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Subordinated Levy Processes and Applications to Crude Oil Options

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Abstract

One approach to oil markets is to treat oil as an asset, besides its role as a commodity. Speculative and nonspeculative activity by investors in the derivatives markets could be responsible for a sizable increase in oil prices. This paper recognizes both the consumption and investment aspects of crude oil and proposes Levy processes for modeling uncertainty and options pricing. Calibration to crude oil futures' options shows high volatility of oil futures prices, fat-tailed, and right-skewed market expectations,implying a higher probability mass on crude oil prices remaining above the futures' level. These findings support the view that demand for futures contracts by investors could lead to excessively high price volatility.

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I. Introduction

A suggested approach to oil markets analysis is to treat oil as an asset, besides its role as a commodity, and to recognize the influence of derivatives markets on oil price behavior. Restricting attention to fundamentals in modeling and forecasting short-term oil prices would be unrealistic and would omit a key aspect of oil markets, which is the asset role of crude oil and the preponderant presence of financial investors, other than the traditional operators, who are the producers and the consumers, in the markets. Sizable demand for futures contracts by institutional investors and speculative activity by hedge funds and commercial entities could exert pressure on prices and cause volatility of oil futures prices to rise to excessive levels. Similarly, in other commodity and currency markets, high volatility would stimulate speculation, which in turn contributes to higher volatility and to volatility clustering.

It is well known that asset markets can experience frequent jumps of different magnitudes and can overshoot or undershoot their equilibrium. Yet policymakers and central bankers would like to monitor regularly these markets, gauge the market sentiment, and forecast price distributions. They turn to derivatives markets, and particularly to options prices, for such forecasts as these markets account for both the consumption and investment aspects of an asset. The purpose of this paper is therefore to model asset prices' uncertainty and option pricing in the context of Levy processes, which are capable of handling discontinuities and are known to behave properly under time aggregation.

Levy processes have gained considerable interest in financial modeling as they were found to overcome many of the shortcomings associated with the Black-Scholes' model (1973) and to offer a more general tool for modeling uncertainty in asset prices. They are seen as a random walk in continuous time with jumps occurring at random times. They include the Brownian motion and the Poisson processes as particular cases. Their general form has a drift, a Brownian motion, and a compound Poisson process which distinguishes small size jumps and large jumps. The probability distributions associated with Levy processes are infinitely divisible and offer more flexibility for fitting financial data, particularly high-frequency data, and can have skewed shapes and slow decaying tails.

While the Black-Scholes model and diffusion processes constitute the main framework for derivatives pricing, they can, nonetheless, have inconsistencies with market data, typically in relation to the implied volatility and to the dynamics of the asset's price. The model's implied volatility tends to vary both in relation to the state and time, exhibiting a smile or a smirk. Volatility could even be seen as stochastic. The dynamics of the asset's price may exhibit jumps of different sizes, with small jumps occurring more often than large jumps, leading both to asymmetries and fat tails in the asset's returns, contradicting thus the Brownian motion assumption underlying the Black-Scholes model. These empirical features of assets' returns were noted by Mandelbrot (1963) and Fama (1965). As the normal distribution did not fit the data, they proposed the use of stable distributions, similar to the Pareto distributions, which are capable of accommodating skewness and the slowly decaying tails of the empirical distribution, in contrast to the thin and rapidly decaying tails of the normal distribution. Clark (1973), however, noted that stable distributions have an infinite variance and heavy tails, whereas assets' returns distributions have a finite variance and semi-heavy tails. He proposed the use of Bochner (1955) concept of subordinated processes. Namely, the price process could be modeled as a Brownian motion, time changed by a random and independent subordinator, which is an increasing positive stochastic process. While Clark proposed the use of volume as subordinator, the number of transactions could also be used for measuring assets' returns. His finding was that, when measured in relation to volume, cotton futures tend to be normally distributed.

This paper addresses option pricing in the context of a Levy market model with an application consisting of deriving crude oil price density forecast from crude oil options. In view of the paramount relevance of Levy processes in financial modeling, Section II describes their key properties and particularly their characteristic function, a major tool for studying their distributional properties. Section III discusses a methodology for constructing Levy processes through subordination and describes examples of probability distributions that are obtained through subordination and that were found to fit adequately financial time series. As Levy processes may exhibit jumps, the markets become incomplete and there is an infinite number of martingale measures compatible with absence of arbitrage that can be used for pricing contingent claims. In this respect, Section IV presents Esscher transform as a procedure for selecting a martingale measure. It is known that probability distributions associated with Levy processes are not always available in closed form and may involve many special functions. The corresponding characteristic function may, however, be readily available. In Section V, the paper presents a methodology for pricing contingent claims based on characteristic functions. In Section VI, the paper discusses an application of the Levy pricing model to crude oil options and attempts to infer the density forecast of future oil prices at a given time horizon. Findings point to high volatility of oil futures prices and a right-skewed market expectations, implying greater probability mass on upward deviations from the mean. Consequently, extracting densities from options prices allows one to analyze the role of crude oil as an asset, besides its role as a commodity, and improves oil market modeling. Section VII concludes.

II. Modeling Uncertainty in ASSETS PRICES: Levy Processes

A. From Random Walks to Levy Processes

A Levy process (LP) is defined as a cadlag (*continu à droite et limite à gauche*, right continuous and left limit (RCLL)) stochastic process $(X_t)_{t\geq 0}$ on a probability space (Ω, F, P) with values in R such that $X_0 = 0$ and possesses the following properties: (i) independent increments: for every increasing sequence of times $t_0, ..., t_n$, the random variables $X_{t_0}, X_{t_1} - X_{t_0}, ..., X_{t_n} - X_{t_{n-1}}$ are independent, (ii) stationary increments: the law of $X_{t+h} - X_t$ does not depend on t, and (iii) stochastic continuity: $\forall \varepsilon > 0$, $\lim_{h\to 0} P(|X_{t+h} - X_t| \geq \varepsilon) = 0$. i.e., discontinuity occurs at random times. Levy processes are limits of random walks and are infinitely divisible into independent and identically distributed (i.i.d.) random variables. For any X_t, $t > 0$, and $n \geq 2$, an LP can be written as:

$X_t = \sum_{k=1}^{n} (X_{tk/n} - X_{t(k-1)/n})$, where the successive increments are independent and stationary. An equivalent way to express the infinite divisibility is: $X_t^{(n)} = Y_1 + ... + Y_n$, where the random variables $Y_1, Y_2, ..., Y_n$ are i.i.d. The probability distribution of X_t is the same as that for $Y_1 + ... + Y_n$. The most common examples of infinitely divisible laws are: the Gaussian, the Gamma, $\alpha -$stable, and the Poisson distributions.

B. The Characteristic Function of the Levy Process: The Levy-Khintchine Formula

Suppose $\phi(u)$ is the characteristic function (CF) of a distribution. If for every positive integer n, $\phi(u)$ is the nth power of a CF, then the distribution is infinitely divisible. For every such distribution there can be defined a stochastic process, $X = \{X_t, t \geq 0\}$, which starts at zero and has independent and stationary increments. If $(X_t)_{t\geq 0}$ is an LP, then for any $t > 0$, the distribution of X_t is infinitely divisible and has a CF $\phi(u) = E[e^{iuX_1}] = e^{-t\psi(u)}$, $u \in R$, $t \geq 0$. The cumulant characteristic function $\psi(u) = \log \phi(u)$ is often called the characteristic exponent and satisfies the Levy-Khintchine formula:[2 3]

$\psi(u) = i\gamma u - \frac{\sigma^2}{2}u^2 + \int_{R\backslash\{0\}} [e^{iux} - 1 - iux1_{|x|<1}(x)]\nu(dx)$, where $\gamma \in R$ is the drift parameter, $\sigma^2 \geq 0$ is the volatility parameter, and ν is a Levy measure on $R\backslash\{0\}$ with $\int_{-\infty}^{\infty} \inf(1, x^2)\nu(dx) < \infty$. The sums of all jumps smaller than some $\varepsilon > 0$ does not converge. However, the sum of the jumps compensated by their mean does converge. This peculiarity leads to the necessity of the compensator term $iux1_{(|x|<1)}$. If the Levy measure is of the form $\nu(dx) = f(x)dx$, then $f(x)$ is called the Levy density. In the same way the instant volatility describes the local uncertainty of a diffusion, the Levy density describes the local uncertainty of a pure jump process.

