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Asset Returns in the Long Run

Ian Martin*

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Abstract

The fundamental equation of asset pricing states that the expected time- and risk-adjusted cumulative return on any asset equals one at all horizons. This paper arrives, via a theorem of Kakutani, at an apparently paradoxical result: for a typical asset, the realized time- and risk-adjusted cumulative return tends to zero with probability one. As a special case, this result strengthens the familiar fact that the growth-optimal portfolio outperforms other assets at long horizons. The apparent paradox is resolved by a further result, which shows that the long-run value of a non-growth-optimal asset is driven by the possibility of extremely good news at the level of the individual asset or extremely bad news at the aggregate level.

*ian.martin@stanford.edu; http://www.people.fas.harvard.edu/∼iwmartin/. First draft: 14 April, 2007. I thank Brandon Bates, John Campbell, John Cochrane, and Lars Hansen for their comments.

The fundamental equation of asset pricing states that the expected time- and risk-adjusted cumulative return on any asset equals one at all horizons. Working in a rather general setting, this paper arrives, via a theorem of Kakutani, at an apparently paradoxical result: for a typical asset, the realized time- and risk-adjusted cumulative return tends to zero with probability one.

Consider the fundamental asset-pricing equation,

$$\mathbb{E}_t M_{t+1} R_{t+1} = 1 \,.$$

I have introduced a stochastic discount factor, M_{t+1}, that prices payoffs at time $t+1$ from the perspective of time t. R_{t+1} is the gross return, from time t to $t+1$, on some arbitrary asset.

The objects of interest in this paper will be the martingale $X_t \equiv M_1 R_1 \cdots M_t R_t$, and the random variable $X_\infty \equiv \lim_{t\to\infty} X_t$. The asset-pricing equation states that $\mathbb{E}X_t = 1$ for all finite t, so it is natural to expect that $\mathbb{E}X_\infty = 1$, too. In Section 1, I show that this may or may not be true; typically, in fact, it is not, and when it is not, $X_\infty = 0$.[1] This dichotomy, together with a diagnostic that determines which of the two cases applies, is the main result of the paper.

The result applies to any valid stochastic discount factor, but to understand it better I consider, in Section 2, the special case in which the stochastic discount factor is the reciprocal of the return on the growth-optimal portfolio. The main result then establishes the familiar observation that the return on the growth-optimal portfolio is, in the long run, arbitrarily larger than the return on a non-growth-optimal[2] asset with probability arbitrarily close to one. I require weaker assumptions than were made by Latané (1959), Samuelson (1971), and Markowitz (1976). Of greater interest,

[1]This statement holds with probability one, or almost surely. Throughout the paper, in the interest of readability, limits of random variables are always almost-sure limits unless otherwise noted.

[2]In a sense to be made precise below.

though, is the fact that I provide a generalization of this traditional result to allow for stochastic discount factors other than the reciprocal of the return on the growth-optimal portfolio. If markets are incomplete, there may be many different stochastic discount factors, and we may wish to think in terms of, say, the marginal utility of wealth, or of consumption.

In Section 3, I consider this more general setting in which M_t is any valid stochastic discount factor. How are we to square the fact that $\mathbb{E}X_t = 1$ with the fact that, for most assets, we have $X_\infty = 0$? I show that such assets derive their value—their $\mathbb{E}X_t = 1$—from low-probability events in which the realized value of X_t is enormous. When such an event happens, either $M_1 \cdots M_t$ is large or $R_1 \cdots R_t$ is large, or both. The former possibility, which is driven by extreme *left*-tail events, can be thought of as representing the importance of rare disasters; this interpretation becomes particularly clear when considering the riskless strategy in which $R_1 \cdots R_t$ is deterministic. The latter possibility represents the importance of the extreme *right* tail of the distribution of returns on the asset in question, and is particularly clear in a risk-neutral world in which $M_1 \cdots M_t$ is deterministic.

Section 4 conducts simulations that illustrate the preceding results in the context of a simple economy featuring two assets—one riskless, the other i.i.d. lognormal. Section 5 concludes.

1 The main result

Time is discrete; today is time 0. I make three assumptions:

(i) There is no arbitrage.

(ii) The asset of interest has limited liability.

(iii) All random variables are independent across periods.

