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General Equilibrium Pricing of Options on the Market Portfolio with Discontinuous Returns

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When the price process for a long-lived asset is of a mixed jump–diffusion type, pricing of options on that asset by arbitrage is not possible if trading is allowed only in the underlying asset and a riskless bond. Using a general equilibrium framework, we derive and analyze option prices when the underlying asset is the market portfolio with discontinuous returns. The premium for the risk of jumps and the diffusion risk forms a significant part of the prices of the options. In this economy, an attempted replication of call and put options by the Black–Scholes type of trading strategies may require substantial infusion of funds when jumps occur. We study the cost and risk implications of such dynamic hedging plans.

In this article, an equilibrium model for pricing options on the market portfolio and an analysis of the

financial implications of continuous hedging strategies when the returns on that portfolio are discontinuous are presented. Empirical evidence of discontinuities in the daily and weekly price changes for diversified equity portfolios can be found in Jarrow and Rosenfeld (1984), Ball and Torous (1985), and Jorion (1988). Press (1967) noted long ago that the analytical characteristics of a Poisson mixture of normal distributions agree with the properties of the empirical distribution of security prices. Diffusion price processes with a significant jump component could actually be one way to model times of high volatility in securities markets, such as we have observed in the past few years. Existing option-pricing formulas are, however, not applicable to the valuation of options on an aggregate portfolio of stocks when its returns contain jumps of random and unpredictable sizes occurring at random times.

We use a general equilibrium framework to price options on the market portfolio with discontinuous returns by embedding the option-pricing problem in a representative agent economy of the Lucas (1978) type. The aggregate dividend process in the economy is exogenously given and is assumed to follow a diffusion process with a jump component, which induces a similar process for the equilibrium price of the market portfolio. The jump times and jump sizes are random. We introduce options on the market portfolio as zero net supply assets in the economy and solve for their equilibrium pricing equations.

For the particular parameterization of the economy that we use, we derive and analyze closed-form solutions for call and put option prices. The formulas of Cox and Ross (1976) (the case of no diffusion uncertainty) and Black and Scholes (1973) (the case of no jump uncertainty) emerge as special cases of the pricing equations derived in this paper. The option prices in our model include the price of the risk of jumps in the underlying asset's value as well as the price of the diffusion risk. We show that the option risk premium forms a significant part of the option's price.

Cox and Ross (1976) develop an option-pricing model for a jump process without a diffusion term. In this model, the only relevant source of uncertainty is the time of the jumps. Jump sizes are of a fixed or predictable magnitude and there is no instantaneous uncertainty about the direction of the jump. In this case, there exists a self-financing strategy consisting of the underlying asset and the riskless bond which replicates the payoff on a given option, and the value of the replicating portfolio at any time is given by the Cox–Ross formula. Merton (1976a) posits asset-price dynamics having a diffusion component as well as jumps of unpredictable sizes occurring at random times. He uses a construct based on the intertemporal capital asset pricing model and a local no-arbitrage argument to solve the pricing

equations. A feature of the Merton model is the assumption that the jumps in security prices are uncorrelated with the return on the market portfolio. Clearly, this assumption is violated if the security under consideration is the market portfolio itself. We show that a model with independent diffusion and jump uncertainties is not complete in the Harrison and Pliska (1981) sense, so that contingent claims in such a model cannot be priced simply by a no-arbitrage argument as has been attempted by Aase (1988). An explicit general equilibrium argument is necessary, which is the approach taken in this article.

Merton (1976b) and Ball and Torous (1985) have analyzed the pricing errors that arise from applying the Black–Scholes formula in a jump–diffusion economy. In this article, however, we consider the consequences of the application of the Black–Scholes trading strategy to replicate options in the jump–diffusion environment. We use our equilibrium pricing equations to analyze the nature of the risks and costs to which investors are exposed if they apply the Black–Scholes strategy of dynamic replication in a jump–diffusion economy. We show that the Black–Scholes strategy results in cash deficits at jump times, the ex ante value of which can be considerable. A continuous hedging plan with an increased level of volatility does not eliminate the jump-time cash outlays. It produces a continuous stream of surplus cash flows instead, but the hedging effectiveness of these additional cash flows against jump-time cash deficits is small.

The article is organized in four sections. In Section 1, we lay down the notation and the basic assumptions. In Section 2, the equilibrium prices of the market portfolio and riskless bonds are first established, and then we consider the pricing of options on the aggregate equity portfolio. We examine the issue of the completeness of the jump–diffusion model and relate our pricing equations to the existing ones in the literature. In Section 3, we discuss the cost and risk implications of dynamic replicating rules, and, in Section 4, we present a brief summary and overall conclusions.

1. The Structure of an Exchange Economy with Discontinuous Prices

Consider a continuous-time variation of the Lucas (1978) pure exchange economy (see, also, Rubinstein, 1976) in a time-homogeneous jump Markov setting. For simplicity, we limit the analysis to a single firm that produces costlessly one perishable consumption good. The firm is completely financed by equity and has one share outstanding. The dividends paid by the firm at an exogenously given stochastic rate $\{\delta_t\}$ are modeled as a Markov process on a given probability space $(\Omega, \mathscr{F}, \mathscr{P})$ with $t \in (0, \infty)$. The process $\{\delta_t\}$ generates the fundamental

uncertainty in the model. The information structure of the agents in the economy is given by the filtration $\mathcal{F}_t \equiv \sigma(\delta_q, 0 \leq q \leq t)$.[1]

Let D_t denote the cumulative dividends of the firm, $\int_0^t \delta_s \, ds$. Since there exists only one firm in the economy, the equity share of the firm is interpreted as the market portfolio and the dividends of the firm have the interpretation of being the aggregate dividends in the economy. The equity share is perfectly divisible and competitively traded at instant t for a price (in terms of the consumption good) S_t. Also available for trading are J claims in zero net supply. The cumulative real dividend process on these claims is given exogenously by a $J \times 1$ vector D_t^J. At instant t, the real prices of these claims are denoted by a $J \times 1$ vector S_t^J.

There exists a representative agent whose problem is to choose an optimal portfolio strategy specifying the number of shares of each traded asset to be held at a given time. Every feasible trading strategy generates an infinite consumption stream. In making his choice, the agent seeks to maximize his expected utility of lifetime consumption

$$E \int_{t=0}^{\infty} U(c_t, t) \, dt, \qquad (1)$$

where the preference function U is continuously differentiable, strictly concave, and strictly increasing in its first argument.

The feasible trading strategies are predictable, locally bounded, self-financing, and satisfy the nonnegative wealth constraint at all times. In addition, at time zero, the value of the agent's portfolio must be less than or equal to the value of the securities with which he is endowed. The agent is assumed to be endowed with one share of the firm and none of the contingent claims. Predictability is an informational constraint on the agent, requiring him to choose portfolios at any time t based only on information available before t. Broadly speaking, local boundedness ensures that the cumulative mean and variance exposure of the investor's portfolio remains finite in finite time, so that the stochastic integral for wealth is well-defined. The self-financing condition is the restriction that portfolio wealth at time t be equal to the initial value of the portfolio plus trading gains, net of the value of consumption between 0 and t. The nonnegative wealth constraint rules out borrowing without repayment and, as indicated in Harrison and Pliska (1981) and shown in Dybvig and Huang (1988), eliminates all arbitrage opportunities in an equilibrium price system.

A competitive equilibrium of the above economy is a set of security-

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[1] By assumption, this filtration satisfies the *usual conditions*. All conditional expectations in the forthcoming analysis are with respect to this filtration.

price processes and a process for the price of the consumption good such that, given the dynamic control problem of maximizing the expected utility function of expression (1) subject to budget and other feasibility constraints, the representative agent optimally chooses to continue to hold one share of the firm and none of the contingent claims and to consume the total amount of dividends issued by the firm so that both the securities and goods markets clear instantaneously. We assume that the process $\{\delta_t\}$, the function $U(\cdot, \cdot)$ and the cumulative dividend processes on the contingent claims are such that the following conditions are satisfied:

$$E \int_0^\infty U(\delta_s, s) \, ds < \infty, \tag{2}$$

$$E \int_0^\infty U_c(\delta_s, s) \, \delta_s \, ds < \infty, \tag{3}$$

$$E \int_0^\infty U_c(\delta_s, s) \, dD_{is} < \infty, \qquad 1 \le i \le J, \tag{4}$$

where D_{it} denotes the ith element of the vector D_t^J. Given this, it can be shown that in the present economy a competitive equilibrium exists, and in this equilibrium the real price at time t of a security with a cumulative real dividend process[2] $\hat{D}_t$ is given by

$$\frac{E_t\{\int_t^\infty U_c(\delta_s, s) \, d\hat{D}_s\}}{U_c(\delta_t, t)}.$$

The above is the usual stochastic Euler equation expressing the price of a security as the sum of its discounted expected dividends with the marginal rate of substitution as the random discount rate.