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[2] The Levy-Khintchine formula is also written as;

$$\psi(u) = -i\gamma u + \frac{\sigma^2}{2}u^2 + \int_{|x|\geq 1} (1 - e^{iux})\nu(dx) + \int_{|x|<1} (1 - e^{iux} + iux)\nu(dx)$$

[3] Recalling the definition of the cumulant generating function (CGF) of a random variable, it follows that ψ is the CGF of X_1: $\psi = \psi_{X_1}$, the CGF of X_t varies linearly in t, $\psi_{X_t} = t\psi_{X_1} = t\psi$. The law of X_t is therefore determined by the knowledge of the law of X_1: the only degree of freedom in specifying a LP is to specify the distribution of X_t for a single time (say, $t = 1$).

The Levy-Khintchine formula allows to study the distributional properties of an LP. Another key concept, the Levy-Ito decomposition theorem, allows one to describe the structure of an LP's sample paths. Let $X = \{X_t, t \geq 0\}$ be an LP, then the distribution of X_1 has as parameters (γ, σ^2, ν) and X decomposes as: $X_t = \gamma t + \sigma W_t + J_t + M_t$, where W_t is a Brownian motion. The instantaneous jump $\Delta X_t = X_t - X_{t-}$ follows an independent Poisson point process with intensity measure ν, $J_t = \sum_{s \leq t} \Delta X_s 1_{\{|\Delta X_s| > 1\}}$, and M is a martingale with jumps $\Delta M_t = \Delta X_t 1_{\{|\Delta X_s| \leq 1\}}$. It is interesting to observe that there is a one-to-one correspondence between an LP and its CF. More precisely, starting with the Levy-Khintchine formula, three processes, $X^{(1)}, X^{(2)}$, and $X^{(3)}$, can be built as follows: denoting $X_t^{(1)} = \gamma t + \sigma W_t$, where W_t is a standard $P-$Brownian motion, the CF of $X^{(1)}$ is straightforward and equal to: $\phi_1(u) = i\gamma u + \frac{1}{2}\sigma^2 u^2$. Now, consider the process $X_t^{(2)} = J_t = \sum_{j=1}^{N_t} Y_j$, where N is a Poisson process whose intensity λ is defined by $\lambda = \int_{|x|>1} \nu(dx)$, and Y_1, Y_2, ..., Y_n are independent random variables, independent of the process N and with common distribution $1_{|x|>1}\nu(dx)$. $X_t^{(2)}$ is a compound Poisson process whose CF is $e^{-\phi_2(u)}$, where $\phi_2(u) = -\int (e^{iux} - 1) 1_{|x|>1} \nu(dx)$. The last term in the Levy-Khintchine formula is the CF of an LP $X^{(3)}$ obtained as the limit of compound Poisson processes (different from $X_t^{(2)}$). Hence: $X = X^{(1)} + X^{(2)} + X^{(3)}$. Each of $X^{(1)}, X^{(2)}, X^{(3)}$ is a semi-martingale, so is X.[4]

An important implication of the Levy-Ito decomposition is that every LP is a combination of a Brownian motion with drift and a possibly infinite sum of independent compound Poisson processes. This also means that every LP can be approximated with arbitrary precision by a jump-diffusion process. In particular, the Levy measure ν describes the arrival rates for jumps of every possible size for each component of X_t. Jumps of sizes in the set A occur

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[4] The transition operator for a Markov process is defined as follows $P_t f(x) = E[f(x + X_t)]$, verifying a semi-group property $P_t P_s = P_{t+s}$. A semi-group P_t can be described by the means of its infinitesimal generator L, which is a linear operator defined by: $Lf = \lim_{t \downarrow 0} \frac{(P_t f - f)}{t}$. Let $(X_t)_{t \geq 0}$ be an LP on R, then the infinitesimal generator of X_t is defined for any $f \in C_0^2(R)$ as $Lf(x) = \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial x^2}(x) + \gamma \frac{\partial f}{\partial x}(x) + \int_R (f(x+y) - f(x) - y \frac{\partial f}{\partial x}(x) 1_{|y| \leq 1}) \nu(dy)$. Computation of expectations of various functionals of Levy processes can be transformed into partial integro-differential equations involving the infinitesimal generator. Due to this fact, infinitesimal generators are important tools in option pricing.

according to a Poisson process with intensity parameter $\int_A \nu(dx)$, where A is an arbitrary interval bounded away from zero. The Levy measure of the process X may also be defined as $\nu(A) = E\{\sum_{0\le s\le 1} 1_A(\Delta X_s)\}$. Integration of the Levy density over a particular spatial domain provides the arrival rates of jumps sized in this domain. Levy process X has infinite activity if the integral of the measure ν on the real line is infinite. This case characterizes a high rate of arrival of jumps of different sizes.

C. Finite-Activity Versus Infinite Activity Levy Jumps

A pure jump LP can display either finite or infinite activity. In the former case, the aggregate jump arrival rate is finite, while in the latter case, an infinite number of jumps can occur in any finite time interval. A pure jump LP exhibits finite activity if the following integral is finite: $\int_{R\setminus\{0\}} \nu(dx) = \lambda < \infty$. The classical example of a finite-activity jump process is the compound Poisson jump process of Merton (1976). For such process, the integral $\int_{R\setminus\{0\}} \nu(dx) = \lambda < \infty$, where λ is the Poisson intensity. Conditional on one jump occurring, the Merton model assumes that the jump magnitude is normally distributed with mean α and variance σ^2. The Levy measure of the Merton process is given by: $\nu(dx) = \lambda \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{(x-\alpha)^2}{2\sigma^2})dx$. Obviously, one can choose any distribution, $F(x)$, for the jump size under the compound Poisson framework and obtain the following Levy measure; $\nu(dx) = \lambda dF(x)$. Kou (2002) assumes a double-exponential conditional distribution for the jump size. The Levy measure in this case is given by $\nu(dx) = \lambda dF(x) = \lambda \frac{1}{2\eta} \exp(-\frac{|x-\alpha|}{\eta})dx$. Eraker et al. (2003) incorporate compound Poisson jumps into the stochastic volatility process; jump size is controlled by one-sided exponential density. The Levy measure in this case is given by $\nu(dx) = \lambda dF(x) = \lambda \frac{1}{\eta} \exp(-\frac{x}{\eta})dx$, $x > 0$.

Based on the Levy-Khintchine formula, the characteristic exponent corresponding to these compound Poisson jump components is given by $\psi(u) = \int_{R\setminus\{0\}} (1-e^{iux})\lambda dF(x) = \lambda(1-\phi(u))$, where $\phi(u)$ denotes the CF of the jump size distribution $F(x)$, $\phi(u) = \int_{R\setminus\{0\}} e^{iux} dF(x)$.

An infinite activity jump process can generate an infinite number of jumps within any finite time interval. The integral of the Levy measure $\int_{R\setminus\{0\}} \nu(dx) = \infty$ is no longer finite. Examples in this class include the normal inverse Gaussian (NIG) model of Barndorff-Nielsen (1998), the generalized hyperbolic class of Eberlein et al. (1998), the variance-gamma (VG) model of Madan and Milne (1991), and the CGMY model of Carr et al. (2002).

III. Subordination and Time Changed-Levy Processes

Three key pieces of evidence on financial securities: jumps, stochastic volatility, and leverage effect are easily addressed when uncertainty in the economy is governed by a time-changed LP. A stochastic time change to the LP amounts to stochastically altering the clock on which the LP is run. Intuitively, one can regard the original clock as a calendar time and the new random clock as a business time. A more active business day implies a faster business clock. Randomness in business activity generates randomness in volatility. Furthermore, if innovations in the LP are correlated with innovations in the random clock on which it is run, this correlation will capture the leverage effect.

