AP
7/10/08
1:00 pm

Can time-varying risk of rare disasters explain aggregate stock market volatility?

Jessica A. Wachter*

First Draft: April 11, 2008

Current Draft: July 2, 2008

Abstract

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*Department of Finance, The Wharton School, University of Pennsylvania, 3620 Locust Walk, Philadelphia, PA 19104 and NBER; Tel: (215) 898-7634; Email: jwachter@wharton.upenn.edu; `http://finance.wharton.upenn.edu/~ jwachter`. I am grateful for helpful comments from Robert Barro, John Campbell, Greg Duffee and Pietro Veronesi. Thomas Plank and Leonid Spesivtsev provided excellent research assistance.

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Abstract

What role does the risk of a rare disaster play in explaining the high equity premium and stock market volatility? This paper investigates this question in a continuous-time model in which the risk of a rare disaster varies over time. I build on the work of Barro (2006), who shows that introducing a small probability of a large negative shock into the endowment process can lead to a large equity premium, even when the representative agent has low risk aversion. I generalize this work along two dimensions: the probability of a rare disaster is allowed to vary over time and the representative agent is allowed to have recursive preferences. These generalizations allow the model to explain the observed volatility of stock returns, At the same time, the model preserves the high equity premium and low riskfree rate found by Barro, while maintaining low volatility for the government bill rate.

1 Introduction

The magnitude of the expected excess return on stocks relative to bonds (the equity premium) constitutes one of the major puzzles in financial economics. As Mehra and Prescott (1985) show, the standard endowment economy model predicts an equity premium that is far too small. One proposed explanation is that the return on equities is high to compensate investors for the risk of a rare disaster (Reitz (1988)). An open question has therefore been whether the risk is sufficiently high, and the rare disaster sufficiently severe, to quantitatively explain the equity premium. Recently, however, Barro (2006) shows that it is possible to explain the equity premium using such a model when the probability of a rare disaster is calibrated to international data on large economic declines.

While the models of Reitz (1988) and Barro (2006) advance our understanding of the equity premium, they fall short in other respects. Most importantly, these models predict that the volatility of stock market returns equals the volatility of dividends. Numerous studies have shown, however, that this is not the case. In fact, there is excess stock market volatility; the volatility of stock returns far exceeds that of dividends (e.g. Shiller (1981), LeRoy and Porter (1981), Keim and Stambaugh (1986), Campbell and Shiller (1988), Cochrane (1992), Hodrick (1992)). While the models of Barro and Reitz address the equity premium puzzle, they do not address this volatility puzzle.

This paper proposes two modifications to the disaster risk model of Barro (2006). First, rather than being constant, the probability of a rare disaster is stochastic and varies over time. Second, the representative agent, rather than having power utility preferences, has recursive preferences. I show that such a model can generate volatility of stock returns close to that in the data at reasonable values of the underlying parameters. Moreover, high stock market volatility does not occur because of high volatility in all assets; the volatility of the government bill rate remains low, as in the data.

Both time-varying disaster probabilities and recursive preferences are necessary to match the model to the data. Time-varying disaster probabilities are important because they produce time-varying discount rates. If the probability of a rare disaster rises, the premium that investors require to hold equities also rises. Thus the rate at which investors discount future cash flows rises, and stock prices fall. Through this mechanism, a time-varying probability of a rare disaster generates excess stock market volatility.

Clearly time-varying disaster probabilities are important; also important, however, is allowing for recursive preferences. These preferences, introduced by Kreps and Porteus (1978) and Epstein and Zin (1989) retain the appealing scale-invariance of power utility, but allow for separation between the willingness to take on risk and the willingness to substitute over time. Power utility requires that these are driven by the same parameter. As I show, power utility implies that the response of the riskfree rate to a change in the disaster probability exceeds the response of the equity premium. This generates counterfactual predictions. Increasing the agent's willingness to substitute over time reduces the effect of the disaster probability on the riskfree rate. With recursive preferences, this can be accomplished without simultaneously reducing the agent's risk aversion.

In what follows, I solve two models. First, to highlight the importance of recursive utility, I solve a model in which the disaster probability is constant and the agent has recursive preferences. To enable comparison with the second model, I assume continuous time. Disaster risk is modeled by introducing a Poisson process for jumps into the standard diffusion model for consumption growth (as in Naik and Lee (1990)) and recursive preferences are modeled following Duffie and Epstein (1992).

The second model is the main contribution of the paper and allows for time-varying disaster probabilities and recursive utility with unit elasticity of intertemporal substitution (EIS). The assumption that the EIS is equal to 1 allows the model to be solved in closed form up to a set of ordinary differential equations. A time-varying disaster probability is modeled by allowing the intensity for jumps to follow a square-root process (Cox, Ingersoll, and Ross (1985)). The solution for the model reveals that allowing the probability of a disaster to vary not only implies a time-varying equity premium, it also increases the level of the equity premium. This is because the risk of an increase in disaster probability leads to a fall in stock prices and therefore itself requires compensation. The dynamic nature of the model therefore leads the equity premium to be higher than what static considerations alone would predict.

Several recent papers also address the potential of rare disasters to explain the aggregate stock market. Gabaix (2007) maintains the power utility assumption of Barro (2006) and assumes the economy is driven by a linearity-generating process that combines the probability of a rare disaster with the degree to which dividends respond to a consumption disaster. However, in the numerical calibration, only the latter is allowed to vary. Therefore the economic mechanism driving stock market volatility in Gabaix's model is quite different than the one considered here. Gourio (2007) extends the framework of Barro to consider the case where a recovery follows a disaster. Barro (2007) and Martin (2007) solve models with a constant disaster probability and recursive utility. In contrast, the model considered here focuses on the case of time-varying disaster probabilities. A different, though related, approach is taken by Veronesi (2004), who assumes that the drift of dividends follows a Markov switching process, with a small probability of falling into a low state. While the physical probability of a low state is constant, the representative investor's subjective probability is time-varying due to learning. Veronesi assumes exponential utility; this allows for the inclusion of learning but makes it difficult to assess the magnitude of the excess volatility generated through this mechanism.

A related literature derives asset pricing results assuming endowment processes that include jumps, with a focus on option pricing. Liu, Pan, and Wang (2005) consider an endowment process in which jumps occur with a constant intensity; their focus is on uncertainty aversion but they also consider recursive utility. My model departs from theirs in that the probability of a jump varies over time. Drechsler and Yaron (2008) show that a model with jumps in the volatility of the consumption growth process can explain the behavior of implied volatility and its relation to excess returns. They focus on the case of EIS greater than one and derive approximate analytical and numerical solutions. Santa-Clara and Yan (2006) consider time-varying jump intensities, but restrict attention to a model with power utility and implications for options. In contrast, the model considered here focuses on recursive utility and implications for the aggregate market.

The outline of the paper is as follows. Section 2 solves a model with constant disaster probabilities and recursive utility with general elasticity of intertemporal substitution. Section 3 solves the model with time-varying disaster risk and unit elasticity of substitution. Section 4 discusses the calibration and simulation of the model, and its fit to aggregate stock market data. Section 5 concludes.

2 Constant disaster risk

2.1 Model assumptions

Assume that aggregate consumption solves the following stochastic differential equation:

$$dC_t = \mu C_{t-}\, dt + \sigma C_{t-}\, dB_t + (e^{Z_t} - 1)C_{t-}\, dN_t.$$

Here, N_t is a Poisson process with constant intensity λ. Z_t is a random variable whose time-invariant distribution ν is independent of N_t and B_t. The notation C_{t-} denotes $\lim_{s\uparrow t} C_s$, while C_t denotes $\lim_{s\downarrow t} C_s$.

The model focuses on negative jumps, i.e. disasters, and for that reason Z_t is assumed to be negative. In what follows, I use the notation E_ν to denote expectations of functions of Z_t taken with respect to the ν-distribution. The t subscript on Z_t will be omitted when not essential for clarity. The diffusion term $\mu C_{t-}\, dt + \sigma C_{t-}\, dB_t$ represents the behavior of consumption in "normal times", and implies that, when no disaster takes place, log consumption growth is normally distributed with mean $\mu - \frac{1}{2}\sigma^2$ and variance σ^2. Rare disasters are captured by the Poisson process N_t. Roughly speaking, λ is the probability of a jump over a given unit of time.[1]

[1]More precisely, the probability of k jumps over the course of a period τ is equal to

Following Duffie and Epstein (1992), I define the utility function V_t for the representative agent using the following recursion:

$$V_t = E_t \int_t^{\infty} f(C_s, V_s)\, ds, \tag{1}$$

where

$$f(C,V) = \frac{\beta}{1-\frac{1}{\psi}} \frac{C^{1-\frac{1}{\psi}} - ((1-\gamma)V)^{\frac{1}{\theta}}}{((1-\gamma)V)^{\frac{1}{\theta}-1}}, \tag{2}$$

and $\theta = (1-\gamma)/(1-\frac{1}{\psi})$. Note that V_t represents continuation utility, i.e. utility of the future consumption stream. Equations (1) and (2) define the continuous-time analogue of the utility function in Epstein and Zin (1989) and Weil (1990). The parameter $\beta > 0$ is the rate of time preference, $\psi > 0$ can be interpreted as the elasticity of intertemporal substitution and $\gamma > 0$ can be interpreted as relative risk aversion.