Having described the basic structure of our economy, we set out the preference and distributional assumptions of the model. The utility function of the agent exhibits constant relative risk aversion. A convenient form is $U(c, t) = \exp[-\phi t]c^\gamma/\gamma$, where $\gamma \le 1$. Aggregate dividends are assumed to follow a compound diffusion–Poisson process under which the stochastic growth function for the ratio of the dividend at time s to time t, $t \le s$, is given by[3]

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[2] $\hat{D}_t$ could be the cumulative dividend process on the share of the firm or on any of the contingent claims.

[3] The stochastic differential equation for the dividend process is given by

$$\frac{d\delta_t}{\delta_{t-}} = (\alpha - \lambda k) \, dt + \sigma \, dz + (\exp[y] - 1) \, dN_t,$$

where y is normally distributed with mean $(\mu_y - 0.5\sigma_y^2)$ and variance σ_y^2.

$$\delta_s = \delta_t \exp\left[\left(\alpha - \frac{\sigma^2}{2} - \lambda k\right)(s - t) + \sigma(z_s - z_t) + \sum_{i=N_t+1}^{N_s} y_i\right], \quad (5)$$

where α is the instantaneous expected rate of change of aggregate dividends, σ^2 is the instantaneous variance of rate of change in aggregate dividends, conditional on the Poisson event not occurring, $\{z_t\}$ is a standard Gauss–Wiener process, $\{N_t\}$ is a Poisson process with parameter λ, and y_i is the random size of the ith jump in the process $\{\log \delta_t\}$. For all i, y_i is normally distributed with mean $(\mu_y - 0.5\sigma_y^2)$ and variance σ_y^2. The expected jump amplitude, k, is equal to $\exp[\mu_y] - 1$. The random variables $\{N_t, t \geq 0\}$, $\{z_t, t \geq 0\}$, and $\{y_i, i \geq 1\}$ are mutually independent and y_i is independent of y_j, for $i \neq j$.

For this structure of preferences and dividends, the conditions ensuring finite values for the equilibrium expected utility and the price of the market portfolio [Equations (2) and (3)] can be simply stated in terms of the parameters of the utility function and the dividend process. We assume that the parameters of the economy are such that the following condition is satisfied:

$$\phi - a(\gamma) > 0, \quad (6)$$

where $a(\cdot)$ as a function of γ is given by

$$a(\gamma) = \gamma(\alpha - \lambda k) + \frac{\gamma(\gamma - 1)}{2}\sigma^2$$
$$+ \lambda\left\{\exp\left(\gamma\mu_y + \frac{\gamma(\gamma - 1)}{2}\sigma_y^2\right) - 1\right\}. \quad (7)$$

Condition (4) ensures finite equilibrium prices for all contingent claims.

Our specification of the dividend process corresponds to an economy that is infrequently subject to real shocks of unpredictable magnitude. Shocks to the equilibrium prices are induced by shocks to the dividends. The shocks to dividends could result from output shocks or shocks due to technological changes. The principal enquiry of this article is structured on this model, where the fundamental uncertainty is the uncertainty about dividends and consumption takes place continuously in time. An alternative construction is to consider a $T (<\infty)$ horizon economy with a representative agent but without intermediate consumption where the process $\{\delta_t\}$ models the flow of information about the terminal cash flow δ_T, and jumps in δ_t correspond to lumpy arrival of information about the final cash flow. The equilibrium equations are similar (but not the same) in the two models. In Appendix B, we solve for the pricing equations for this alternative model.

The assumption of constant relative risk aversion and the multiplicative structure of the $\{\delta_t\}$ process make the marginal rates of substitution between current and future consumption independent of the current level of aggregate consumption or wealth, which allows us to take account of the wealth effects on equilibrium prices in a simple manner. This eventually allows us to derive option-pricing formulas that nest the arbitrage-based Black and Scholes (1973) and Cox and Ross (1976) option models, and to relate conveniently to these models. Also, since we are interested in the prices of claims on the market portfolio, we want our formulas to depend on economy-wide risk aversion because we believe risk aversion to be an important determinant of the values of such securities. The isoelastic parameterization permits us to do that precisely.

2. Equilibrium Security Prices in a Jump–Diffusion Economy with Risk-Averse Agents

For constant relative risk-averse preferences, the Euler equation of the previous section simplifies to give the following pricing equation for the market portfolio:

$$S_t \delta_t^{\gamma-1} = \int_t^\infty E_t \exp[-\phi(s - t)]\delta_s^{\gamma} \, ds. \tag{8}$$

Similarly, if a (zero net supply) contingent claim pays off[4] $h(\delta_T)$ on date T, its price for any $t \leq T$ is given by S_t^h, where

$$S_t^h \delta_t^{\gamma-1} = E_t \exp[-\phi(T - t)]\delta_T^{\gamma-1} h(\delta_T). \tag{9}$$

The contingent claims that are of interest to us are riskless bonds of various maturities, and call and put options on the market portfolio. As shown below, the equilibrium prices of the market portfolio, riskless bonds and call and put options on the market portfolio can be derived in closed form in the present model.

Proposition 1. *Let S_t be the equilibrium price at time t of the market portfolio in the economy described in Section 1 where the aggregate dividends evolve as in Equation (5), and the agent has a utility function with constant relative risk aversion and a constant time discount factor ϕ. S_t is given by $S(\delta,t) \equiv S(\delta)$ with*

$$S(\delta) \equiv \frac{\delta}{\phi - a(\gamma)}, \tag{10}$$

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[4] It is straightforward to introduce claims with more general payoffs.

where $a(\cdot)$ is as defined in Equation (7).

The proof follows from the substitution of (5) into (8) and the evaluation of the resulting integrals and expectations.

From Equations (10) and (5), it is clear that the equilibrium stochastic evolution of S_t is given by

$$S_s = S_t \exp\left[\left(\alpha - \frac{\sigma^2}{2} - \lambda k\right)(s - t) + \sigma(z_s - z_t) + \sum_{i=N_t+1}^{N_s} y_i\right], \quad (11)$$

where $s > t$ and all the variables are defined as in Equation (5).

This, then, endogenizes a mixed jump–diffusion process for the price of the market portfolio. A process of this type was taken as the primitive in Merton (1976a) and was the basis of empirical investigations by Jarrow and Rosenfeld (1984), Ball and Torous (1985), and Jorion (1988). Jorion[5] notes (p. 434) that, ". . . the jump–diffusion model is a significant improvement over the simple diffusion model in both the foreign exchange and stock markets." Equation (10) also shows that the endogenously derived dividend yield $\pi \ [\equiv \phi - a(\gamma)]$ on the market portfolio in the above economy is constant.

The equilibrium price of a pure discount bond that pays one unit of the consumption good at its maturity date T is given in this economy by

$$S_t^r(T) \equiv \exp[\{-\phi + a(\gamma - 1)\}(T - t)], \quad (12)$$

where $a(\cdot)$ is as defined in Equation (7). This implies that the term structure in the present economy is flat with the instantaneous riskless interest rate, r, given by

$$r = \phi + (1 - \gamma)(\alpha - \lambda k) + (\sigma^2/2)(\gamma - 1)(2 - \gamma)$$
$$+ \lambda[1 - \exp((\gamma - 1)\mu_y + (\sigma_y^2/2)(1 - \gamma)(2 - \gamma))].$$

The deterministic term structure of interest rates and the constant dividend yield are consequences of the fact that the investment opportunity set is constant over time in the present economy.[6]

The market portfolio and the riskless bond constitute the basic securities in our model. Before we direct our attention to the problem of the valuation of options on the market portfolio, we address the more fundamental question of whether claims contingent on the realization of the process $\{S_t\}$ can be priced in the present model

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simply by a no-arbitrage argument—that is, whether the present model is complete in the Harrison and Pliska (1981) sense.