A. Construction of Levy Processes by Subordination

Clark (1973) proposed Bochner's (1955) concept of subordinated stochastic process as a model to account for non-normality of returns. He showed that finite-variance distributions subordinate to the normal distribution fit cotton futures better than stable distributions. Writing the return process $X(t)$ as a subordinated process $X(t) = Z(T(t))$, where the subordinator $T(t)$ is an increasing Levy process with independent and stationary increments, and using historical data on returns (represented by $X(t)$) and volume (represented by $T(t)$), he was able to show that the distribution of Z computed in relation to T did satisfy classical normality tests. He also showed that the kurtosis of the increments of $Z(T(t))$ is an increasing function of the variance of the increments of $T(t)$.[5]

Monroe (1978) proved that every semi-martingale X_t can be written as a time-changed Brownian motion, where the random time T_t is a positive and increasing semi-martingale. By such result, there exists a Brownian motion $(W(u), u \geq 0)$ and a random time change $T(t)$ where $T(t)$ is an increasing stochastic process such that: $X(t) = W(T(t))$. As an implication, every semi-martingale can also be written as a time-changed LP $X_t = Z(T(t))$. The distribution of increments, $\Delta Z(T(t))$, is said to be subordinate to the distribution of increments, $\Delta Z(t)$; $T(t)$ is a clock measuring the speed of the evolution. Furthermore, every semi-martingale X_t starting at zero ($X_0 = 0$), can be uniquely represented in the form:

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[5] In Clark (1973), if $T(t)$ is a lognormal with independent increments distributed as $N(\mu, \sigma_1^2)$ and Z is a normal process with independent increments distributed as $N(0, \sigma_2^2)$, then $X(t) = Z(T(t))$ has the following lognormal-normal increments:

$$f_{LNN}(y) = \frac{1}{2\pi\sigma_1^2\sigma_2^2}\int_0^\infty \upsilon^{-3/2}\exp\left(\frac{-(\log\upsilon - \mu)^2}{2\sigma_1^2}\right)\exp\left(\frac{-y^2}{2\upsilon\sigma_2^2}\right)d\upsilon .$$

$X_t = \alpha_t + X_t^c + \int_0^t \int_{|x|>1} xd\mu + \int_0^t \int_{|x|\le 1} xd(\mu - \nu)$, where X_t^c is the continuous component, μ is the counting measure of the semi-martingale, and ν is its compensator.

An important result from Monroe's theorem relates to modeling the return distribution as a mixture of normals with a view to account for the observed fat tails of the return. Choosing for simplicity a discrete f_T for T, the following holds:
$P(X(t) \in dx) = \sum_{\zeta} P(Z(t) \in dx \,|\, T(t) = \zeta)\, f_T(\zeta)$. Now let $Z(t) = W(t)$, and assuming the independence of the processes W and T, this yields: $P(X(t) \in dx) = \sum_{\zeta} P(W(\zeta) \in dx)\, f_T(\zeta)$.
Hence the distribution of X appears as a mixture of normal distributions, where the mixing factor is the density of the time change, which itself accounts for the market activity measured by volume or number of trades.

A simple example of a subordinated LP is a compound Poisson process with a finite arrival rate, i.e., a random walk time changed by a Poisson process expressed as follows:
$X(t) = \sum_{i=1}^{N(t)} Y_i$, where $N(t)$ is a Poisson process with arrival rate λt, and the sequence of Y_i is i.i.d. with density $f(y) = \dfrac{\sqrt{2}\exp(-\frac{y^2}{2\sigma^2})}{\sigma\sqrt{\pi}}$ for $y > 0$. The CF of X_t is
$\phi_X(u) = \exp(2\lambda t(\exp(-\frac{\sigma^2 u^2}{2}) - 1))$.

Another example of a time changed LP is to subordinate a Brownian motion $(W_t)_{t\ge 0}$ with drift μ by an independent positive process $(T_t)_{t\ge 0}$, yielding a new LP: $X_t = \sigma W(T_t) + \mu T_t$. This process is a Brownian motion if it is observed on a new time scale, which is the stochastic time scale given by T_t. It is worth noticing that the constant volatility in the arithmetic Brownian motion W is going to give rise to stochastic volatility for the stochastic price X when W is compounded with a stochastic time T. Time changes appear a natural tool to handle stochastic volatility. The interpretation of an LP as a subordinated Brownian motion is easier to understand than general Levy models.

B. The Characteristic Function of a Subordinated Process

The CF of a subordinated process X is obtained by composition of the Laplace exponent[6] of T with the characteristic exponent of Z. Since the time-changed process $X_t = Z_{T_t}$ is a stochastic process evaluated at a stochastic time, its CF involves expectation over two sources of randomness, $\phi_{X_t}(u) = E[e^{iuZ_{T_t}}] = E[E[e^{iuZ_\xi} \mid T_t = \xi]]$, where the inside expectation is taken on Z_{T_t}, conditional on a fixed value of $T_t = \xi$, and the outside expectation is on all possible values of T_t. If the random time T_t is independent of Z_t, the randomness due to the LP can be integrated out using equation: $\phi_{Z_t}(u) = E[e^{iuZ_t}] = e^{-t\psi_z(u)}$. Simple computation yields: $\phi_{X_t}(u) = E[e^{iuZ_{T_t}}] = E[E[e^{iuZ_\xi} \mid T_t = \xi]] = Ee^{-T_t\psi_z(u)} = L_{T_t}(\psi_z(u))$. Under independence, the CF of $X_t = Z_{T_t}$ is just the Laplace transform of T_t evaluated at the characteristic component of Z_t.[7] We can introduce asymmetry to the distribution of $X_t = Z_{T_t}$ by introducing correlation between the time change and the Levy innovations in Z.

C. Examples of Subordinated LP

A simple approach to defining a parametric LP is to obtain an LP by subordinating a Brownian motion with an independent increasing LP. Here the CF of the resulting process can be obtained immediately, but there may not be always an explicit formula for the Levy measure. Due to the conditionally Gaussian structure of the process, simulation and computation can be considerably simplified.

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[6] Let $(T_t)_{t\geq 0}$ be a subordinator, that is an LP whose trajectories are increasing. Since T_t is a positive random variable for all t, it is described by its Laplace transform rather than the Fourier transform. Let the characteristic triplet of T be $(0, \rho, a)$. Then the moment generating function of T_t is: $E[e^{uT_t}] = e^{tl(u)}$ $\forall u \leq 0$, where $L(u) = au + \int_0^\infty (e^{ux} - 1)\rho(dx)$, and $L(u)$ is the Laplace exponent of T.

[7] Let $(Z_t)_{t\geq 0}$ be an LP on R with characteristic exponent $\psi(u)$ and triplet (σ, ν, γ) and let $(T_t)_{t\geq 0}$ be a subordinator with Laplace exponent $L(u)$ and triplet $(0, \rho, a)$. Then $X(t) = Z(T(t))$ is an LP with CF $E[e^{iuX_t}] = e^{tL(\psi(u))}$, and triplet $(\sigma^X, \nu^X, \gamma^X)$ is given by: $\sigma^Y = a\sigma$, $\nu^X(B) = a\nu(B) + \int_0^\infty p_s^Z(B)\rho(ds)$, $\forall B \in B(R)$, $\gamma^X = a\gamma + \int_0^\infty \rho(ds) \int_{|z|\leq 1} z p_s^z(dz)$, where p_t^Z is the probability density of Z_t.

The Inverse Gaussian Process

Let $T^{(a,b)}$ be the first time a Brownian motion with drift $b>0$, i.e., $(W_s+bs, s\geq 0)$, reaches the positive level $a>0$. It is well known that this random time follows the so-called Inverse Gaussian law, $IG(a,b)$, and has a CF: $\phi_{IG}(u;a,b)=\exp(-a(\sqrt{-2iu+b^2}-b))$. The IG distribution is infinitely divisible and the IG process $X^{(IG)}=\{X_t^{(IG)}, t\geq 0\}$, with parameters $a,b>0$, is defined as the process which starts at zero and has independent and stationary increments over the interval $[s,s+t]$, $s,t\geq 0$ such that:
$E(\exp(iuX_t^{(IG)}))=\phi_{IG}(u;at,b)=\exp(-at(\sqrt{-2iu+b^2}-b))$. The density function of the $IG(a,b)$ law is explicitly given by: $f_{IG}(x;a,b)=\dfrac{a}{\sqrt{2\pi}}\exp(ab)x^{-3/2}\exp(-\dfrac{1}{2}(a^2x^{-1}+b^2x))$, $x>0$. Its Levy measure is given by $\nu_{IG}(dx)=(2\pi)^{-1/2}ax^{-3/2}\exp(-\dfrac{1}{2}b^2x)1_{(x>0)}dx$.