2.2 Solution

In what follows, I solve for asset prices using the state-price density. The advantage of this method is that it generalizes to the case of time-varying disaster probabilities. A necessary first step when assuming recursive utility is to solve for the value function. Accordingly, Section 2.2.1 describes the solution for the value function. The solution for the riskfree rate, the wealth-consumption ratio and the risk premium on the consumption claim follows easily. Section 2.2.2 describes the solution for the price-dividend ratio for the dividend claim and Section 2.2.3 describes the solution for the expected rate of return on government bills, allowing for partial default.

2.2.1 The value function

The first step in solving the model is to solve for the value function, i.e. utility as a function of wealth. I conjecture that

$$J(W) = \frac{W^{1-\gamma}}{1-\gamma} j^{1-\gamma} \tag{3}$$

$e^{-\lambda\tau}\frac{(\lambda\tau)^k}{k!}$, where τ will be measured in units of years. In the calibrations that follow, the parameter λ will be set to equal 0.0170, implying a 0.0167 probability of a single jump over the course of a year, a 0.00014 probability of two jumps, and so forth.

for a constant j. Let S_t denote the value of the claim to aggregate consumption, and conjecture that the price-dividend ratio for this claim is constant:

$$\frac{S_t}{C_t} = l. \tag{4}$$

The process for consumption and the conjecture (4) imply that S_t satisfies

$$dS_t = \mu S_{t-}\, dt + \sigma S_{t-}\, dB_t + (e^{Z_t} - 1)S_{t-} dN_t. \tag{5}$$

Furthermore, conjecture that the riskfree rate is a constant r. The Bellman equation for an investor who allocates wealth between the consumption claim and the riskfree asset is

$$\sup_{\alpha, C} \Bigg\{ J_W(W\alpha(\mu - r + l^{-1}) + Wr - C) + \frac{1}{2} J_{WW} W^2 \alpha^2 \sigma^2 + \\ \lambda E_\nu \left[J(W(1 + \alpha(e^Z - 1))) - J(W) \right] \Bigg\} + f(C, J) = 0, \tag{6}$$

where J_W is the first derivative of J with respect to wealth and J_{WW} is the second derivative. This formula follows from an application of Ito's Lemma to J (see Duffie (2001, Appendix F) for Ito's Lemma with jumps).

Substituting (3) into (6) and taking the derivative with respect to α implies the following first order condition:

$$W^{1-\gamma} j^{1-\gamma} (\mu - r + l^{-1}) - \gamma W^{1-\gamma} j^{1-\gamma} \alpha \sigma^2 + \\ W^{1-\gamma} j^{1-\gamma} \lambda E_\nu \left[(1 + \alpha(e^Z - 1))^{-\gamma} (e^Z - 1) \right] = 0.$$

In equilibrium, α must equal 1. Therefore,

$$\mu + l^{-1} - r = \gamma \sigma^2 - \lambda E_\nu \left[e^{-\gamma Z} (e^Z - 1) \right]. \tag{7}$$

Let r^C denote the instantaneous expected return on the consumption claim, defined as the drift in the price, plus the dividend, plus the expectation of the jump in the price, as a proportion of the current price:

$$r^C \equiv \mu + l^{-1} + \lambda E_\nu [e^Z - 1].$$

It follows from (7) that the instantaneous equity premium on the consumption claim is given by

$$r^C - r = \gamma \sigma^2 + \lambda E_\nu \left[-e^{-\gamma Z} (e^Z - 1) + e^Z - 1 \right]. \tag{8}$$

Equation (7) also has an interpretation: it is the instantaneous premium on the consumption claim conditional on no disasters.

There are several noteworthy features of (8). The first term, equal to risk aversion times consumption volatility, is the equity premium under the standard diffusion model. The second term is therefore the compensation for the possibility of a rare disaster. This term can be rearranged to equal $-E_\nu\left[(e^{-\gamma Z}-1)(e^{Z}-1)\right]$. Because Z is negative, this expression is positive. Therefore the contribution of the disaster risk term to the equity premium is positive and increasing in the disaster intensity λ. Finally, the risk premium depends only on risk aversion, not on the elasticity of intertemporal substitution. The risk premium, including the disaster risk term, is therefore the same for recursive utility as for power utility, a result also found by Liu, Pan, and Wang (2005) and Barro (2007).

In equilibrium, $W = S$. Therefore, the conjecture (4) is equivalent to

$$\frac{W}{C} = l, \tag{9}$$

Let f_C denote the first derivative of $f(C,V)$ with respect to C. As shown in Appendix A.1, the envelope condition

$$J_W = f_C \tag{10}$$

together with the form of J, (3), and the conjecture that the wealth-consumption ratio is constant, implies that

$$l = \beta^{-\psi} j^{\psi-1}. \tag{11}$$

An equation for j follows from the Bellman equation (6) by substituting in for J and derivatives from (3), consumption from (9), l from (11) and $\alpha = 1$. Appendix A.1 provides details of this calculation. It follows that

$$j = \left[\left(-\mu + \frac{1}{2}\gamma\sigma^2 - \lambda(1-\gamma)^{-1}E_\nu\left[e^{(1-\gamma)Z}-1\right] + \frac{\beta}{1-\frac{1}{\psi}}\right)\frac{1-\frac{1}{\psi}}{\beta^{\psi}}\right]^{\frac{1}{1-\psi}}, \tag{12}$$

and therefore that the wealth-consumption ratio l equals:

$$l = \left(-\mu + \frac{1}{2}\gamma\sigma^2 - \lambda(1-\gamma)^{-1}E_\nu\left[e^{(1-\gamma)Z}-1\right] + \frac{\beta}{1-\frac{1}{\psi}}\right)^{-1}\left(1-\frac{1}{\psi}\right)^{-1}. \tag{13}$$

This equation is discussed in Section 2.2.2.

The equation for the riskfree rate follows from substituting in the equation for l into (7). Rearranging implies that

$$r = \beta + \frac{1}{\psi}\mu - \frac{1}{2}\left(\gamma + \frac{\gamma}{\psi}\right)\sigma^2 + \\ \lambda E_\nu\left[-(e^{-\gamma Z} - 1) + \left(1 - \frac{1}{\theta}\right)(e^{(1-\gamma)Z} - 1)\right]. \quad (14)$$

The first three terms in (14) are the same as in the standard model without disaster risk and reflect the roles of the discount rate, intertemporal smoothing and precautionary savings respectively. The last term arises from the risk of rare disasters. To understand this term, first consider the special case of power utility ($\theta = 1$). Because $e^{-\gamma Z} > 1$, the riskfree rate is decreasing in λ. An increase in the risk of rare disasters causes the representative investor to desire to save more, and thus lowers the riskfree rate. The greater is risk aversion, the greater is this effect.

In the case of recursive utility, the interpretation of (14) is more complicated. To fix ideas, assume $\gamma > 1$. For $1 - \frac{1}{\theta} > 0$, or equivalently, $\gamma > \frac{1}{\psi}$, an increase in λ has a smaller impact on the riskfree rate for recursive utility than for power utility. That is, increasing the agent's willingness to substitute over time reduces the dependence of r_t on λ.

2.2.2 The state-price density and the dividend claim

Armed with the value function, it is possible to compute the state-price density. Given the state-price density, the price of any risky asset follows from a no-arbitrage condition. Duffie and Skiadas (1994) show that the state-price density π_t relates to the value function via the formula

$$\pi_t = \exp\left\{\int_0^t f_V(C_s, V_s)\, ds\right\} f_C(C_t, V_t). \quad (15)$$

I model dividends as levered consumption, i.e. $Y_t = C_t^\phi$ as in Abel (1999) and Campbell (2003). Ito's Lemma then implies

$$dY_t = \mu_Y Y_{t-}\, dt + \phi\sigma Y_{t-}\, dB_t + (e^{\phi Z_t} - 1)Y_{t-}\, dN_t, \quad (16)$$

where

$$\mu_Y = \phi\mu + \frac{1}{2}\phi(\phi - 1)\sigma^2.$$

Let $F(Y_t)$ denote the price of the dividend claim (we later verify that this price is a function of Y_t). No arbitrage implies that

$$F(Y_t) = E_t\left[\int_t^\infty Y_s \frac{\pi_s}{\pi_t}\, ds.\right] \tag{17}$$