Let us consider claims that mature on or before some date T. Notice that there are three independent sources of uncertainty in the present model—the process $\{z_t, 0 \le t \le T\}$, the process $\{N_t, 0 \le t \le T\}$, and the process $\{y_i, i \ge 1\}$. If there were no Poisson uncertainty, we would have the Black–Scholes (1973) setup in which, as is well known, all claims can be priced by arbitrage. If there were no diffusion uncertainty and if the jump sizes were constant or even predictable, we would be in the Cox and Ross (1976) setup, and it can be shown that that model is complete as well. However, if we have both types of uncertainty, even if there were no uncertainty about the jump sizes, the minimum number of securities needed to complete the model is 3 because the multiplicity of the filtration generated by independent processes $\{z_t\}$ and $\{N_t\}$ is 2. Duffie and Huang (1985) provide a complete treatment of the issues of multiplicity and completeness.

If the securities markets model consists of only two securities, one with a price process of the kind given in Equation (11) and the other a riskless bond with a constant interest rate, we show in Appendix A that the model is not complete and thus it is not true that all claims can be priced by a replication argument alone in such a model. Aase (1988) considers the pricing of options on an asset whose price dynamics is of the kind specified in Equation (11). It is suggested in his article that a model with this price process is complete, with trading in only a riskless bond and the underlying asset. The argument is that, since a model with only the diffusion uncertainty is complete with two assets, and the model with only uncertainty in jump times but with predictable jump sizes is complete with two assets, a model that has both these sources of uncertainty is also complete with two assets. This final implication is stated without proof there, and our counterexample shows that it need not hold.

2.1. Equilibrium valuation of options

Given the above considerations, we use the Euler condition (9) to value options contingent on $\{S_t\}$, the price process for the market portfolio in our economy. This is accomplished in the following propositions for European call and put options.

Proposition 2. *Consider a European call option on the market portfolio maturing at date T with exercise price K. The value of the option at time t (t ≤ T), S_t^c, is given by*

$$S_t^c = \sum_{n=0}^{\infty} p(n) \, W(S_t, \pi_n, r_n, \sigma_n, (T-t)), \qquad (13)$$

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where

$$p(n) = \frac{\exp[-\lambda(T - t)][\lambda(T - t)]^n}{n!}$$

and

$$W(S_t, \pi_n, r_n, \sigma_n, (T - t)) = S_t \exp[-\pi_n(T - t)]N(d_{1n})$$
$$- K \exp[-r_n(T - t)]N(d_{2n}), \quad (14)$$

with

$$r_n = \phi + (1 - \gamma)\left\{\alpha + \mu_y \frac{n}{(T - t)} - \lambda k\right\}$$
$$+ \frac{1}{2}\left\{\sigma^2 + \sigma_y^2 \frac{n}{(T - t)}\right\}\{(\gamma - 1)(2 - \gamma)\},$$

$$\pi_n = \phi - \gamma\left\{\alpha + \mu_y \frac{n}{(T - t)} - \lambda k\right\}$$
$$- \frac{1}{2}\left\{\sigma^2 + \sigma_y^2 \frac{n}{(T - t)}\right\}\{\gamma(\gamma - 1)\},$$

$$\sigma_n^2 = \sigma^2 + \sigma_y^2 \frac{n}{(T - t)},$$

$$d_{1n} = \frac{\ln(S_t/K) + (r_n - \pi_n + \frac{1}{2}\sigma_n^2)(T - t)}{\sigma_n\sqrt{T - t}},$$

$$d_{2n} = \frac{\ln(S_t/K) + (r_n - \pi_n - \frac{1}{2}\sigma_n^2)(T - t)}{\sigma_n\sqrt{T - t}}.$$

Proposition 2 is an application of Equation (9) using the payoff function

$$\max[S_T - K, 0] \equiv \max\left[\frac{\delta_T}{\phi - a(\gamma)} - K, 0\right].$$

To gain some insight into the pricing formula (13), note that $p(n)S_t \times \exp[-\pi_n(T - t)]$ is the price at time t of a security that promises to deliver one unit of the market portfolio at time T if and only if n jumps occur in the interval (t, T). Similarly, the price at time t of a security that promises to deliver one unit of the consumption good at time T conditional on n jumps occurring between t and T is

$p(n)\exp[-r_n(T - t)]$. Call the first security the conditional market portfolio, and the second one a conditional bond. Also recall that the variance of $\log(S_T/S_t)$ conditional on n jumps in (t, T) is $\sigma_n^2(T - t)$. Now, the nth term in the sum in (13) gives the price at time t of a conditional option (i.e., of an option that promises to pay $\max[S_T - K, 0]$ at T if n jumps occur between t and T). According to the pricing formula (13), the price of the conditional option is given by the application of the Black–Scholes formula to the prices of the con-ditional market portfolio and the conditional bond using the con-ditional variance. The price of the unconditional call is the sum of the prices of all possible conditional calls.

In the following proposition we state the price of a European put in the present economy.

Proposition 3. *Let S_t^p be the price at time t of a European put option maturing at date T and with exercise price K, which is written on the market portfolio. Then the put price at time t is*

$$S_t^p = \sum_{n=0}^{\infty} p(n) V(S_t, \pi_n, r_n, \sigma_n, (T - t)), \qquad (15)$$

where

$$V(S_t, \pi_n, r_n, \sigma_n, (T - t)) = K \exp[-r_n(T - t)]N(-d_{2n}) \\ - S_t \exp[-\pi_n(T - t)]N(-d_{1n}) \quad (16)$$

and $p(n)$, r_n, π_n, d_{1n}, d_{2n}, and σ_n are as defined in Proposition 2.

The above call and put formulas satisfy the put–call parity for Euro-pean options on assets with a constant dividend yield.[7] As $\lambda \to 0$, the process $\{S_t\}$ converges to a lognormal diffusion and the above pricing formula for the call option converges to the usual Black–Scholes formula with a constant dividend yield. As $(\sigma, \sigma_y) \to 0$, the underlying price process converges to a pure jump process with a constant jump size, which is a special case of the processes considered in Cox and Ross (1976). In this case, the above call formula converges to the Cox–Ross pure jump formula with a constant dividend yield. Setting (σ, σ_y) equal to zero in the formula (13) for a call's price, we get, after simplification,

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[7] Comparative statics exercises reveal that, typically, put prices are higher and call prices are lower with increasing risk aversion, ceteris paribus. Under risk neutrality and when $\mu_y = 0$, put and call prices are increasing in λ and σ_y. When agents are risk averse, in-the-money calls tend to lose value with increasing jump intensity and jump variance. Some of these results are obtained from numerical analysis.

$$\sum_{n=0}^{\infty} p(n)\exp[-r_n(T-t)]$$

$$\times \max(S_t \exp[(\alpha - \lambda k)(T-t) + \mu_y n] - K, 0). \qquad (17)$$

Using the expressions for the equilibrium interest rate, r, and the dividend yield, π, the above formula can be rewritten as

$$\exp[-r(T-t)] \sum_{n=0}^{\infty} p^*(n)\max(S_t \exp[(\alpha - \lambda k)$$

$$\cdot (T-t) + \mu_y n] - K, 0), \qquad (18)$$

where

$$p^*(n) = \exp[-\lambda^*(T-t)]\frac{(\lambda^*(T-t))^n}{n!}$$

and

$$\lambda^* = -\frac{(\alpha - \lambda k + \pi - r)}{(\exp[\mu_y] - 1)}.$$

This is the Cox–Ross pure jump formula for a call on a risky asset with a constant dividend yield.

2.2. Risk premium implicit in option prices

The above formulas price the risk in option cash flows arising from the possibility of continuous as well as discrete random changes in the price of the underlying asset by restricting investors' preferences. This is the reason why the risk-aversion parameter γ enters the option-pricing formulas. It is in this respect that the call formula (13) differs from the formula derived in Merton (1976a), in spite of the apparent similarities that arise on account of the similarity of our distributional assumptions to those in that study.