Assumption (i) implies that for all $t \geq 1$ we can define M_t to be a stochastic discount factor which prices payoffs at time t from the perspective of time $t-1$, and then we have

$$M_t > 0 \quad \text{and } \mathbb{E}\left(M_1 R_1 \cdot M_2 R_2 \cdot \ldots \cdot M_t R_t\right) = 1\,. \tag{1}$$

Assumption (ii) implies that for any $t \geq 1$

$$R_t \geq 0\,, \tag{2}$$

where R_t is the gross realized return from time $t-1$ to time t on some arbitrary asset or investment strategy. M_t and R_t are random variables that only become known at time t.

To simplify notation, define the random variables X_t, $t = 1, 2, \ldots$, by

$$X_t \equiv M_1 R_1 \cdot M_2 R_2 \cdot \ldots \cdot M_t R_t\,,$$

so $\mathbb{E} X_t = 1$. X_t is a non-negative martingale, because

$$\begin{aligned}
\mathbb{E}_{t-1} X_t &= \mathbb{E}_{t-1}\left(M_1 R_1 \cdots M_t R_t\right) \\
&= M_1 R_1 \cdots M_{t-1} R_{t-1}\, \mathbb{E}_{t-1}\left(M_t R_t\right) \\
&= M_1 R_1 \cdots M_{t-1} R_{t-1} \\
&= X_{t-1}\,.
\end{aligned}$$

Two definitions will be convenient. First, let

$$X_\infty \equiv \lim_{t\to\infty} X_t = \lim_{t\to\infty} M_1 R_1 \cdot M_2 R_2 \cdot \ldots \cdot M_t R_t\,.$$

As X_t is a non-negative martingale, the random variable X_∞ exists almost surely by the martingale convergence theorem.

Since, for any finite t,

$$\mathbb{E} X_t = 1,$$

it is tempting—but wrong—to conclude that in general

$$\mathbb{E}X_\infty = \mathbb{E}\lim_{t\to\infty} X_t \stackrel{?}{=} \lim_{t\to\infty}\mathbb{E}X_t = \lim_{t\to\infty} 1 = 1.$$

The interchange of expectation and limit is the weak link in this chain, as demonstrated by

Proposition 1. *Under assumptions (i)–(iii), either*

$$\sum_{t=1}^{\infty} var\sqrt{M_t R_t} < \infty \quad and \quad \mathbb{E}X_\infty = 1 \tag{3}$$

or

$$\sum_{t=1}^{\infty} var\sqrt{M_t R_t} = \infty \quad and \quad X_\infty = 0\,. \tag{4}$$

Proof. I adapt the proof of Kakutani's (1948) product martingale theorem in Williams (1995, pp. 144–5).

Absence of arbitrage implies that $\mathbb{E}M_t R_t = 1$, so by Jensen's inequality, $\mathbb{E}\sqrt{M_t R_t} \leq 1$; it also implies that $\mathbb{E}\sqrt{M_t R_t} > 0$. For notational convenience, write $a_t = \mathbb{E}\sqrt{M_t R_t}$; so we have $a_t \in (0, 1]$.

First, suppose that $\sum \operatorname{var}\sqrt{M_t R_t} < \infty$. As $\mathbb{E}M_t R_t = 1$, this is equivalent to $\sum (1 - a_t^2) < \infty$. By Lemma 1 (a standard result proved, for convenience, in the Appendix) this implies $\prod a_t^2 > 0$; hence also $\prod a_t > 0$. Now, define a new martingale

$$Y_t = \frac{\sqrt{M_1 R_1}}{a_1}\frac{\sqrt{M_2 R_2}}{a_2}\cdots\frac{\sqrt{M_t R_t}}{a_t}.$$

We have $\mathbb{E}Y_t^2 = 1/(a_1 a_2 \cdots a_t)^2 \leq 1/\left(\prod a_s\right)^2 < \infty$, so the martingale Y_t is uniformly bounded in second moment. We then have

$$\mathbb{E}\left(\sup_t X_t\right) \leq \mathbb{E}\left(\sup_t Y_t^2\right) \leq 4\sup_t \mathbb{E}\left(Y_t^2\right) < \infty.$$

(The second inequality is Doob's $\mathscr{L}^2$ inequality—see Williams (1995, pp. 143–4).) That is, the random variable $\sup_t X_t$ is integrable. Since $\sup_t X_t$ dominates X_t, it follows that X_t is uniformly integrable, and so $\mathbb{E}X_\infty = 1$.