Given a jump-diffusion process x_t, let Dx_t denote the drift of that process, δx_t the diffusion, and $\mathcal{J}(x_t)$ the expected jump in the process, provided that a jump occurs. As shown in Appendix A.2, the no-arbitrage condition (17) implies that F satisfies the following:

$$\pi_t(DF_t) + F_t(D\pi_t) + Y_t\pi_t + (\delta\pi_t)(\delta F_t) + \lambda\mathcal{J}(\pi_t F_t) = 0, \tag{18}$$

where $F_t = F(Y_t)$. Conjecture that

$$F(Y_t) = l_Y Y_t \tag{19}$$

for a constant l_Y (which, by definition, equals the price-dividend ratio on the dividend claim). Appendix A.2 shows that substituting (19) into (18) implies

$$\mu_Y + l_Y^{-1} - r = \phi\gamma\sigma^2 + \lambda E_\nu\left[e^{-\gamma Z} - 1\right] - \lambda E_\nu\left[e^{(\phi-\gamma)Z} - 1\right]. \tag{20}$$

Let r^e denote the instantaneous expected return on the dividend claim. Analogously to the expected return on the consumption claim, r^e equals the drift in the price, plus the dividend, plus the expectation of the jump in the price, as a proportion of the current price. The conjecture that the price-dividend ratio is a constant implies that (as proportions) the drift in the price is the same as the drift in the dividend process, and the expectation of the jump in the price is the same as the expectation of the jump in the dividend process. Therefore

$$r^e = \mu_Y + l_Y + \lambda E_\nu\left[e^{\phi Z} - 1\right].$$

It follows from (20) that the instantaneous equity premium equals

$$\begin{aligned} r^e - r &= \phi\gamma\sigma^2 + \lambda E_\nu\left[\left(e^{-\gamma Z} - 1\right) - \left(e^{(\phi-\gamma)Z} - 1\right) + \left(e^{\phi Z} - 1\right)\right] \\ &= \phi\gamma\sigma^2 + \lambda E_\nu\left[e^{-\gamma Z}\left(1 - e^{\phi Z}\right) + e^{\phi Z} - 1\right]. \end{aligned}$$

The first term appears in the standard diffusion model, and equals risk aversion multiplied by the instantaneous covariance of the endowment process with the dividend process. The next terms account for jump risk. The second expression shows that the term multiplying λ is positive and therefore that disaster risk raises the risk premium and is increasing in λ.[2]

Equations (14) and (20) imply that the price-dividend ratio equals

$$l_Y = \left(\beta - \mu_Y + \frac{1}{\psi}\mu - \frac{1}{2}\left(\gamma + \frac{\gamma}{\psi} - 2\phi\gamma\right)\sigma^2 + \right. \\ \left. \lambda E_\nu\left[\left(1 - \frac{1}{\theta}\right)\left(e^{(1-\gamma)Z} - 1\right) - \left(e^{(\phi-\gamma)Z} - 1\right)\right]\right)^{-1}. \quad (21)$$

The first three terms inside the outer parentheses in (21) appear in the standard diffusion model. To understand the last term, it is useful to consider several special cases. For the remainder of this section, I assume γ is greater than 1.

First consider the consumption claim, corresponding to $\phi = 1$. In this case, the disaster risk term inside of the parentheses in (21) reduces to $\lambda E_\nu\left[-\frac{1}{\theta}e^{(1-\gamma)Z}\right]$. The term in the exponent is positive under the maintained assumption that $\gamma > 1$, so $\lambda E_\nu\left[-\frac{1}{\theta}e^{(1-\gamma)Z}\right]$ takes the sign opposite to that of θ. Because of the inverse, it follows that the price-dividend ratio decreases in the probability of a disaster if and only if $\theta < 0$, i.e. if and only if the EIS is greater than 1. This observation is also made by Barro (2007).

Next consider the case of unit EIS. In the limit as the EIS approaches one, $1 - 1/\theta = 1$ and the disaster risk term inside the parentheses reduces to $\lambda E_\nu\left[e^{(1-\gamma)Z} - e^{(\phi-\gamma)Z}\right]$. Because $Z < 0$, this term is positive if and only if $\phi > 1$. Therefore the price-dividend ratio decreases in λ if and only if equity is levered, i.e. if $\phi > 1$.

Finally consider power utility, for which $\theta = 1$. The disaster risk term reduces to $\lambda E_\nu\left[1 - e^{(\phi-\gamma)Z}\right]$, which is positive if and only if $\phi > \gamma$. Therefore, for power utility, the price-dividend ratio decreases in the disaster probability if and only if leverage exceeds risk aversion.

[2] Note that $e^{\phi Z}$ is between 0 and 1. Therefore $e^{\phi Z} + \left(1 - e^{\phi Z}\right)e^{-\gamma Z}$ is a weighted average of $e^{-\gamma Z} > 1$ and 1, and is therefore greater than 1.

This last point suggests difficulties that the power utility model might encounter in a dynamic setting in which λ is allowed to vary. For values of risk aversion exceeding leverage, a low price-dividend ratio would indicate that disaster risk is low and the equity premium is low as well. When risk aversion exceeds leverage, the price-dividend ratio would predict excess returns with a positive, not a negative sign, contradicting the data. Moreover, the riskfree rate would be more volatile than the equity premium in such a model.

Equation (21) suggests that recursive utility can help solve this problem. For $\gamma > 1$, $1 - \frac{1}{\theta} > 0$ if and only if $\gamma > \frac{1}{\psi}$. Assuming parameter values in these ranges, the disaster risk term in (21) arising from recursive utility, $\lambda E_\nu \left[\left(1 - \frac{1}{\theta}\right)\left(e^{(1-\gamma)Z} - 1\right)\right]$, is positive. This term therefore leads the price-dividend ratio to increase less, or possibly to decrease, in the disaster probability.

The net effect of an increase in the disaster probability on the price-dividend ratio reflects the interplay between the effects on the equity premium, on expected future cash flows, and on the riskfree rate. As shown above, the equity premium is increasing in λ regardless of the choice of parameters (provided, of course, that γ and ϕ are positive). Increasing λ decreases future expected cash flows; this effect is relatively small because the probability of a rare disaster is low, and unlike the effect on the equity premium or riskfree rate, it is not amplified by the curvature in the utility function. Finally, the riskfree rate is decreasing in λ for power utility. It is also decreasing for recursive utility provided that $\theta > 0$. For power preferences with $\gamma > \phi$, the riskfree rate effect dominates the other two effects, and the price-dividend ratio increases in the probability of a disaster. For recursive preferences, it is possible for this effect to reverse because the riskfree rate effect is weaker.

2.2.3 Risk of default

As discussed in Barro (2006), disasters often coincide with at least a partial default on government securities. This point is of empirical relevance if one tries to match the behavior of the riskfree asset to the rate of return on government securities in the data. I therefore allow for partial default on government debt, and consider the rate of return on this defaultable security.

Let L_t be the price process for government liabilities and assume that

$$\frac{dL_t}{L_t} = r^L\,dt + \left(e^{Z_{L,t}} - 1\right) dN_t,$$

where r^L is the "face value" of government debt (i.e. the amount investors receive if there is no default), $Z_{L,t}$ is a random variable whose distribution will be described shortly and N_t is the same Poisson process considered above. Assume that, in event of a disaster, there is a probability q that there will be a default on government liabilities. I follow Barro (2006) and assume that in the event of default, the percent loss is equal to the percent fall in consumption. Therefore,

$$Z_{L,t} = \begin{cases} Z_t & \text{with probability } q \\ 0 & \text{otherwise} \end{cases}$$

Arguments similar to those used to price the dividend claim imply that

$$r^L = r + \lambda E_\nu\left[e^{-\gamma Z} - 1\right] - \lambda\left((1-q)E_\nu\left[e^{-\gamma Z} - 1\right] + qE_\nu\left[e^{(1-\gamma)Z} - 1\right]\right). \quad (22)$$

Let r^b denote the instantaneous expected return on government debt. Then

$$r^b = r^L + \lambda q E_\nu\left[e^Z - 1\right], \quad (23)$$

with r^L given in (22). The instantaneous equity premium relative to the government bill rate is therefore (21) plus r, minus (23).

While the the presence of default for government debt affects the equity premium, it does not affect the price-dividend ratio (21), nor for that matter the total expected return on equities. It only affects how this expected return is decomposed into the equity premium and the government bill rate. Much of the discussion in Section 2.2.2 therefore is unaffected by default. That is, the problems with the power utility model, namely that the price-dividend ratio increases rather than increases in λ, cannot be solved by allowing default on government bills.

Of course, the possibility of default does affect the government bill rate. The premium attached to government bills is increasing in λ; the greater the risk of disaster, the greater the risk of default, and the more compensation investors require to hold bills. On the other hand, the riskfree rate is decreasing in λ for a wide range of parameter values. These effects offset each other, so that the expected return on government debt varies less with λ than the riskfree rate.

3 Time-varying disaster risk

Now consider the case in which the disaster intensity λ is not constant, but rather follows the process

$$d\lambda_t = \kappa(\bar{\lambda} - \lambda_t)\, dt + \sigma_\lambda \sqrt{\lambda_t}\, dB_{\lambda,t}.$$

For analytical convenience, the Brownian motion $B_{\lambda,t}$ is assumed to be independent of B_t, the Brownian motion driving the consumption and dividend processes. Given that the risk of disasters has a much greater impact on the equity premium than does diffusion risk, this assumption is unlikely to significantly affect the results.

