.9350	0.0108
Options prices (*across moneyness*)			
Mean	−10.1421	−4.6770	0.9504
Median	−10.2171	−4.7927	0.9500
Std err	1.0117	1.2921	0.0159
RMSE	1.0212	1.3312	0.0159

Note: The moments used in the estimation in Tables 1, 2 and 3 are the mean, the variance and the autocovariance of the respective series. For options in Table 1, we also used the means of three options with different moneyness. The true values are $\rho = -10$, $\alpha = -5$ and $\beta = 0.95$ for the preferences and $p_{11} = 0.9$, $p_{22} = 0.6$, $m_{X1} = 0.0015$, $m_{X2} = -0.0009$, $\sigma_{X1} = \sigma_{X2} = 0.003$, $m_{Y1} = m_{Y2} = 0$, $\sigma_{Y1} = 0.02$, $\sigma_{Y2} = 0.12$ and $\rho_{XY} = 0.6$ for the state variable and consumption and dividend processes. The results are reported for options with maturity of one period. The results are based on 1000 replications of the experiment.

additional error term makes for a fair comparison of the informational content of stock returns vis-à-vis option prices.

Another possibility to construct moment conditions is to choose a particular option, say at the money, and compute the moments based on the mean, variance and co-variance of a time series of prices for this particular option (always normalized by a given moneyness for stationarity). We will pursue both avenues to infer preference parameters from option prices.

3.2.3. Simulation results about estimated preference parameters

We investigate the properties of the estimators for the preference parameters while holding the other parameters of the model fixed at their true values.[11] In Tables 1–3, we report the results of this simulation experiment in terms of mean, median, standard error and root mean square error (RMSE) for the three parameters. We report the results for the method-of-moments estimators based on option prices (from a time series and an across-moneyness perspective), stock returns, and price–dividend ratios,[12] respectively. First, we notice that the estimators based on stock returns are more biased than the estimators based on moment conditions for options. It is the case even if we use comparable moments computed on the time series of one particular option. The bias is

[11] The values of the endowment process are similar to those estimated from actual data by Bonomo and Garcia (1996). Other values, such as the ones used in David and Veronesi (1999), yielded the same conclusions.

[12] The informational content of price–dividend ratios was suggested by Bansal and Lundblad (1999).

Table 2
Descriptive statistics for the method-of-moments estimator of preference parameters based on simulated stock returns

Stock returns	ρ	α	β
Mean	−11.0711	−2.4557	0.9950
Median	−10.9812	−1.8966	0.9955
Std err	1.0457	1.6153	0.0035
RMSE	1.4965	3.0134	0.0451

Table 3
Descriptive statistic for the method-of-moments estimator of preference parameters based on simulated price-dividend ratios

Price–dividend ratio	ρ	α	β
Mean	−10.5537	−3.5051	0.9501
Median	−10.0003	−4.9861	0.9497
Std err	1.2742	2.1530	0.0017
RMSE	1.3887	2.6202	0.0017

more pronounced for the parameters ρ and α than for the subjective discount factor β. A possible reason for this finite sample bias could be the non-linearity in parameters present in the model.[13] It is possible that the non-linear nature of the option payoffs helps in this regard. Improvements in terms of RMSE can be obtained in two directions, one for options, the other for the stock.

First, by using a set of three options with different moneyness, we can see that the RMSE is reduced at least for ρ and α. The main difference in the information base of the sets of estimators is that in one case we use a time series of a unique asset, while in the other we use a panel of option prices.[14] To estimate well the preference parameters, it is necessary to recover well the stochastic discount factor or the underlying risk neutral probability distribution. This is easier with a panel of option prices than with a time series on the underlying asset or one particular option. The second direction of improvement is to use moments on the price–dividend ratio of the stock instead of stock returns to estimate the parameters. The RMSE is reduced for the three parameters compared to the estimates obtained with the stock returns. It should be emphasized that in the true model used to simulate the prices, the price–dividend ratio takes two values, one for each state, as it is the case for option prices. We therefore added noise to $\log(\varphi(U_t)$ as $\log(\varphi(U_t)) + \sigma_Y(U_t)v_t$ where v_t is an i.i.d.

[13] This is not a numerical issue. In fact, we gave an advantage to the stock returns conditions in the sense that we started the optimization at the true parameter values, while for the options the initial values were taken in a random neighborhood of the true values.

[14] On the one hand, we use more information by having three price series, on the other hand we do not use this information as efficiently since we limit ourselves to first moments in the estimation to obtain the three moment conditions needed to estimate β, ρ and α.

N(0, 1) process, in the same way we did for normalized option prices. However the RMSEs remain higher than the RMSEs obtained with option prices.

Several conclusions can be drawn from the simulation results. First, it seems fair to state that stock return data provide poor estimates of the preference parameters. While price–dividend ratios produce better estimates, the standard errors for ρ and α are higher than for stock returns. Moreover, the distribution of α is dramatically skewed to the right, producing a marked underestimation in average of the risk aversion coefficient $1-\alpha$. The superior inference produced by option prices is all the more remarkable that we have used not the Euler equations but the generalized Black–Scholes formula, which already incorporates the information conveyed by observed bond and stock prices, for estimating the preference parameters. In other words, in such an exercise, option pricing formulas that are close to Hull and White preference-free formula would have led to option price data without any informational content about preference parameters. What we have captured in our estimation with moments on options is the marginal information provided by option prices in excess of the information provided by bond and stock prices.

3.2.4. Estimating all model parameters with simulated prices

In reality, we cannot consider that we know any of the parameters of the model. In order to estimate simultaneously all the structural parameters of our model we combine moment conditions from the stock returns and option price series. It should be emphasized that the only observables are the stock and option price data, and the dividend series (to construct the stock returns including dividends). The consumption series does not need to be observed. The estimation method will allow us to infer values for the means and variance of consumption growth from financial market data as it was done by Bonomo and Garcia (1996) using a maximum likelihood approach. We also need to compute from Euler equations, given values for the other model parameters, the price–consumption ratios λ_1 and λ_2 for the market portfolio and the price–dividend ratios φ_1 and φ_2 for the stock.