The Generalized Inverse Gaussian Process

The Inverse Gaussian $IG(a,b)$ law can be generalized to the Generalized Inverse Gaussian distribution $GIG(\lambda,a,b)$. This distribution is given by

$$f_{GIG}(x;\lambda,a,b)=\frac{(b/a)^{\lambda}}{2K_{\lambda}(ab)}x^{\lambda-1}\exp(-\frac{1}{2}(a^2x^{-1}+b^2x)),\ x>0.$$

The CF is given by $\phi_{GIG}(u;\lambda,a,b)=\dfrac{1}{K_{\lambda}(ab)}(1-2iu/b^2)^{\lambda/2}K_{\lambda}(ab\sqrt{1-2iub^{-2}})$,

where $K_{\lambda}(x)$ denotes the modified Bessel function of the third kind with index λ.
The $GIG(\lambda,a,b)$ distribution is infinitely divisible. The GIG process is defined as the LP where the increment over the interval $[s,s+t]$, $s,t\geq 0$ has the CF $(\phi_{GIG}(u;\lambda,a,b))^t$. The Levy measure has a density: $\nu(x)=x^{-1}\exp(-\dfrac{1}{2}b^2x)(a^2\int_0^{\infty}\exp(-xz)g(z)dz+\max[0,\lambda])$, where $g(z)=(\pi^2a^2z(J_{|\lambda|}^2(a\sqrt{2z})+N_{|\lambda|}^2(a\sqrt{2z})))^{-1}$ and J and N are Bessel functions.

The Variance Gamma Process

A Variance Gamma (VG) process is defined as a Brownian motion with drift θ and volatility σ time-changed by a Gamma process. More precisely, let $G=\{G_t, t\geq 0\}$ be a Gamma process with parameters $a=1/\upsilon>0$ and $b=1/\upsilon>0$.[8] Let $W=\{W_t, t\geq 0\}$ denote a Brownian

[8] The probability density of the Gamma process with mean rate t and variance υt is well known: $f(u)=u^{\frac{1}{\upsilon}-1}e^{-\frac{u}{\upsilon}}/\upsilon^{\frac{t}{\upsilon}}\Gamma(\dfrac{t}{\upsilon})$. Its Laplace transform is $E[\exp(-\lambda G_t^{\upsilon})]=(1+\lambda\upsilon)^{-\frac{t}{\upsilon}}$.

(continued…)

motion, and let $\sigma > 0$ and $\theta \in R$; then the VG process $X^{(VG)} = \{X_t^{(VG)}, t \geq 0\}$, with parameters $\sigma > 0$, $\upsilon > 0$, and θ, can be defined as $X_t^{(VG)} = \theta G_t + \sigma W_{G_t}$. The CF is given by $\phi_{VG}(u;\sigma,\nu,\theta) = (1 - iu\theta\upsilon + \frac{1}{2}\sigma^2\upsilon u^2)^{-\frac{1}{\upsilon}}$. The distribution is infinitively divisible and the process $X^{(VG)} = \{X_t^{(VG)}, t \geq 0\}$ is defined as the process which starts at zero, has independent and stationary increments and for which $X_{s+t}^{(VG)} - X_s^{(VG)}$ follows a $VG(\sigma\sqrt{t}, \upsilon / t, t\theta)$ law over the interval $[s, t+s]$. The CF is:

$$E[\exp(iuX_t^{(VG)})] = \phi_{VG}(u;\sigma\sqrt{t}, \upsilon / t, t\theta) = \phi_{VG}(u;\sigma,\upsilon,\theta)^t = (1 - iu\theta\upsilon + \frac{1}{2}\sigma^2\upsilon u^2)^{-\frac{t}{\upsilon}}.$$

The two additional parameters in the VG distribution, which are the drift of the Brownian motion, θ, and the volatility of the time change, υ, provide control over skewness and kurtosis, respectively. Namely, when $\theta < 0$, the distribution is negatively skewed, and vice versa. Moreover, larger values of υ indicate frequent jumps and contribute to fatter tails. The Levy measure has infinite mass, and hence a VG process has infinitely many jumps in any finite time interval. The moments of $VG(\sigma,\upsilon,\theta)$ are: the mean $= \theta$; the variance $= \sigma^2 + \upsilon\theta^2$; skewness = $\frac{\theta\upsilon(3\sigma^2 + 2\upsilon\theta^2)}{(\sigma^2 + \upsilon\theta^2)^{3/2}}$; and kurtosis = $3(1 + 2\upsilon - \upsilon\sigma^4(\sigma^2 + \upsilon\theta^2)^{-2})$. Clearly, skewness is influenced by θ, and kurtosis by υ.

The Normal Inverse Gaussian Process

The Normal Inverse Gaussian (NIG) process can be related to a Brownian motion time-changed by an Inverse Gaussian process. Let $W = \{W_t, t \geq 0\}$ be a Brownian motion and let $IG = \{IG_t, t \geq 0\}$ be an IG process with parameters $a = 1$ and $b = \delta\sqrt{\alpha^2 - \beta^2}$, with $\alpha > 0$, $-\alpha < \beta < \alpha$ and $\delta > 0$; then the process: $X_t = \beta\delta^2 IG_t + \delta W_t$ is an NIG process with parameters α, β, δ.[9] $NIG(\alpha,\beta,\delta)$ has a CF given by $\phi_{NIG}(u;\alpha,\beta,\delta) = \exp(-\delta(\sqrt{\alpha^2 - (\beta + iu)^2} - \sqrt{\alpha^2 - \beta^2}))$. This is an infinitely divisible CF. Therefore, the NIG $X^{(NIG)} = \{X_t^{(NIG)}, t \geq 0\}$, with $X_0^{(NIG)} = 0$ has stationary and independent

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It results that the VG process has a simple CF $\phi_{VG}(u) = (1 - i\theta\upsilon u + \frac{\sigma^2\upsilon}{2}u^2)^{-\frac{t}{\upsilon}}$.

[9] An equivalent parameterization of the NIG process is a Brownian motion with drift θ and volatility σ computed at a random time given by an $IG(1,\upsilon)$ process:
$X_{NIG}(t;\sigma,\upsilon,\theta) = \theta IG_t^{\upsilon} + \sigma W(IG_t^{\upsilon})$. The CF is $E[e^{iuX_{NIG}(t)}] = E[\exp(i\theta u - \frac{\sigma^2 u^2}{2}) IG_t^{\upsilon}]$.

increments. $X_t^{(NIG)}$ has a $NIG(\alpha,\beta,t\delta)$ law with a CF:
$\phi_{NIG}(u;\alpha,\beta,t\delta)=\exp[-t\delta(\sqrt{\alpha^2-(\beta+iu)^2}-\sqrt{\alpha^2-\beta^2})]$
The Levy measure is given by $\nu_{NIG}(dx)=\frac{\delta\alpha}{\pi}\frac{\exp(\beta x)K_1(\alpha|x|)}{|x|}dx$, where $K_\lambda(x)$ denotes the modified Bessel function of the third kind with index λ. The integral of ν_{NIG} over the real line is infinite; hence, the NIG process has infinite activity. The density of the $NIG(\alpha,\beta,\delta)$ distribution is given by

$f_{NIG}(x;\alpha,\beta,\delta)=\frac{\alpha\delta}{\pi}\exp(\delta\sqrt{\alpha^2-\beta^2}+\beta x)\frac{K_1(\alpha\sqrt{\delta^2+x^2})}{\sqrt{\delta^2+x^2}}$. The moments are: mean $=\delta\beta\sqrt{\alpha^2-\beta^2}$; variance $=\alpha^2\delta(\alpha^2-\beta^2)^{-3/2}$; skewness $=3\beta\alpha^{-1}\delta^{-1/2}(\alpha^2-\beta^2)^{-1/4}$; and kurtosis $=3(1+\frac{\alpha^2+4\beta^2}{\delta\alpha^2\sqrt{\alpha^2-\beta^2}})$. The NIG has semi-heavy tails, in particular $f_{NIG}(x;\alpha,\beta,\delta)\sim|x|^{-3/2}\exp((\pm\alpha+\beta)x)$ as $x\to\pm\infty$ up to a multiplicative constant.

The Generalized Hyperbolic Process

The Generalized Hyperbolic (GH) distribution can be represented as a normal variance-mean mixture. Let T be a GIG random variable and W be an independent standard normal variable. Then the law of $\sqrt{T}W+\mu T$, where μ is a constant, is called normal variance-mean mixture with mixing distribution GIG.