Figure 1 plots the probability density function for λ_t, assuming the parameter values discussed below. The unconditional mean of the process is $\bar{\lambda}$ and is calibrated to 0.017 per annum (see Barro (2006)). However, the distribution is highly skewed; there is a long right tail of high values for λ. This occurs because of the square root term. A high realization of λ_t makes the process more volatile, and thus high values are more likely than they would be under a standard auto-regressive process. Thus the model implies that there are times when "rare" disasters can occur with high probability, but that these times are themselves unlikely.

Further assume that the EIS is equal to 1. This has the advantage of leading to closed-form solutions. Moreover, independent empirical evidence suggests that this value is not unreasonable. Duffie and Epstein (1992) show that, for the limiting case of $\psi = 1$, preferences can be expressed using the aggregator

$$f(C, V) = \beta(1-\gamma)V \left(\log C - \frac{1}{1-\gamma} \log((1-\gamma)V) \right). \tag{24}$$

The model for consumption and dividends is the same as in the previous section, except that the disaster intensity is time-varying.

3.1 The value function

I follow the blueprint of the solution in the constant disaster risk case, first solving for the value function and the wealth-consumption ratio. These will allow the state-price density to be expressed in terms of the model's primitives.

Conjecture that the value function takes the form

$$J(\lambda, W) = I(\lambda)\frac{W^{1-\gamma}}{1-\gamma}, \tag{25}$$

where I is a function of λ that will be derived in this section. Also conjecture that the price-dividend ratio for the claim to aggregate consumption is a constant l, as in (4). The price of the consumption claim therefore follows the process given by (5) with time-varying disaster intensity λ_t. Let r_t denote the riskfree rate. The Bellman equation for an investor who allocates wealth between the consumption claim and the riskfree asset is therefore

$$\sup_{\alpha_t, C_t} \Big\{ J_W W_t \alpha_t(\mu - r_t + l^{-1}) + J_W W_t r_t - J_W C_t + J_\lambda \kappa(\bar{\lambda} - \lambda_t) +$$
$$\frac{1}{2} J_{WW} W_t^2 \alpha_t^2 \sigma^2 + \frac{1}{2} J_{\lambda\lambda} \sigma_\lambda^2 \lambda_t + \lambda_t E_\nu \left[J(W_t(1 + \alpha_t(e^{Z_t} - 1)), \lambda_t) - J(W_t, \lambda_t) \right] \Big\}$$
$$+ f(C_t, J) = 0, \tag{26}$$

where J_λ denotes the first derivative of J with respect to λ and $J_{\lambda\lambda}$ the second derivative of J with respect to λ.

Reasoning identical to that in Section 2.2.1 implies that, in equilibrium,

$$\mu - r_t + l^{-1} = \gamma\sigma^2 - \lambda_t E_\nu \left[e^{-\gamma Z}(e^Z - 1) \right]. \tag{27}$$

Let r_t^C denote the instantaneous expected return on the consumption claim. As in Section 2.2.1, this is defined as the drift in the price, plus the dividend, plus the expected jump in the price, all as a proportion of the current price:

$$r_t^C \equiv \mu + l^{-1} + \lambda_t E_\nu \left[e^Z - 1 \right].$$

It then follows from (27) that the instantaneous equity premium on the consumption claim is given by

$$r_t^C - r_t = \gamma\sigma^2 + \lambda_t E_\nu \left[-e^{-\gamma Z}(e^Z - 1) + e^Z - 1\right], \tag{28}$$

while (27) is the instantaneous premium conditional on no disasters. Both terms reduce to their counterparts in Section 2.2.1 for constant λ_t. Because the EIS equals one, the dynamic nature of the model does not effect the premium for the consumption claim.

The value of the wealth-consumption ratio follows from the equilibrium condition $W = S$ (and therefore $W/C = l$), and the envelope condition (10). Note that

$$f_C(C, V) = \beta(1-\gamma)\frac{V}{C}. \tag{29}$$

At the optimum, V is given by (25). The envelope condition therefore implies

$$I(\lambda)W^{-\gamma} = \beta(1-\gamma)I(\lambda)\frac{W^{1-\gamma}}{1-\gamma}\frac{1}{l^{-1}W}.$$

Solving for l yields $l = \beta^{-1}$, which equals the limit of (13) as ψ approaches one. The equation for the riskfree rate follows from (27):

$$r_t = \mu + \beta - \gamma\sigma^2 + \lambda_t E_\nu \left[e^{-\gamma Z}\left(e^Z - 1\right)\right], \tag{30}$$

For λ_t constant, this equation reduces to (14) in the case of $\psi = 1$.

Finally, to solve for $I(\lambda)$, I substitute (25) and the optimal policy functions into (26). Algebraic computations in Appendix B.1 verify that (25) is a solution to (26), with I given by

$$I(\lambda) = e^{a+b\lambda}, \tag{31}$$

where

$$\begin{aligned} a &= \frac{1-\gamma}{\beta}\left(\mu - \frac{1}{2}\gamma\sigma^2\right) + (1-\gamma)\log\beta + b\frac{\kappa\bar{\lambda}}{\beta}, \\ b &= \frac{\kappa+\beta}{2\sigma_\lambda^2} - \sqrt{\left(\frac{\kappa+\beta}{2\sigma_\lambda^2}\right)^2 - \frac{E_\nu\left[e^{(1-\gamma)Z} - 1\right]}{\sigma_\lambda^2}}, \end{aligned}$$

and that this solution corresponds to that derived in Section 2.2.1 for the special case of constant λ.

Unlike the value function for the model with constant disaster risk, the value function given by (25) and (31) depends on λ_t. In the calibration below, $b > 0$. Because $\gamma > 1$, this implies that an increase in disaster risk reduces utility for the representative agent. As the following section shows, the price of the dividend claim falls when the disaster probability rises. The agent requires compensation for this risk (because utility is recursive, marginal utility depends on the value function), and thus time-varying disaster risk increases the equity premium.

3.2 The state-price density and the dividend claim

Given the value function, it is possible to compute the process for the state-price density, and therefore to price any risky asset using the no-arbitrage condition. The state-price density π_t is given by (15) for both the constant disaster risk model and the time-varying disaster risk model. However, the processes C_t and V_t are different.

As in Section 2.2.2, I derive the price of the dividend claim for dividends $Y_t = C_t^{\phi}$. Y_t follows the process (16), where the intensity λ varies over time. I conjecture that the price of this claim can be written as a function F of λ_t and Y_t. As shown in Appendix B.2, the no-arbitrage condition (17) implies that

$$\pi_t(DF_t) + F_t(D\pi_t) + Y_t\pi_t + (\delta\pi_t)^{\top}(\delta F_t) + \lambda_t \mathcal{J}(\pi_t F_t) = 0, \tag{32}$$

where $F_t = F(\lambda_t, Y_t)$. Because there are two (independent) sources of uncertainty, the diffusion terms δF_t and $\delta\pi_t$ are represented by 2×1 vectors.

In Appendix B.2, I show that (32) is solved by a function of the form

$$F(\lambda, Y) = G(\lambda)Y \tag{33}$$

where

$$G(\lambda) = \int_0^{\infty} \exp\left\{a_\phi(t) + b_\phi(t)\lambda\right\}\, dt, \tag{34}$$

and where a_ϕ and b_ϕ satisfy the ordinary differential equations

$$\begin{aligned} a'_\phi(t) &= \mu_Y - \mu - \beta + \gamma\sigma^2 - \gamma\sigma^2\phi + \kappa\bar{\lambda} b_\phi(t) && (35) \\ b'_\phi(t) &= \frac{1}{2}\sigma_\lambda^2 b_\phi(t)^2 + (b\sigma_\lambda^2 - \kappa) b_\phi(t) + E_\nu\left[e^{(\phi-\gamma)Z} - e^{(1-\gamma)Z}\right] && (36) \end{aligned}$$

with boundary conditions $a_\phi(0) = b_\phi(0) = 0$.

The price of the claim to total dividends can be understood as the integral of prices of claims to dividends at specific points in time. The function $a_\phi(t)$ represents the effect of maturity on the price of these claims for λ_t equal to zero. Equation (35) shows that as maturity increases, a_ϕ is incremented by the value of expected dividend growth and decremented by the values of the riskfree rate and the equity premium when $\lambda_t = 0$. The final term in (35) represents the effect of future changes in λ_t on the price. It depends on the effect of λ_t on the price ($b_\phi(t)$), on the average value of λ_t ($\bar{\lambda}$) and on the speed at which λ_t reverts to this average value (κ).

The function $b_\phi(t)$ represents the effect of maturity interacted with λ_t. The term $E_\nu[e^{(\phi-\gamma)Z} - e^{(1-\gamma)Z}]$ in (36) summarizes the effect of λ on the price-dividend ratio in the static model for $\psi = 1$ (see (21)); that is, it represents a combination of the equity premium, riskfree rate, and cash flow effect. Equation (36) shows that there are three additional terms. The term $b\sigma_\lambda^2 b_\phi(t)$ is an additional component of the equity premium arising from the dynamic nature of the model, as discussed further below. The remaining two terms, $\frac{1}{2}\sigma_\lambda^2 b_\phi(t)^2$ and $-\kappa b_\phi(t)$ represent the effect of future changes in λ_t. The former is a Jensen's inequality term; all else equal, a more volatile equity premium increases the price-dividend ratio. The latter represents the fact that, if λ_t is high in the present, λ_t is likely to decrease in the future on account of mean reversion.