Alternatively, suppose that $\sum \operatorname{var} \sqrt{M_t R_t} < \infty$. Then $\prod a_t = 0$. Defining Y_t as before, Y_t is a non-negative martingale, and so has an almost-sure limit, Y_∞. But since $Y_\infty = \sqrt{X_\infty}/\prod a_t$, and $\prod a_t = 0$, it must be the case that $X_\infty = 0$. $\square$

To get some intuition for the result, note that (3) only prevails if the variance of $\sqrt{M_t R_t}$ declines rapidly to zero as $t \to \infty$: in other words if $M_t R_t$ is roughly constant for large t. The following result makes this idea formal.

Proposition 2. $\mathbb{E}X_\infty = 1$ *can only occur if* $M_t R_t \to 1$ *as* $t \to \infty$*. That is, the case* $\mathbb{E}X_\infty = 1$ *can only prevail if*

(i) the period return, R_t*, tends to the growth-optimal return, and*

(ii) the period SDF, M_t*, tends to the reciprocal of the growth-optimal return.*

Proof. By Chebyshev's inequality, we have, for arbitrary $\varepsilon > 0$,

$$\mathbb{P}\left(\left|\sqrt{M_t R_t} - \mathbb{E}\sqrt{M_t R_t}\right| > \varepsilon\right) \leq \frac{\operatorname{var}\sqrt{M_t R_t}}{\varepsilon^2}.$$

If $\mathbb{E}X_\infty = 1$, Proposition 1 tells us that $\sum \operatorname{var} \sqrt{M_t R_t} < \infty$; combining these facts, we have

$$\sum_{t=1}^{\infty} \mathbb{P}\left(\left|\sqrt{M_t R_t} - \mathbb{E}\sqrt{M_t R_t}\right| > \varepsilon\right) < \infty. \tag{5}$$

It follows from (5) and the first Borel-Cantelli lemma that

$$\sqrt{M_t R_t} - \mathbb{E}\sqrt{M_t R_t} \to 0. \tag{6}$$

Next, I show that $\mathbb{E}\sqrt{M_t R_t} \to 1$. Suppose, for a contradiction, that this is not the case. Then, since $a_t \equiv \mathbb{E}\sqrt{M_t R_t} \leq 1$, we must have $\prod a_t = 0$, and hence also $\prod a_t^2 = 0$.

It follows from Lemma 1 that $\sum(1-a_t^2)=\infty$. Equivalently, $\sum \operatorname{var}\sqrt{M_t R_t}=\infty$. But this contradicts our starting assumption, so we must indeed have

$$\mathbb{E}\sqrt{M_t R_t}\to 1. \tag{7}$$

Combining (6) and (7), we have $\sqrt{M_t R_t}\to 1$, and hence also $M_t R_t\to 1$, as required.

For the second statement, observe that for a general SDF M and return R, if we have $MR=1$ then R is the growth-optimal return, since—applying Jensen's inequality to $\mathbb{E}MR=1$—we have $\mathbb{E}\log R\le\mathbb{E}\log(1/M)$, with equality if and only if $R=1/M$. □

We also have a third result that throws further light on the properties of $\{X_t\}$ in the two cases:

Proposition 3. *If $\mathbb{E}X_\infty=1$, then $\mathbb{E}\sup X_t<\infty$. If $X_\infty=0$, then $\mathbb{E}\sup X_t=\infty$ (and see also the new result I've added to the later proposition).*

Proof. The first statement was shown in the course of the proof of Proposition 1. The second statement follows because if $\mathbb{E}\sup X_t$ were *not* infinite, then X_t could be bounded by the integrable random variable $\sup X_t$. It would then follow that X_t was a uniformly integrable martingale, and so $\mathbb{E}X_\infty=1$, a contradiction. □

And a fourth:

Proposition 4. *If $X_\infty=0$, then we have*

$$\mathbb{E}\left[X_t\log\left(1+X_t\right)\right]\to\infty\quad as\quad t\to\infty\,, \tag{8}$$

even though, of course,

$$\mathbb{E}X_t=1\quad \textit{for all } t.$$

Proof. Write something here. . . □

Equation (8) is to be contrasted with the starting observation that $\mathbb{E}X_t = 1$ for all t. The important feature of the equation is that $\log(1 + X_t)$ grows *very* slowly with X_t. The fact that the expectation in (8) tends to infinity expresses a sense in which X_t is *enormous* in some states of the world.

