Disasters and asset pricing: evidence from option markets

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Abstract

There is renewed interest in the fact that rare disasters can have a significant impact on asset prices, and in particular can contribute to an explanation of the equity premium puzzle. If such rare disasters are indeed important, their influence should be detected in option prices. Working in a consumption-based framework, I provide a pricing formula for options on equities. The formula can be interpreted as a weighted average of terms that look like Black's formula for options on commodities. As is well known, the presence of jumps produces a volatility smile with particularly high implied volatilities of low strike options. Of greater interest, unconditional expected excess returns on long positions in at- or near-the-money options are extremely negative, as in the data. The model also predicts an upward-sloping term structure of implied volatility. As regards this last prediction, the evidence is mixed: in a dataset of implied volatilities on S&P500 index options with between 3 months and 10 years to expiry, I document that the term structure of implied volatility was upward-sloping on 96.4% of trading days between April 22, 2003 and May 30, 2006, and on 30.0% of trading days between May 31, 2001 and April 17, 2003.

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1 Introduction

Barro (2006) has revived interest in Rietz's (1988) peso problem explanation of the equity premium puzzle. A central aim of this paper is to bring new evidence to bear on the rare disaster story. In particular, I investigate what such an explanation would entail for option prices, and the extent to which the evidence lines up with the predictions of such a model. It is desirable to do so because the consumption-based asset pricing literature has largely chosen to ignore the information available in option prices.

The model I consider features a representative agent whose exogenously specified consumption process follows a geometric Brownian motion (GBM) overlaid with jumps of random sizes which take place at random times. I calibrate the model so that the jumps are typically large and downward, and take place extremely infrequently—on the order of once a century.

I solve for the prices of the riskless asset (assumed to be in zero net supply) and of the "market" (interpreted as a claim to the consumption stream); the framework under consideration generates a high equity premium and low riskless rate, as in Barro (2006).

When trying to draw broad conclusions about asset markets from option prices, the traditional modelling strategy that identifies equities with the consumption claim hits trouble (in view of the equity volatility puzzle) because of the central importance of volatility in option pricing. That is, options on equities are far more valuable than options on the consumption claim because the volatility of equities is far higher than the volatility of consumption. It is therefore not obvious how to link the information in real-world option prices to discussion which centers on more macro-level issues such as the equity premium puzzle, and which has traditionally been grounded in a consumption-based framework.

My solution to this problem is to follow Campbell (1986) and Abel (1999) in modelling equities as a leveraged claim on consumption in a certain sense which I make precise below. In this way, I can model equities with realistic levels of volatility.

I provide a pricing formula for options on equities which generalizes the formula presented by Naik and Lee (1990) and is reminiscent of that in Merton (1976). An important difference is that Merton had to assume that jump risk was idiosyncratic—that is, diversifiable and hence unpriced—in order to derive his option pricing formula. In my model, the jumps affect *consumption*, so are systematic and thus have a risk price. I give an intuitive decomposition of the option price as an infinite weighted average of modified Black-Scholes prices.

The predictions of the model line up with available empirical evidence along three dimensions. First, the model generates a volatility smile: out-of-the-money options—especially out-of-the-money puts—have high implied volatilities. This phenomenon has been documented in a wide range of option markets.

Second, and of greater interest, the model reveals that expected excess returns on long option positions, conditional on jumps not occurring within the sample period, can be extremely negative even when jumps only occur very rarely. Although this is intuitive, qualitatively speaking, a contribution of this paper is to show how extremely *quantitatively* important jumps are. The effect is particularly strong in options with strikes near the forward price of the underlying—such as straddles with betas close to zero, which might be expected to earn excess returns close to zero if the possibility of jumps were neglected.[1] This model therefore may offer at least a partial explanation for the results of Coval and Shumway (2001), who find that zero-beta (long) option positions on the S&P 500 index lose approximately three per cent per week. Coval and Shumway (2001) document that "crash-neutral" long option positions also earn extremely negative excess returns, and suggest that rare disasters cannot explain this fact. However, I show that the presence of rare disasters is so important for option prices that even at-the-money *call* options can have negative expected excess returns.

Third, I investigate the term structure of implied volatility.[2] The model generates an

[1] A straddle position consists of being long a call and a put with the same strike. In the absence of jumps, the Black-Scholes model holds, and straddles with strikes at the forward price earn zero expected excess returns.

[2] *Implied volatility* is always to be understood with reference to the Black-Scholes model; that is, given an option's characteristics and price, its implied volatility is the volatility which when plugged into the

upward-sloping term structure of implied volatility for at-the-money-forward options, which is a feature of recent implied volatility data.

A premise of this paper is that the attention lavished on geometrical Brownian motion as a driving stochastic process for finance models is due more to its mathematical tractability than to any fundamental economic reasoning. On the contrary: the simplest economic model suggests that aggregate consumption should *jump* when (public) information arrives that, say, technological progress has increased.[3] For the purposes of this paper, though, it is not necessary to specify the cause of the jumps.

2 Setup

I work in a consumption-based asset pricing framework, and take as given a stochastic process for consumption. This need not be an endowment economy: the consumption process can be thought of as emerging in the equilibrium of an arbitrarily complicated economy.

I suppose that log consumption follows a Lévy process of the following form:

$$d \ln C = \mu dt + \sigma dZ + Y dJ \,, \tag{1}$$

where dZ is a standard Brownian increment, dJ is a jump component which, intuitively, we can think of as taking the values

$$dJ = \begin{cases} 0 \text{ w.p. } 1 - \omega dt \\ 1 \text{ w.p. } \omega dt \end{cases} ,$$

and Y is a Normal random variable

$$Y = -b + \psi \cdot N(0, 1) \tag{2}$$

Black-Scholes formula recovers the price. This provides a simple way of comparing the prices of options with different strikes and expiry dates.

[3]In an endowment economy, consumption cannot adjust so prices have to change in order to leave people content to consume the endowment stream; but this statement will be true in a model with endogenous investment or in which there is the possibility of international borrowing, for example.

So, the jumps happen at rate ω and each time a jump occurs, a new iid realization of the random variable Y is drawn. Equivalently, working in levels rather than in logs, we can write

$$\frac{dC}{C} = (\mu + \frac{1}{2}\sigma^2)dt + \sigma dZ + (e^Y - 1)dJ.$$

It follows from (1) that consumption growth to time t can be represented as

$$C_t = C_0 e^{\mu t + \sigma Z_t + \sum_{j=1}^{N(t)} Y_j} \tag{3}$$

The Y_j are iid and distributed like Y of equation (2). $N(t)$, the number of jumps that have taken place by time t, is distributed according to a Poisson distribution with parameter ωt.