Figure 2 plots $a_\phi(t)$ and $b_\phi(t)$ as functions of time for parameter values described below. The top panel shows that a_ϕ asymptotes to a decreasing linear function. The fact that this is decreasing is necessary for convergence if λ_t varies over time.[3] The asymptote is linear because $b_\phi(t)$ asymptotes to a constant. The figure also shows that $b_\phi(t)$ is negative, and thus the price-dividend ratio is decreasing in λ_t. Thus the equity premium effect, together with the cash flow effect, dominates the riskfree rate effect at these parameter values. The magnitude of $b_\phi(t)$ increases in t. This is a duration effect. The

[3]When λ_t does not vary over time, then convergence requires that $a_\phi(t) + b_\phi(t)\bar{\lambda}$ is decreasing.

further out the cash flows occur, the more the price of the claim varies with the discount rate.

As discussed in Section 2.2, the instantaneous expected return is defined as the drift in the price (as a proportion), plus the dividend yield, plus the expected jump in in the price (as a proportion). For this more general model, this implies

$$r_t^e \equiv \frac{1}{F_t}\left(DF_t + Y_t + \lambda_t \mathcal{J}(F_t)\right). \tag{37}$$

Equation (32) can be rearranged to give a convenient characterization of the risk premium. No-arbitrage implies that the riskfree rate is characterized by

$$r_t = -\frac{D\pi_t}{\pi_t} - \lambda_t \frac{\mathcal{J}(\pi_t)}{\pi_t}.$$

Combining (37) with this expression and applying (32) implies

$$r_t^e - r_t = -\left(\frac{\delta\pi_t}{\pi_t}\right)^\top \left(\frac{\delta F_t}{F_t}\right) - \lambda_t \left(\frac{\mathcal{J}(F_t\pi_t)}{F_t\pi_t} - \frac{\mathcal{J}(F_t)}{F_t} - \frac{\mathcal{J}(\pi_t)}{\pi_t}\right).$$

The first term represents the portion of the equity premium that is compensation for diffusion risk (which includes time-varying λ_t), while the second represents the portion from jump risk. The second term can be understood as jump equivalent of a covariance. The greater the covariance of the price F_t with the marginal utility process π_t, the more the asset provides a hedge and the lower is the risk premium.

To write this equation in terms of the model primitives, first note that Ito's Lemma applied to the state-price density implies that

$$\frac{\delta\pi_t}{\pi_t} = \begin{bmatrix} -\gamma\sigma \\ b\sigma_\lambda\sqrt{\lambda_t} \end{bmatrix} \tag{38}$$

(Appendix B.2 gives details). The negative of the first element of the vector (38) is the price of diffusion risk, where the negative of the second element is the price of risk associated with time-varying λ_t. Second, note that Ito's Lemma applied to $F_t = F(\lambda_t, Y_t)$ implies that

$$\delta F_t = \begin{bmatrix} Y_t G(\lambda_t)\phi\sigma \\ Y_t G'(\lambda_t)\sigma_\lambda\sqrt{\lambda_t} \end{bmatrix}. \tag{39}$$

Finally, $\frac{\mathcal{J}(F_t)}{F_t} = E_\nu\left[e^{\phi Z} - 1\right]$ because λ_t follows a diffusion process. Combining (38), (39), (B.8), (B.11) implies that the equity premium equals

$$r_t^e - r_t = \phi\gamma\sigma^2 - b\sigma_\lambda^2 \frac{G'}{G}\lambda_t \; + \lambda_t E_\nu\left[e^{\phi Z} - 1 + e^{-\gamma Z} - 1 - (e^{(\phi-\gamma)Z} - 1)\right]. \quad (40)$$

The new term in this expression relative to the static model is $-b\sigma_\lambda^2 \frac{G'}{G}\lambda_t$. This is the contribution to the risk premium due to changes in future expected returns. Because b_ϕ is negative, G' is also negative. Moreover, b is positive, so this term represents a positive contribution to the risk premium.

3.3 Risk of default

The calculation for the government bill rate is similar to the corresponding calculation in the case of constant disaster risk. The face value of the government bill is given by

$$r_t^L = r_t + \lambda_t E_\nu\left[e^{-\gamma Z_t} - 1\right] - \lambda_t\left((1-q)E_\nu\left[e^{-\gamma Z_t} - 1\right] + q\left[e^{(1-\gamma)Z_t} - 1\right]\right).$$

This is also the expected return, conditional on no disasters occurring. The instantaneous expected return on government debt is

$$r_t^b = r_t^L + \lambda_t q E_\nu\left[e^Z - 1\right]. \quad (41)$$

Figure 3 shows the face value of government debt, r_t^L, the instantaneous expected return on government debt r_t^b and the riskfree rate r_t as a function of λ_t. Because of the required compensation for default, r_t^L lies above r_t. The expected return lies between the two because the actual cash flow that investors receive from the government bill will be below r_t^L if default occurs.

All three rates decrease in λ_t because, at these parameter values, a higher λ_t induces a greater desire to save. However, r_t^L and r_t^b are less sensitive to changes in λ than r_t because of an opposing effect: the greater is λ_t, the greater is the risk of default, and therefore the greater the return investors demand for holding the government bill. Because of a small cash flow effect, r_t^b decreases more than r_t^L, but still less than r_t.

The instantaneous equity premium relative to the government bill rate is equal to (40) plus r_t, minus r_t^b. This instantaneous equity premium is shown in

Figure 4 (solid line). The difference between the dashed line and the solid line represents the dynamic component of the equity premium, namely $-b\sigma_\lambda^2 \frac{G'}{G}\lambda_t$, and shows that this term is large in magnitude. The dashed line represents the equity premium in the standard diffusion model without disaster risk and is negligible compared with the disaster risk component.

4 Calibration and Simulation

This section first describes the calibration of the time-varying disaster risk model and results from simulated data. I then compare these results with those obtained from a model with constant disaster risk.

4.1 Calibration

Table 1 describes the parameters in the main case. Most parameters are set to values considered by Barro (2006) to highlight this model's novel implications. In the model, time is measured in units of years and parameter values should be interpreted accordingly. The drift rate μ is calibrated so that in normal periods, the expected growth rate of log consumption is 2.5% per annum.[4] The standard deviation of log consumption σ is 2% per annum. These parameters are chosen as in Barro to match postwar data in G7 countries. The average disaster intensity is $\bar{\lambda}$, set equal to 0.017. The decline in consumption when a disaster does occur, Z_t, is calibrated to the empirical distribution of declines in GDP. The probability of default given disaster, q, is set equal to 0.4, which is the probability of default given a disaster. These values are calculated by Barro based on data for 35 countries over the period 1900–2000.

Leverage, ϕ, is set equal to 3, in line with values used elsewhere in calibrated models of the equity premium (e.g. Bansal and Yaron (2004)). Given the high ratio of dividend volatility to consumption volatility, this value is conservative. Barro considers values of risk aversion equal to 3 and 4 and values of the rate of time preference equal to 0.02 and 0.03. I choose risk aversion equal to 3 and

[4] The value $\mu = 2.52\%$ reflects an adjustment for Jensen's inequality.

rate of time preference equal to 0.02 because, given other parameter choices, these deliver the closest match to the equity premium and the riskfree rate in the present model.

The novel parameters are the EIS ψ, the mean reversion of the disaster intensity, κ, and the volatility parameter for the disaster intensity, σ_λ. The EIS is set equal to 1 for tractability. A number of studies have concluded that reasonable values for this parameter lie in a range close to one, or slightly lower than 1 (see Campbell (2003) for a discussion). Mean reversion κ is chosen to match the autocorrelation of the price-dividend ratio in annual U.S. data. Because λ_t is the single state variable, the autocorrelation of price-dividend ratio implied by the model will approximately equal the autocorrelation of λ_t. Setting κ equal to 0.142 generates an autocorrelation for the price-dividend ratio equal to 0.865, its value in the data. The volatility parameter σ_λ is chosen to be 0.07; this generates a reasonable level of volatility in stock returns.

4.2 Results for the time-varying disaster risk model

Table 2 describes moments from a simulation of the model as well as moments from annual U.S. data. Annual U.S. data come from Robert Shiller's website. Data are from 1890 to 2004 and are described in detail in Shiller (1989, Chap. 26).

The model is discretized using an Euler approximation (e.g. Lord, Koekkoek, and van Dijk (2006)) and simulated at a monthly frequency for 50,000 years; simulating the model at higher frequencies produces negligible differences in the results.[5] The monthly results are then compounded to an annual frequency to compare with the annual data set. Two types of moments are reported. The first type (referred to as "population" in the tables) are calculated based on all years in the simulation. The second type (referred to as "conditional" in the tables) are calculated after first eliminating years in which one or more disasters took place. This second type therefore conditions on no disasters

[5] The discrete-time approximation requires setting λ_t to equal zero in the square root when it is negative. However, this occurs in less than 1% of the simulated draws.

occurring. Neither represents an exact comparison with U.S. data, and for this reason, the data should be viewed as an approximate benchmark.