In Table 4, we proceed to estimate jointly all the parameters of the model, again with an exact method of moments applied to the simulated asset prices as above. We use enough moment conditions from option prices and stock returns to estimate the 12 parameters of interest. We get 9 moment conditions on options by considering 3 different moneynesses (1.1, 1 and 0.9) and times to maturity (1, 2 and 3 periods) and three moment conditions from the stock returns (mean, variance and covariance). At each stage of the estimation, given the current set of values for the model parameters, the λ and the φ parameters are computed by solving the Euler equations. The results indicate that the preference parameters, the transition probabilities and the consumption–dividend correlation parameter are estimated without bias and rather precisely. It is not the case for the means and variance of the consumption process, which are biased. The variance parameters of the dividend process are slightly biased upward.[15] Acknowledging these potential problems in recovering some parameters, we

[15] However, it should be noted that the median bias of the estimators for the consumption and dividend parameters is acceptable.

Table 4
Descriptive statistics for the joint estimation of the structural parameters by an exact method-of-moments based on simulated asset prices

	β	ρ	α	p_{11}	p_{22}	ρ_{XY}
Mean	0.9164	−10.0517	−4.9728	0.8983	0.5916	0.5954
Median	0.9504	−9.9903	−5.0177	0.9010	0.5983	0.5997
Std err	0.1119	1.4381	1.3672	0.0507	0.0749	0.0980
RMSE	0.1168	1.4383	1.3667	0.0507	0.0753	0.0981
	m_{X1}	m_{X2}	σ_X	m_Y	σ_{Y1}	σ_{Y2}
Mean	0.0520	0.0500	0.0068	−0.0780	0.0462	0.1849
Median	0.0013	−0.0052	0.0031	−0.0088	0.0193	0.1249
Std err	1.0176	0.8822	0.0267	0.5529	0.3704	0.2028
RMSE	1.0183	0.8832	0.0269	0.5581	0.3711	0.2128

Note: The true values are $\rho = -10$, $\alpha = -5$ and $\beta = 0.95$ for the preferences and $p_{11} = 0.9$, $p_{22} = 0.6$, $m_{X1} = 0.0015$, $m_{X2} = -0.0009$, $\sigma_{X1} = \sigma_{X2} = 0.003$, $m_{Y1} = m_{Y2} = 0$, $\sigma_{Y1} = 0.02$, $\sigma_{Y2} = 0.12$ and $\rho_{XY} = 0.6$ for the state variable and consumption and dividend processes. The results are based on 1000 replications of the experiment.

will nevertheless proceed with this method for estimating the parameters of the model with actual data since it allows to recover well the preference parameters which are the main focus of our analysis.

3.3. *Is there evidence of preference parameters in S&P 500 option prices?*

The simulation experiments of the last section lay the ground for a general estimation of the model with option price and stock return data. To estimate the parameters and assess the out-of-sample pricing performance of the various models, we use daily price data for S&P 500 Index call European options obtained from the Chicago Board Options Exchange for the period January 1991 to December 1995.[16] The S&P 500 index option market is extremely liquid and it is one of the most active options markets in the United States. This market is the closest to the theoretical setting of the Black–Scholes model and the extensions proposed in this paper. We also used daily return data for the S&P 500 Index.

3.3.1. *Estimation of the parameters*

We used the following method of estimation. At time t, the GBS model is estimated by the method of moments using the moments defined in the simulation study. By estimating parameters for options of different maturities and moneyness we take the model to the letter. The same preferences should apply to the pricing of all assets. Therefore, we include options of all maturities and moneyness. Also, the same preference parameters will apply to the risk premia associated with the state variable

[16] Rosenberg and Engle (2002) use the same 1991–1995 period for estimating empirical pricing kernels, allowing for comparisons.

that makes the mean of the consumption process or the volatility of the dividend jump. In that sense our approach distinguishes itself from arbitrage-based methods as developed in Pan (2002), where risk premia are estimated as if investors had different risk attitudes towards the various types of risk. To compute the empirical moments, we use a 3-month window prior to the time of estimation. This last feature also pushes in the direction of estimating the structural parameters of the model. Often option pricing models are estimated with a window as short as a day making the process more like a calibrating exercise than an estimation one. We will pursue further such a calibration exercise in the next section.

More precisely, the parameters are estimated based on matching the following moments for the options:

$$f\left(\frac{S_t}{K},(T-t);\theta\right)=\mathrm{E}\left[GBS\left(U_t,\frac{S_t}{K},(T-t)\right)\right]-\frac{1}{M_{S_t/K}}\sum_{\tau=t-h}^{t}\pi_\tau\left(\frac{S_t}{K},(T-t)\right), \tag{3.9}$$

where the expectation is with respect to U_t, h equals 3 months, and θ regroups all the parameters. The notation $\pi_\tau(S_t/K,(T-t))$ denotes a call option on the underlying stock at time τ, with a moneyness equal to S_t/K and a maturity equal to $(T-t)$. The quantity $M_{S_t/K}$ represents the number of options over the period h with a moneyness equal to S_t/K. We proceeded by partitioning the options into moneyness categories based on S_t/K and maturity categories based on $(T-t)$. It should be noted that we take an unconditional expectation of the GBS formula to build unconditional moments.

We then minimized

$$\sum\sum f\left(\frac{S_t}{K},(T-t);\theta\right)^2, \tag{3.10}$$

where the first summation is over moneyness categories and the second over the maturity categories. We also included some moment conditions based on the stock returns and conditions based on the Euler equations for the identification of λ and φ, in order to obtain as many moment conditions as there were parameters to estimate.

For the estimation, we start each trading day with a set of initial values and use first a simplex algorithm to obtain initial estimates followed by a DFP routine.[17] The same strategy is also applied to the expected utility model where γ is constrained to a value of 1. We conduct this experiment for 5 years, from 1991 to 1995. Table 5 reports the average values of the preference parameters that we obtained in each of the 5 years and over the 5-year period. Looking first at the GBS model, we can say that the estimates of the risk aversion and intertemporal substitution parameters appear reasonable. Over the 5-year period, the coefficient of relative risk aversion is equal to 0.6838 on average and the elasticity of intertemporal substitution has a mean value of 0.8532. This is a result that conforms with intuition since one generally expects that the inverse of the elasticity of substitution should be greater than the coefficient of relative risk aversion, as emphasized in Weil (1989). As the yearly means and standard errors indicate, the

[17] To make sure that we explore well the parameter space in the optimization, and especially the preference parameters, we also used a grid of initial values for the preference parameters. The final results were similar.