Finally, suppose that there is a representative agent who has CRRA utility with parameter γ and therefore maximizes

$$\mathbb{E}_0 \left(\int_{t=0}^{\infty} e^{-\rho t} \frac{C_t^{1-\gamma} - 1}{1-\gamma} dt \right).$$

3 The moments of consumption growth

The introduction of jumps into the model permits an assessment of the importance of higher moments of log consumption growth. Write M_n for the nth central moment of log consumption growth $\mathbb{E}(\ln C_1 - \overline{\ln C_1})^n$. Then, define

$$\text{standard deviation:} \quad s \equiv (M_2)^{1/2} \tag{4}$$

$$a_1 \equiv \frac{\mathbb{E}\left(\ln C_1 - \ln C_0\right)}{s} \tag{5}$$

$$a_2 \equiv \frac{M_2}{s^2} \tag{6}$$

$$\text{skewness:} \quad a_3 \equiv \frac{M_3}{s^3} \tag{7}$$

$$\text{kurtosis:} \quad a_4 \equiv \frac{M_4}{s^4} - 3 \tag{8}$$

I define a_2 in this apparently redundant way to make subsequent formulas clearer. With these definitions, the baseline case of GBM has zero skew and zero kurtosis.

In the the case with jumps considered here,

$$
\begin{aligned}
s &= \left[\sigma^2 + \omega\left(b^2 + \psi^2\right)\right]^{1/2} \\
a_1 &= \frac{\mu - \omega b}{s} \\
a_2 &= 1 \\
a_3 &= -\frac{\omega\left(b^3 + 3b\psi^2\right)}{s^3} \\
a_4 &= \frac{\omega\left(b^4 + 6b^2\psi^2 + 3\psi^4\right)}{s^4}.
\end{aligned}
$$

Note that whenever $b > 0$ log consumption growth is negatively skewed. Whether or not $b > 0$, the presence of jumps implies positive kurtosis ($a_4 > 0$).

4 Pricing assets: bonds and equities

The time-0 price of a claim to an arbitrary dividend stream $\{d_t : t \geq 0\}$ is

$$
P(D) \equiv \mathbb{E}_0\left(\int_{t=0}^{\infty} e^{-\rho t}\left(\frac{C_t}{C_0}\right)^{-\gamma} d_t \, dt\right) \tag{9}
$$

I start by considering an asset which pays dividend stream $D_t \equiv (C_t)^{\lambda}$ for some constant λ; this simplifies the analysis because substituting $\lambda = 0$ gives the price of the riskless bond, which pays constant dividend 1, and substituting $\lambda = 1$ gives the price of the consumption claim. I follow Campbell (1986) and Abel (1999) and interpret $\lambda > 1$ as a tractable way of approximating levered equity claims. Below, I write P_λ for the price of the asset at time 0 when λ is allowed to take any value. When I set $\lambda = 0$, I use subscript f to denote the riskless asset, and when $\lambda = 1$, I use subscript c to denote the consumption claim.

From equation (9),

$$
\begin{aligned}
P_\lambda &= \mathbb{E}_0\left(\int_{t=0}^{\infty} e^{-\rho t}\left(\frac{C_t}{C_0}\right)^{-\gamma} (C_t)^{\lambda} \, dt\right) \\
&= (C_0)^{\lambda}\mathbb{E}_0\left(\int_{t=0}^{\infty} e^{-\rho t}\left(\frac{C_t}{C_0}\right)^{-(\gamma-\lambda)} dt\right)
\end{aligned}
$$

Substituting in for C_t/C_0 from equation (3), it follows that

$$\begin{aligned} P_\lambda &= D_0 \cdot \mathbb{E}_0 \int_0^\infty e^{-\rho t} e^{\mu(\lambda-\gamma)t+\sigma(\lambda-\gamma)Z_t} \cdot \prod_{j=1}^{N(t)} e^{(\lambda-\gamma)Y_j}\, dt \\ &= D_0 \cdot \int_0^\infty e^{-\rho t} e^{\mu(\lambda-\gamma)t} \mathbb{E}_0 \left(e^{\sigma(\lambda-\gamma)Z_t}\right) \mathbb{E}_0 \left(\prod_{j=1}^{N(t)} e^{(\lambda-\gamma)Y_j}\right) dt \qquad (10) \end{aligned}$$

The last equality follows from the independence of $N(t)$, $\{Y_j\}$ and the Brownian motion Z.

Now,

$$\begin{aligned} \mathbb{E}_0 \left(\prod_{j=1}^{N(t)} e^{(\lambda-\gamma)Y_j}\right) &= \mathbb{E}_0 \left(\mathbb{E}\left(\prod_{j=1}^{N(t)} e^{(\lambda-\gamma)Y_j} | N(t)\right)\right) \\ &= \sum_{n=0}^{\infty} \frac{e^{-\omega t}(\omega t)^n}{n!} \mathbb{E}\left(\prod_{j=1}^{n} e^{(\lambda-\gamma)Y_j}\right) \\ &= \sum_{n=0}^{\infty} \frac{e^{-\omega t}(\omega t)^n}{n!} \left(\mathbb{E} e^{(\lambda-\gamma)Y_j}\right)^n \\ &= \sum_{n=0}^{\infty} \frac{e^{-\omega t}(\omega t)^n}{n!} \underbrace{\left(e^{-b(\lambda-\gamma)+\frac{1}{2}(\lambda-\gamma)^2\psi^2}\right)^n}_{\equiv \xi} \\ &= e^{-\omega t} \sum_{n=0}^{\infty} \frac{(\omega \xi t)^n}{n!} \\ &= e^{\omega(\xi-1)t} \qquad (11) \end{aligned}$$

Substituting (11) into (10), we get

$$\begin{aligned} P_\lambda &= D_0 \cdot \int_0^\infty e^{-\{\rho+(\gamma-\lambda)\mu-\frac{1}{2}(\gamma-\lambda)^2\sigma^2-\omega(\xi-1)\}t} dt \\ &= D_0 \cdot \left\{\rho + (\gamma-\lambda)\mu - \frac{1}{2}(\gamma-\lambda)^2\sigma^2 - \omega(\xi-1)\right\}^{-1} \qquad (12) \\ &= D_0 \cdot \left\{\rho + (\gamma-\lambda)\mu - \frac{1}{2}(\gamma-\lambda)^2\sigma^2 - \omega(e^{b(\gamma-\lambda)+\frac{1}{2}(\gamma-\lambda)^2\psi^2} - 1)\right\}^{-1} \end{aligned}$$

Using equation (12), I can find the return on the asset, R_λ:

$$\begin{aligned} R_\lambda dt &= \frac{dP_\lambda}{P_\lambda} + \frac{D}{P_\lambda} dt \\ &= \frac{dD}{D} + \frac{D}{P_\lambda} dt \end{aligned}$$

Applying the jump-diffusion version of Ito's lemma to equation (1), we have that for general λ,

$$\frac{dD}{D} = \left(\lambda\mu + \frac{1}{2}\lambda^2\sigma^2\right)dt + \lambda\sigma dZ + \left(e^{\lambda Y} - 1\right)dJ \tag{13}$$

Thus, after simplifying,

$$\begin{aligned} R_\lambda dt &= \lambda\sigma dZ + \left(e^{\lambda Y} - 1\right)dJ + \\ &+ \left[\rho + \gamma\mu - \frac{1}{2}\gamma(\gamma - 2\lambda)\sigma^2 - \omega\left(e^{b(\gamma-\lambda)+\frac{1}{2}(\gamma-\lambda)^2\psi^2} - 1\right)\right]dt \end{aligned} \tag{14}$$

If we want to condition on no jumps taking place, the term in dJ can be dropped. This will enable me to calculate the expected equity premium for sample periods in which jumps do not occur.