Table 2 shows that the model generates a realistic equity premium. In population, the equity premium is 5.6%, while, conditional on no disasters, the equity premium is 6.5%. In these data it is 5.8%. The expected return on the government bill is 2.5% in population, 2.8% conditional on no disasters, and 2.0% in the data. The model predicts equity volatility of 17.4% per annum in population and 15.1% conditional on no disasters. The observed volatility is 18.5%. The Sharpe ratio is 0.33 in population, 0.42 conditional on no disasters and 0.32 in the data (the Sharpe ratio is substantially higher over the postwar period that in the long data set).

The model is able to generate reasonable volatility for the stock market without generating excessive volatility for the government bill or for consumption and dividends. The volatility of the government bill is 3.2% in population, most of which is due to disasters; it is only 1.7% conditional on no disasters. This compares with a volatility of 5.9% in the data. Given that interest rate volatility in the data arises largely from unexpected inflation that is not captured by the model, the data volatility should be viewed as an upper bound on reasonable model volatility.

The volatilities for consumption and dividends predicted by the model for periods of no disasters are also below their data counterparts. Conditional on no disasters, consumption volatility is 2%, compared with 3.6% in the data. Dividend volatility is 6.0%, compared with 11.5% in the data. Including rare disasters in the data simulated from the model has a large effect on dividend volatility. When the rare disasters are included, dividend volatility is 17.7%. The difference between the effect of including rare disasters on returns as compared with the effect on fundamentals is striking. Unlike dividends, returns exhibit a relatively small difference in volatility when calculated with and without rare disasters: 17.4% versus 15.1%. This is because a large amount of the volatility in returns arises from variation in the equity premium. Risk premia are equally variable regardless of the whether disasters actually occur in the simulated data or not.

The model also generates excess return predictability, as shown in Table 3. I regress long-horizon excess returns (the log return on equity minus the log return on the government bill) on the price-dividend ratio in simulated data. I calculate this regression for returns measured over horizons ranging from 1 to 10 years. Table 3 reports results for the entire simulated data set ("population moments") for periods in the simulation in which no disasters occur ("conditional moments") and for the historical sample.

Panel A of Table 3 shows population moments from simulated data. The coefficients on the price-dividend ratio are negative: a high price-dividend ratio corresponds to low disaster risk and therefore predicts low future expected returns on stocks relative to bonds. The R^2 for the long-horizon regression is 3% at a horizon of 1 year, rising to 9% at a horizon of 10 years. Panel B reports conditional moments. The conditional R^2s are larger: 13% at a horizon of 1 year, rising to 50% at a horizon of 10 years. The unconditional R^2 values are much lower because, when a disaster occurs, nearly all of the unexpected return is due to the shock to cash flows.

The data moments fall in between the population and conditional moments. As demonstrated in a number of studies (e.g. Campbell and Shiller (1988), Cochrane (1992), Fama and French (1989), Keim and Stambaugh (1986)) and replicated in this sample, high price-dividend ratios predict low excess returns. While returns exhibit predictability over a wide range of sample periods, the high persistence of the price-dividend ratio leads sample statistics to be unstable (see, for example, Lettau and Wachter (2007) for calculations of long-horizon predictability using this data set but for differing sample periods), and unusually low when calculated over recent years. For this reason, the R^2 statistics in the data should be viewed as an approximate benchmark.

Another potential source of variation in returns is variation in expected future consumption growth. According to the model, a low price-dividend ratio indicates not only that the equity premium is likely to be high in the future, but also that consumption growth is likely to be low because of the increased probability of a disaster. However, a number of studies (e.g. Campbell (2003), Cochrane (1994), Hall (1988), Lettau and Ludvigson (2001)) have found that consumption growth exhibits little predictability at long horizons, a finding replicated in Panel B of Table 4. It is therefore of interest to quantify the amount of consumption growth predictability implied by model.

Table 4 reports the results of running long-horizon regressions of consumption growth on the price-dividend ratio in data simulated from the model and in historical data. Panel A shows the population moments implied by the model. The model does imply some predictability in consumption growth, but the effect is very small. The R^2 values never rise about 4%, even at long horizons. This predictability arises entirely from the realization of a rare disaster. When these rare disasters are conditioned out, there is zero predictability because consumption follows a random walk (in simulated data, the coefficients with values of less than .001 and R^2 values less than .0001). Thus the model accounts for both the predictability in long-horizon returns and the absence of predictability in consumption growth.

4.3 Comparison with the constant disaster risk model

It is instructive to contrast the results in the previous section with results when the probability of a rare disaster does not vary over time. Table 5 calculates moments corresponding to those in Table 2 for the constant disaster risk model. The long-horizon regressions in Tables 3 and 4 are not repeated because, when disaster risk is constant, the predictability coefficients and R^2 statistics are zero at all horizons. The first calibration (results reported in Panel A) corresponds to the parameters in Barro (2006), and replicates the results in that paper.[6] The second calibration (results reported in Panel B) alters these parameters slightly: rather than assuming power utility, the EIS is set to 1. This highlights the role of recursive utility when nothing else changes. The third calibration (results reported in Panel C) maintains the assumptions of the second set, but raises leverage from 1.5 to 3. This shows the effect of an increase in leverage when nothing else changes.

[6]The model for leverage in this paper differs slightly from Barro's. However, the effects of the differences are second-order.

As Panel A shows, the power utility model with constant disaster probabilities is capable of replicating the equity premium in the data, and reconciling it with a low riskfree rate. This model is not, however, capable of replicating the volatility of the stock market. Stock return volatility is about equal to dividend volatility.[7] That is, the conditional volatility of stock returns is 3.3%, compared to the dividend volatility of 3.0%. The population volatility of stock returns is 6.7%, compared with the dividend volatility of 8.7%. As a consequence, the Sharpe ratio predicted by the model is 1.84 (conditional on no disasters), much higher than the observed value (0.32).

Panel B demonstrates the effect of recursive utility. This simulation is identical to the above, except the EIS is set equal to 1, the value from the base case. As this simulation verifies, recursive utility makes little difference to the equity premium. The volatility of stock returns and the Sharpe ratio are also nearly the same. However, the government bill rate is substantially lower; 1.7% per annum rather than 3.9% conditional on no disasters. The reason is that the inverse of the EIS multiplies the growth rate of the economy in the formula for the riskfree rate. Because growth is positive, power utility, with an EIS of 0.25, generates a higher riskfree rate than recursive utility with an EIS of 1.

Panel C demonstrates the effect of increasing leverage from 1.5 to 3. While increasing leverage increases volatility, it is not nearly sufficient to match the volatility of stock returns. Not surprisingly, raising leverage from 1.5 to 3 doubles the log volatility of log dividends. However, this does not generate nearly enough volatility in either the conditional or full population to match the volatility of returns in the data. These results contrast with the model that incorporates both recursive utility and time-varying disaster risk, which can generate a realistic amount of stock market volatility.

[7] Stock market volatility and dividend volatility are not exactly because the table reports the volatility of stock returns measured in levels, while dividends are measured in logs. The volatility of log returns is identical to the volatility of log dividends, both in the full and the conditional populations.

5 Conclusion

This paper has shown that a continuous-time endowment model in which there is time-varying risk of a rare disaster can explain many features of the aggregate stock market. It can explain the high equity premium without assuming a high value of risk aversion for the representative investor. Assuming, realistically, that the probability of the rare disaster varies over time, it can also explain the high level of stock market volatility. The volatility of the government bill rate remains low due to a tradeoff between an increased desire to save and a greater risk of default. The model therefore offers a novel explanation of volatility in the aggregate stock market that is consistent with other macroeconomic data. Finally, the model can be solved in closed form, allowing for straightforward computation and for potential extensions. While this paper has focused on the aggregate stock market, the model could be extended to price additional asset classes, such as long-term government bonds, options and exchange rates.

Appendix

A Constant disaster risk model

A.1 The value function

The envelope condition (10) can be used to derive the relation between l (the wealth-consumption ratio) and j (the constant in the value function) in (11). Equation (3) implies

$$\begin{aligned} J_W(W(C)) &= (W(C))^{-\gamma} j^{1-\gamma} \\ &= (lC)^{-\gamma} j^{1-\gamma}. \end{aligned} \tag{A.1}$$

Moreover,

$$f_C(C,V) = \frac{\beta C^{-\frac{1}{\psi}}}{((1-\gamma)V)^{\frac{1}{\theta}-1}}, \tag{A.2}$$

where it follows from (4) that

$$V(C) = J(W(C)) = \frac{(lC)^{1-\gamma}}{1-\gamma} j^{1-\gamma}. \tag{A.3}$$

Substituting (A.3) into (A.2) implies

$$\begin{aligned} f_C(C,V(C)) &= \beta C^{-\frac{1}{\psi}} C^{(1-\frac{1}{\theta})} (lj)^{(1-\frac{1}{\theta})(1-\gamma)} \\ &= \beta C^{-\gamma} (lj)^{\frac{1}{\psi}-\gamma}. \end{aligned} \tag{A.4}$$

Equating (A.4) with (A.1) and solving for l implies (11).