I now turn to the interpretation of these results. [****What follows is in urgent need of an overhaul... Should mention that it generalizes the Samuelson et al results, but only briefly. Then move on to discuss implications for more interesting, non-inverse-growth-optimal SDFs. I think it generalizes the paper of Chamberlain and Wilson (2000) showing that consumption goes to infinity in a variety of situations. Also related to Weitzman's papers on long run rates. Important feature is that we can choose the SDF and return in question optimally to get strong and interpretable results; eg marginal utility is much easier to think about than inverse of return on growth-optimal portfolio... Two leading cases that will often be useful: $R_t = R_{f,t}$ or $R_t = R_t^*$]

2 The growth-optimal portfolio

As a first step, it is helpful to consider a particular SDF: the reciprocal of the growth-optimal return, $R_t^* = \arg\max_{R_t} \mathbb{E} \log R_t$. To see that this *is* an SDF, suppose that there are N assets with returns $R_t^{(i)}$, $i = 1, \ldots, N$. The growth-optimal portfolio is obtained by picking $\alpha_i, i = 1, \ldots, N$ to solve

$$\max_{\{\alpha_i\}} \mathbb{E} \log \sum \alpha_i R_t^{(i)} \quad \text{s.t.} \quad \sum \alpha_i = 1 \,.$$

The first-order conditions are that, for each i,

$$\mathbb{E} \frac{R_t^{(i)}}{\sum \alpha_j R_t^{(j)}} = \lambda \,.$$

Multiplying both sides of this equation by α_i and summing over i, we find $\lambda = 1$, so

$$\mathbb{E}\, \frac{R_t^{(i)}}{\sum \alpha_j R_t^{(j)}} = 1 \quad \text{for all } i\,,$$

which exhibits $1/\sum \alpha_j R_t^{(j)} = 1/R_t^*$ as a valid SDF.

[CONTINUE EDITING FROM HERE!!!!] If markets are complete, the stochastic discount factor is unique and, necessarily, $M_t = 1/R_t^*$. In the incomplete market case, the main result applies to any valid stochastic discount factor, but we can get further insight into Proposition 1 by choosing to focus on the stochastic discount factor $1/R_t^*$. Doing so, we have

$$X_T = \frac{R_1 \cdot R_2 \cdot \ldots \cdot R_T}{R_1^* \cdot R_2^* \cdot \ldots \cdot R_T^*}\,,$$

so in this case X_T has a simple interpretation as the *relative performance* of the asset in question by comparison with the G-OP.

We can conclude the section by rephrasing Proposition 1 as follows.

Proposition 5. *Suppose that assumptions (i)–(iii) hold, and define* $a_t = \mathbb{E}\sqrt{R_t/R_t^*}$. *Either*

$$\prod_{t=1}^{\infty} a_t > 0 \quad \textit{and} \quad \mathbb{E}\left[\lim_{T\to\infty} \frac{R_1 \cdot R_2 \cdot \ldots \cdot R_T}{R_1^* \cdot R_2^* \cdot \ldots \cdot R_T^*}\right] = 1 \tag{9}$$

or

$$\prod_{t=1}^{\infty} a_t = 0 \quad \textit{and} \quad \lim_{T\to\infty} \frac{R_1 \cdot R_2 \cdot \ldots \cdot R_T}{R_1^* \cdot R_2^* \cdot \ldots \cdot R_T^*} = 0\,. \tag{10}$$

Proof. Follows from Proposition 1, after setting $M_t = 1/R_t^*$. □

2.1 Examples

Example 1: the G-OP. Suppose the asset in question *is* the G-OP. Then R_t/R_t^* is trivially equal to 1, so

$$a_t = 1 \quad \text{and hence} \quad \prod_{t=1}^{\infty} a_t = 1\,.$$

The growth-optimal portfolio is the archetypal Type 1 asset with $\mathbb{E}X_\infty = 1$.

Example 2: an i.i.d. world. Fix any asset other than the G-OP. Since the world is i.i.d., a_t equals a, some constant. Since the asset in question is not the G-OP and the world is nondeterministic, the mean-1 random variable $M_t R_t$ is nonconstant, so a Jensen's inequality argument delivers the *strict* inequality $a < 1$. It follows that

$$\prod_{t=1}^{\infty} a_t = \prod_{t=1}^{\infty} a = 0\,.$$

Any fixed asset which is not the growth-optimal portfolio is of Type 2: $X_\infty = 0$ and equation (10) holds.