4.1 The riskless asset

Armed with (12) and (14), I can calculate the price and return of the riskless asset by substituting $\lambda = 0$:

$$P_f = \left\{\rho + \gamma\mu - \frac{1}{2}\gamma^2\sigma^2 - \omega(e^{b\gamma+\frac{1}{2}\gamma^2\psi^2} - 1)\right\}^{-1}$$

$$R_f dt = \left[\rho + \gamma\mu - \frac{1}{2}\gamma^2\sigma^2 - \omega\left(e^{b\gamma+\frac{1}{2}\gamma^2\psi^2} - 1\right)\right]dt \tag{15}$$

This expression for R_f can alternatively be written as a power series, in a way that demonstrates surprisingly neatly how the riskless rate depends on higher monents of consumption growth:

$$R_f = \rho + \frac{(\gamma s)}{1!}\cdot a_1 - \frac{(\gamma s)^2}{2!}\cdot a_2 + \frac{(\gamma s)^3}{3!}\cdot a_3 - \frac{(\gamma s)^4}{4!}\cdot a_4 + \ldots \tag{16}$$

a_1 through a_4 are the various scaled moments of log consumption growth defined in equations (4)–(8). The skewness term (in $(\gamma s)^3$) and the kurtosis term (in $(\gamma s)^4$) are not present in the GBM case. Each drives the riskless rate down, in this case, since the skewness is negative and the kurtosis is positive. Martin (2007) explains why (16) takes such a neat form.

4.2 The consumption claim

Substituting $\lambda = 1$, we get

$$\begin{aligned} P_c &= C_0 \cdot \left\{ \rho + (\gamma - 1)\mu - \frac{1}{2}(\gamma-1)^2\sigma^2 - \omega(e^{b(\gamma-1)+\frac{1}{2}(\gamma-1)^2\psi^2} - 1) \right\}^{-1} \\ R_c dt &= \sigma dZ + \left(e^Y - 1\right) dJ + \\ &\quad + \left[\rho + \gamma\mu - \frac{1}{2}\gamma(\gamma-2)\sigma^2 - \omega\left(e^{b(\gamma-1)+\frac{1}{2}(\gamma-1)^2\psi^2} - 1\right) \right] dt \end{aligned}$$

So, the expected excess return on the consumption claim $EP_c = \mathbb{E}(R_c) - R_f$ is

$$EP_c = \gamma\sigma^2 + \omega e^{-b+\frac{1}{2}\psi^2} - \omega e^{b(\gamma-1)+\frac{1}{2}(\gamma-1)^2\psi^2} + \omega(e^{b\gamma+\frac{1}{2}\gamma^2\psi^2} - 1) \tag{17}$$

Equation (17) captures Rietz's point. When $b > 0$, the possibility of disasters gives the equity premium a huge kick.

If jumps do not occur, the above equation is modified to become

$$EP_c^{no\ jumps} = \gamma\sigma^2 + \omega e^{b\gamma+\frac{1}{2}\gamma^2\psi^2}\left(1 - e^{-b+\frac{1}{2}\gamma(\gamma-2)\psi^2}\right)$$

Although conditioning on no jumps raises the equity premium by a small amount, the effect is relatively slight. This model does not explain the equity premium by arguing that there are out-of-sample disasters which, when taken into account, would depress the equity premium to "reasonable" levels; rather, it predicts that the equity premium is high even in a sample which contains a representative number of jumps, because jumps constitute a risk which the representative agent demands a large premium to bear.

4.3 Levered equity

Using equations (14) and (15), the levered equity premium is

$$EP_\lambda = \lambda\gamma\sigma^2 + \omega(e^{-b\lambda+\frac{1}{2}\lambda^2\psi^2} - 1) + \omega(e^{b\gamma+\frac{1}{2}\gamma^2\psi^2} - 1) - \omega(e^{b(\gamma-\lambda)+\frac{1}{2}(\gamma-\lambda)^2\psi^2} - 1)$$

If jumps do not occur, the above equation is modified to become

$$EP_\lambda^{no\ jumps} = \lambda\gamma\sigma^2 + \omega(e^{b\gamma+\frac{1}{2}\gamma^2\psi^2} - 1) - \omega(e^{b(\gamma-\lambda)+\frac{1}{2}(\gamma-\lambda)^2\psi^2} - 1)$$

Again, the equity premium is not greatly affected whether or not one conditions on jumps not taking place in sample.

5 Pricing assets: forwards and options

5.1 Forwards

The time-t forward price of date T consumption is defined to be the value of Fwd_t that solves the equation

$$\mathbb{E}_t\left(e^{-\rho(T-t)}\left(\frac{C_T}{C_t}\right)^{-\gamma}(C_T - Fwd_t)\right) = 0 \tag{18}$$

That is, the forward price is defined so that if two agents agree to transact C_T at time T at price Fwd_t (agreed upon at t), no money needs to change hands at time t because the value of the contract is zero, as expressed in equation (18).

Solving equation (18), we get

$$\begin{aligned} Fwd_t &= C_t \cdot \frac{\mathbb{E}_t\left(\frac{C_T}{C_t}\right)^{1-\gamma}}{\mathbb{E}_t\left(\frac{C_T}{C_t}\right)^{-\gamma}} \\ &= C_t \exp\left\{\left(\mu - \frac{1}{2}(2\gamma-1)\sigma^2 - \omega e^{b\gamma+\frac{1}{2}\gamma^2\psi^2}\left(1-e^{-b-\frac{1}{2}(2\gamma-1)\psi^2}\right)\right)(T-t)\right\} \end{aligned}$$

Similarly, the time-t forward price of the date T dividend $D_T = (C_T)^\lambda$ is

$$Fwd_{\lambda,t} = D_t \exp\left\{\left(\lambda\mu - \frac{1}{2}\lambda(2\gamma-\lambda)\sigma^2 - \omega e^{b\gamma+\frac{1}{2}\gamma^2\psi^2}\left(1-e^{-b\lambda-\frac{1}{2}\psi^2\lambda(2\gamma-\lambda)}\right)\right)(T-t)\right\}$$

The forward prices of the corresponding dividend *streams* are just multiples of these two expressions since, as shown above, price-dividend ratios are constant in this economy.

5.2 Options

If a peso problem is behind the equity premium puzzle, it should be possible to find evidence in option prices—in particular, in out-of-the-money put prices, which will appear high if the peso problem story is correct.

It is traditional in the equity premium literature to interpret equities as a claim to consumption. When one wants to consider options, this assumption is particularly strained: the volatility of the underlying asset is a critical parameter, and it is well-known that consumption is extremely smooth by comparison with market prices. (Campbell (2003) provides an overview of this equity volatility puzzle.) This fact complicates attempts to link evidence from option prices with macro-level stories about the equity premium. It is therefore particularly important to price options on the "equity" asset which sets $\lambda > 1$ and therefore has higher volatility, as shown in equation (13).