Given (11), the an expression for j follows from the fact that $V(C(W)) = J(W)$, and so

$$f(C(W),J(W)) = \frac{\beta}{1-\frac{1}{\psi}} W^{1-\gamma} \frac{l^{\frac{1}{\psi}-1} - j^{1-\frac{1}{\psi}}}{j^{\gamma-\frac{1}{\psi}}}. \tag{A.5}$$

Substituting into the Bellman equation (6) and dividing through by $W^{1-\gamma}$ yields:

$$\left(\mu - \frac{1}{2}\gamma\sigma^2 + \lambda(1-\gamma)^{-1} E_\nu \left[e^{(1-\gamma)Z} - 1\right]\right) j^{1-\gamma} + \\ \frac{\beta}{1-\frac{1}{\psi}} \frac{\beta^{\psi-1} j^{(1-\frac{1}{\psi})(1-\psi)} - j^{1-\frac{1}{\psi}}}{j^{\gamma-\frac{1}{\psi}}} = 0,$$

and therefore

$$\mu - \frac{1}{2}\gamma\sigma^2 + \lambda(1-\gamma)^{-1}E_\nu\left[e^{(1-\gamma)Z} - 1\right] - \frac{\beta}{1-\frac{1}{\psi}} + \frac{\beta^\psi}{1-\frac{1}{\psi}}j^{1-\psi} = 0,$$

Rearranging implies (12).

A.2 The dividend claim

By Ito's Lemma,

$$\begin{aligned} d(F_t\pi_t) + Y_t\pi_t\,dt = {} & \pi_t(DF_t)\,dt + F_t(D\pi_t)\,dt + Y_t\pi_t\,dt + \\ & \pi_t(\delta F_t)\,dB_t + F_t(\delta\pi_t)\,dB_t + (\delta\pi_t)(\delta F_t)\,dt + \\ & \lambda\mathcal{J}(\pi_t F_t)\,dt + \left((\pi_t F_t - \pi_{t^-}F_{t^-})dN_t - \lambda\mathcal{J}(\pi_t F_t)\right)\,dt, \end{aligned} \quad \text{(A.6)}$$

Under mild regularity conditions (see Duffie, Pan, and Singleton (2000)), the no-arbitrage condition (17) implies that the instantaneous expectation of (A.6) is equal to zero. This establishes (18).

Ito's Lemma applied to (15) implies that the diffusion term for the state-price density is given by

$$\begin{aligned} \delta\pi_t &= \frac{\pi_t}{f_C}\delta f_C \\ &= -\frac{\pi_t}{f_C}\beta\gamma C^{-\gamma-1}(lj)^{\frac{1}{\psi}-\gamma}C\sigma \\ &= -\pi_t\gamma\sigma, \end{aligned} \quad \text{(A.7)}$$

where the second line follows from (A.4). No-arbitrage implies that the drift of the state-price density must equal

$$D\pi_t = -r\pi_t - \lambda\mathcal{J}(\pi_t).$$

Let

$$H_t = \exp\left\{\int_0^t f_V(C_s, V_s)\,ds\right\}. \quad \text{(A.8)}$$

Then

$$\begin{aligned} \mathcal{J}(\pi_t) &= H_t E_\nu\left[f_C(Ce^{Z_t}, V(Ce^{Z_t})) - f_C(C, V(C))\right] \\ &= H_t E_\nu\left[\beta(Ce^{Z_t})^{-\gamma}(lj)^{\frac{1}{\psi}-\gamma} - \beta C^{-\gamma}(lj)^{\frac{1}{\psi}-\gamma}\right] \\ &= \pi_t E_\nu\left[e^{-\gamma Z_t} - 1\right]. \end{aligned} \quad \text{(A.9)}$$

By Ito's Lemma,

$$\begin{aligned} DF_t &= F_Y\mu_Y Y_t + \frac{1}{2}F_{YY}Y_t\phi^2\sigma^2 = l_Y\mu_Y Y_t && \text{(A.10)} \\ \delta F_t &= F_Y\phi\sigma Y_t = l_Y\phi\sigma Y_t && \text{(A.11)} \end{aligned}$$

Finally,

$$\begin{aligned} \mathcal{J}(\pi_t F_t) &= H_t E_\nu\left[f_C(Ce^{Z_t}, V(Ce^{Z_t}))F(Y_t e^{\phi Z_t}) - f_C(C, V(C))F(Y_t)\right] \\ &= \pi_t E_\nu\left[e^{-\gamma Z_t}F(Y_t e^{\phi Z_t}) - F(Y_t)\right] && \text{(A.12)} \\ &= l_Y E\nu\left[e^{(\phi-\gamma)Z} - 1\right]\pi_t Y_t && \text{(A.13)} \end{aligned}$$

Substituting (A.7 – A.13) into (18) verifies the conjecture (19), for l_Y defined implicitly by (20).

B Time-varying disaster risk model

B.1 Value function

Substituting the optimal policies $\alpha = 1$ and $W = lC$ into the Bellman equation (26) implies

$$\begin{aligned} J_W W_t\mu + J_\lambda\kappa(\bar\lambda - \lambda_t) + \frac{1}{2}J_{WW}W_t^2\sigma^2 + \frac{1}{2}J_{\lambda\lambda}\sigma_\lambda^2\lambda_t + \\ \lambda_t E_\nu\left[J(W_t e^{Z_t}, \lambda_t) - J(W_t, \lambda_t)\right] + f(C_t, J) = 0. \quad \text{(B.1)} \end{aligned}$$

Using (25), the last term can be rewritten as follows:

$$\begin{aligned} f(C(W), J(\lambda, W)) &= \beta I(\lambda)W^{1-\gamma}\left(\log(\beta W) - \frac{1}{1-\gamma}\log(I(\lambda)W^{1-\gamma})\right) \\ &= \beta I(\lambda)W^{1-\gamma}\left(\log\beta - \frac{\log I(\lambda)}{1-\gamma}\right). \end{aligned}$$

Therefore, substituting (25) into (B.1) and dividing through by $W^{1-\gamma}$ implies

$$\begin{aligned} I(\lambda_t)\mu + I'(\lambda_t)(1-\gamma)^{-1}\kappa(\bar\lambda - \lambda_t) - I(\lambda_t)\frac{1}{2}\gamma\sigma^2 \\ + \frac{1}{2}I''(\lambda_t)\sigma_\lambda^2\lambda_t(1-\gamma)^{-1} + \lambda_t I(\lambda_t)(1-\gamma)^{-1}E_\nu\left[e^{(1-\gamma)Z} - 1\right] \\ + \beta I(\lambda_t)\left[\log\beta - \frac{\log I(\lambda)}{1-\gamma}\right] = 0. \quad \text{(B.2)} \end{aligned}$$

Conjecture that $I(\lambda)$ is given by (31), implying that

$$\begin{aligned} I'(\lambda) &= bI(\lambda) \\ I''(\lambda) &= b^2 I(\lambda). \end{aligned}$$

Substituting into (B.2), I find

$$\mu + b(1-\gamma)^{-1}\kappa(\bar{\lambda} - \lambda_t) - \frac{1}{2}\gamma\sigma^2 + b^2\sigma_\lambda^2\lambda_t(1-\gamma)^{-1} + \\ \lambda_t(1-\gamma)^{-1}E_\nu\left[e^{(1-\gamma)Z} - 1\right] + \beta\left(\log\beta - (1-\gamma)^{-1}(a + b\lambda_t)\right) = 0.$$

Collecting terms in λ_t results in the following quadratic equation for b:

$$b^2 - \frac{\kappa+\beta}{\sigma_\lambda^2}b + \frac{1}{\sigma_\lambda^2}E_\nu\left[e^{(1-\gamma)Z} - 1\right] = 0,$$

so

$$b = \frac{\kappa+\beta}{2\sigma_\lambda^2} \pm \sqrt{\left(\frac{\kappa+\beta}{2\sigma_\lambda^2}\right)^2 - \frac{E_\nu\left[e^{(1-\gamma)Z} - 1\right]}{\sigma_\lambda^2}}, \tag{B.3}$$

while

$$a = \frac{1-\gamma}{\beta}\left(\mu - \frac{1}{2}\gamma\sigma^2\right) + (1-\gamma)\log\beta + b\frac{\kappa\bar{\lambda}}{\beta}.$$

Following Tauchen (2005), I focus on the solution corresponding to the negative root in (B.3). This solution has the desirable property of approaching a well-defined limit as σ_λ approaches 0, as shown below. However, given the parameter choices in Table 1, the discriminant in (B.3) is of the order of 10^{-5}, and therefore the asset pricing implications of this choice are negligible.