Table 5
Yearly means and standard errors of daily estimated preference parameters from S&P 500 option and stock price data

GBS model

	ρ	γ	β	CRRA $(1-\alpha)$	EIS
1991	−0.2048 (0.0904)	−1.6637 (0.9144)	0.9397 (0.0372)	0.6885 (0.0987)	0.8342 (0.0564)
1992	−0.0936 (0.0400)	−1.9975 (0.4171)	0.9783 (0.0180)	0.8201 (0.0646)	0.9156 (0.0321)
1993	−0.2007 (0.0737)	−2.4294 (1.1218)	0.9413 (0.0380)	0.5509 (0.1269)	0.8358 (0.0494)
1994	−0.2110 (0.1211)	−1.7369 (0.6011)	0.9142 (0.0437)	0.6706 (0.1366)	0.8334 (0.0778)
1995	−0.1963 (0.1504)	−1.8744 (0.7700)	0.9029 (0.0377)	0.6884 (0.1559)	0.8466 (0.0870)
1991–1995	−0.1812 (0.1114)	−1.9406 (0.8458)	0.9353 (0.0444)	0.6838 (0.1478)	0.8532 (0.0710)

Expected utility model

	ρ	β	CRRA $(1-\alpha)$
1991	−8.7505 (1.7685)	0.9513 (0.0229)	9.7505 (1.7685)
1992	−6.2337 (3.7156)	0.8401 (0.1259)	7.2337 (3.7156)
1993	−4.9742 (1.8897)	0.9710 (0.0275)	5.9742 (1.8897)
1994	−5.1044 (7.0187)	0.8321 (0.1026)	6.1044 (7.0187)
1995	−5.7259 (6.1479)	0.8172 (0.1230)	6.7259 (6.1479)
1991–1995	−6.1590 (4.8260)	0.8824 (0.1130)	7.1590 (4.8260)

Note: The estimation is based on the same exact method of moments used in Table 4. CRRA denotes the coefficient of relative risk aversion, EIS the elasticity of intertemporal substitution.

values obtained are remarkably stable over time, a reassuring fact for a structural model with a representative investor. It is interesting to compare these estimates with the values obtained when we constrain the parameter γ to be equal to 1. Similarly to what was obtained with stock returns series in various studies aimed at solving the equity premium puzzle, we obtain a high average value of 7.16 for the coefficient of relative risk aversion, with a standard deviation of 4.83. It is interesting to note that over the same 1991–1995 period, Rosenberg and Engle (2002) find an empirical risk aversion of 7.36 with a power utility function defined over wealth (measured by the S&P 500 index), based on S&P 500 option price data. Bakshi et al. (2003) also estimate by GMM the coefficient of relative risk aversion in a power utility setting based on a relation between the risk-neutral skewness of index returns and conditional moments of the physical index distribution. Depending on the set of instruments, estimates are in the range 1.76–11.39.[18] Therefore, relaxing the constraint $\gamma=1$ allows for a more reasonable value for the elasticity of intertemporal substitution. The value found for β in the expected utility case is somewhat low (0.88 on average), while it appears more reasonable (0.94 average) when γ is not constrained to be equal to one.

Another way to assess the reasonableness of the model estimates for the pricing of the assets is to look at the estimates obtained for the other parameters of the model,

[18] However, most of the estimated values are in the neighborhood of 2. This is obtained for short- and medium-term options. It is close to the average value of 2.35 that we obtain with short-term options in Section 4.

Table 6
Yearly means and standard errors of daily estimated parameters for the state variable and consumption and dividend processes from S&P 500 option price data over the period 1991–1995

λ_1	λ_2	φ_1	φ_2	p_{11}	p_{22}	ρ_{XY}
8.1182	10.6924	12.5700	18.8356	0.9758	0.8078	−0.3178
(0.6675)	(0.8453)	(0.7349)	(1.5631)	(0.0243)	(0.1266)	(0.5120)
	m_{X1}	m_{X2}	σ_X	m_Y	σ_{Y1}	σ_{Y2}
	−0.3216	0.0623	0.0202	−0.0688	0.0365	0.1139
	(0.1654)	(0.2270)	(0.0427)	(0.0076)	(0.0176)	(0.0802)

Note: The estimation is based on the same exact method of moments used in Table 4.

both the fundamental processes and the state variable. The averages over the 1991–1995 period are given in Table 6. As we saw in the theoretical formulas in Section 2, the price–consumption (λ) and the price–dividend (φ) ratios play a fundamental role in the pricing of the assets. We found averages of around 8 and 11 for the price–consumption ratio and 13 and 19 for the price–dividend ratio with little variability over the 5-year period (standard deviations of 0.67 and 0.85 respectively). These values also appear reasonable.

In terms of the state variable we find average values of 0.9758 and 0.8078 for the transition probabilities in states 1 and 2, respectively, implying values of 0.89 and 0.11, respectively, for the unconditional probabilities. State 1 is, in fact, a crash-like state with a very negative mean for consumption growth (−0.32 on average), but one should not forget however that the state variable also controls the volatility of dividends. So state 1 is in fact a low volatility of dividends and low-consumption state. Given the negative mean value of dividends, it appears that the representative investor attributes an unreasonably high probability to the bad state, while as we just saw the inferred preference parameters are reasonable. This is in contrast with the results obtained by Jackwerth (2000) with a non-parametric methodology. In a parametric framework, Rosenberg and Engle (2002) find results that differ from Jackwerth (2000), in particular they do not find negative risk aversions when they use a power pricing kernel. Rosenberg and Engle (2002) find results similar to Jackwerth's results when they use an orthogonal polynomial pricing kernel. In particular, they find that there is a region of negative risk aversion over the range from −4% to 2% for returns. Our estimates of the model parameters suggest that the potential mispricing comes from a very pessimistic assessment of the fundamentals of the economy and not from unreasonable preferences.[19] In any case, this exercise illustrates the difficulty of disentangling the subjective probability assessments of the states from the preferences. In the non-parametric framework of Jackwerth (2000), the risk aversion function is recovered by treating as given both the option prices and the stock index prices to estimate non-parametrically the risk-neutral

[19] The unreasonable parameters for the fundamentals process may also result from a misspecification of the growth rate equations. We could increase the number of states as in Bonomo and Garcia (1996) where a three-state bivariate Markov switching model is estimated on an annual frequency over the last century or so. It should also be noted that our state variable captures both the jump and the stochastic volatility effects. A way to disentangle the two would be to introduce a GARCH specification in the volatility of dividends.

and subjective probabilities respectively and by taking their ratios. If prices were generated from our economy with state variables it is possible that one could recover a bimodal graph for preferences as Jackwerth (2000) does even though preferences are here constant. The values extracted for the probabilities, say from an implied binomial tree, are pseudo-true values (since the tree is likely to be misspecified) and will depend on all the parameters of the economy, including preference parameters. Therefore, the separation between probabilities and preferences is not as obvious as it seems. Finally, the values estimated for the volatility parameters, both consumption and dividends, appear quite reasonable.