A European call option on the dividend D_T expiring at time T with strike price K pays $\max\{0, D_T - K\}$ at time T. Its price is

$$P_t^{call,\lambda} \equiv \mathbb{E}_t \left(e^{-\rho\tau} \left(\frac{C_T}{C_t} \right)^{-\gamma} (D_T - K)_+ \right)$$

where $\tau \equiv T - t$ is the amount of time remaining before expiry and $(x)_+$ is notation for $\max\{0, x\}$.

I show in the appendix that

$$P_t^{call,\lambda} = \sum_{n=0}^{\infty} \frac{e^{-\omega\tau} (\omega\tau)^n}{n!} \cdot \Theta(F_n, K, r_n, \lambda\sigma_n, \tau) \tag{19}$$

where

$$\begin{aligned}
\Theta(F_n, K, r_n, \lambda\sigma_n, \tau) &\equiv F_n e^{-r_n\tau} \Phi\left(\frac{\ln(F_n/K) + \frac{1}{2}\lambda^2\sigma_n^2\tau}{\lambda\sigma_n\sqrt{\tau}} \right) - K e^{-r_n\tau} \Phi\left(\frac{\ln(F_n/K) - \frac{1}{2}\lambda^2\sigma_n^2\tau}{\lambda\sigma_n\sqrt{\tau}} \right) \\
\sigma_n^2 &\equiv \sigma^2 + \psi^2 \frac{n}{\tau} \\
\mu_n &\equiv \mu - b\frac{n}{\tau} \\
F_n &\equiv D_t e^{\left(\lambda\mu_n - \frac{1}{2}\lambda(2\gamma-\lambda)\sigma_n^2\right)\tau} \\
r_n &\equiv \rho + \gamma\mu_n - \frac{1}{2}\gamma^2\sigma_n^2
\end{aligned}$$

F_n is the forward price at time t of D_T, in a world with no jumps and mean consumption growth rate and variance rate given by μ_n and σ_n^2 respectively. Correspondingly, r_n is the riskless rate that would prevail in such a world.

The expression $\Theta(F, K, r, \sigma, \tau)$ is a version of the Black-Scholes formula that was first derived by Black (1976) in the context of commodity option pricing. So, we can characterize the option price in the presence of jumps as a weighted sum of Black-Scholes-type option prices with appropriately adjusted forward rates F_n, riskless rates r_n, and volatilities $\lambda\sigma_n$.

Note that in the case where $\omega = 0$—so there are no jumps—the pricing formula (19) reduces to $P_{call} = \Theta(F_0, K, r_0, \lambda\sigma_0, \tau)$, which is just the standard Black(-Scholes) formula.

Thus in this case, option prices can be calculated with reference to observable market prices and σ^2 alone. In the general case in which $\omega > 0$, however, preference parameters enter unavoidably.

Again, since the value of the equity claim is just a constant multiple of the current dividend, the value of a call option on a levered equity *claim* takes the same form, once $P_{\lambda,t}$ is substituted for D_t in the definition of F_n. Since the two are so similar, I will talk in terms of options on a dividend, but everything goes through for the case of options on the equity claim.

Equation (19) is similar to the equation derived in Merton (1976) for an option price when the underlying is subject to jumps. To derive his formula, however, Merton must assume that jump risk is idiosyncratic, and hence has zero price of risk. This is *not* the case in the derivation of (19): jump risk has a positive price, because the jumps are shocks to consumption, and so are systematic by assumption. This is why the preference parameter γ appears in the formula.

A pricing formula equivalent to (19) was derived by Naik and Lee (1990) in the case where $\lambda = 1$. However, as argued above, restricting to $\lambda = 1$ prevents one from linking to the consumption-based asset pricing literature, and, in particular, from assessing the merits of the peso problem story.

Finally, having priced a call option, it is a simple matter to price a put option via put-call parity:

$$P_t^{put,\lambda} = P_t^{call,\lambda} + (K - Fwd_{\lambda,t})e^{-R_f(T-t)}$$

6 Empirical regularities in option prices

Barro (2006) argues that Rietz's peso problem explanation of the equity premium puzzle can account for the equity premium puzzle. I aim to provide independent evidence for or against this claim by focussing attention on the predictions of the model above for option markets.

For numerical calculations, I truncate equation (19), only using values of n between 0 and 6 inclusive. That is, the possibility of seven or more jumps occurring before expiry is discounted. Doing so underestimates the price of an option; but it is easy to see that for the parameter values I consider, this truncation does not have a significant effect on option prices. I explain in the appendix why truncating creates pricing errors which are smaller than 4×10^{-6} even for the most extreme values of the various parameters considered here, and in general are less than 10^{-12}.

In my calculations, I set $\rho = 0.03, \gamma = 4, \mu = 0.025, \sigma = 0.02, \psi = 0.24, b = 0.39, \omega = 0.017$. All parameters other than b and ψ are taken directly from Barro (2006), and b and ψ are chosen to fit the mean and variance of the log jump sizes used in that paper. With these parameters, the riskless rate is 1.6% and the risk premium on the consumption claim is 5.4%.

6.1 The volatility smile

One way of understanding the implications of jumps for option prices is to compare the price produced by equation (19) with the price that would be produced by plugging the forward price of consumption (which is assumed to be observable) and the riskless rate into the Black-Scholes formula,[4] and using σ for the volatility—that is, not making any allowance for the possibility of jumps.

A problem with this comparison is that it is hard to compare option prices across strikes. A better approach is to think in terms of implied volatilities.

Definition 1 *Given the observable forward rate F, observable riskless rate r, and a call option with strike K, price P and time τ to expiry, the* Black-Scholes implied volatility *$\sigma_{BS}(F, K, r, \tau, P)$ is defined by*

$$\Theta(F, K, r, \sigma_{BS}(F, K, r, \tau, P), \tau) = P, \tag{20}$$

where $\Theta(F, K, r, \sigma, \tau)$ is Black's formula for the price of a call option, as defined above.

[4] The forward price of consumption and riskless rate, of course, do take into account the fact that jumps are a part of the model.

That is, $\sigma_{BS}(F, K, r, \tau, P)$ is the volatility that, when plugged into the Black-Scholes formula, recovers the option prices that are observed in the market. This is a natural metric for comparing option prices across strikes, and it is widely used by market practitioners. Finally, the solution to (20) is unique—*existence* of a solution is assured, assuming that simple arbitrage bounds are satisfied by market prices—since $\Theta(\cdot)$ is monotonic increasing in σ. Roughly speaking, we can say that when an option price is "high", σ is high.

Although the Black-Scholes formula cannot be inverted to give a closed-form expression for σ_{BS} as a function of option price (and other fundamental parameters), the monotonicity of price with respect to volatility means that it is easy to calculate the Black-Scholes implied volatility numerically.