$f_{GH}(x;\alpha,\beta,\delta,\upsilon,\mu)=\int_0^\infty f_{Normal}(x;\mu+\beta w,w)f_{GIG}(w;\upsilon,\delta,\sqrt{\alpha^2-\beta^2})dw$. The GH distribution, $GH(\alpha,\beta,\delta,\upsilon)$, is defined in Barndorff-Nielsen (1998) through its CF

$$\phi_{GH}(u;\alpha,\beta,\delta,\upsilon,\mu)=e^{i\mu u}\left(\frac{\alpha^2-\beta^2}{\alpha^2-(\beta+iu)^2}\right)^{\upsilon/2}\frac{K_\upsilon(\delta\sqrt{\alpha^2-(\beta+iu)^2})}{K_\upsilon(\delta\sqrt{\alpha^2-\beta^2})},$$

where K_υ is the modified Bessel function of the third kind. This is an infinitely divisible CF. The GH Levy process $X^{(GH)}=\{X_t^{(GH)},t\ge 0\}$, with $X_0^{(GH)}=0$, stationary and independently distributed increments, has the CF: $E[\exp(iuX_t^{(GH)})]=(\phi_{GH}(u;\alpha,\beta,\delta,\upsilon,\mu))^t$. The density of $GH(\alpha,\beta,\delta,\upsilon,\mu)$ distribution is given by

$$f_{GH}(x;\alpha,\beta,\delta,\upsilon,\mu)=a(\alpha,\beta,\delta,\upsilon)(\delta^2+(x-\mu)^2)^{(\upsilon-1/2)/2}K_{(\upsilon-1/2)}(\alpha\sqrt{\delta^2+(x-\mu)^2})\exp(\beta(x-\mu)),$$

$a(\alpha,\beta,\delta,\upsilon)=\frac{(\alpha^2-\beta^2)^{\upsilon/2}}{\sqrt{2\pi}\alpha^{\upsilon-\frac{1}{2}}\delta^\upsilon K_\upsilon(\delta\sqrt{\alpha^2-\beta^2})}$. The GH distribution has semi-heavy tails: in particular $f_{GH}(x;\alpha,\beta,\delta,\upsilon)\sim|x|^{\upsilon-1}\exp((\pm\alpha+\beta)x)$ as $x\to\pm\infty$ up to a multiplicative constant.

Some of the above processes are special cases of the GH process. The Variance-Gamma process can be obtained from the GH process by taking: $\upsilon = \sigma^2/\nu$, $\alpha = \sqrt{(2/\nu)+(\theta^2/\sigma^2)}$, $\beta = \theta/\sigma^2$ and $\delta \to 0$. The Hyperbolic process (HYP) is obtained when $\upsilon = 1$. In this case, $X_1^{(HYP)}$ follows $HYP(\alpha,\beta,\delta)$ with CF:

$$\phi_{HYP}(u;\alpha,\beta,\delta) = \left(\frac{\alpha^2-\beta^2}{\alpha^2-(\beta+iu)^2}\right)^{1/2} \frac{K_1(\delta\sqrt{\alpha^2-(\beta+iu)^2})}{K_1(\delta\sqrt{\alpha^2-\beta^2})}.$$

The density reduces to $f_{HYP}(x;\alpha,\beta,\delta) = \frac{(\alpha^2-\beta^2)^{1/2}}{\sqrt{2\pi}\delta K_1(\delta\sqrt{\alpha^2-\beta^2})})\exp(-\alpha\sqrt{\delta^2+x^2}+\beta x)$,

The Normal Inverse Gaussian process is obtained when $\upsilon = -1/2$, thus: $GH(\alpha,\beta,\delta,-1/2) = NIG(\alpha,\beta,\delta)$.

The normal inverse Gaussian, the Variance-Gamma, and the hyperbolic motion are Levy processes which share the property of being pure jump and infinite activity models. Their empirical performance in modeling skewness, leptokurtosis, and the implied volatility smile in option prices made them more appealing than the classical diffusions or jump-diffusion models. Their representation as time-changed Brown motions allows to model the time change which itself reflects the intensity of economic activity through news arrival and trades. The tractability of their CF allows to recover option prices through fast Fourier transform (FFT). Eberlein et al. (1998) showed that the hyperbolic distribution allows an almost perfect fit to financial data, both in spot and derivatives markets. The knowledge of the CF enables to recover the probability distribution through numerical inversion as: $F(x) = \frac{1}{2} + \frac{1}{2\pi}\int_0^\infty \frac{e^{iux}\phi(-u) - e^{-iux}\phi(u)}{iu}du$.

The Appendix describes the estimation of the parameters of an LP using the CF.

IV. Market Incompleteness and Esscher Transform

When uncertainty is modeled by an LP, except when X is a Brownian motion or a Poisson process, the Levy process is an incomplete model. A perfect hedge cannot be obtained and there is always a residual risk which cannot be hedged. In a Levy market, there are many different equivalent martingale measures under which the discounted asset price process is a martingale. The existence of a martingale measure is related to the absence of arbitrage, while the uniqueness of the equivalent martingale measure is related to market completeness, i.e., perfect hedging. A contingent claim can be perfectly hedged if there exists a predictable strategy which can replicate the claim in the sense that there is a dynamic portfolio, investing in a riskless bond and the asset, such that at every time point the value of the portfolio matches the value of the claim. The portfolio must be self-financing. A market is called complete if for every integral contingent claim there exists an admissible self-financing strategy replicating the claim.[10]

[10] The question of completeness is linked with the uniqueness of the martingale measure, which is in turn linked with the mathematical predictable representation property (PRP) of a

(continued…)

One approach for finding an equivalent martingale measure is the Esscher transform proposed by Gerber and Shiu (1994). An Esscher transform of a stock-price process induces an equivalent probability measure on the process. The Esscher parameter is determined so that the discounted price of a security is a martingale under the new probability measure. Let $S(t) = S(0)e^{X(t)}$,[11] where $\{X(t)\}_{t\geq 0}$ is an LP defined on a probability space (Ω, F, P), with stationary and independent increments and $X(0) = 0$. For each t, the random variable $X(t)$, seen as the continuous compounded rate of return over the t periods, has an infinitely divisible distribution with a probability density under P given by $f(x,t)$, $t > 0$. The moment generating function (MGF) is assumed to exist and is defined as $M(u,t) = E[e^{uX(t)}] = \int_{-\infty}^{\infty} e^{ux} f(x,t)dx$. By assuming that $M(u,t)$ is continuous at $t = 0$, it follows from the infinite divisibility that $M(u,t) = [M(u,1)]^t$. Let h be a real number such that $M(h) = \int_{-\infty}^{\infty} e^{hx} f(x)dx$ exists. The Esscher transform (with parameter h) of the original distribution is defined as $f(x;h) = \frac{e^{hx} f(x)}{M(h)}$. The Esscher transform (with parameter h) of the process $\{X(t)\}_{t\geq 0}$ is defined as an LP process with stationary and independent increments, where the new probability density of $X(t)$, $t > 0$, is $f(x,t;h) = \frac{e^{hx} f(x,t)}{\int_{-\infty}^{\infty} e^{hy} f(y,t)dy} = \frac{e^{hx} f(x,t)}{M(h,t)}$.

The corresponding MGF is $M(u,t;h) = \int_{-\infty}^{\infty} e^{ux} f(x,t;h)dx = \frac{M(u+h,t)}{M(h,t)}$ and $M(u,t;h) = [M(u,1;h)]^t$. The parameter h is determined so that the modified probability measure, denoted by Q, is an equivalent martingale measure. The idea is to find $h = h^*$, so that the discounted stock price process $\{e^{-rt}S(t)\}_{t\geq 0}$ is a martingale with respect to the probability measure corresponding to h^*. The martingale condition is:

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martingale. In probability theory a martingale M is said to have the PRP if, for any square-integrable random variable H, we have $H = E[H] + \int_0^T a_s dM_s$ for some predictable process $a = \{a_s, 0 \leq s \leq T\}$. If such a representation exists, the predictable process a_s will give the necessary self-financing admissible strategy.

[11] The asset price can alternatively be specified as a stochastic differential equation: $dS_t = \mu S_{t-}dt + \sigma S_{t-}dX_t$, where S_{t-} is left limit and X_t is an LP. The solution to this equation is the well-known Doleans-Dade or stochastic exponential given by: $S_t = S_0 \exp(\mu t + \sigma X_t)\prod_{s\leq t}(1+\sigma\Delta X_s)e^{-\sigma\Delta X_s}$, with $\Delta X_s = X_s - X_{s-}$ and $(1+\sigma\Delta X_s) > 0$.