3.3.2. Pricing errors

In this section, we will assess the pricing errors associated with our generalized non-expected utility option pricing formula and compare them with the errors obtained with the expected utility model and a preference-free stochastic volatility model. Using the estimates obtained each trading day following the estimation method described in the previous section,[20] we forecast the prices for all the options of the following day separated in long (more than 180 days), medium (between 180 and 60 days) and short (less than 60 days) maturities irrespective of moneyness. We average the daily forecast errors over each year for the corresponding categories and compare the performance with the absolute and the relative errors for various maturity categories for all three models. Christoffersen and Jacobs (2001) have recently emphasized that the loss function used in parameter estimation and model evaluation should be the same. We use an absolute dollar measure which is consistent with the mean square criterion used in estimation, but we add also a relative measure to give an idea of the magnitude of the error. The results are shown in Table 7. The absolute errors appear to be roughly uniform across maturities, but the relative loss is much smaller for the expensive long-term options than for the cheap short-term options. However, the ranking of the models is the same for both measures.

We compare three models: the most general option model for the non-separable recursive utility model given by formula (2.21), the expected utility model obtained by setting γ equal to one in (2.21) to judge the importance of non-separabilities, and finally the preference-free stochastic volatility model which results from (2.21) when $Q_{XY}(t,T)=1$ to gauge the importance of preferences for option prices. It should be emphasized that the objective of this forecasting exercise is to assess the relative performance of the three models. One cannot hope to obtain errors of small magnitude by using only unconditional moments in the estimation and a rather long window in the past. Conditional information needs to be incorporated in some way to achieve more sensible pricing performances. This can be done in a structural way by using the option pricing formula as a function of the unobserved state and by filtering the current value of the latent state variable. We will take a simpler approach in the next section by incorporating conditioning information such as BS implied volatilities in an ad hoc way in the model.

[20] The parameters for the stochastic volatility model are estimated with the same moment conditions as the two preference models but we impose the constraint that $Q_{XY}(t,T)$ is equal to one.

Table 7
Yearly means of absolute and relative pricing errors for short, medium and long-term call options averaged over moneyness

Short term	GBS	EU	SV
Relative errors			
1991 (3132)	0.8588	1.4995	1.5798
1992 (2928)	1.3303	1.8417	1.9287
1993 (2921)	1.7720	1.7636	1.7769
1994 (3365)	1.4821	1.9350	2.3282
1995 (4022)	1.4664	1.3508	2.1910
1991–1995 average	1.3800	1.6800	1.9600
Absolute errors			
1991 (3132)	3.1444	4.4779	4.8473
1992 (2928)	3.6726	4.2741	5.2431
1993 (2921)	4.2028	3.8674	4.2968
1994 (3365)	3.1141	3.8733	4.4483
1995 (4022)	4.0907	4.2658	5.6873
1991–1995 average	3.6400	4.1500	4.9000

Relative errors				Absolute errors			
Medium term	GBS	EU	SV	Medium term	GBS	EU	SV
1991 (2187)	0.3436	0.7731	0.7669	1991 (2187)	2.8921	3.9251	4.4258
1992 (2379)	0.7215	1.1348	1.2831	1992 (2379)	3.3759	4.6117	5.4437
1993 (2163)	1.2042	1.3287	1.3471	1993 (2163)	4.4210	4.4138	4.9754
1994 (2897)	1.2097	1.5967	1.9032	1994 (2897)	3.6488	4.4388	4.9771
1995 (2991)	0.8658	0.9799	1.4150	1995 (2991)	4.5432	5.2378	6.4743
1991–1995 average	0.8700	1.1600	1.3400	1991–1995 average	3.7800	4.5300	5.2600

Relative errors				Absolute errors			
Long term	GBS	EU	SV	Long term	GBS	EU	SV
1991 (694)	0.0036	0.1946	0.2374	1991 (694)	2.5882	3.0367	3.4266
1992 (538)	−0.0170	0.2246	0.2128	1992 (538)	3.0306	3.9870	3.3401
1993 (492)	0.2138	0.2969	0.2278	1993 (492)	2.5911	2.9982	2.8856
1994 (910)	−0.0006	0.0864	0.2543	1994 (910)	3.5838	4.4165	3.5591
1995 (1053)	0.1212	0.2417	0.5201	1995 (1053)	3.3501	4.4264	4.4793
1991–1995 average	0.0600	0.2100	0.2900	1991–1995 average	3.0300	3.7700	3.5400

Note: GBS refers to the generalized Black–Scholes formula in (2.21); EU to the same formula special case where the parameter γ is equal to 1; SV to the stochastic volatility formula (special case of (2.21) with $Q_{XY}(t,T)=1$). The numbers in parentheses besides the years indicate the number of options considered.

The results are clear. For all maturities, GBS does better that the specification where γ is equal to one which in turn is better than the SV specification.[21] Compared to

[21] We do not carry a formal statistical test of the equality of errors between the models as most papers in the literature, but tests of predictive accuracy (as in Diebold and Mariano, 1995 or West, 1996) could be applied (see Dumas et al., 1998).

the stochastic volatility model, the relative error for GBS is reduced by up to 50% for short- and medium-term options. This shows that preferences are important in pricing options on the index. Moreover, the data seem to indicate that preferences are of the non-separable type since the restricted value of γ generally increases the relative error. Of course, as we advance in maturity, the relative error falls for all models since the volatility smile flattens and pricing tends to approach Black–Scholes. However, for long-term options, the GBS model performs significantly better than the other two. These results parallel the simulation results reported in Garcia et al. (2001) about the smile effect. First, it was shown that a non-preference free framework was able to reproduce the various asymmetries observed in the implied volatility curve inferred from option price data. Second, the parameter γ was seen to be more important than the risk aversion parameter α in calibrating the smile.

To conclude this section, it seems fair to say that we have obtained reasonable values of the preference parameters based on price data of all options, irrespective of their moneyness and maturity, but that the pricing errors are very large. In the next section, we take some liberty with the model and show that by incorporating conditioning information, focussing on short-term options specifically and reducing the estimation window, the pricing errors are reduced considerably.