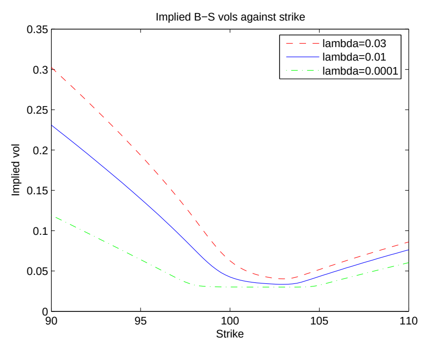


Figure 1: Implied volatility that would be calculated from the prices of calls with varying strikes by someone using the Black-Scholes formula, for different values of ω. $\lambda = 1$. Parameter values: $\rho = 0.02, \gamma = 4, \mu = 0.02, \sigma = 0.03, \psi = 0.2, b = 0.4, T = 0.1$.

In figure 1 I plot the Black-Scholes implied volatilities across a range of strikes for three different values of the jump arrival rate ω. Increasing the jump probability makes options more expensive across the whole range of strikes. (If we set $\omega = 0$ so that there are no jumps, we would be in a Black-Scholes world, and the Black-Scholes implied volatilities would equal σ, which I set equal to 0.03 for the purpose of the figure.)

Furthermore, implied volatilities increase for far out-of-the-money options, and particularly so for out-of-the-money *puts*; this is the well-known volatility smile which, as documented by Rubinstein (1994) and Jackwerth and Rubinstein (1996), has been a feature of index option prices since the crash of October 1987.[5] Derman (2003) reports that the smile in options on gold slopes the other way: implied volatility is highest on out-of-the-money *call* options. This fits nicely with the story, since in times when the market drops sharply, the price of gold tends to rise.

6.2 Expected excess returns on options

It is a robust and intriguing fact that options earn significantly lower excess returns than are predicted by standard models. For example, Coval and Shumway (2001) present evidence that zero-beta S&P index straddle positions lose approximately three per cent per week.[6] Coval and Shumway's sample consists of straddles with "roughly between 20 and 50 days to expiration." Below, I report the $\tau = 0.1$ and $\tau = 1$ cases.

A natural approach is to examine the expected excess returns on what I will call a *benchmark straddle* with strike price equal to the forward price of equity. In a Black-Scholes world with $\omega = 0$, such benchmark straddles have deltas of zero and thus, following the CAPM logic of Black and Scholes (1973),[7] must earn zero excess return.

I therefore examine the impact of the possibility of jumps on straddle returns, with particular attention being paid to the returns on benchmark straddles. Since forward prices are observable in the market, we have the appealing feature that the identity of the benchmark straddle is model-independent.

Conditional on no jumps occurring, the expected return on an option with price $P(D,t)$ (D is the price of the underlying; for notational convenience I have suppressed the dependence of P on other variables) can be found by applying Itô's lemma:[8]

$$\begin{aligned}\frac{\mathbb{E}(dP)}{P} &= \frac{1}{P}\cdot\left\{\frac{\partial P}{\partial D}\left(\lambda\mu+\frac{1}{2}\lambda^2\sigma^2\right)D+\frac{1}{2}\frac{\partial^2 P}{\partial D^2}\lambda^2\sigma^2 D^2+\frac{\partial P}{\partial t}\right.\\ &\qquad \left. +\,\omega\,\mathbb{E}\left[P(e^{\lambda Y}D,t)-P(D,t)\right]\right\}dt \qquad (21)\end{aligned}$$

[5] These papers also demonstrate that prior to 1987 there was no volatility smile, suggesting that the possibility of a crash was not anticipated in option markets.

[6] A straddle position consists of a call and a put with the same maturity and strike price.

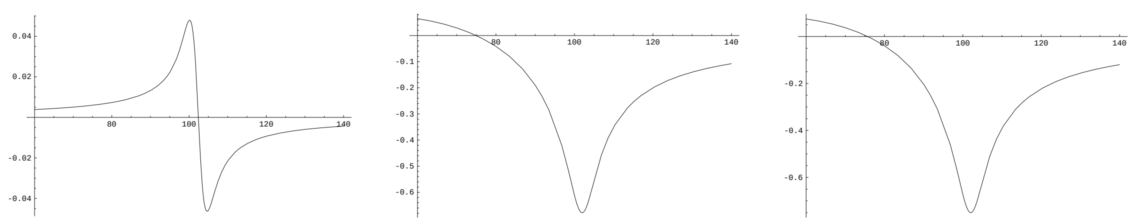
[7] See the section titled "An Alternative Derivation."

[8] The expectation on the right-hand side of (21) is with respect to Y, which is $N(-b,\psi^2)$.

We can therefore decompose the expected return on an option into two parts: one due to the diffusion component of the driving stochastic process, the other to the jump component, as follows:

$$ER_{opt} \equiv \underbrace{\frac{\frac{\partial P}{\partial D}\left(\lambda\mu + \frac{1}{2}\lambda^2\sigma^2\right)D + \frac{1}{2}\frac{\partial^2 P}{\partial D^2}\lambda^2\sigma^2 D^2 + \frac{\partial P}{\partial t}}{P}}_{\text{diffusion component}} + \underbrace{\omega\,\frac{\mathbb{E}\left[P(e^{\lambda Y}D,t) - P(D,t)\right]}{P}}_{\text{jump component}} \qquad (22)$$

I used *Mathematica* to calculate the two components of (22) for the truncated pricing formula. The diffusion component can be found in closed form, although the resulting expression is so complicated that it is uninformative. The jump component must be calculated numerically. I thereby find instantaneous expected excess returns on at-the-money-forward straddles, both unconditionally as in (22) and conditional on the disaster event not taking place (that is, dropping the jump component of expected returns).



(a) $\omega = 0$: Black-Scholes world: forward price is 102.39.

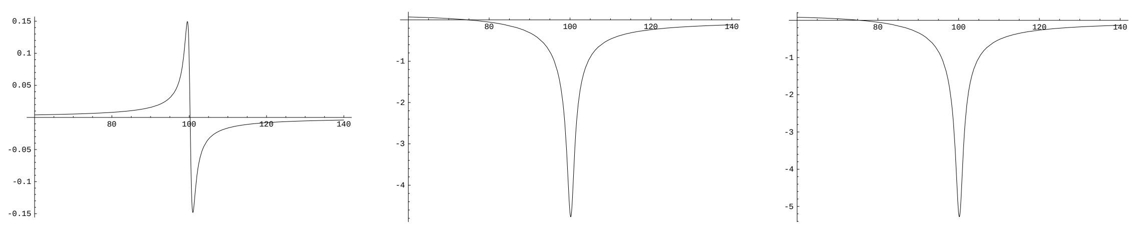
(b) $\omega = 0.017$: forward price is 96.69.

(c) $\omega = 0.017$: forward price is 96.69: conditional on no disasters happening in sample.

Figure 2: The instantaneous expected excess return on straddles with one year to expiry, plotted against strike. Parameter values: $\gamma = 4, \sigma = 0.02, \rho = 0.03, \mu = 0.025, \lambda = 1$.

As discussed above, when $\omega = 0$, the expected excess return on a benchmark straddle is zero. (See Figures 2a and 3a below, which illustrate the expected returns on straddles in a Black-Scholes world with $\omega = 0$.) If $\omega > 0$, however, the unconditional expected excess return on a benchmark straddle is *extremely negative* for reasonable parameter values, and the effect is most marked in short-dated options. This theoretical prediction is consistent with the results of Coval and Shumway (2001). Figure 2 shows the case with one year to expiry for various different assumptions. Figure 2a is the jump-free Black-Scholes case which has already been discussed. Figure 3b shows that the instantaneous expected return is extremely negative in the presence of jumps, *even in samples in which the representative number of crashes occurs*. Figure 3c confirms that conditioning on no jumps taking place in sample makes the expected return on an option position even lower.