$S(0)=E^{Q}[e^{-rt}S(t)]=e^{-rt}E^{Q}[S(t)]$. The parameter h^{*} is a solution to

$$S(0)=E^{Q}[e^{-rt}S(t)]=e^{-rt}E^{Q}[S(0)e^{X(t)}]=e^{-rt}S(0)\frac{E^{P}[e^{(h+1)X(t)}]}{E^{P}[e^{hX(t)}]}=e^{-rt}S(0)\frac{M(u+h,t)}{M(h,t)}.$$

This condition is equivalent to the following equation: $1=e^{-rt}E^{Q}[e^{X(t)}]$, or $e^{rt}=M(1,t;h^{*})$. The solution does not depend on t. Therefore, setting $t=1$, yields $e^{r}=M(1,1;h^{*})$, or in logarithm form: $r=\log[M(1,1;h^{*})]=\log[M(1+h^{*},1)]-\log[M(h^{*},1)]$. The risk-neutral probability measure obtained as an Esscher equivalent measure is given by:

$\frac{dQ}{dP}\,|\,F_{t}=\frac{e^{hX_{t}}}{E(e^{hX_{t}})}=\exp\left(hX_{t}-t\log(M(h))\right)$. The parameter h is the solution to

$r=\log M(h+1)-\log M(h)$.

Let $\phi(u)=E\{\exp(uiX_{1})\}$ denote the CF of X_{1}. Since the MGF is, up to a change of variable $u\leftrightarrow -iu$, a CF, yielding $M(u)=\phi(-iu)$ or equivalently $M(iu)=\phi(u)$, the CF $\phi^{(h)}$ of the Esscher transform measure is given by $\phi^{(h)}(u)=\frac{\phi(u-ih)}{\phi(-ih)}$ and remains infinitely divisible. The condition on MGF yields an identical condition on the CF, namely for the discounted price to be a martingale, the following has to hold: $e^{r}=\frac{\phi(-i(h+1))}{\phi(-ih)}$, or

$r=\log\phi(-i(h+1))-\log\phi(-ih)$. Moreover, if the characteristic triplet of the process X_{t} under P are $(\sigma^{2},\nu(dx),b)$, then they become, under Q, $(\sigma^{(h)2},\nu^{(h)}(dx),b^{(h)})$, with $\sigma^{(h)2}=\sigma^{2}$, $\nu^{(h)}(dx)=\nu^{h^{*}x}(dx)$, and $b^{(h)}=b+h^{*}\sigma^{2}+\int_{\{|x|\le 1\}}x(e^{h^{*}x}-1)\nu(dx)$. Miyahara (2004) showed that Esscher transform could be identified with the minimum entropy martingale measure.

An alternative approach for computing a risk-neutral measure, similar to the Esscher transform, can also be proposed (Carr et al., 2003). Let $(X_{t})_{t\ge 0}$ be a real-valued process with independent increments. Then $\left(\frac{e^{iuX_{t}}}{E[e^{iuX_{t}}]}\right)_{t\ge 0}$ is a martingale $\forall u\in R$. For example, if the asset price S_{t} is modeled as $S_{t}=S_{0}\exp[rt+X_{t}]$ where X_{t} is an LP. The resulting risk-neutral process for the log price is: $\log S(t)=(\log S(0)+rt-\log E[\exp(X(t)])+X(t))$. The CF of the log price is: $E[\exp(iu\log(S(t)))]=\exp(iu((\log S(0)+rt-\log E[\exp(X(t)])E[\exp(iuX(t))]$.

V. Option Pricing Using Characteristic Functions

Characteristic functions were useful in simplifying the complexity of option pricing. Under martingale pricing, the value of an option is a convolution of a discounted pay-off function with the state price density. Using the Feynman-Kac formula, which stipulates that if $f(S,t)$ satisfies a partial integro-differential equation (PIDE), with final condition: $f(S,T)=g(S)$ for all S , then the solution is given by:[12]

$$f(S_t,t)=E_t^Q[g(S_T)\exp\{-\int_t^T r(\varsigma)d\varsigma\}\mid S_t=S]=\int_0^\infty \exp\{-\int_t^T r(\varsigma)d\varsigma\}g(S_T)p(S_T\mid S_t)dS_T$$

The conditional expectation is computed with respect to a risk-neutral transition probability density $p(S_T\mid S_t)$. From Breeden and Litzenberger (1978), the state price density is: $p(S_T\mid S_t)\exp(-\int_t^{t+\tau} r(\varsigma)d\varsigma)=\frac{\partial^2}{\partial K^2}|_{K=S}\ C(t;K,T)$, where $C(t;K,T)$ is the value of a call option.[13] However, for many stochastic processes, particularly Levy processes, the transition densities are often complicated and may not be readily available in closed form. The CF of the underlying stochastic process may be readily available in closed form. Using the CF, it turns out to be much easier to compute option prices as an integral in the Fourier space and interpret this integral as a Parseval identity. Let $\phi(z;t,\tau)$ be the conditional CF of the state price density $\phi(z,t)=\int_0^\infty e^{izS_T}\exp\{-\int_t^T r(\varsigma)d\varsigma\}p(S_T\mid S_t)dS_T$, Lewis (2001) has shown that the option value can be expressed as a convolution in the Fourier space:

$C(S_t,\tau;K)=\frac{e^{-r\tau}}{2\pi}\int_{i\varpi-\infty}^{i\varpi+\infty} e^{-izX}\phi(-z)\hat{g}(z)dz$,[14] where $\ln S_t=X_t+r\tau$, $\hat{g}(z)=\int_{-\infty}^{\infty}\exp(izs)g(s)ds$, $z=u+iw$, and $s_t=\ln S_t$. Being expressed as a complex-valued integral, the option value can thus be computed using residue calculus.

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[12] When the asset price S_t is modeled as an exponential LP, Ito's formula applied to $f(S_t)$ shows that under the risk neutral measure S_t has the following infinitesimal generator:

$$L_t f(x)=\frac{1}{2}c^2\sigma_t^2x^2 f''(x)+r_t x f'(x)+\int_R[f(x+\sigma_t xy)-f(x)-\sigma_t xyf'(x)]\tilde{\nu}_t(dy),$$

where $[\sigma_t,\tilde{\nu}(dx),r_t]$ are the triplet under the risk-neutral measure. Let $f(t,S)$ be the solution of the following Cauchy problem: $\frac{\partial f}{\partial t}+L_t f-r_t f=0$, $f(T,S)=g(S)$. Then $f(t,S)$ admits the representation $f(t,S)=E^Q\left[\exp\left\{-\int_t^T r_\varsigma d\varsigma\right\}g(S_T)\mid F_t\right]$.

[13] The risk-neutral probabilities, discounted at the risk-free rate of interest, are interpreted economically as the prices of Arrow-Debreu (AD) securities, or the state prices. An AD security is a primitive security associated with a particular future state of the world; it pays $1 if that state occurs, and nothing otherwise. All contingent claims and derivatives can be expressed in terms of a portfolio of AD securities and priced accordingly. Given a vector of state prices, the price of any contingent claim may be determined by multiplying the claim's payoff in each state by the corresponding state price, and then summing over all states.

[14] This formula is based on Parseval's identity: $\int_{-\infty}^{\infty} g(S_T)p(S_T)dS_T=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{g}(z)\phi^*(z)dz$,

where $\hat{g}(z)$ is the Fourier transform of $g(S_T)$ and $\phi^*(z)$ is the conjugate of $\phi(z)$.