Example 3: an i.i.d. risk-neutral world. In this case, the stochastic discount factor is deterministic, so the G-OP has deterministic returns and so must be invested in the riskless asset. As a result of the previous example, we can say that any strategy which invests in the same risky asset each period must eventually have returns which satisfy

$$\frac{R_1 \cdot R_2 \cdot \dots \cdot R_T}{R_{f,1} \cdot R_{f,2} \cdot \dots \cdot R_{f,T}} < \varepsilon \tag{11}$$

where $R_{f,t}$ indicates the riskless rate from time $t-1$ to time t. The realized return on any risky asset is ultimately negligible by comparison with the riskless return.

Example 4: eventually-growth-optimal strategies. Consider the strategy of investing in arbitrary fashion until some fixed finite time T' and then investing in the G-OP. Such a strategy is *eventually-growth-optimal.* Since $a_t = 1$ for t larger than T', we have

$$\prod_{t=1}^{\infty} a_t = \prod_{t=1}^{T'} a_t > 0$$

and so eventually-growth-optimal strategies are of Type 1, and satisfy $\mathbb{E}X_\infty = 1$.

Example 5: fixed strategies that invest in the G-OP infinitely often. A trading strategy which invests in the G-OP infinitely often may nonetheless be of Type 2. Suppose, for example, that the strategy invests in the G-OP during time periods 1, 3, 5, . . . , and in some other i.i.d. asset during time periods 2, 4, 6, . . .; write a for the value of a_t during these even periods and note that $a < 1$ by Jensen's inequality. We have

$$\prod_{t=1}^{\infty} a_t = \prod_{t=1}^{\infty} a_{2t} = \prod_{t=1}^{\infty} a = 0$$

and so $X_\infty = 0$.

Example 6: strategies that are never growth-optimal but which satisfy $\mathbb{E}X_\infty = 1$. If a trading strategy becomes increasingly similar to the G-OP over time, it may be possible to sustain the case $\mathbb{E}X_\infty = 1$. Suppose for example that we have

$$a_t = 1 - 1/t^2$$

for all t. It follows that $\sum_{t=1}^{\infty}(1-a_t) < \infty$ and this condition implies that $\prod_{t=1}^{\infty} a_t > 0$.

2.2 Relationship with previous results

Various authors have obtained results similar to Proposition 5. Latané (1959) and Samuelson (1971) assume that the world is i.i.d., and rely on the Weak Law of Large Numbers and the Central Limit Theorem respectively. They show that

$$\mathbb{P}\left[\frac{R_1 \cdot R_2 \cdot \cdots \cdot R_T}{R_1^* \cdot R_2^* \cdot \cdots \cdot R_T^*} < 1\right] \longrightarrow 1 \text{ as } T \to \infty. .$$

This conclusion is weaker than the conclusion presented above for three reasons. First, the result holds only *at* the time horizon T, and gives no guarantee about what happens thereafter. Second, the result shows only that the G-OP outperforms, rather than that it overwhelmingly outperforms. Third, the result holds with probability approaching one, rather than with probability equal to one.

Markowitz (1976) also assumes that the world is i.i.d., and shows that the Strong Law of Large Numbers delivers the conclusion of Proposition 5,[3]

$$\lim_{T\to\infty} \frac{R_1 \cdot R_2 \cdot \ldots \cdot R_T}{R_1^* \cdot R_2^* \cdot \ldots \cdot R_T^*} = 0\,.$$

[3] In fact, under the i.i.d. assumption of Markowitz's paper, a stronger result can be obtained,

Each result, weak or strong, can be derived by applying the appropriate Law of Large Numbers, Weak or Strong, to the random variables $\log M_t + \log R_t$, which are i.i.d. by assumption and which have mean $\mathbb{E} \log M_t + \log R_t \equiv \mu < 0$ by Jensen's inequality.

Breiman (1960) does not require that random variables are independent across time. But he does assume that returns are bounded away from zero and from infinity, thereby ruling out lognormality of returns or the possibility of bankruptcy, for example. Given this boundedness assumption, he shows that Proposition 5 holds if a condition equivalent to

$$\prod_{t=1}^{\infty} e^{\mathbb{E}_{t-1} \log(R_t/R_t^*)} = 0$$

holds.

3 Where's the value in a Type 2 asset?

Who would buy a Type 2 asset, if a dollar placed in the growth-optimal portfolio will outperform it, in the long run, with probability one? Why aren't such assets *cheaper*?