4. Calibrating the model for practical option pricing

In the last section, the goal was to obtain estimates of the structural parameters of the model. In this section, we aim at minimizing the out-of-sample pricing errors in the spirit of Bakshi et al. (1997). In this type of exercise one typically makes concessions with the structural model. The window for estimating the parameters is generally very short (from a day to a week) and conditioning information is included in an ad hoc way, usually inconsistent with the model. The best example of this ad hoc approach is to use the BS formula to extract implied volatility on a given day for a certain maturity and moneyness and to use this volatility to price options the next day with the same maturity and moneyness. In so doing, practitioners completely ignore the fact that the assumption of constant volatility underlying the BS model is obviously violated since the implied volatility may vary widely from one day to the next. Yet they use the formula as a tool and the performance of this rather crude method is difficult to beat by more sophisticated models unless one is ready to recognize that the parameters of the model are unstable and their estimates need to be updated.[22]

In what follows, we have decided to adapt our model in an ad hoc way in order to improve its out-of-sample pricing performance. First, we reduce the window over which

[22] Heston and Nandi (2000) claim that their closed-form GARCH option pricing formula outperform the ad hoc BS model of Dumas et al. (1998) even without updating but a closer look at the results (Table 7) shows that this is not true for short-term options. Even with updating the GARCH model does not outperform the ad hoc BS approach for close-to-the-money short-term options. Moreover, a more precise procedure, using a model estimated with the same criterion as in the out-of-sample performance evaluation, produces a smaller error for the practitioner Black–Scholes methodology, as pointed out by Christoffersen and Jacobs (2001).

we estimate the parameters to 5 days instead of 3 months. Second, we incorporate the option's implied volatility σ_t^* into the dividend volatility process as[23]

$$\sigma_Y(U_t = j) = \delta_{0j} + \delta_{1j}\sigma_t^* \sqrt{(T-t)}, \tag{4.1}$$

for $j = 1, 2$.

Parameter estimates were then based on a modification of the estimation method where, for a given maturity $(T-t)$, we minimized

$$\frac{1}{M_{S_t/K}} \sum_{\tau=t-h}^{t} \left[\mathrm{E}\left[GBS\left(U_t, \frac{S_t}{K}, (T-t), \sigma_t^*\right)\right] - \pi_\tau\left(\frac{S_t}{K}, (T-t)\right)\right]^2. \tag{4.2}$$

Note that we maintain the use of unconditional moments.[24] Therefore, there is a different set of parameter estimates for each maturity category. (This relaxes the implicit constraints of the form $p_{ik}^{(2)} = p_{ij}^{(1)} p_{jk}^{(1)}$, where $p_{..}^{(s)}$ are s-period transition probabilities). Also, we now leave aside the moment conditions based on stock returns and use only the moment conditions associated with the options.

In addition, we impose the following constraints:

$$\mathrm{E}_t[Q_{XY}(t,T)] = 1, \tag{4.3}$$

$$\mathrm{E}_t[\tilde{B}(t,T)] = \exp(-r(T-t)), \tag{4.4}$$

where r is the observed interest rate. These constraints were implicitly embodied in the original estimation method in Section 3 through the Euler equations for λ and φ. With the modified method, it is no longer (numerically) feasible to enter the Euler equations into the estimation problem since φ depends on σ_Y, which now depends on σ_t^*. Therefore, λ and φ are treated as free parameters to be explicitly estimated along with the other model parameters; hence the first constraint. The second constraint serves to incorporate information on the interest rate.

Let us start with pricing errors since the goal of the calibration exercise is to improve the out-of-sample performance. We now use estimates obtained every day to forecast the prices for all the short maturity (less than 60 days) options of the next day irrespective of moneyness. We average the daily forecast errors over each year and compare the performance of the previous three models (non-expected utility, expected-utility and preference-free stochastic volatility) to which we added a practitioner BS model in the spirit of Dumas et al. (1998).[25] Table 8A reports the absolute and relative pricing errors for each year in the 1991–1995 period. First, note that the relative errors associated with the preference-based GBS or EU formulas have fallen considerably and vary between 1% and 9%. It is interesting to note that the two preference-based models

[23] The $\sqrt{T-t}$ in the formula below refers to the exact maturity of each option used to extract the corresponding implied volatility.

[24] We could have probably improve further the out-of-sample pricing performance by filtering the state probability at time t and use this information to compute the expectation of the GBS formula.

[25] Note that we do not estimate a volatility function as in Dumas et al. (1998). We simply group options by moneyness categories and forecast the volatility of an option one day ahead by the implied volatility of the moneyness category the day before. The procedure of Christoffersen and Jacobs (2001) might have produced a lower error for the practitioners' BS model.

Table 8
Yearly means of absolute and relative errors for short-term call options averaged over moneyness with conditional pricing based on implied volatility

(a) *One-day ahead forecast*

Relative errors

Short term	GBS	EU	SV	PBS
1991 (3132)	0.0068	0.0078	0.0573	−0.0065
1992 (2928)	0.0212	0.0214	0.0728	0.0022
1993 (2921)	0.0221	0.0216	0.0775	−0.0034
1994 (3365)	0.0886	0.0888	0.1914	0.0473
1995 (4022)	0.0626	0.0611	0.1619	0.0092
1991–1995 average	0.0400	0.2007	0.1100	0.0100

Absolute errors

Short term	GBS	EU	SV	PBS
1991 (3132)	0.9223	0.9214	1.0630	0.8019
1992 (2928)	0.7828	0.7829	0.8834	0.6899
1993 (2921)	0.7441	0.7456	0.8540	0.6616
1994 (3365)	0.6991	0.6987	0.8763	0.5959
1995 (4022)	0.9637	0.9656	1.2545	0.6802
1991–1995 average	0.8200	0.8200	0.9900	0.6900

(b) *Five-day ahead forecast*

Relative errors			Absolute errors		
Short term	GBS	PBS	Short term	GBS	PBS
1991 (3085)	0.032	0.015	1991 (3085)	1.0432	0.9335
1992 (2861)	0.017	0.002	1992 (2861)	0.8226	0.7438
1993 (2871)	0.017	0.009	1993 (2871)	0.8151	0.7259
1994 (3329)	0.087	0.044	1994 (3329)	0.8642	0.7370
1995 (3969)	0.068	0.0085	1995 (3969)	1.1004	0.7635
1991–1995 average	0.040	0.020	1991–1995 average	0.9300	0.7800

Note: GBS refers to the generalized Black–Scholes formula in (2.21); EU to the same formula special case where the parameter γ is equal to 1; SV to the stochastic volatility formula (special case of (2.21) with $Q_{XY}(t,T)=1$); PBS refers to the practitioners' BS model. The numbers in parentheses besides the years indicate the number of options considered.

produce now very similar errors. However, the relative errors of the BS ad hoc model remain smaller. We also report in Table 8B out-of-sample pricing errors at a horizon of 5 days for the GBS and PBS models. The gap between the two models is mostly maintained, except in the beginning of the sample where the BS absolute error tends to increase faster than the GBS one.[26]

[26] As expected, this trend is even more accentuated for longer horizons such as $t+10$ and $t+15$. The results are not reported here for space considerations.

