Qualifier Question 2010

Return Bounds
For any asset with a dividend stream D_t and whose price process P_t^j, define the returns

$$r_{t+1}^j = \frac{P_{t+1}^j + D_{t+1}^j}{P_t^j}$$

These are gross returns; think of numbers like 1.06 for equity. Therefore, we can restrict attention to random variables that are strictly positive. The risk free asset has return r_{t+1}^1 earned from period t to $t+1$. Again, this is a gross-rate so think 1.01 (The risk-free asset has $D_{t+1}^1 = 1$ and pays 0 at all other periods and has price B_t.) The absence of arbitrage implies there exists a positive random variable, m_t, called the stochastic discount factor such that $E_t[m_t r_{t+1}^j] = 1$ for all $j = 1, 2...$.

[Hints: Lots of parts here. Skim it all before you start. Write something(!) for each part. Some algebra properties at the end might help.]

(a) Consider a complete markets economy and a representative agent with CRRA utility. Preferences are:

$$U(c_0, c_1, ...) = E_0 \sum_{t=0}^{\infty} \beta^t c_t^\alpha / \alpha$$

log of endowment growth $\log x_{t+1} = \log(e_{t+1}/e_t) \sim NID(\bar{x}, \sigma_x^2)$. What is the pricing kernel, m_{t+1} in this economy? [Be brief. You can state this without much derivation. This is just to get us rolling.]

(b) Alvarez and Jerman: Suppose there was an asset (call it $*$) in the economy whose return happened to be $r_{t+1}^* = m_{t+1}^{-1}$. Show that the absence of arbitrage implies that for all assets in the economy, $E_t[\log r_{t+1}^*] \geq E_t[\log r_{t+1}^j]$ (where all returns are a strictly positive random variable).

[Hint: Time is short, so let us simplify. Consider a one-period version of the model with a finite number of states, $z \in \{1, ..., Z\}$ so returns are $r(z)$ and discount factor $m(z)$. In this setting, think of "choosing" the random $r(z)$ with the largest expected log return.]

(c) Alvarez and Jerman (AJ Bound). Show that absence of arbitrage implies that

$$\log E_t[m_{t+1}] - E_t[\log m_{t+1}] = E_t[\log r_{t+1}^j - \log r_{t+1}^1]$$

[Hint: Start with the previous result]

[0]Version-December 18, 2009- 12 : 59

(d) Hansen-Jagannathan (HJ Bound). Show that the absence of arbitrage implies:

$$\frac{V_t(m_{t+1})^{0.5}}{E_t[m_{t+1}]} \geq \frac{E_t[r^j_{t+1} - r^1_t]}{V_t[r_{t+1}]^{0.5}}$$

where V_t and E_t are conditional moments. [Hint: Nothing fancy is needed here. A couple of lines is all that is needed. If you get stuck, explain your logic. You can also restrict yourself to the one-period finite-state case as in (b) if you like.]

(e) In the setting like (a), where $\log m_{t+1} \sim NID(\bar{m}, \sigma^2_m)$ and $\log r^j_{t+1} \sim NID(\bar{r}^j, \sigma^2_j)$. Compare AJ and HJ bounds. Why are they (roughly) pointing to the same conclusion? [Hint: Calculate the left-hand side of HJ and AJ]

(f) In what setting would AJ and HJ provide quantitatively different implications? Why? [Open ended discussion here. Keep it brief.]

FYI:

- $\log z \sim N(\bar{z}, \sigma^2_z)$ implies $E[\exp(z)] = \exp(\bar{x} + 0.5\sigma^2_z)$
- $\log z \sim N(\bar{z}, \sigma^2_z)$ implies $\frac{V(\exp(z))^{0.5}}{E[\exp(z)]} = (\exp(\sigma^2_z) - 1)^{0.5} \approx \sigma_z$
- For any z and x random variables, $\text{cov}(x, z) = E[xz] - E[x]E[z]$

SOLUTIONS – DO NOT INCLUDE WITH EXAM, of course

(a) The question did not ask much here, so just the first line is all that is needed.

$$\begin{aligned} m_{t+1} &= \beta x_{t+1}^{\alpha-1} \\ \log m_{t+1} &= \log\beta + (\alpha-1)\log x_{t+1} \\ \log m_{t+1} &\sim NIID(\bar{m}, \sigma_m^2) \\ & \quad \bar{m} = \log\beta + (\alpha-1)\bar{x} \\ & \quad \sigma_m^2 = (\alpha-1)^2\sigma_x^2 \end{aligned}$$