Merton (1976a) assumes that jumps in the prices of the underlying asset are uncorrelated with the changes in the price of the market portfolio, so that the jump risk is not priced in equilibrium. This assumption is combined with a local no-arbitrage argument to arrive at a pricing equation which asserts that, if the price of the underlying asset at time t is S, then the price of a European call maturing at T and with exercise price K is equal to[8]

$$\exp[-r(T-t)]E(\max[\check{S}_T - K, 0]|\check{S}_t = S), \qquad (19)$$

where $\{\check{S}_t\}$ is a stochastic process that evolves according to

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[8] This follows from a rewriting of equation (16) in Merton (1976a).

$$\check{S}_T = \check{S}_t \exp\left[\left(r - \frac{\sigma^2}{2} - \lambda k\right)(T - t) + \sigma(z_T - z_t) + \sum_{i=N_t+1}^{N_T} y_i\right],$$

where r is the instantaneous riskless interest rate (assumed to be constant) and the other variables have the same definition as ours. Thus, the Merton (1976a) formula equates the call-option price to the expected value (discounted at the riskless interest rate) of the option cash flow under the assumption that the underlying asset earns an instantaneous expected return equal to r. This parallels the derivation of the Black–Scholes formula in Cox and Ross (1976) using "risk-neutral" pricing. $\check{S}_T$ is the price of the underlying asset at time T under the "risk-neutral" probability measure. The formula is justified by (i) a local no-arbitrage condition, which, it is argued, allows a portfolio, having no diffusion risk and consisting of the underlying asset, riskless bonds, and the option in question to be formed; and (ii) the assumption of the uncorrelated jumps, which implies that the jump risk is not priced in equilibrium.

We use a fully stated economic equilibrium to price the options in which we are interested. Both the jump risk and the diffusion risk are priced in this equilibrium. In contrast with Equation (19) above, our formula (13) for the price of a call option on the market portfolio is equivalent to

$$S_t^c = \exp[-r(T - t)]E_t \max[S_T - K, 0]$$
$$+ \text{Cov}_t(U_c(S_T, T)/U_c(S_t, t), \max[S_T - K, 0]), \qquad (20)$$

where

$$S_T = S_t \exp\left[\left(\alpha - \frac{\sigma^2}{2} - \lambda k\right)(T - t) + \sigma(z_T - z_t) + \sum_{i=N_t+1}^{N_T} y_i\right].$$

Thus, the call price is a sum of two terms. The first term is similar to (not the same as) the Merton (1976a) formula: it is the expected cash flow on the option (discounted at the riskless interest rate which is shown to be constant in our equilibrium) when the underlying asset earns its equilibrium instantaneous expected exdividend return of α. Since we do not use the local no-arbitrage argument, we do not conclude that the option is priced as if the underlying asset earns an expected return equal to the riskless interest rate. The prices of options in our model depend on the endogenously determined price process for the underlying asset.

The second term in Equation (20) is the equilibrium price of the jump and diffusion risks implicit in the option's price: it equals the covariance of the option's payoff with the change in the marginal

utility of equilibrium aggregate wealth.[9] Under our parameterization, we can explicitly evaluate that term and do not need to assume that the jump correlation with returns on aggregate wealth is zero. That assumption would be unacceptable when the underlying asset is the market portfolio. As shown below, the covariance term is too large to be ignored in pricing options on the market portfolio.

Table 1 shows the absolute value of the covariance term [the second term of Equation (20)] as a percentage of the option prices for some representative parameter values.[10]

In Table 1, we consider combinations of low and high values for λ and σ_y. Even for a relatively moderate volatility of jump sizes ($\sigma_y = 0.05$) and a low jump frequency (λ of 0.25 translates into 1 jump in 4 years on average), the risk premiums are about 20 percent of the option prices for near-the-money options. These risk premiums increase to as much as 30 to 40 percent when the frequency of jumps or the volatility of jump sizes is high. As we would intuitively expect, our numerical analysis shows that the option risk premiums increase as the level of risk aversion in the economy or the intensity of jumps increases or as the options move out of the money.

3. Implications of Continuous Hedging in the Jump–Diffusion Economy

In a pure diffusion economy in which the Black–Scholes assumptions are met, there exists a dynamic self-financing trading strategy comprising the market portfolio and the riskless bond that replicates any claim whose cash flows are contingent on the price of the market portfolio. It follows that in an arbitrage-free economy the value of the replicating portfolio must equal the price of the claim.

Suppose now that the underlying price process has jumps of random amplitudes occurring at random time intervals and takes the form of Equation (11), and that riskless bonds of various maturities are also available for trading. As pointed out in Section 2, a dynamic hedging plan, involving only the underlying asset and riskless bonds, no longer exists as there are more sources of uncertainty than can be hedged

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[9] In equilibrium, the agent holds only the market portfolio and thus his aggregate wealth at any time t is S_t.

[10] In the numerical analysis here and elsewhere, we select a set of plausible parameter values which ensure positive equilibrium interest rates and dividend yields in the exchange economy for coefficients of risk aversion ranging from -1 to 1. These values of the risk aversion parameter cover the possible range of investor types that we wish to explore: $\gamma = 1$, for risk neutrality; $\gamma = 0$ for an agent with logarithmic utility; and for the agent who is substantially risk-averse, $\gamma = -1$. The set of parameter values fixed in the analysis are $\alpha = 0.05$, $\phi = 0.07$, $\sigma = 0.15$, $\mu_y = 0$, $S_0 = 1$, time to maturity $= 1$, and the infinite sum of the cumulative Poisson distribution is truncated at $n = 10$. For the values of λ that we consider, the option prices are not affected by truncation at higher values of n. The unit of time is one year.

Table 1
Risk premiums on calls and puts in the jump economy

	Calls		Puts	
	(% of call price)		(% of put price)	
K	$\lambda = 1, \sigma_y = 0.15$	$\lambda = 0.25, \sigma_y = 0.05$	$\lambda = 1, \sigma_y = 0.15$	$\lambda = 0.25, \sigma_y = 0.05$
0.9	24.3	14.2	30.1	23.9
1.0	31.7	19.9	24.2	18.3
1.1	40.4	26.7	19.3	13.8

$T = 1$; $S_0 = 1$; $\gamma = 0$; $\phi = 0.07$; $\alpha = 0.05$; $\sigma = 0.15$; $\mu_y = 0$. The risk premium of an option is defined to be the difference between the price of the option and the expected value of its cash flow discounted at the riskless interest rate expressed as a percentage of the price of the option. T is the time to maturity of the option; K is the exercise price; S_0 is the current price of the market portfolio; γ is the risk-aversion parameter ($\gamma = 0$ corresponds to logarithmic utility); and ϕ is the rate of time preference. α is the instantaneous expected exdividend rate of return on the market portfolio, and σ^2 is the instantaneous variance of the return on the market portfolio conditional on no jumps occurring. λ is the intensity of jumps, σ_y^2 is the variance of jumps in the process for the logarithm of the price of the market portfolio, and $\mu_y - 0.5\sigma_y^2$ is the mean size of the jumps in that process.

with two assets. In general, the Black–Scholes plan will neither be self-financing nor replicate the claim. In this section, we investigate the cost and risk implications of such continuous replication plans as applied to the market portfolio with discontinuous returns.

Consider first the case in which the agent attempts to replicate dynamically the payoff on a claim using the Black–Scholes trading strategy, with the volatility parameter assumed to be equal to the volatility of the continuous part of the asset's price movement, σ^2. In this case, the hedging plan fails to be self-financing at the times when a jump occurs (and only at such times); at these jump times, either infusion of funds is needed to accomplish the intended replication or funds can be withdrawn.

Suppose that the market supplies the agent with a jump-financing security which, at the time of a jump, reimburses him the exact amount of additional cash needed (or charges him the extra cash generated) to bring the agent's portfolio back to a position from which dynamic replication by the Black–Scholes hedging strategy can be continued. Then, this jump-financing security supports the dynamic hedging. The price of this security at any time is equal to the difference between the price of the claim (which is being replicated) and the value of the (intended) Black–Scholes replication portfolio held at that time. This price can be called the ex ante cost of the potential infusions and withdrawals of funds that this strategy will need to actually accomplish the replication. The following proposition makes the above idea rigorous.

Proposition 4. *Consider an economy in which the price of the market portfolio, S_s, evolves according to Equation (11), the market portfolio*

507

yields dividends at time t δ_t equal to πS_t, where π is a constant, and the price at time t ($t \leq T$) of a pure discount bond paying 1 unit of the numeraire risklessly at time T is $B_t \equiv \exp[-r(T - t)]$, for some constant r. Consider a claim that matures at time T and pays off $g(S_T)$ at that time for some twice differentiable $g(\cdot)$.[11]

Let $C(S_t, t)$ be the unique solution (which is twice continuously differentiable in its first argument and once continuously differentiable in its second argument) to the following partial differential equation:[12]

$$C_2 + (r - \pi)C_1 S + \tfrac{1}{2}C_{11}\sigma^2 S^2 = rC,$$

subject to the boundary condition that $C(S_T, T)$ equals $g(S_T)$.