The reduction in pricing errors with respect to Section 3 can come from three sources. We assess the contribution of each source for the year 1991. First, in this calibration exercise, we focus the estimation on short-term options, compared with all options in Section 3. Reestimating the same GBS model only for short options reduces by half the absolute error for the year 1991 (from 3.14 to 1.52 for the absolute error and from 0.859 to 0.285 for the relative error). The second source of error reduction is the reduced span of the data to carry out the estimation, from 3 months to 5 days. This brings down the absolute error to 1.41 and the relative error to 0.0935, but these are still higher than the highest errors of Table 8A for the SV model. Indeed, the introduction of the implied volatility information reduces substantially the errors, bringing down the absolute error to 0.92 and the relative error to 0.0068 as reported in Table 8A.

The errors of the SV model are definitely higher than the ones of the PBS model yet they both use implied volatility. The main difference comes from the fact that the PBS method uses the implied volatility of the day before while the SV method smoothes the implied volatility of the last five days. This underlines the penalty imposed by the averaging over the past values for out-of-sample forecasting. The risk aversion parameter appears therefore of prime importance since it reduces the error considerably despite the averaging effect over five days for the implied volatility.

A reassuring result is that the preference parameter estimates obtained for the GBS model, although much more variable than before both within and between the years, are close to the estimates we obtained with the 3-month window estimation, as illustrated in Table 9. Both the relative risk aversion coefficient and the elasticity of intertemporal substitution are slightly lower on average (0.42 and 0.66, respectively). The risk aversion parameter in the expected utility model is now estimated at a lower more reasonable value of 2.35. Notice also that the estimates of β in the expected utility model are much more reasonable than with the previous method both for the EU and the GBS models. While the parameters are difficult to interpret in this less structural model, it is worth noting that we obtained more reasonable parameters for the consumption mean parameters (1% and -17% in states 1 and 2, where state 1 is also the low dividend volatility parameter, and where p_{11} and p_{22} are 0.80 and 0.22, respectively. Therefore, the good state (high level of consumption growth and low volatility of dividends [27]) appears to be more frequent as one should expect.

5. Conclusion

In this paper, we contribute to the empirical asset pricing literature by estimating a recursive utility model with option prices. Not only do we show that preferences matter for option pricing but also that option prices help distinguish between the expected

[27] This is in line with usual empirical evidence. It is hard to say what explains the changes in parameters estimates with respect to the unintuitive results obtained in Section 3.3.1 where options of all maturities and moneyness were considered all together.

Table 9
Yearly means and standard errors of daily estimated preference parameters from S&P 500 option price data

GBS model	ρ	γ	β	CRRA$(1-\alpha)$	EIS
1991	−0.9010 (0.3821)	−0.8324 (0.3887)	0.9150 (0.0135)	0.3864 (0.07534)	0.5500 (0.1207)
1992	−0.9522 (0.5600)	−0.4948 (0.4557)	0.8704 (0.0512)	0.7000 (0.1617)	0.5617 (0.1736)
1993	−0.3631 (0.2426)	−2.9782 (1.2942)	0.9448 (0.0082)	0.1849 (0.0312)	0.7518 (0.1043)
1994	−0.6221 (0.4469)	−1.8325 (0.8712)	0.9471 (1.0620)	0.2068 (0.0088)	0.6541 (0.1393)
1995	−0.3040 (0.0941)	−1.2201 (0.3075)	0.9526 (0.0086)	0.6396 (0.1101)	0.7706 (0.0512)
1991–1995	−0.6290 (0.4656)	−1.4707 (1.1597)	0.9259 (0.0395)	0.4238 (0.2340)	0.6575 (0.1549)

Expected utility model	ρ	β	CRRA $(1-\alpha)$
1991	−1.5242 (2.5058)	0.9804 (0.0198)	2.5242 (2.5107)
1992	0.1664 (1.3060)	0.9620 (1.5749)	0.8336 (1.3085)
1993	−1.1387 (1.1143)	0.9458 (0.0140)	2.1387 (1.1166)
1994	−2.0040 (1.3927)	0.9871 (0.0066)	3.0040 (1.3955)
1995	−2.2802 (1.7051)	0.9681 (0.0008)	3.2802 (1.7085)
1991–1995	−1.3537 (1.8847)	0.9687 (0.020)	2.3537 (1.8847)

Note: The parameters are estimated with an exact method-of-moments applied to short-term S&P 500 call option prices and a dividend process incorporating implied volatility information. CRRA denotes the coefficient of relative risk aversion, EIS the elasticity of intertemporal substitution.

and the non-expected utility models. The informativeness of option price data about preference parameters is confirmed in a simulation experiment. The estimates we obtain for the preference parameters are quite reasonable. This is in contrast with recent results of Jackwerth (2000) who infers risk-aversion functions that are at odds with usual theoretical assumptions. It should be emphasized that in our method preference parameters enter consistently in the equilibrium pricing of all assets.

Of course, given the simplicity of the practitioners' Black–Scholes approach and its good predictive performance, our structural model faces a tough challenge as a competitor. However, we consider that both our simulation experiments and the estimation performed with S&P 500 option price data strongly support the claim that preference parameters are important in option pricing. To better understand the structure of index option prices, one can think of several possible extensions in terms of preference specifications or distributions for the state variables. One potential weakness of our model for fundamentals is the modeling of volatility. Our specification captures only sudden changes in volatility in dividends. The model will gain by adding some GARCH effects to the switching regimes governing the dividend process.

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Appendix.

The proof of the option pricing formula is based on the following lemma.