Figures 2a to 3c plot expected excess returns on the y-axis and strike on the x-axis. As the strike gets further from the forward, the straddle becomes more and more like a long position (if the strike is low) or a short position (if the strike is high) in the underlying asset. Moving away from the center of the x-axis, we therefore see the expected excess return tending towards plus or minus the risk premium on the consumption claim: plus to the left, minus to the right.



(a) $\omega = 0$: Black-Scholes world: forward price is 100.24.

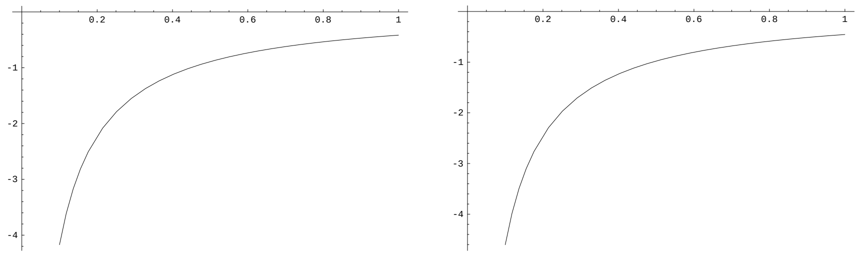
(b) $\omega = 0.017$: forward price is 99.66.

(c) $\omega = 0.017$: forward price is 99.66: conditional on no disasters happening in sample.

Figure 3: The instantaneous expected excess return on straddles with 0.1 years to expiry, plotted against strike. Parameter values: $\gamma = 4, \sigma = 0.02, \rho = 0.03, \mu = 0.025, \lambda = 1$.

The results are even more stark for short-dated options such as those considered by Coval and Shumway (2001), as in figure 3. Figure 3b shows the unconditional expected excess return on an option with 0.1 years to expiry; figure 3c shows the expected return on such an option, conditional on no jumps taking place in sample. The results are qualitatively similar but quantitatively even more striking than in the one-year cases.

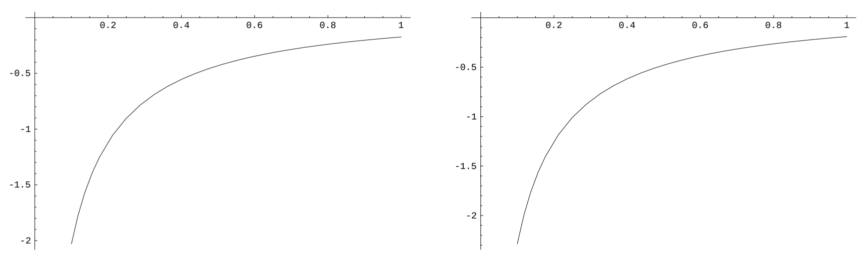
I also plot graphs of expected excess returns on benchmark straddles against option time to expiry, both unconditionally and conditional on no jumps taking place in sample. See Figure 4.



(a) Unconditional. (b) Conditional on no jumps in sample.

Figure 4: The instantaneous expected excess return on straddles plotted against time to expiry. Parameter values: $\gamma = 4, \sigma = 0.02, \rho = 0.03, \mu = 0.025, \lambda = 1$.

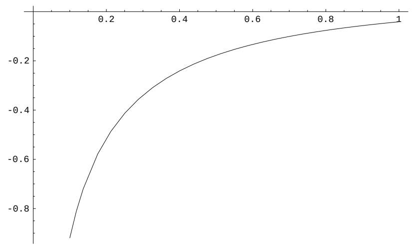
Coval and Shumway also show that what they term *crash-neutral* straddles deliver similarly poor expected returns, despite being constructed to have betas of zero. As a first attempt to see how returns on crash-neutral straddles vary with ω, I follow their lead and calculate the expected returns on positions which are long at-the-money-forward straddles and short out-of-the-money put options with strikes equal to 85 per cent of the forward price. I repeat the exercise with the out-of-the-money put strikes equal to 95 per cent of the exercise price. The results are in Figures 5 and 6. Once again, expected excess returns are extremely low whether or not one conditions on jumps not taking place in sample.



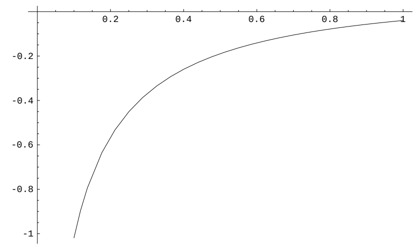
(a) Unconditional. (b) Conditional on no jumps in sample.

Figure 5: The instantaneous expected excess return on a crash-neutral forward straddle—with downside gains capped by shorting a put with strike equal to 85 per cent of the forward price—against time to expiry. Parameter values: $\gamma = 4, \sigma = 0.02, \rho = 0.03, \mu = 0.025, \lambda = 1$.

As is apparent from the two figures, excess returns can be extremely low even for crash neutral straddles.



(a) Unconditional.



(b) Conditional on no jumps in sample.

Figure 6: The instantaneous expected excess return on a crash-neutral forward straddle—with downside gains capped by shorting a put with strike equal to 95 per cent of the forward price—against time to expiry. Parameter values: $\gamma = 4, \sigma = 0.02, \rho = 0.03, \mu = 0.025, \lambda = 1$.

The evidence is that the size of expected excess returns, conditional on no jumps, can be negative and economically significant, though perhaps not *as* negative as was found by Coval and Shumway. On the other hand, as mentioned above, the finding of Rubinstein (1994) and Jackwerth and Rubinstein (1996) that the smile has only been a feature of index option markets since the crash may contribute to explaining this phenomenon. Had there been a smile before 1987, the out-of-the-money puts sold by the Coval-Shumway crash-neutral strategy would have claimed a higher price, and historical excess returns on the crash-neutral strategy would have been less negative. Although this does not explain why it was that option markets failed to entertain the possibility of a crash before 1987, it does suggest that if the (now present) index option volatility smile persists, rationally expected returns on crash-neutral strategies will in future be less negative than Coval and Shumway found.

6.3 The term structure of implied volatility

Figure 10 shows the term structure of implied volatility for a range of different strikes. The figure reveals that for far out-of-the-money options, the term structure of volatility predicted by the model is (at least initially) downward-sloping. An important caveat is that holding the strike constant as the time to expiry, T, increases is somewhat misleading insofar as the forward price to time T varies with T, in the sense that the delta of an option with fixed strike will change as T increases.

To address this concern, I plot the implied vol on options with strikes set equal to the forward price for each time horizon. Figure 11 how the term structure of implied vol changes as the disaster arrival rate ω increases. When ω equals zero, we are in the world of Black-Scholes and thus, unsurprisingly, the volatility term structure is flat and equal to 0.03 at all expiry horizons. As ω increases, the term structure becomes upward sloping, and the slope is greatest at the short end.