Suppose that agents can also buy and sell a jump-financing security,[13] which has the following cash flows:

$$C(S_q, q) - C(S_{q-}, q-) - C_1(S_{q-}, q-)[S_q - S_{q-}], \qquad (21)$$

for all times q for which S_q does not equal S_{q-}, that is, the times for which there is a jump in the price process of the underlying asset. Call this security the jump-financing security associated with claim $g(S_T)$ and strategy $C(S, t)$.[14]

If there are no arbitrage opportunities in the economy, the price at time t ($t \leq T$) of a security that pays off $g(S_T)$ at time T is equal to $C(S_t, t)$ plus the value at time t of the jump-financing security associated with claim $g(\cdot)$ and strategy $C(S, t)$.

Proof. See Appendix A.

Recall that the Black–Scholes continuous hedging plan would require holding $C_1(S_{q-}, q-)$ units of the underlying asset and ($C(S_{q-}, q-) - C_1(S_{q-}, q-)S_{q-})/B_q$ units of the riskless bond at any time q. The Black–Scholes plan is not self-financing as at every jump time the portfolio that is required to be held for replication to succeed and the portfolio that can actually be purchased with the value accumulated till then do not have the same price. Infusion or withdrawal of funds is needed at the jump times. It can be directly verified that the cash flows on the jump-financing security are exactly the infusions (or withdrawals) of cash that are needed. If $g(S_T)$, the cash flow at

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[11] Later on, we will apply this and similar results to options whose terminal payoffs are not differentiable functions of S. However, these payoffs can be approximated arbitrarily closely by smooth functions. See Duffie (1988).

[12] Subscripts denote partial derivatives.

[13] This security will make the replication of claim $g(\cdot)$ possible using a trading strategy based on $C(\cdot, \cdot)$. It does not make the jump-diffusion model complete in the Harrison and Pliska (1981) sense.

[14] S_{q-} equals $\lim_{u\downarrow q} S_u$. This limit is to be understood in an almost sure sense.

Table 2
Black–Scholes replication of a put in the jump economy (assumed volatility, σ^2): Cost of cash deficits during continuous replication of a put in the jump economy

| | $\gamma = -1$ | | | | $\gamma = 1$ | | | |
| | $\lambda = 1$ | | $\lambda = 0.25$ | | $\lambda = 1$ | | $\lambda = 0.25$ | |
K	$\sigma_y = 0.15$	$\sigma_y = 0.05$	$\sigma_y = 0.15$	$\sigma_y = 0.05$	$\sigma_y = 0.15$	$\sigma_y = 0.05$	$\sigma_y = 0.15$	$\sigma_y = 0.05$
0.9	74.7	12.2	28.4	3.2	138.2	16.6	35.9	4.2
1.0	28.3	5.4	10.6	2.1	56.3	7.9	15.2	2.0
1.1	11.3	2.2	4.1	0.6	25.3	3.6	6.8	0.9

$T = 1$; $S_0 = 1$; $\phi = 0.07$; $\alpha = 0.05$; $\sigma = 0.15$; $\mu_y = 0$. The cost of cash deficits during continuous hedging of a put in the jump economy is the difference between the equilibrium value of the put [Equation (15)] and the value of the assets in the Black–Scholes portfolio for that put as a percentage of the value of the Black–Scholes portfolio. T is the time to maturity of the option; K is the exercise price; S_0 is the current price of the market portfolio; γ is the risk-aversion parameter; and ϕ is the rate of time preference. α is the instantaneous expected exdividend rate of return on the market portfolio, and σ^2 is the instantaneous variance of the return on the market portfolio conditional on no jumps occurring. λ is the intensity of jumps, σ_y^2 is the variance of jumps in the process for the logarithm of the price of the market portfolio, and $\mu_y - 0.5\sigma_y^2$ is the mean size of the jumps in that process.

maturity on the claim being replicated, is convex in S_T, then all the jump cash flows as defined in expression (21) are positive. Thus, if a call or a put option was being replicated using the Black–Scholes plan, at all jump times fund infusion would be needed for the replication to succeed, irrespective of whether the jumps were upward or downward.[15]

The equilibrium we have specified in the preceding sections meets the requirements of Proposition 4, and in that equilibrium we can price a security with cash flows $g(S_T)$ at time T. As explained above, given that equilibrium, the value of the jump-financing security for claim $g(\cdot)$ and strategy $C(\cdot, \cdot)$ at time t is simply the difference between the equilibrium value of the claim $g(\cdot)$ at time t and the value of assets in the Black–Scholes portfolio at time t, $C(S_t, t)$. Table 2 shows the values of the jump-financing security for a put option on the market portfolio (expressed as a percentage of the initial value of the Black–Scholes replication portfolio) at some representative parameter values.

The ex ante cost of cash deficits at jump times can be more than 50 percent of the initial value of the assets in the Black–Scholes replication strategy. It is apparent that the volatility of jump sizes is a more influential determinant of the ex ante cost of hedging than is the jump intensity. This suggests that it is much less expensive to maintain the Black–Scholes hedging strategy in a market with frequent jumps of small magnitude than in a market in which infrequent

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[15] It can also be shown that, for a fixed time of the jump and for a given price of the underlying asset immediately before the jump, the funds infusion is a convex function of the jump size.

jumps of large magnitude can happen. As expected, as the probability of jumps or the maturity of the option increases so does this cost.

The above discussion relates to the situation in which the volatility assumed for continuous hedging is the volatility of the continuous part of the asset's price movement, σ^2. The question arises naturally whether the cash deficits arising at jump times could be eliminated if the volatility assumed in the continuous hedging plan were higher (say, equal to $\sigma^2 + \lambda\sigma_y^2$).[16]

Let $C^*(S, t)$ be the Black–Scholes solution to the following partial differential equation:

$$C_2^* + (r - \pi)C_1^*S + \tfrac{1}{2}(\sigma^2 + \lambda\sigma_y^2)C_{11}^*S^2 = rC^*,$$

subject to the boundary condition that $C^*(S, T)$ equals $g(S_T)$.

By a natural extension of Proposition 4, it can be shown that the Black–Scholes strategy based on $C^*(S, t)$ is equivalent to a portfolio comprising one unit of the claim $g(S_T)$; long one unit of a security that pays continuously a cash flow at the stochastic rate that equals $0.5\lambda\sigma_y^2 C_{11}^*(S_{q-}, q-)S_{q-}^2$ at time q (we label this security the jump-variance security); and short one unit of a jump-financing security that pays $C^*(S_q, q) - C^*(S_{q-}, q-) - C^*_1(S_{q-}, q-)[S_q - S_{q-}]$, for all times q for which S_q does not equal S_{q-}.

Since $C_{11}^*(\cdot, \cdot)$ is positive when the claim being replicated is a put or a call, the cash flow from the jump variance security is always positive. By the same reasoning, the short position in the jump-financing security involves a cash outflow at all jump times. Thus, the Black–Scholes hedging plan with a higher assumed volatility is still not self-financing and does not accomplish the intended replication. The investment in the incremental volatility seeks to make up for the cash deficits at jump times by creating a continuous stream of cash inflows. A numerical analysis of our equilibrium option pricing formulas shows that the ex ante value of the cash deficits at jump times from strategy $C^*(S, t)$ is almost the same as the value of the continuous stream of cash inflows arising from the incremental volatility. However, this does not imply that jump time cash deficits are fully offset by the accumulation of the surplus cash inflow stream. The two cash flow streams have different risk characteristics as we show below.

The probability of substantial cash deficits at jump times is significant irrespective of whether the hedging scheme is based on C or C^*. To show this, we derive below a lower bound on the probabilities that the cash deficits at jump times will exceed a given value. Let t_1 denote the time of the first jump of the process $\{N_t\}$, and let Y_q denote

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[16] Variance per unit time of $\log(S_t/S_0)$, conditional on n jumps occurring in the interval $(0, t)$ is $\sigma^2 + n\sigma_y^2$. The expectation of this conditional variance is $\sigma^2 + \lambda\sigma_y^2$.

the difference between the change in the value of C^* and the change in the value of its Black–Scholes replication portfolio when a jump occurs at time q:

$$Y_q \equiv C^*(S_q, q) - C^*(S_{q-}, q-) - C_1^*(S_{q-}, q-)(S_q - S_{q-}).$$

If the security being replicated is a European put or a call, then we have, for $x > 0$ and with Y_q defined above,

$$\mathscr{P}\left(\sum_{0 < q \le T} Y_q > x\right) \ge \mathscr{P}(\{Y_{t_1} > x\} \cap \{N_T > 0\}).$$

The probability on the right-hand side of the above equation can be evaluated in closed form as shown in Appendix A. Table 3 displays the values of the resulting lower bound on the probability that the jump cash outflows will be greater than 1 percent of the initial value of the underlying asset, when the security being replicated is a European put. This computation is done for the Black–Scholes strategy with assumed volatility of $\sigma^2 + \lambda\sigma_y^2$ and σ^2. In the latter case, the computations for the lower-bound probability are the same except that C^* is replaced by C in the definition of Y_q.