Lemma. *If* $\binom{Z_1}{Z_2}$ *is a bivariate Gaussian vector with*

$$\mathrm{E}\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, \quad Var\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} \omega_1^2 & \rho\omega_1\omega_2 \\ \rho\omega_1\omega_2 & \omega_2^2 \end{pmatrix},$$

then $\mathrm{E}[\exp(Z_1)\mathbf{1}_{[Z_2 \geqslant 0]}] = \exp(m_1 + \omega_1^2/2)\Phi(m_2/\omega_2 + \rho\omega_1)$, *where* Φ *is the cumulative normal distribution function.*

Proof. Let us denote by Q the probability measure corresponding to the above-specified Gaussian distribution of (Z_1, Z_2) and define the probability $\tilde{Q}$ by

$$\frac{\mathrm{d}\tilde{Q}}{\mathrm{d}Q}(Z) = \exp\left[(Z - m_1) - \frac{\omega_1^2}{2}\right].$$

Then, with obvious notation:

$$\mathrm{E}[(\exp Z_1)(\mathbf{1}_{[Z_2 \geqslant 0]})] = \exp\left(m_1 + \frac{\omega_1^2}{2}\right)\tilde{Q}[Z_2 \geqslant 0]$$

But by Girsanov's theorem, we know that under $\tilde{Q}$, Z_2 is a Gaussian variable with mean $m_2 + \rho\omega_1\omega_2$ and variance ω_2^2. Therefore,

$$\tilde{Q}[Z_2 \geqslant 0] = 1 - \Phi\left[\frac{-m_2 - \rho\omega_1\omega_2}{\omega_2}\right] = \Phi\left[\frac{m_2}{\omega_2} + \rho\omega_1\right]. \qquad \square$$

Proof of option pricing formula: From the Euler equation, we have that the price of an option on the dividend-paying stock is

$$\frac{\pi_t}{S_t} = \mathrm{E}_t\left[\beta^{\gamma(T-t)}\left(\frac{C_T}{C_t}\right)^{\alpha-1}\prod_{\tau=t}^{T-1}\left[\frac{(1+\lambda(U_1^{\tau+1})}{\lambda(U_1^\tau)}\right]^{\gamma-1}\max\left[0,\frac{(S_T+D_T)}{S_t}-\frac{K}{S_t}\right]\right].$$

Let

$$\frac{G_t}{S_t} = \mathrm{E}_t\left[\beta^{\gamma(T-t)}\left(\frac{C_T}{C_t}\right)^{\alpha-1}a_t^T(\gamma)\frac{(S_T+D_T)}{S_t}\mathbf{1}_{[(S_T+D_T)/S_t\geqslant K/S_t]}\right],$$

$$\frac{H_t}{S_t} = \mathrm{E}_t\left[\beta^{\gamma(T-t)}\left(\frac{C_T}{C_t}\right)^{\alpha-1}a_t^T(\gamma)\frac{K}{S_t}\mathbf{1}_{[(S_T+D_T)/S_t\geqslant K/S_t]}\right],$$

where $a_t^T(\gamma) = \prod_{\tau=t}^{T-1}[(1+\lambda(U_1^{\tau+1}))/\lambda(U_1^\tau)]^{\gamma-1}$ such that $\pi_t/S_t = G_t/S_t - H_t/S_t$.

In order to arrive at the generalized Black–Scholes formula, we will prove that

$$\frac{G_t}{S_t} = \mathrm{E}_t[Q_{XY}(t,T)\Phi(d_1)]$$

and

$$\frac{H_t}{S_t} = \mathrm{E}_t\left[\frac{K\tilde{B}(t,T)}{S_t}\Phi(d_2)\right].$$

First, given that

$$\log\frac{C_T}{C_t} = \sum_{\tau=t+1}^{T} X_\tau,$$

and

$$\log\frac{S_T+D_T}{S_t} = \log\frac{\varphi(U_1^T)+1}{\varphi(U_1^t)} + \sum_{\tau=t+1}^{T} Y_\tau,$$

G_t and H_t can be rewritten as

$$\frac{G_t}{S_t} = \mathrm{E}_t\left\{\beta^{\gamma(T-t)}a_t^T(\gamma)\frac{\varphi(U_1^T)+1}{\varphi(U_1^t)}\exp\left[(\alpha-1)\sum_{\tau=t+1}^{T}X_\tau + \sum_{\tau=t+1}^{T}Y_\tau\right]\right.$$
$$\left.\mathbf{1}_{[\sum_{\tau=t+1}^{T}Y_\tau\geqslant\log(K/S_t)\varphi(U_1^t)/\varphi(U_1^T)+1]}\right\}$$

$$H_t = \mathrm{E}_t\left\{\beta^{\gamma(T-t)}a_t^T(\gamma)\exp\left[(\alpha-1)\sum_{\tau=t+1}^{T}X_\tau\right]\mathbf{1}_{[\sum_{\tau=t+1}^{T}Y_\tau\geqslant\log K/S_t\varphi(U_1^t)/\varphi(U_1^T)+1]}\right\}$$

By the law of iterated expectations:

$$\mathrm{E}_t(\cdot) = \mathrm{E}_t[\mathrm{E}_t(\cdot|U_1^T)],$$

we are led to compute some expectations of the form $\mathrm{E}[\exp(Z_1)\mathbf{1}_{[Z_2\geqslant 0]}]$, where $(Z_1,Z_2)'$ is a bivariate Gaussian vector.

(a) *Proof of first part of the formula* $G_t/S_t = \mathrm{E}_t[Q_{XY}(t,T)\Phi(d_1)]$: We apply the above lemma with

$$Z_1 = (\alpha-1)\sum_{\tau=t+1}^{T} X_\tau + \sum_{\tau=t+1}^{T} Y_\tau,$$

$$Z_2 = \sum_{\tau=t+1}^{T} Y_\tau - \log\frac{K}{S_t}\,\frac{\varphi(U_1^t)}{\varphi(U_1^T)+1}. \tag{A.1}$$

We know that, given U_1^T, $(Z_1, Z_2)'$ is a bivariate Gaussian vector with the following moments:

$$m_1 = (\alpha-1)\sum_{\tau=t+1}^{T} m_{X_\tau} + \sum_{\tau=t+1}^{T} m_{Y_\tau},$$

$$m_2 = \sum_{\tau=t+1}^{T} m_{Y_\tau} - \log\frac{K}{S_t}\,\frac{\varphi(U_1^t)}{\varphi(U_1^T)+1},$$

$$\omega_1^2 = (\alpha-1)^2\sum_{\tau=t+1}^{T}\sigma_{X_\tau}^2 + \sum_{\tau=t+1}^{T}\sigma_{Y\tau}^2 + 2(\alpha-1)\sum_{\tau=t+1}^{T}\sigma_{XY\tau},$$

$$\omega_2^2 = \sum_{\tau=t+1}^{T}\sigma_{Y\tau}^2,$$

$$\rho\omega_1\omega_2 = (\alpha-1)\sum_{\tau=t+1}^{T}\sigma_{XY\tau} + \sum_{\tau=t+1}^{T}\sigma_{Y\tau}^2.$$