I have obtained, from a major Wall Street bank, historical time series of implied volatility on the S&P500 for expiry horizons of 3 months, 6 months, 1 year, 5 years and 10 years. I have complete daily data on each for the period 22 April 2003 to 30 May 2006. For the short end of the vol curve (3 months up to 1 year) I have a longer sample, from 31 May 2001 until 30 May 2006. Table 1 summarizes the data.

	3mo	6mo	1yr	5yr	10yr	upward	downward
5/31/01–4/17/03	24.0	23.0	22.8			30.0%	46.1%
	(4.62)	(3.66)	(2.83)				
4/22/03–5/30/06	14.5	15.2	16.0	18.9	21.7	96.4%	0.0%
	(2.62)	(2.35)	(2.08)	(1.59)	(1.81)		
5/31/06–5/30/06	18.0	18.1	18.5			–	–
	(5.76)	(4.73)	(4.08)				

Table 1: Means, (standard deviations), and the proportion of days on which the term structure of implied volatility was monotonically increasing ("upward") or monotonically decreasing ("downward").

Figures 7 and 8 plot implied volatility over time. I break the data into two time periods. In the first, I only have data for options with expiry horizons of one year or less. In the second, I have data for all five expiry horizons.[9]

[9]The figures also demonstrate a failure of the model, which predicts that the implied vol at each horizon

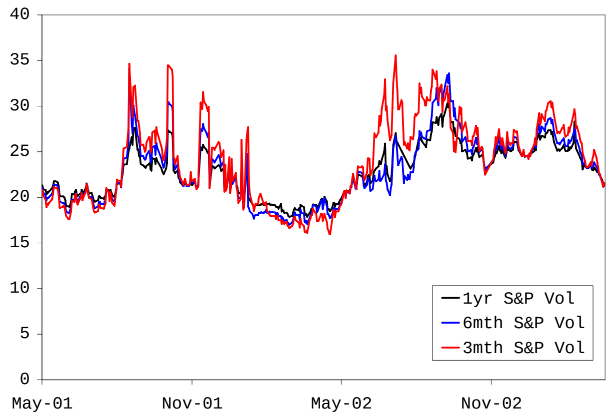


Figure 7: Graph shows the historical time series of at-the-money implied volatilities on the S&P 500 for options with 3 months, 6 months, and 1 year to expiry.

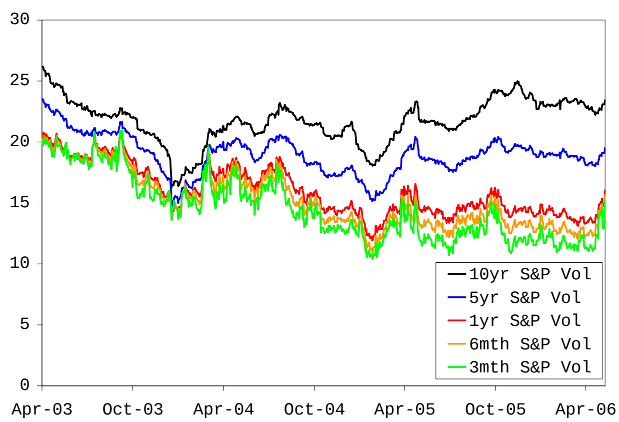


Figure 8: Graph shows the historical time series of at-the-money implied volatilities on the S&P 500 for options with 3 months, 6 months, 1 year, 5 years, and 10 years to expiry.

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should remain constant. Another notable feature of the graph is that the implied volatility on longer-dated options appears to track the implied volatility on short-dated options surprisingly—excessively?—closely. This recalls the findings of Stein (1989), who documented a similar phenomenon in the very short end of the vol curve for a data set running from December 1983 to September 1987. Working in a jump-free framework, Stein interpreted this as indicative of overreactions in the options market. Figure 11 suggests that for small ω, variation in ω causes roughly parallel moves in the vol term structure. It is therefore tempting to speculate that (roughly) parallel moves in the volatility term structure *might* reflect changes in ω, the perceived arrival rate of disasters. On the other hand, since the model presented here does not allow for changing ω, this can only be suggestive.

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A Appendix

A.1 Deriving the option pricing formula (19)

I reduce the complexity of the problem in stages. I use $\widetilde{Z}_1, \widetilde{Z}_2, \widetilde{Z}_3, \ldots$ to denote independent standard Normal random variables.

We have that

$$\begin{aligned}
P_t^{call,\lambda} &= \mathbb{E}_t\left(e^{-\rho\tau}\left(\frac{C_T}{C_t}\right)^{-\gamma}(D_T-K)_+\right) \\
&= \mathbb{E}_t\left(e^{-\rho\tau}\left(\frac{C_T}{C_t}\right)^{-\gamma}\mathbf{1}\left[(C_T)^\lambda \geq K\right]\left((C_T)^\lambda - K\right)\right) \\
&= \mathbb{E}_t\left(e^{-\rho\tau}\left(\frac{C_T}{C_t}\right)^{-\gamma}(C_T)^\lambda\mathbf{1}\left[C_T \geq K^{1/\lambda}\right]\right) - Ke^{-\rho\tau}\mathbb{E}_t\left(\left(\frac{C_T}{C_t}\right)^{-\gamma}\mathbf{1}\left[C_T \geq K^{1/\lambda}\right]\right) \\
&= (C_t)^\lambda e^{-\rho\tau}\mathbb{E}_t\left(\left(\frac{C_T}{C_t}\right)^{-\gamma-\lambda}\mathbf{1}\left[C_T \geq K^{1/\lambda}\right]\right) - Ke^{-\rho\tau}\mathbb{E}_t\left(\left(\frac{C_T}{C_t}\right)^{-\gamma}\mathbf{1}\left[C_T \geq K^{1/\lambda}\right]\right)
\end{aligned}$$

Noting the similarity between the two expectation terms, the new target is to find

$$\mathbb{E}_t\left(\left(\frac{C_T}{C_t}\right)^{-\gamma}\mathbf{1}\left[C_T \geq K^{1/\lambda}\right]\right)$$

Now, by the law of iterated expectations, this expression can be written as

$$\sum_{n=0}^{\infty} \mathbb{P}(n \text{ jumps}) \cdot \mathbb{E}_t \left(\left(\frac{C_T}{C_t} \right)^{-\gamma} \mathbf{1} \left[C_T \geq K^{1/\lambda} \right] \middle| n \text{ jumps} \right)$$

The problem is reduced again: the new, final, target is to find

$$\mathbb{E}_t \left(\left(\frac{C_T}{C_t} \right)^{-\gamma} \mathbf{1} \left[\frac{C_T}{C_t} \geq \frac{K^{1/\lambda}}{C_t} \right] \middle| n \text{ jumps} \right) \tag{23}$$

Conditional on n jumps taking place,

$$\frac{C_T}{C_t} = \exp \left\{ \mu\tau + \sigma \left(Z_T - Z_t \right) + \sum_{1}^{n} Y_j \right\} .$$