Fix, for the sake of argument, some particular Type 2 asset. How are we to square the fact that X_t tends to zero with the fact that $\mathbb{E}X_t = 1$ for all finite t? It seems intuitively clear that there must be some unlikely states of the world in which X_t is *very large*, and that the value of the Type 2 asset in question is driven by these unlikely states of the world.

The following Proposition makes this idea formal.

namely that

$$\lim_{T\to\infty} \left(\frac{R_1 \cdot R_2 \cdot \ldots \cdot R_T}{R_1^* \cdot R_2^* \cdot \ldots \cdot R_T^*} \right)^{1/T} < 1 \,,$$

where R_t is the return on a fixed non-growth-optimal asset.

Proposition 6. *For Type 1 assets, we have*

$$\mathbb{E} \sup_{t \geq 1} X_t < \infty \,, \tag{12}$$

while for Type 2 assets, we have

$$\mathbb{E} \sup_{t \geq 1} X_t = \infty \quad \textit{and} \quad \sup_{t \geq 1} \mathbb{E}\left(X_t \log^+ X_t\right) = \infty \quad \textit{and} \quad \lim_{t \to \infty} \mathbb{E}\left[X_t \log\left(1 + X_t\right)\right] = \infty \,, \tag{13}$$

where $\log^+(x) \equiv \max\{\log x, 0\}$.

Proof. Equation (12) is established in the course of the proof of Kakutani's product martingale theorem in Williams (1995, pp. 144–5). (It leads to the conclusion that for Type 1 assets, the family $\{X_t\}$ is uniformly integrable,[4] from which Proposition 1 follows.)

Both parts of (13) can be established by contradiction. If $\mathbb{E} \sup_t X_t < \infty$ then it would follow that the family of random variables $\{X_t\}$, being dominated by the integrable random variable $\sup_t X_t$, would be uniformly integrable, and hence that $\mathbb{E}X_\infty = \mathbb{E}X_1 = 1$. But this contradicts the conclusion of Proposition 1.

Similarly, if $\sup_t \mathbb{E}\left(X_t \log^+ X_t\right) < \infty$ it would follow, by Proposition IV-2-10 of Neveu (1975, p. 70), that $\sup_t X_t$ would be integrable, and hence, as in the previous paragraph, that $\{X_t\}$ would be a uniformly integrable family of random variables. Again, this contradicts Proposition 1, and the result follows. □

When contemplating (12) and (13), it is helpful to keep in mind the fact that

$$\sup_{t \geq 0} \mathbb{E}X_t = \sup_{t \geq 1} 1 = 1 \,.$$

[4]A family $\{X_t\}$ of random variables is *uniformly integrable* if

$$\sup_{t \geq 1} \mathbb{E}\left(|X_t| \, \mathbf{1}[|X_t| > a]\right) \longrightarrow 0 \quad \text{as } a \to \infty.$$

Although the expected value of X_t is equal to 1 for all t, and the expected value of the supremum of X_t is finite for Type 1 assets, the expected value of the supremum of X_t is infinite in the case of Type 2 assets: there are rare states of the world in which X_t becomes very large indeed.

In such states, we have

$$M_1 R_1 \cdot M_2 R_2 \cdots M_t R_t \quad \text{very large,}$$

and so we must have some combination of large $M_1 \cdots M_t$ and large $R_1 \cdots R_t$. The former possibility, large $M_1 \cdots M_t$, corresponds roughly to the realization of a disastrously bad state of the world. In a consumption-based model with time-separable utility, for example, $M_1 \cdots M_t$ is large when marginal utility at time t is high. The latter possibility, large $R_1 \cdots R_t$, corresponds to a particularly favorable return realization for the asset in question.

In some sense, therefore, the value in Type 2 assets derives either from aggregate disasters (large $M_1 \cdots M_t$) or asset-specific triumphs (large $R_1 \cdots R_t$). At a general level, we can say no more. Nonetheless, for the sake of intuition, it is interesting to consider simple special cases that focus attention on each of the two channels separately.

Suppose, first, that we are in a risk-neutral i.i.d. world, as in Example 3 above, and consider a risky Type 2 asset. Since $M_1 \cdots M_t = (1/R_f)^t$ is deterministic, the value of the asset is driven by very occasional asset-specific triumphs—explosions in $R_1 \cdots R_t$—that is, by extreme *right*-tail events.