Table 3 shows that the probability that the dynamic hedging plan calls for a funds infusion greater than 1 percent of the initial value of the underlying portfolio can be as much as 10–30 percent and remains nonnegligible even for low values of λ and σ_y. More notably, the table shows that the difference between the values of the lower-bound probabilities for the Black–Scholes plan with higher volatility and those for the plan with volatility equal to σ^2 is small. This demonstrates that the effect of incremental volatility in reducing the cash deficits is not very pronounced. The probability figures in this table reaffirm the earlier observation that the variance of jump sizes has a far greater effect on cash deficits at jump times than does the frequency of jumps.

Since the incremental volatility does not seem to reduce the probability of large cash deficits appreciably, the only advantage of the dynamic hedging plan with increased volatility lies in the surplus cash inflow stream that it generates. Recall that this inflow is at a stochastic rate of $0.5\lambda\sigma_y^2 C_{11}^*(S_{q-}, q-)S_{q-}^2 \equiv X_q$ at time q. In general, this stream of cash inflows does not offset the cash deficits incurred at jump times. The key determinant of the size of a cash shortfall at a given jump time is the size of the jump, which is independent of past innovations due to the Brownian motion. Therefore, the amount of the cash deficit at the jump time is unrelated to the sum of the incremental cash inflows that have been accumulated till the time of the jump. The investment in the incremental volatility does not pro-

Table 3
Probability of large funds infusion during continuous hedging in the jump economy

σ_y	BS hedging with $\sigma^2 + \lambda\sigma_y^2$		BS hedging with σ^2	
	$\lambda = 1.00$	$\lambda = 0.25$	$\lambda = 1.00$	$\lambda = 0.25$
.15	.30	.11	.31	.11
.05	.04	.02	.05	.02

$T = 1$; $S_0 = 1$; $K = 1$; $\gamma = 1$; $\phi = 0.07$; $\alpha = 0.05$; $\sigma = 0.15$; $\mu_y = 0$. BS hedging is Black–Scholes hedging strategy for a European put. The entries in the cells are the lower-bound probabilities that funds infusion at jump times exceeds 1 percent of the initial portfolio value. T is the time to maturity of the option; K is the exercise price; S_0 is the current price of the market portfolio; γ is the risk-aversion parameter; and ϕ is the rate of time preference. α is the instantaneous expected exdividend rate of return on the market portfolio, and σ^2 is the instantaneous variance of the return on the market portfolio conditional on no jumps occurring. λ is the intensity of jumps, σ_y^2 is the variance of jumps in the process for the logarithm of the price of the market portfolio, and $\mu_y - 0.5\sigma_y^2$ is the mean size of the jumps in that process.

vide any insurance against the random jump size. Moreover, jumps of a considerable magnitude can occur at any time before the claim matures. Obviously, the continuous stream of surplus cash inflows does not provide very much of a cushion against cash deficits at jump times if the jumps occur early in the life of the claim. These observations are confirmed by a simulation analysis.[17] Not surprisingly, our simulations also show that the surplus cash flows from the incremental volatility are always strictly positive but do not usually take on large values. The jump-time deficits, however, are zero with a strictly positive probability (the probability of no jumps) but when they do occur they can be quite large.

4. Conclusions

In the seminal articles by Black and Scholes (1973) and Cox and Ross (1976), the prices of contingent claims are determined in complete markets. Merton (1976a) considers pricing options in an incomplete market, generalizing the pure jump model of Cox and Ross to include uncertainties in jump sizes and jump times and encompassing the diffusion model of Black and Scholes. Recognizing that the principle of arbitrage behind the Black–Scholes derivation is inapplicable when there are multiple sources of uncertainty, Merton assumes that the jump component of an asset's return is uncorrelated with the return on the market portfolio before applying a local no-arbitrage argument to derive an option-pricing formula. This article dealt with the valuation of options on the portfolio of all risky assets in the economy.

[17] The simulation is for puts on the market portfolio maturing in six months. Five hundred sample paths of the process given by Equation (11) are drawn. Given a price path for the market portfolio, the discrete counterparts of the integral for the surplus cash stream and the cash deficits are computed using the closed-form expressions for C^*, C_T^*, and C_{11}^*.

The risk of jumps in the underlying asset's return is clearly not diversifiable in this case. As neither the no-arbitrage argument nor Merton's approach is appropriate in this context, we derived the pricing equations for the European call and put options on the market portfolio by placing restrictions on investors' preferences. The option formulas are internally consistent and are reducible to the Black–Scholes formula and the Cox–Ross formula.

The pricing equation for the European put was employed to analyze the Black–Scholes hedging strategy in a jump–diffusion environment in which it is not, in general, self-financing. We showed that to maintain the Black–Scholes hedging strategy in a jump–diffusion model always requires additional investments at jump times, irrespective of whether the jumps in asset prices are upward or downward. If the Black–Scholes replicating portfolio is constructed for a volatility equal to the volatility of the diffusion component, the additional investments required at jump times can be large and the ex ante cost of these investments is considerable. To construct the replicating portfolio at a volatility inclusive of part or all of the volatility due to the jump component, additional initial investment is necessary. This additional investment generates surplus cash inflows along the continuous portion of the sample path in the asset's price. Since the jumps in the asset's price might be potentially big, net cash outlays may still be needed at jump times. In volatile markets the surplus cash stream is likely to provide a poor hedge against the risk of large cash outlays needed at jump times for the replicating portfolio to succeed. Our analysis also revealed that the variance of jump sizes is a more important determinant of the cost and risk of large jump-time cash deficits than is the frequency of jumps; and, thus, jump effects are expected to be important in economies with considerable uncertainty about jump sizes even if the probability of a jump occurring is small.

Appendix A

Incompleteness of the jump–diffusion model with two traded assets

We show that a model in which the uncertainty is generated by a Brownian motion and an independent Poisson process is not complete in the Harrison and Pliska (1981) sense, if trading can take place only in a riskless asset and a risky asset with a diffusion price process with jumps. Our example below is similar to the one given in Harrison and Pliska (1981, Section 6.3).

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a complete probability space on which a standard Brownian motion $\{z_t, 0 \leq t \leq T\}$ and a Poisson process $\{N_t, 0 \leq t \leq T\}$ of rate λ are defined. $\{N_t\}$ and $\{z_t\}$ are mutually independent. Let

$$x_t \equiv z_t + n_t \equiv z_t + N_t - \lambda t. \tag{A1}$$

Consider an economy in which two securities are available for continuous trading between 0 and T: a riskless bond maturing at T whose price at time t is $B_t \equiv 1$, $0 \le t \le T$; and a risky asset whose price at time t, S_t, is given by $\mathcal{E}(x)_t$, the exponential process associated with $\{x_t\}$. It can be checked that

$$S_t \equiv \mathcal{E}(x)_t = \exp[z_t - 0.5t - \lambda t + \log(2) N_t]. \tag{A2}$$

This price process is in the general class of processes considered in Aase (1988).

The investors are endowed with the filtration $\mathcal{G}_t = \sigma(S_q, 0 \le q \le t)$. We first observe that $\mathcal{G}_t \equiv \mathcal{F}_t \equiv \sigma(z_q, N_q, 0 \le q \le t)$. This follows as $N_t = \Sigma_{0 < q \le t} 1_{[\{S_q - S_{q^-}\} \ne 0]}$ and z_t can be inferred from S_t and N_t. Here, 1_A for a set A denotes the indicator variable of that set. That is to say that the observation of the paths of the process $\{S_t\}$ is equivalent to observing the paths of $\{z_t\}$ and $\{N_t\}$ since the jumps of $\{S_t\}$ coincide with the jumps of $\{N_t\}$ and S_t is functionally related to N_t and z_t. It is also immediate that $\mathcal{G}_t \equiv \sigma(S_q, 0 \le q \le t) \equiv \sigma(x_q, 0 \le q \le t)$.