Substituting in the definitions of random variables Y_j and Z_T, we get

$$\begin{aligned} \frac{C_T}{C_t} &\sim \exp \left[\mu\tau - bn + \sigma\sqrt{\tau}\widetilde{Z}_1 + \psi\sqrt{n}\widetilde{Z}_2 \right] \\ &\sim \exp \left[\mu\tau - bn + \theta\widetilde{Z}_3 \right] \end{aligned} \tag{24}$$

where $x \sim y$ indicates that the random variables x and y have the same distributions, and

$$\theta^2 \equiv \sigma^2\tau + \psi^2 n \, .$$

Thus, using (24), we can see that (23) amounts to

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} e^{-\gamma(\mu\tau - bn - \frac{1}{2}\theta^2 + \theta z)} \mathbf{1} \left[z \geq \frac{\ln\left(\frac{K^{1/\lambda}}{C_t}\right) - \mu\tau + bn}{\theta} \right] dz$$

which is easily seen to equal

$$e^{-\gamma(\mu\tau - bn) + \frac{1}{2}\gamma^2\theta^2} \Phi \left(\frac{1/\lambda \cdot \ln\left(\frac{D_t}{K}\right) + \mu\tau - bn - \theta^2\gamma}{\theta} \right) .$$

Substituting back gives the option pricing formula (19).

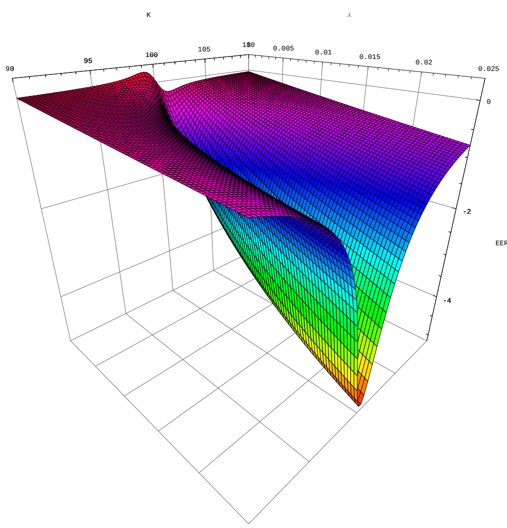
A.2 Bounding the pricing errors induced by truncating (19)

Truncating equation (19) evidently causes the option to be underpriced, since each of the infinitely many terms which are deleted is strictly positive. I now provide an upper bound for the mispricing, and show that the upper bound is negligibly small.

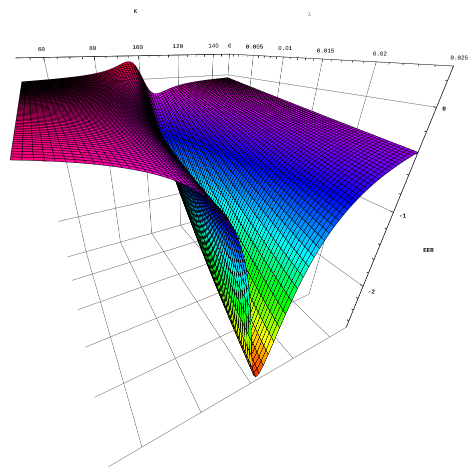
A simple no-arbitrage argument shows that independent of the number of jumps that take place before the maturity of a call option with strike K, its value at time t is less than $F_t e^{-R_f \tau}$, where F_t is the appropriate forward price (to time T) of the underlying asset (say, the dividend D_T). For, an arbitrageur can agree to buy the underlying at the forward price F_t, and set aside $F_t e^{-R_f \tau}$ in the riskless asset to enable her to make the payment at time T. Doing so guarantees the final payoff D_T which is greater than or equal to the option's final payoff $\max\{0, D_T - K\}$. Thus, in particular, the value of the option *conditional* on n jumps taking place is less than or equal to $F_t e^{-R_f \tau}$. So, an upper bound on the error induced by truncation is

$$F_t e^{-R_f \tau} \cdot \mathbb{P}(7 \text{ or more jumps}) = F_t e^{-R_f \tau} \left(1 - \sum_{n=0}^{6} \frac{e^{-\omega \tau} (\omega \tau)^n}{n!} \right)$$

Since I choose values of ω and τ which are small—that is, the probability of a jump taking place during the life of the option is small—the term in brackets is almost zero. In my calculations, upper extremes for ω and τ are $\omega = 0.03$ and $\tau = 10$; with these values, the term in brackets is less than 3.4×10^{-8}. For smaller values of ω and τ the error from truncation is even smaller. Finally, in all my calculations, $F_t e^{-R_f \tau} \approx 100$. Thus a conservative upper bound for the errors present due to truncation is 4×10^{-6}.



(a) $\lambda = 1$.



(b) $\lambda = 6$.

Figure 9: Expected excess returns on straddles, conditional on no jumps taking place during the life of the straddle, for different strike prices K and disaster arrival rates ω. Parameter values: $\rho = 0.02, \gamma = 4, \mu = 0.02, \sigma = 0.03, \psi = 0.2, b = 0.4, T = 0.1$.

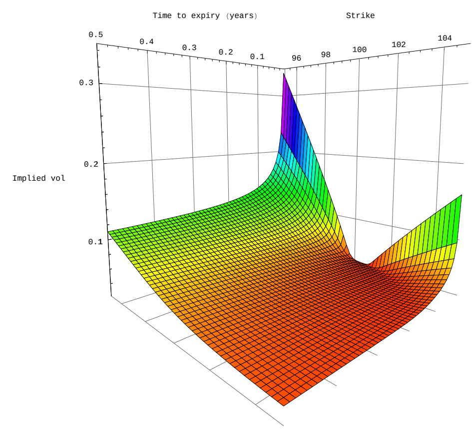


Figure 10: Graph shows Black-Scholes implied volatility surface generated by the model with jumps, across a range of strikes and times to expiry. Taking slices through the time-to-expiry dimension, this graph shows volatility smiles or smirks; taking slices through a particular strike shows the term structure of implied volatility for options of a particular strike. In this graph, I use $\lambda = 1$. Parameter values: $\rho = 0.02, \gamma = 4, \mu = 0.02, \sigma = 0.03, \psi = 0.2, b = 0.4, \omega = 0.01$.

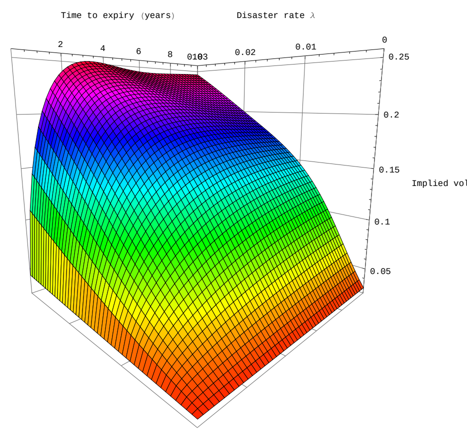


Figure 11: The graph shows how the term structure of implied vol changes as ω increases from zero (in which case the term structure is flat and the implied vol is 0.03 as expected). In this graph, I use $\lambda = 1$. Parameter values: $\rho = 0.02, \gamma = 4, \mu = 0.02, \sigma = 0.03, \psi = 0.2, b = 0.4, \omega = 0.01$.