Conversely, suppose that the world is i.i.d. but not risk-neutral, so that M_t is not constant, and that we are considering the riskless strategy that rolls cash over in the riskless asset. Now, $R_1 \cdots R_t = (R_f)^t$ is deterministic. Again, this is a Type 2 strategy, but now the value is derived from aggregate disasters—states of the world which occur with very low probability, but in which $M_1 \cdots M_t$ is far larger than its expected value. In other words, the value of this strategy, in the long run, is driven by the presence of extreme *left*-tail events. Weitzman (2004) emphasizes the importance of this effect.

4 An example

Consider an i.i.d. economy with two assets, a riskless asset which pays the certain return $R_{f,t} \equiv e^{r_f}$ and a risky asset which pays the lognormal return $R_t \equiv e^{\mu-\sigma^2/2+\sigma Z_t}$, where Z_t is a standard Normal random variable. It is easy to check that the stochastic discount factor $M_t \equiv e^{-r_f-\lambda^2/2-\lambda Z_t}$ prices the assets, where λ is the Sharpe ratio $(\mu - r_f)/\sigma$. (Notice that M_t so defined is not the reciprocal of the return on the growth-optimal portfolio, since the latter is not lognormally distributed.) Each asset is of Type 2, as is easily checked.

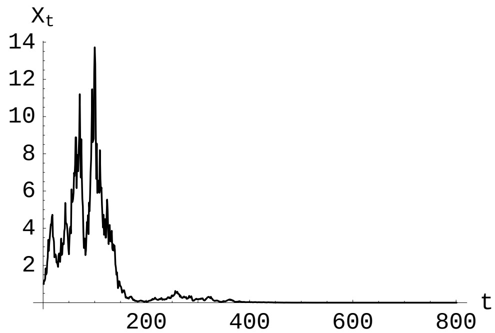
Writing $X_{f,t} \equiv M_1 R_{f,1} \cdots M_t R_{f,t}$ and $X_t \equiv M_1 R_1 \cdots M_t R_t$, we have

$$X_{f,t} = e^{-\lambda(Z_1+\cdots+Z_t)-\lambda^2 t/2} \tag{14}$$

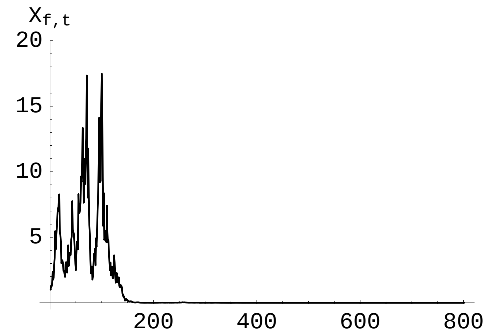
$$X_t = e^{(\sigma-\lambda)(Z_1+\cdots+Z_t)-(\sigma-\lambda)^2 t/2} \tag{15}$$

Notice that in this example with just one kind of shock, realistic values of σ and λ imply that X_t is large when $(Z_1 + \cdots + Z_t)$ is small: in the long run, only disasters matter.

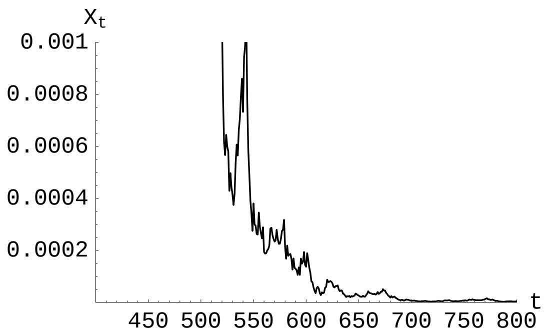
Figure 1 plots a realization of X_t and $X_{f,t}$. Each time period represents one quarter. I have set $\sigma = 0.08$ and $\lambda = 0.25$, which corresponds to an annualized standard deviation of 16% and Sharpe ratio of 50% for the risky asset. Proposition 1 states that X_t and $X_{f,t}$ tend to zero as t tends to infinity; along this particular sample path, X_t and $X_{f,t}$ are indistinguishable from zero, even in the zoomed-in graphs, after about 700 quarters. Note also the occasional spikes, which Proposition 6 led us to expect.