Since x_t is a $\mathcal{F}_t$ martingale, it is a $\mathcal{G}_t$ martingale. This implies that the discounted risky asset price process $Z_t \equiv S_t/B_t \equiv S_t$ is a $\mathcal{G}_t$ martingale, and so $\mathcal{P}$ itself can be taken as the reference probability measure. Also $y_t \equiv z_t + \log(2) n_t$ is a $\mathcal{F}_t$ and, therefore, a $\mathcal{G}_t$ martingale. S_t, x_t, and y_t are all square integrable.

We show that y_t cannot be written as a stochastic integral with respect to x_t. This will imply that y_t cannot be written as a stochastic integral against Z_t so that, by the theorem in Harrison and Pliska (1983), the above model is not complete. The proof is in two steps.

Proof. (1) y_t cannot be written as a stochastic integral against x_t, that is, there does not exist a process $\{\alpha_t\}$, which is $\mathcal{G}_t$ predictable and with $E \int_0^T \alpha_t^2 \, d[x]_t < \infty$, such that

$$y_t = y_0 + \int_0^t \alpha_s \, dx_s = \int_0^t \alpha_s \, dx_s. \tag{A3}$$

Note that $z_0 = n_0 = 0$. The square integrability restriction on integrands corresponds to equations (4.1) and (2.2) in Aase (1988). See also Duffie and Huang (1985, section 4).

The proof is by contradiction. Suppose that $\{\alpha_t\}$ with the above properties exists. Then

$$E \int_0^T \alpha_t^2 \, d[z]_t < \infty \quad \text{and} \quad E \int_0^T \alpha_t^2 \, d[n]_t < \infty, \tag{A4}$$

since $d[x]_t = d[z]_t + d[n]_t$. Also,

$$\int_0^t \alpha_s \, dx_s = \int_0^t \alpha_s \, dz_s + \int_0^t \alpha_s \, dn_s$$

$$= 1 \int_0^t dz_s + \log(2) \int_0^t dn_s. \tag{A5}$$

The last equality in (A5) follows from the definition of y_t. Equation (A5) implies, by lemma A.2 in Duffie and Huang (1985),

$$\mathcal{P}\,[\exists\, t \in [0,\, T]\colon \alpha_t = 1 \text{ and } \alpha_t = \log(2)] > 0, \tag{A6}$$

which is not possible.

Conditions of lemma A.2 in Duffie and Huang (1985) are satisfied as $\{z_t\}$ and $\{n_t\}$ constitute a finite set of elements of $\mathcal{M}^2_{\mathcal{P}}$, the set of square-integrable $(\mathcal{F}_t, \mathcal{P})$ martingales, with the representation property of their Theorem 4.1 and as (A4) holds. The proof of this representation property can be found in proposition 7.3 of Wong and Hajek (1985).

(2) If $\{y_t\}$ cannot be written as a stochastic integral with respect to $\{x_t\}$, then it cannot be written as a stochastic integral against $S_t = \mathcal{E}(x)_t$, since, if it could be, then $y_t = \int_0^t \theta_s \, dS_s$ would hold for some predictable process $\{\theta_t\}$, such that $E \int_0^T \theta_s^2 \, d[S]_s < \infty$. But $E \int_0^T \theta_s^2 \, d[S]_s < \infty$ and the predictability of processes $\{\theta_t\}$ and $\{S_{t-}\}$ imply that $\{\theta_t S_{t-}\}$ is a valid integrand against $\{x_t\}$ and $\int_0^t \theta_s \, dS_s = \int_0^t \theta_s S_{s-} \, dx_s$. This contradicts Part (1) above. Therefore, this implies that in the above economy, it is not possible to replicate a claim maturing at time T with the payoff given by $y_T \equiv \log(S_T \exp[\theta T])$ for $\theta = 0.5 + \lambda - \lambda \log(2)$.

The price process postulated in this example is not arbitrary. It can be supported in a general equilibrium. It is the equilibrium price process in the representative agent economy of the kind described in Appendix B (the economy with no intermediate consumption) in which $\gamma = 1$, $\alpha = 0$, $\sigma = 1$, $\mu_y = \log(2)$, $\sigma_y = 0$, and $\delta_0 = 1$.

Proof of Proposition 4

The filtration underlying all the computations for this proposition is $\{\mathcal{F}_t\}$, where $\mathcal{F}_t = \sigma(S_s, 0 \le s \le t)$. We will show that there exists a predictable trading strategy that replicates the claim and is self-financing.

Consider the predictable strategy that has at time q, $C_1(S_{q-}, q-)$ units of the underlying asset, $(C(S_{q-}, q-) - C_1(S_{q-}, q-)S_{q-})/B_q$ units of the pure discount riskless bond maturing at T and 1 unit of the jump-financing security associated with claim $g(S_T)$ and strategy $C(S, t)$. Let J_t be the value of the jump-financing security at time t. The value at any time $t \le T$ of assets in the trading strategy is $C_t +$

J_t by construction. The strategy would be self-financing if the value of assets in the strategy at any time t equals the value of the assets at time 0 plus any capital gains or losses (including the cash flows on the jump-financing security) and dividends realized between 0 and t. That is to say, letting $G_t = S_t + \int_0^t \delta_q \, dq$, we must have

$$C_t + J_t = C_0 + J_0 + \int_0^t C_1(S_{q-}, q-) \, dG_q$$

$$+ \int_0^t \frac{[C(S_{q-}, q-) - C_1(S_{q-}, q-)S_{q-}]}{B_q} \, dB_q$$

$$+ \int_0^t dJ_q$$

$$+ \sum_{0 < q \le t} [C(S_q, q) - C(S_{q-}, q-)$$

$$- C_1(S_{q-}, q-)[S_q - S_{q-}]]. \tag{A7}$$

The last term in the above equation is the cumulative cash flow on the jump-financing security up to time t. However, since δ_q equals πS_q,

$$\int_0^t C_1(S_{q-}, q-) \, dG_q = \int_0^t C_1(S_{q-}, q-) \, dS_q$$

$$+ \int_0^t C_1(S_{q-}, q-)\pi S_{q-} \, dq. \tag{A8}$$

Also, as $dB_q = rB_q \, dq$, and from the partial differential equation stated in the proposition,

$$r[C(S_{q-}, q-) - C_1(S_{q-}, q-)S_{q-}]$$

$$= C_2(S_{q-}, q-) + \tfrac{1}{2}\sigma^2 C_{11}(S_{q-}, q-)S_{q-}^2 - \pi C_1(S_{q-}, q-)S_{q-},$$

we have that

$$\int_0^t \frac{[C(S_{q-}, q-) - C_1(S_{q-}, q-)S_{q-}]}{B_q} \, dB_q$$

$$= \int_0^t C_2(S_{q-}, q-) \, dq$$

$$+ \int_0^t \left[\frac{1}{2}\sigma^2 C_{11}(S_{q-}, q-)S_{q-}^2 - \pi C_1(S_{q-}, q-)S_{q-}\right] dq. \tag{A9}$$

Equations (A8) and (A9), and the fact the J_t equals $J_0 + \int_0^t dJ_q$, imply that for the trading strategy to be self-financing we must have

$$
\begin{aligned}
C_t = C_0 &+ \int_0^t C_1(S_{q-}, q-) \, dS_q \\
&+ \int_0^t \left[C_2(S_{q-}, q-) + \frac{1}{2}\sigma^2 C_{11}(S_{q-}, q-)S_{q-}^2 \right] dq \\
&+ \sum_{0<q\leq t} \Big[C(S_q, q) - C(S_{q-}, q-) \\
&\qquad\qquad - C_1(S_{q-}, q-)[S_q - S_{q-}] \Big].
\end{aligned}
\tag{A10}
$$

However, by the Doleans–Dade–Meyer generalization of Ito's formula to functions of semimartingale processes [see Elliott (1982, theorem 12.13)], we have that C_t equals

$$
\begin{aligned}
C_0 &+ \int_0^t C_1(S_{q-}, q-) \, dS_q \\
&+ \int_0^t C_2(S_{q-}, q-) \, dq \\
&+ \frac{1}{2} \int_0^t C_{11}(S_{q-}, q-) \, d\langle S^c \rangle_q \\
&+ \sum_{0<q\leq t} [C(S_q, q) - C(S_{q-}, q-) - C_1(S_{q-}, q-)[S_q - S_{q-}]],
\end{aligned}
\tag{A11}
$$

where the process $\langle S^c \rangle_q$ is the predictable quadratic variation process associated with the continuous martingale part of the process S_t. [See Elliot (1982, chapters 10–12) for a detailed description.]