Therefore, by application of the lemma:

$$\begin{aligned}
&\mathrm{E}\left[\exp\left[(\alpha-1)\sum_{\tau=t+1}^{T} X_\tau + \sum_{\tau=t+1}^{T} Y_\tau\right]\mathbf{1}_{[\sum_{\tau=t+1}^{T} Y_\tau \geqslant \log(K/S_t)\varphi(U_1^t)/\varphi(U_1^T)+1]}\big| U_1^T\right]\\
&\quad= \exp\left[(\alpha-1)\sum_{\tau=t+1}^{T} m_{X_\tau} + \sum_{\tau=t+1}^{T} m_{Y_\tau}\right.\\
&\qquad\left. + \frac{1}{2}(\alpha-1)^2\sum_{\tau=t+1}^{T}\sigma_{X_\tau}^2 + \frac{1}{2}\sum_{\tau=t+1}^{T}\sigma_{Y\tau}^2 + (\alpha-1)\sum_{\tau=t+1}^{T}\sigma_{XY\tau})\right]\\
&\qquad\times\Phi\left[\frac{1}{(\sum_{\tau=t+1}^{T}\sigma_{Y\tau}^2)^{1/2}}\left[A_t + \sum_{\tau=t+1}^{T}\sigma_{Y\tau}^2\right]\right]
\end{aligned}$$

with $A_t = \sum_{\tau=t+1}^{T} m_{Y_\tau} - \log(K/S_t)\varphi(U_1^t)/(\varphi(U_1^T)+1) + (\alpha-1)\sum_{\tau=t+1}^{T}\sigma_{XY\tau}$.

It is worth noticing at this stage that

$$\mathrm{E}_t\left[\frac{S_T+D_T}{S_t}|U_1^T\right]=\frac{\varphi(U_1^T)+1}{\varphi(U_1^t)}\exp\left[\sum_{\tau=t+1}^{T}m_{Y_\tau}+\frac{1}{2}\sum_{\tau=t+1}^{T}\sigma_{Y\tau}^2\right]$$

and in turn

$$A_t=\log\mathrm{E}\left[\frac{S_T+D_T}{S_t}|U_1^T\right]-\log\frac{S_t}{K}+(\alpha-1)\sum_{\tau=t+1}^{T}\sigma_{XY\tau}-\frac{1}{2}\sum_{\tau=t+1}^{T}\sigma_{Y\tau}^2$$

$$=\log\frac{S_tQ_{XY}(t,T)}{K\tilde{B}(t,T)}-\frac{1}{2}\sum_{\tau=t+1}^{T}\sigma_{Y\tau}^2-\log b_t^T+\log\frac{\varphi(U_1^T)+1}{\varphi(U_1^t)}.$$

The last two terms cancel when one takes out the intermediate dividends (all except T) which do not accrue to the option holder. Therefore,

$$A_t=\log\frac{S_tQ_{XY}(t,T)}{K\tilde{B}(t,T)}-\frac{1}{2}\sum_{\tau=t+1}^{T}\sigma_{Y\tau}^2. \tag{A.2}$$

Therefore, the above application of the lemma proves that

$$\frac{G_t}{S_t}=\mathrm{E}_t\left\{\beta^{\gamma(T-t)}a_t^T(\gamma)\exp\left[(\alpha-1)\sum_{\tau=t+1}^{T}m_{X_\tau}\right.\right.$$
$$\left.\left.+\frac{1}{2}(\alpha-1)^2\sum_{\tau=t+1}^{T}\sigma_{X\tau}^2+(\alpha-1)\sum_{\tau=t+1}^{T}\sigma_{XY\tau}\right]\mathrm{E}\left[\frac{S_T+D_T}{S_t}|U_1^T\right]\Phi(d_1)\right\},$$

where

$$d_1=\frac{1}{(\sum_{\tau=t+1}^{T}\sigma_{Y\tau}^2)^{1/2}}\left[\log\frac{S_tQ_{XY}(t,T)}{K\tilde{B}(t,T)}+\frac{1}{2}\sum_{\tau=t+1}^{T}\sigma_{Y\tau}^2\right].$$

In other words, again by realizing that without intermediate dividends $b_t^T=\varphi(U_1^T)+1/\varphi(U_1^t)$, we have proven that

$$\frac{G_t}{S_t}=\mathrm{E}_t[Q_{XY}(t,T)\Phi(d_1)]$$

which is the required result.

(b) *Proof of second part of the formula* $H_t=K\mathrm{E}_t[\tilde{B}(t,T)\Phi(d_2)]$: We apply the lemma with:

$$Z_1=(\alpha-1)\sum_{\tau=t+1}^{T}X_\tau,$$

$$Z_2=\sum_{\tau=t+1}^{T}Y_\tau-\log\frac{K}{S_t}\frac{\varphi(U_1^t)}{\varphi(U_1^T)}.$$

Therefore, (m_2, ω_2^2) are unchanged with respect to (a) above, but now:

$$m_1 = (\alpha - 1) \sum_{\tau=t+1}^{T} m_{X_\tau},$$

$$\omega_1^2 = (\alpha - 1)^2 \sum_{\tau=t+1}^{T} \sigma_{X\tau}^2,$$

$$\rho\omega_1\omega_2 = (\alpha - 1) \sum_{\tau=t+1}^{T} \sigma_{XY\tau}.$$

Therefore, by application of the lemma:

$$\mathrm{E}\left[\exp\left[(\alpha - 1)\sum_{\tau=t+1}^{T} X_\tau\right] \mathbf{1}_{[\sum_{\tau=t+1}^{T} Y_\tau \geqslant \log(K/S_t)\varphi(U_1^t)/\varphi(U_1^T)+1]} | U_1^T\right]$$

$$= \exp\left[(\alpha - 1)\sum_{\tau=t+1}^{T} m_{X_\tau} + \frac{1}{2}(\alpha - 1)^2 \sum_{\tau=t+1}^{T} \sigma_{X\tau}^2\right] \Phi\left[\frac{1}{(\sum_{\tau=t+1}^{T} \sigma_{Y\tau}^2)^{1/2}} A_t\right].$$

By the same argument as above, we then obtain

$$\frac{H_t}{K} = \mathrm{E}_t\left\{\beta^{\gamma(T-t)} a_t^T(\gamma) \exp\left[(\alpha - 1)\sum_{\tau=t+1}^{T} m_{X_\tau} + \frac{1}{2}(\alpha - 1)^2 \sum_{\tau=t+1}^{T} \sigma_{X\tau}^2\right] \Phi(d_2)\right\}$$

with

$$d_2 = d_1 - \frac{(\sum_{\tau=t+1}^{T} \sigma_{Y\tau}^2)^{1/2}}{2}.$$

This provides the required result:

$$H_t = K\mathrm{E}_t[\tilde{B}(t, T)\Phi(d_2)].$$

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