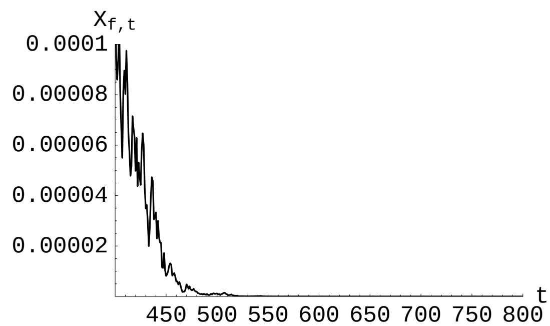
(a) The evolution of X_t over time



(b) The evolution of $X_{f,t}$ over time

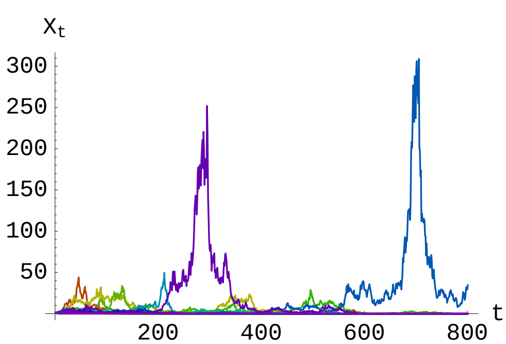


(c) Zooming in on X_t

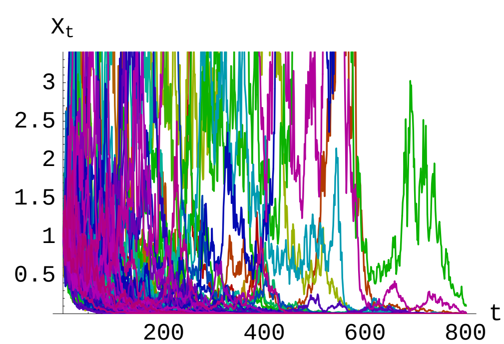


(d) Zooming in on $X_{f,t}$

Figure 1: Realizations of X_t and $X_{f,t}$ on one particular sample path.



(a) 100 sample paths of X_t



(b) Zooming in

Figure 2: Realizations of X_t on 100 sample paths.

Figure 2 shows realizations of X_t along 100 different sample paths. As before, each period represents one quarter. On two sample paths, X_t spikes above 250. These spikes are so large, compared with the values of X_t attained on the vast majority of sample paths, that only about six of the sample paths are visible on the first, unzoomed, diagram. Despite these spikes, after 800 quarters only one sample path remains above 0.5.

In the continuous-time limit, the analogue of X_t would follow a geometric Brownian martingale of the form $e^{\alpha W_t - \alpha^2 t/2}$, where α is some constant and W_t is a Brownian motion. In this special case, we can see directly that $e^{\alpha W_t - \alpha^2 t/2} \to 0$, because $\alpha W_t - \alpha^2 t/2 \to -\infty$ as $t \to \infty$; this follows, in turn, from the fact that $W_t/t \to 0$ as $t \to \infty$ (Karatzas and Shreve (1991, p. 104)).

5 Conclusion

Although expected time- and risk-adjusted cumulative returns on any asset equal one at all horizons, *realized* time- and risk-adjusted cumulative returns on Type 2 assets tend to zero with probability one.

This apparent paradox is resolved in Section 3, which demonstrates that the value of such an asset is driven by the possibility of two types of rare events: spectacular outperformance of the asset itself, and occasional aggregate disasters. Only the first is relevant for the valuation of risky assets in a risk-neutral economy; only the second is relevant for the valuation of riskless strategies in a risky, risk-averse world.

Just three assumptions underpin these results. Two of these—no arbitrage and limited liability—are uncontroversial. The third—independence across time of the relevant random variables—is less desirable. Ritter (1979) presents a generalization of Kakutani's theorem that relaxes the independence assumption, and it may be that the ideas in that paper can be used to improve Proposition 1; in the interests of simplicity, I have not pursued such an extension here.

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A Appendix

Lemma 1. *For any infinite sequence of numbers* $\{a_t\}$*, with* $a_t \in (0,1]$ *for all* t*,* $\prod a_t > 0$ *if and only if* $\sum(1-a_t) < \infty$*.*

Proof. For either to hold, we must have $a_t \to 1$. Furthermore, $\prod a_t > 0$ if and only if $\sum \log(1/a_t) < \infty$. But $\sum \log(1/a_t)$ converges if and only if $\sum(1-a_t)$ converges, by the limit comparison test, since

$$\frac{\log(1/a_t)}{1-a_t} \to 1 \quad \text{as} \quad a_t \to 1$$

by l'Hôpital's rule. $\square$