(b) Drop all the "t" subscripts and focus on the case of $z \in \{1, ..., Z\}$ states with probabilities $p(z)$. Choose a random variable $r(z) > 0$ to solve:

$$\begin{aligned} \max_{r(z)} & \quad E\log r(z) \\ s.t. & \quad E[m(z)r(z)] = 1 \end{aligned}$$

The first order condition is

$$p(z)r(z)^{-1} = \lambda p(z)m(z)$$

This implies that $r(z) = m(z)^{-1}\lambda^{-1}$. Just plug $r(z) = m(z)^{-1}$ into the constraint and notice it holds; hence the multiplier $\lambda = 1$ (this last bit is not too important).

(c) (1) Start with the previous result; (2) The risk-free rate is not random; (3) $r_{t+1}^1 = (E[m_{t+1}])^{-1}$

$$\begin{aligned} E_t[\log r_{t+1}^*] &\geq E_t[\log r_{t+1}^j] \\ E_t[\log r_{t+1}^*] - \log r_{t+1}^1 &\geq E_t[\log r_{t+1}^j] - \log r_{t+1}^1 \\ E_t[\log m_{t+1}^{-1}] - \log(E[m_{t+1}]^{-1}) &\geq E_t[\log r_{t+1}^j - \log r_{t+1}^1] \\ \log E[m_{t+1}] - E_t[\log m_{t+1}] &\geq E_t[\log r_{t+1}^j - \log r_{t+1}^1] \end{aligned}$$

(d) I am hoping this one is easier. Start with $E[mr^j] = 1$ and $E[mr^1] = 1$ to get $E[m(r^j - r^1)] = 0$. (We can ignore all the time subscripts for the moment)

$$\begin{aligned} E[m(r^j - r^1)] &= 0 \\ \text{cov}(r^j, m) + E[r^j - r^1]E[m] &= 0 \\ \frac{\text{cov}(r^j, m)}{V[m]^{0.5}V[r^j]^{0.5}} + \frac{E[r^j - r^1]}{V[r^j]^{0.5}}\frac{E[m]}{V[m]^{0.5}} &= 0 \\ \frac{E[r^j - r^1]}{V[r^j]^{0.5}}\frac{E[m]}{V[m]^{0.5}} &= -\frac{\text{cov}(r^j, m)}{V[m]^{0.5}V[r^j]^{0.5}} \end{aligned}$$

But $|\frac{\text{cov}(r^j,m)}{V[m]^{0.5}V[r^j]^{0.5}}| < 1$ and focus on the case of $E[r^j - r^1] > 0$

$$\begin{aligned}
\frac{E[r^j - r^1]}{V[r^j]^{0.5}} \frac{E[m]}{V[m]^{0.5}} &= \left|\frac{\text{cov}(r^j, m)}{V[m]^{0.5}V[r^j]^{0.5}}\right| \\
\frac{E[r^j - r^1]}{V[r^j]^{0.5}} \frac{E[m]}{V[m]^{0.5}} &\leq 1 \\
\frac{E[r^j - r^1]}{V[r^j]^{0.5}} &\leq \frac{V[m]^{0.5}}{E[m]}
\end{aligned}$$

(Projection argument works as well)

(e) In the log-normal setting, both AJ and HJ are pointing to the same thing: large excess returns imply a high volatility to the pricing kernel. For HJ:

$$\frac{V[m]^{0.5}}{E[m]} \approx \sigma_m$$

(see "FYI's"). For AJ,

$$\log E[m_{t+1}] - E_t[\log m_{t+1}] = \bar{m} + 0.5\sigma_m^2 - \bar{m} = 0.5\sigma_m^2$$

(f) The AJ bound $\log E[m_{t+1}] - E_t[\log m_{t+1}]$ will pick up higher order moments of the distribution of m (Note that $L(m) = \log E[m_{t+1}] - E_t[\log m_{t+1}] > 0$ is an Entropy measure of the dispersion in the random variable m). So any modeling assumption that implies a pricing kernel that departure to from log-normality will make AJ and HJ bounds, arguably, quantitatively different. For example, stochastic volatility, disaster risk, etc. (Of course this will remain a quantitative question since all of these features do show up in the second moment of the kernel as well). And, if you want bonus points, mention non-expected utility preferences. These also may have a big impact on the higher order moments of the equilibrium pricing kernel.