It can be checked that the predictable quadratic variation of the continuous martingale part of the process S_t as defined in the proposition is

$$
\langle S^c \rangle_q = \sigma^2 \int_0^q S_{s-}^2 \, ds.
\tag{A12}
$$

Using relation (A12) to substitute for $d\langle S^c \rangle_q$ in expression (A11) we see that by the generalized Ito rule the right-hand side of Equation (A10) is indeed equal to the left-hand side of that equation. This shows that the strategy is self-financing. The fact that the strategy replicates the claim can be directly verified.

Computation of $P(\{Y_{t_1} > x\} \cap \{N_T > 0\})$

$P(\{Y_{t_1} > x\} \cap \{N_T > 0\})$ equals

$$\int_0^T \int_{-\infty}^{+\infty} [P(Y_{t_1} > x \,|\, N_T > 0,\ t_1 = t,\ S_{t_{1-}} = \exp[z])$$

$$\times f(z \,|\, N_T > 0,\ t_1 = t) g(t \,|\, N_T > 0)] \, dz \, dt, \tag{A13}$$

where Y_{t_1} is the jump cash flow at time t_1, $f(\cdot \,|\, t_1 = t)$ is the normal density function with mean $(\alpha - \lambda k - 0.5\sigma^2)t$ and variance $\sigma^2 t$,

$$g(t \,|\, N_T > 0) = \frac{\lambda \exp[-\lambda t]}{1 - \exp[-\lambda T]},$$

and

$$\mathscr{P}(Y_{t_1} > x \,|\, N_T > 0,\ t_1 = t,\ S_{t_{1-}} = \exp[z])$$

$$= 1 - \mathscr{N}[\log y_u(z,t)] + \mathscr{N}[\log y_d(z,t)].$$

Here $y_u(z,t)$ and $y_d(z,t)$ are unique solutions to

$$C^*(y \exp[z],\ t) - C^*(\exp[z],\ t)$$

$$- C_1^*(\exp[z],\ t)(y - 1)\exp[z] = x, \tag{A14}$$

over the ranges $(1, \infty)$ and $(0, 1)$, respectively. $\mathscr{N}(\cdot)$ is the cumulative probability distribution of a normal random variable with mean $(\mu_y - 0.5\sigma_y^2)$ and variance σ_y^2. The left-hand side of Equation (A14) is monotonically decreasing in y over $(0, 1]$ and monotonically increasing in y over $(1, \infty)$; hence, unique solutions to the said equations exist for all positive x.

The probability in the integral in Equation (A13) is the probability that the absolute jump size at the time of the first jump is greater than what would have caused the jump cash flow to equal x. This probability is conditional on $\{N_T > 0\}$, the time of first jump, t_1, being equal to t and the asset price just before the first jump being equal to $\exp[z]$. $f(\cdot \,|\, \cdot)$ is the density of $\log(S_{t_{2-}})$ conditional on $\{N_T > 0\}$ and on the first jump time being equal to t. $g(\cdot)$ is the density of the first jump time conditional on $\{N_T > 0\}$.

Appendix B: Interpreting $\{\delta_t\}$ as Information about Terminal Cash Flows

Below, we briefly describe an alternative method of pricing options on the share of a representative firm in the economy, where the jumps in the price of the share are induced by jumps in the information about the terminal cash flows paid off by the firm. We consider an

exchange economy in which economic activity takes place in the time interval $[0, T]$. There is one firm and a representative agent. The firm is financed by one equity share and it pays out a terminal divided δ_T at time T, and the agents in the economy receive continuous information about δ_T by an exogenously given stochastic process $\{\delta_t\}$ on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$. The share of the firm, a riskless bond maturing at T and options on the share maturing at T are competitively traded at every $t \in [0, T]$. The riskless bond serves as the numeraire, so all prices are relative to the price of a riskless security.

The representative agent maximizes expected utility of consumption at time T,

$$EU(c_T), \tag{B1}$$

subject to a budget constraint, where c_T equals his wealth at time T. $U(\cdot)$ is assumed, as before, to be of the form $(1/\gamma)c^\gamma$. The maximization takes place over all $\mathcal{F}_t$ predictable, locally bounded, and self-financing trading strategies that are budget feasible at time 0 and satisfy the nonnegative wealth constraint at every instant.

Now suppose that the information arrival about the terminal dividend is given by the following compound diffusion–Poisson process:

$$\delta_t = \delta_0 \exp\left[\left(\alpha - \frac{\sigma^2}{2} - \lambda k\right)t + \sigma z_t + \sum_{i=1}^{N_t} y_i\right], \tag{B2}$$

where $\{z_t\}$ is a standard Gauss–Wiener process, $\{N_t\}$ is a Poisson process with parameter λ, and y_i is the random size of the ith jump in the process $\{\log \delta_t\}$. For all i, y_i is normally distributed with mean $(\mu_y - 0.5\sigma_y^2)$ and variance σ_y^2. The mean jump displacement k is equal to $\exp[\mu_y] - 1$. The random variables $\{N_t, t \geq 0\}$, $\{z_t, t \geq 0\}$, and $\{y_i, i \geq 1\}$ are mutually independent and y_i is independent of y_j, for $i \neq j$.

The Euler equations for the above economy yield the equilibrium price of the equity share at time t, S_t, to be $S_t = E_t\delta_T\gamma / E_t\delta_T^{\gamma-1}$, and the price at time t of a call option maturing at T and having an exercise price of K is given by $S_t^c = E_t\delta_T^{\gamma-1} \max[\delta_T - K, 0]/E_t\delta_T^{\gamma-1}$. Recall that given our assumptions, $\delta_T^{\gamma-1}$ is almost surely strictly positive. Evaluating the expectations in these pricing equations for the information process specified in Equation (B2), we get $S_t = \delta_t \exp[c(T - t)]$, where

$$c = (\alpha - \lambda k) + (\gamma - 1)\sigma^2 + \lambda \left[\exp[\gamma\mu_y + 0.5\gamma(\gamma - 1)\sigma_y^2] \right.$$
$$\left. - \exp[(\gamma - 1)\mu_y + 0.5(1 - \gamma)(2 - \gamma)\sigma_y^2]\right]$$

and

$$S_t^c = \sum_{n=0}^{\infty} p(n) W(S_t, k_n, f_n, \sigma_n, (T - t)), \tag{B3}$$

where

$$p(n) = \frac{\exp[-\lambda(T - t)][\lambda(T - t)]^n}{n!}$$

and

$$W(S_t, k_n, f_n, \sigma_n, (T - t)) = S_t \exp[(-k_n)(T - t)]N(d_{1n})$$
$$- K \exp[(-f_n)(T - t)]N(d_{2n}),$$

with

$$k_n = \lambda\{\exp[\gamma\mu_y + 0.5\gamma(\gamma - 1)\sigma_y^2] - 1\}$$

$$- (\gamma\mu_y + 0.5\gamma(\gamma - 1)\sigma_y^2)\frac{n}{T - t},$$

$$f_n = \lambda\{\exp[(\gamma - 1)\mu_y + 0.5(1 - \gamma)(2 - \gamma)\sigma_y^2] - 1\}$$

$$- ((\gamma - 1)\mu_y + 0.5(\gamma - 2)(\gamma - 1)\sigma_y^2)\frac{n}{T - t},$$

$$\sigma_n^2 = \sigma^2 + \sigma_y^2\frac{n}{T - t},$$

$$d_{1n} = \frac{\ln(S_t/K) + (f_n - k_n + \frac{1}{2}\sigma_n^2)(T - t)}{\sigma_n\sqrt{T - t}},$$

$$d_{2n} = \frac{\ln(S_t/K) + (f_n - k_n - \frac{1}{2}\sigma_n^2)(T - t)}{\sigma_n\sqrt{T - t}}.$$

The formula for the price of a put option has the same functional form as that in Proposition 3 with the parameters redefined as in Equation (B3).

The above formula reduces to the Black–Scholes formula (with zero interest rate and no dividends) when $\lambda = 0$, and reduces to the Cox–Ross formula (with zero interest rate and no dividends) when σ and σ_y are both equal to zero.

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