Carr and Madan (1999) proposed the use of fast Fourier transform (FFT) for pricing options. Let $k = \ln(K)$, where K is the strike price, $C_T(k)$ = value of a $T-$maturity call option with strike K, and $c_T(k) \equiv \exp(\alpha k) C_T(k)$ for $\alpha > 0$, the damped option price for which a Fourier transform exists. This transform is expressed as: $\psi_T(z) = \int_{-\infty}^{\infty} e^{izk} c_T(k) dk$. Carr and Madan (1999) showed that transforms are related as follows: $\psi_T(z) = \dfrac{e^{-rT}\phi_T(z-(\alpha+1)i)}{\alpha^2+\alpha-z^2+i(2\alpha+1)z}$.
Knowledge of the CF of $\phi_T(z-(\alpha+1)i)$, which is the CF of the log of the asset price under the risk neutral-measure, implies knowledge of the Fourier transform of the value of the option. The option price can therefore be computed via Fourier inversion as:

$$C_T(k) = \frac{\exp(-\alpha k)}{2\pi} \int_{-\infty}^{\infty} e^{-izk}\psi(z)dz = \frac{\exp(-\alpha k)}{\pi}\int_{0}^{\infty} e^{-izk}\psi(z)dz .$$

The Fourier inversion can be approximated discretely via an N-point sum with a grid spacing of Δ in the Fourier domain. The inversion integral can be approximated using an integration rule, such as Simpson's or the trapezoidal rule, as $\int_0^{\infty} e^{-ixz}\psi(z)dz \approx \sum_{j=0}^{N-1} e^{-i(\frac{2\pi}{N})xz_j}\tilde{\psi}_j\Delta$.
The points z_j are equidistant with grid spacing Δ, $z_j = j\Delta$. The value of Δ should be sufficiently small to approximate the integral well enough, while the value of $N\Delta$ should be large enough to assume the CF is equal to zero for $z > \bar{z} = N\Delta$. In general, the values $\tilde{\psi}_j$ are set equal to $\tilde{\psi}_j = \psi(z_j)w_j$, where w_j are the weights of the integration rule. Appropriate values of the coefficient α are chosen to ensure the boundedness of the truncation error.

VI. Application to Crude Oil Options: The Inverse Problem

An application of the above analysis to crude oil options is undertaken in this section with the objective of estimating, from observed options' market values, density forecast for crude oil prices at a given maturity date. The estimation of the risk-neutral distribution is known as the inverse problem in option pricing models. While the pricing problem is concerned with computing values of options given model's parameters, the inverse problem consists of backing out the parameters describing risk-neutral dynamics from observed prices. The inverse problem is also known as model calibration, whereby parameters are extracted from observed market values for the options. In accordance with the above analysis, crude oil prices are assumed to follow an exponential Levy process with triplet $(\sigma(\Phi), \nu(\Phi), \gamma(\Phi))$ under the risk-neutral measure and Φ denotes the parameters of distribution. The calibration procedure is based on minimizing the quadratic pricing error: $\hat{\Phi} = \arg\min_{\Phi} \dfrac{1}{N}\sum_{i=1}^{N}\left(C^{\Phi}(T,K_i) - C_i(T,K_i)\right)^2$, under the put-call parity constraint: $S_t + Put_t - Call_t = Ke^{-r(T-t)}$, where C^{Φ} denotes the call option computed for the exponential

Levy model with triplet $(\sigma(\Phi), \nu(\Phi), \gamma(\Phi))$, C_i denotes the observed prices of call options for maturity T and strikes K_i. The put option values are the observed market values.[15]

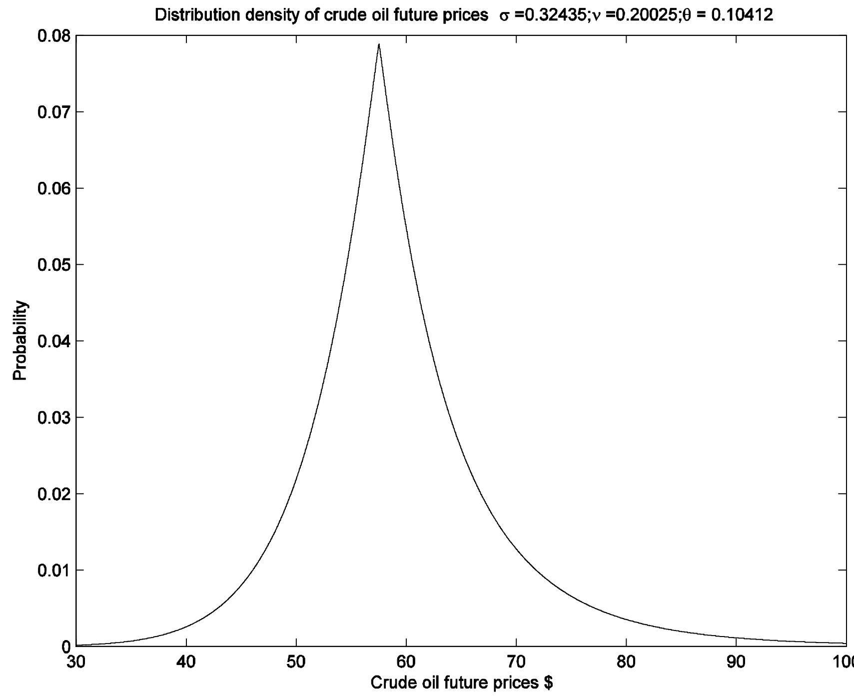
The calibration exercise is applied to the Variance-Gamma model. The data set consists of observed crude oil options on July 5, 2005 for maturity end-September 2005; the risk-free interest was the three-month U.S. treasury bill rate; and the crude futures price was US$ 59.81 per barrel. The estimation yielded the following parameters: $\hat{\sigma}^2 = 0.32, \hat{\upsilon} = 0.20$ $\hat{\theta} = 0.10$. The estimated $\hat{\sigma}^2$ indicates high volatility characterizing the oil market; $\hat{\upsilon}$ shows fat tails, implying higher probability than the normal distribution for important deviations of prices from the futures level; finally, $\hat{\theta}$ indicates positive skewness, meaning that the market was according higher probability for upward deviations from the expected mean (Figure 1). The robustness of the results is confirmed by the use of the following linear model: $A = D.q$, where A is a vector of call and put options prices, D is a pay-off matrix, and q is a vector of Arrow-Debreu prices. Owing to the high volatility, market expectations can change dramatically during intra-day trading or from one day to the other and thus can change dramatically the density forecast for a given maturity time. For this reason, the calibration results need to be interpreted with caution.

These findings support the asset's view to crude oil markets. Traders in derivatives markets are hedgers, arbitrageurs, and speculators. Many types of investors participate in the crude oil futures market, including speculative and non speculative traders. The latter group includes institutional investors (e.g., pension funds) who seek to diversify their portfolios with less correlated assets, whereas the former group includes hedge funds and commercial entities registered with the Commodity Futures Trading Commission (CFTC). The aggregate of all large-traders' position reported to the CFTC usually represents 70-90 percent of total open interests in any given derivatives market.[16] Commercial traders occasionally take speculative short-term positions during periods of large price swings. High volatility and volatility clustering increase the speculative activity and add pressure on futures prices. Furthermore, very low interest rates reduce considerably the cost of shorting bonds as well as the cost of margin requirements and increase the volume of activity in the futures market. For instance, increased demand for long contracts would exert an upward pressure on futures prices. A significant portion in crude oil price increase could be attributed to derivatives markets and to the role of crude oil as an asset rather than as a commodity.

[15] Cont and Tankov (2004) argued that the calibration problem could be an ill-posed problem and proposed the use of relative entropy, which is the Kullback-Leibler distance for measuring the proximity of two equivalent probability measures, as a regularization method with the prior distribution estimated from the statistical data via the maximum likelihood method. This regularization will enable to find a unique martingale measure.

[16] Data for February 1, 2005 indicated that commercial traders held 67.1 percent of the open long positions and 69.2 percent of the short positions in crude oil futures on the New York Mercantile Exchange (NYMEX).

Figure 1. Crude oil prices density forecast on July, 5, 2005 for end-September 2005.



Subsequently, the forecast of oil prices relying only on the role of crude oil as a commodity would certainly omit the significant impact of derivatives markets. The use of Levy processes and their corresponding inverse problem would allow one to study the role of asset markets in the behavior of crude oil prices. The commodity aspect is also important. To the extent that demand is acting against a short-term fixed crude oil supply and bottlenecks in refining and distribution capacity, it causes frequent and large jumps in crude oil prices and suggests the use of Levy processes for modeling these jumps.

VII. CONCLUSIONS

The paper has addressed option pricing models from the perspective of Levy processes, which offer better tools for analyzing skewness, fat tails, and stochastic volatility in high-frequency financial data than the classical diffusions or jump-diffusion models. The concept of subordination plays an important role in building an LP and amounts to measuring returns in relation to the level of activity and news, instead of calendar time. High level of activity or important news may cause higher volatility in returns. The Normal Inverse Gaussian, the Variance-Gamma, and the General Hyperbolic motions are Levy processes which are time-changed Brown motions and share the property of being pure jump and infinite activity models. Their empirical performance in modeling skewness, leptokurtosis, and the implied volatility smile in option prices was deemed consistent with data. Levy processes, however, lead to incomplete markets and an infinite number of martingale measures that are compatible with no arbitrage. The Esscher measure constitutes a procedure, among many others, for obtaining a martingale measure. The role of characteristic functions in option pricing has become prominent, particularly in the context of processes that do not have easily available distribution functions. Fourier transforms offer an efficient tool for option pricing when CFs are available in closed forms.

The paper has addressed the inverse problem and attempted to extract a risk-neutral distribution from crude oil options. The results indicate that market expectations were positively skewed, namely the market put a higher probability mass on crude oil prices remaining above the futures' level. This outcome is in conformity with the sustained pressure on oil prices in the recent past. The Levy market model described in this paper is highly relevant to the work of the Fund. It provides an adequate tool for analyzing high-frequency data, gauge market sentiment, and design appropriate policy responses. The findings of the calibration could be relevant for policymaking. This may require assessing factors causing pressure on crude oil demand, including low interest rates and depreciating currencies, and seeking greater energy efficiency and inter-energy substitution. They also point to the importance of derivatives markets in influencing crude oil prices. A high speculative activity associated with high volatility in futures prices would lead to volatility clustering and hence greater uncertainty in crude oil futures prices. Energy modeling would need therefore to look at the role of crude oil as an asset besides its role as a commodity. The latter aspect remains important. To the extent that demand pressure is acting against a short-term fixed crude oil supply and bottlenecks in the refining and distribution capacity, frequent and sizable jumps in crude oil prices will take place. Levy processes and their corresponding inverse problems would provide a framework for assessing both the asset and commodity aspects in crude oil prices behavior.

Empirical Characteristic Function and Estimation in the Frequency Domain

The lack of a tractable form of the probability density function makes estimation via maximum likelihood of the parameters of the distribution extremely difficult. Moreover, the likelihood function can be unbounded over the parameter space. Consequently, alternative methods, based on the characteristic function, were proposed, (e.g., Parzen (1962), and Feuerverger and McDunnough (1981a and 1981b)), to deal with inference problems involving such distribution. Being a Fourier transform of the probability density function the characteristic function (CF) is always bounded; it can have a closed form expression; and it retains all the information in the sample. The basic idea of the estimation in the frequency domain, called also the empirical characteristic function (ECF) procedure, is to match the CF derived from the model and the ECF obtained from the data. Because the minimization of the distance between the ECF and CF over a grid of points in the Fourier domain is equivalent to matching a finite number of moments, the ECF method is in essence equivalent to the Generalized Method of Moments (GMM). Feuerverger (1990) proves that, under some regularity conditions, the resulting estimate can be made to have arbitrarily high asymptotic efficiency provided that the sample of observations is sufficiently large and the grid of points is sufficiently fine and extended.

Suppose $X_1,....,X_n$ are i.i.d. realization of the same variable X with density $f(x;\theta)$ and distribution function $F_\theta(x)$. The parameter $\theta \in R^l$ is the parameter of interest with true value θ_0. We wish to estimate θ from a realization $\{X_1, X_2,..., X_n\}$. Define the theoretical CF as : $\phi_\theta(u) = \int e^{iux} dF_\theta(x)$ and its empirical counterpart (ECF) as

$$\phi_n(u) = \int e^{iux} dF_n(x) = \frac{1}{n}\sum_1^n \exp(iuX_j) = \frac{1}{n}\sum_{j=1}^n \cos(uX_j) + i\frac{1}{n}\sum_{j=1}^n \sin(uX_j)$$

The focus here is on moment conditions of the form: $h(u, X_j;\theta) = \exp(iuX_j) - \phi(u,\theta)$. Obviously, $E(h(u, X_j;\theta_0)) = 0$, $\forall u$. Suppose q discrete points $u_1, u_2,....,u_q$ in the Fourier space are used and define

$$m(X_j;\theta) = \left(\mathrm{Re}\left[h(u_1, X_j;\theta)\right],....,\mathrm{Re}\left[h(u_q, X_j;\theta)\right], \mathrm{Im}\left[h(u_1, X_j;\theta)\right],....,\mathrm{Im}\left[h(u_q, X_j;\theta)\right]\right)'.$$

By construction $E(m(X_j;\theta_0)) = 0$. This forms $2q$ (usually larger than l) moment conditions. Evaluating the ECF and the CF at the grid points yields:

$V_n = \left(\mathrm{Re}[\phi_n(u_1)],....,\mathrm{Re}[\phi_n(u_q)], \mathrm{Im}[\phi_n(u_1)],....,\mathrm{Im}[\phi_n(u_q)]\right)'$ and

$V_\theta = \left(\mathrm{Re}[\phi(u_1;\theta)],....,\mathrm{Re}[\phi(u_q;\theta)], \mathrm{Im}[\phi(u_1;\theta)],....,\mathrm{Im}[\phi(u_q;\theta)]\right)'$. Obviously:

$\frac{1}{n}\sum_{j=1}^n m(X_j;\theta) = V_n - V_\theta$. From Feuerverger and McDunnough (1981a, Theorem 2.1) we have that $\sqrt{T}(V_T - V_\theta)$ converges in distribution to a $2q-$dimensional normal distribution with zero mean and covariance matrix: $\Omega = \begin{pmatrix} \Omega_{RR} & \Omega_{RI} \\ \Omega_{IR} & \Omega_{II} \end{pmatrix}$

where the elements in the partitions associated with u_j and u_k are given by:

$$\left(\Omega_{RR}\right)_{jk} = \frac{1}{2}\left(\operatorname{Re}\phi(u_j+u_k)+\operatorname{Re}\phi(u_j-u_k)\right)-\operatorname{Re}\phi(u_j)\operatorname{Re}\phi(u_k)$$

$$\left(\Omega_{RI}\right)_{jk} = \frac{1}{2}\left(\operatorname{Im}\phi(u_j+u_k)-\operatorname{Im}\phi(u_j-u_k)\right)-\operatorname{Re}\phi(u_j)\operatorname{Im}\phi(u_k)$$

$$\left(\Omega_{II}\right)_{jk} = \frac{1}{2}\left(\operatorname{Re}\phi(u_j-u_k)-\operatorname{Re}\phi(u_j+u_k)\right)-\operatorname{Im}\phi(u_j)\operatorname{Im}\phi(u_k)\,.$$

The ECF method estimates θ by finding the θ that minimizes $(V_T-V_\theta)'\hat{\Omega}^{-1}(V_T-V_\theta)$, where $\hat{\Omega}$ is a consistent estimator of Ω. This procedure can be thought of as the second stage GMM estimation or the non-linear regression of V_n on V_θ and hence yields GMM efficient estimators. The asymptotic properties of the ECF estimator $\hat{\theta}$ have been examined by Feuerverger and McDunnough (1981a). The basic result is that $\hat{\theta}$ is strongly consistent and asymptotically normal with covariance $\frac{1}{n}\left[\left(\frac{\partial V_\theta}{\partial\theta}\right)'\hat{\Omega}^{-1}\left(\frac{\partial V_\theta}{\partial\theta}\right)\right]^{-1}$. The asymptotic efficiency of the ECF procedure depends essentially on the choice of $\left\{u_j\right\}$. Feuerverger and McDunnough (1981a) argued that in some cases one can obtain full asymptotic efficiency of the procedure (in terms of achieving the Cramer-Rao lower bound) by selecting the grid of points $\left\{u_j\right\}$ to be sufficiently fine and extended.

Besides being a GMM method, the ECF procedure is also equivalent to the maximum likelihood method. The likelihood equation is $\int \frac{\partial \log f_\theta(x)}{\partial\theta} dF_n(x)=0$, or alternatively $\int \frac{\partial \log f_\theta(x)}{\partial\theta} d[F_n(x)-F_\theta(x)]=0$. This last equation may be transformed. Using Parseval identity, we obtain a Fourier domain version of the likelihood equation: $\int w_\theta(u)[\phi_n(u)-\phi_\theta(u)]du=0$, where $w_\theta(u)=\frac{1}{2\pi}\int \exp(-iux)\frac{\partial \log f_\theta(x)}{\partial\theta}$, the Fourier transform of the score function $\frac{\partial \log f_\theta(x)}{\partial\theta}$, is the optimal weight. Obviously, when the likelihood function has no closed form expression, the optimal weight is unknown.

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