I now check that this solution for the value function is consistent with the solution in the constant disaster risk case. To do so, I first evaluate a and b in the limit as σ_λ approaches zero, and then evaluate the solution for the constant disaster risk model in the limit as ψ approaches 1. First note that, under the choice of the negative root, b can be written as

$$b = \frac{1}{\sigma_\lambda^2}\left(\frac{\kappa+\beta}{2} - \sqrt{\left(\frac{\kappa+\beta}{2}\right)^2 - E_\nu\left[e^{(1-\gamma)Z} - 1\right]\sigma_\lambda^2}\right).$$

For any constants $c_1 \neq 0$ and c_2,

$$\lim_{x\to 0}\frac{1}{x}\left(c_1 - \sqrt{c_1^2 + xc_2}\right) = \frac{d}{dx}\left(c_1 - \sqrt{c_1^2 + xc_2}\right)\Big|_{x=0} = -\frac{c_2}{2c_1}.$$

Choosing $c_1 = (\kappa + \beta)/2$ and $c_2 = -E_\nu \left[e^{(1-\gamma)Z-1}\right]$ implies that

$$\lim_{\sigma_\lambda \to 0} b = \frac{E_\nu \left[e^{(1-\gamma)Z} - 1\right]}{\kappa + \beta},$$

and therefore that

$$\lim_{\sigma_\lambda \to 0, \kappa \to 0} (a + b\bar{\lambda}) = \frac{1-\gamma}{\beta}\left(\mu - \frac{1}{2}\gamma\sigma^2\right) + (1-\gamma)\log\beta + \bar{\lambda}\frac{E_\nu \left[e^{(1-\gamma)Z} - 1\right]}{\beta}.$$

I now evaluate the solution for the constant disaster risk model in the limit as ψ approaches 1. Let

$$A = -\mu + \frac{1}{2}\gamma\sigma^2 - \lambda(1-\gamma)^{-1}E_\nu \left[e^{(1-\gamma)Z} - 1\right].$$

Then, by (12),

$$\begin{aligned}
j &= \left[\left(A + \frac{\beta}{1-\frac{1}{\psi}}\right)\frac{1-\frac{1}{\psi}}{\beta^\psi}\right]^{\frac{1}{1-\psi}} \\
&= \left[A\left(1 - \frac{1}{\psi}\right)\beta^{-\psi} + \beta^{1-\psi}\right]^{\frac{1}{1-\psi}} \\
&= \left[A\beta^{-1}\left(1 - \frac{1}{\psi}\right) + 1\right]^{\frac{1}{1-\psi}}\beta \\
&= \exp\left\{-\frac{1}{\psi}\log\left(\left[A\beta^{-1}\left(1-\frac{1}{\psi}\right)+1\right]^{\frac{1}{1-\frac{1}{\psi}}}\right) + \log\beta\right\}.
\end{aligned}$$

From the definition of the exponential, it follows that

$$\lim_{\psi \to 1}\left[A\beta^{-1}\left(1 - \frac{1}{\psi}\right) + 1\right]^{\frac{1}{1-\frac{1}{\psi}}} = \exp\left\{A\beta^{-1}\right\}.$$

Since both this limit and the limit of $1/\psi$ are well-defined as ψ approaches 1, I conclude that

$$\lim_{\psi \to 1} j = \exp\left\{-\beta^{-1}A + \log\beta\right\}.$$

Thus the solution for the time-varying case is consistent with that for the constant disaster risk case as long as the positive root is chosen.

B.2 Dividend claim

Applying Ito's Lemma to the product $F(\lambda, Y)\pi$ in the case of time-varying disaster risk yields

$$\begin{aligned} d(F_t\pi_t) + Y_t\pi_t\,dt = \pi_t(DF_t)\,dt + F_t(D\pi_t)\,dt + Y_t\pi_t\,dt + \\ \pi_t(\delta F_t)^\top\left[dB_t\,dB_{\lambda,t}\right]^\top + F_t(\delta\pi_t)^\top\left[dB_t\,dB_{\lambda,t}\right]^\top + (\delta\pi_t)^\top(\delta F_t)\,dt + \\ \lambda\mathcal{J}(\pi_t F_t)\,dt + \left((\pi_t F_t - \pi_{t^-}F_{t^-})dN_t - \lambda\mathcal{J}(\pi_t F_t)\right)\,dt, \quad \text{(B.4)} \end{aligned}$$

where I have used the fact that B_t and $B_{\lambda,t}$ are orthogonal. Equation (32) then follows from the argument given in Appendix A.2.

As in the case of constant disaster risk, Ito's Lemma applied to (15) implies that the diffusion term for the state-price density is given by

$$\delta\pi_t = \frac{\pi_t}{f_C}\delta f_C$$

It follows from (25) that continuation utility equals

$$V(\lambda, C) = J(\lambda, W(C)) = J(\lambda, lC) = l^{1-\gamma}I(\lambda)\frac{C^{1-\gamma}}{1-\gamma}.$$

Therefore, by (29),

$$f_C(C, V(\lambda, C)) = \beta l^{1-\gamma}I(\lambda)C^{-\gamma} = \beta^\gamma I(\lambda)C^{-\gamma} \quad \text{(B.5)}$$

It follows from Ito's Lemma that

$$\frac{\delta f_C}{f_C} = \begin{bmatrix} -\gamma\sigma \\ b\sigma_\lambda\sqrt{\lambda_t} \end{bmatrix}.$$

and therefore

$$\frac{\delta\pi_t}{\pi_t} = \begin{bmatrix} -\gamma\sigma \\ b\sigma_\lambda\sqrt{\lambda_t} \end{bmatrix}. \quad \text{(B.6)}$$

No-arbitrage implies that the diffusion for the state-price density is given by

$$D\pi_t = -r_t\pi_t - \lambda\mathcal{J}(\pi_t), \quad \text{(B.7)}$$

where, defining H_t as in (A.8), it follows that

$$\begin{aligned} \mathcal{J}(\pi_t) &= H_t E_\nu\left[f_C(C_t e^{Z_t}, V(\lambda_t, C_t e^{Z_t})) - f_C(C_t, V(\lambda_t, C_t))\right] \\ &= H_t\beta^\gamma I(\lambda_t)E_\nu\left[C_t^{-\gamma}e^{-\gamma Z_t} - C_t^{-\gamma}\right] \\ &= \pi_t E_\nu\left[e^{-\gamma Z_t} - 1\right], \quad \text{(B.8)} \end{aligned}$$

and where r_t is given by (30).

It follows from (33) and Ito's Lemma that

$$\delta F_t = \begin{bmatrix} Y_t G(\lambda_t)\phi\sigma \\ Y_t G'(\lambda_t)\sigma_\lambda\sqrt{\lambda_t} \end{bmatrix}. \tag{B.9}$$

and

$$DF_t = G(\lambda_t)Y_t\mu_Y + G'(\lambda_t)Y_t\kappa(\bar{\lambda} - \lambda_t) + \frac{1}{2}G''(\lambda_t)Y_t\sigma_\lambda^2\lambda_t. \tag{B.10}$$

Finally,

$$\begin{aligned} \mathcal{J}(\pi_t F_t) &= H_t G(\lambda_t) E_\nu \left[f_C(Ce^Z, V(\lambda_t, Ce^Z))Y_t e^{\phi Z} - f_C(C_t, V(\lambda_t, C_t))Y_t \right] \\ &= H_t G(\lambda_t) Y_t E_\nu \left[\beta^\gamma I(\lambda_t) C_t^{-\gamma} e^{(\phi-\gamma)Z_t} - \beta^\gamma I(\lambda_t) C_t^{-\gamma} \right] \\ &= \pi_t F_t E_\nu \left[e^{(\phi-\gamma)Z} - 1 \right]. \end{aligned} \tag{B.11}$$

Substituting (B.6 – B.11) into (32) implies

$$\begin{aligned} &G\mu_Y + G'\kappa(\bar{\lambda} - \lambda_t) + \frac{1}{2}G''\sigma_\lambda^2\lambda_t - Gr_t \\ &\qquad - G\lambda_t E_\nu \left[e^{-\gamma Z} - 1 \right] + 1 - G\gamma\sigma^2\phi + G'b\sigma_\lambda^2\lambda_t \\ &\qquad\qquad + G\lambda_t E_\nu \left[e^{(\phi-\gamma)Z} - 1 \right] = 0. \end{aligned} \tag{B.12}$$

Substituting (30) in for r_t implies

$$\begin{aligned} &G\mu_Y + G'\kappa(\bar{\lambda} - \lambda_t) + \frac{1}{2}G''\sigma_\lambda^2\lambda_t - G\left(\mu + \beta - \gamma\sigma^2 + \lambda_t E_\nu \left[e^{-\gamma Z}\left(e^Z - 1 \right) \right] \right) \\ &\qquad - G\lambda_t E_\nu \left[e^{-\gamma Z} - 1 \right] + 1 - G\gamma\sigma^2\phi + G'b\sigma_\lambda^2\lambda_t \\ &\qquad\qquad + G\lambda_t E_\nu \left[e^{(\phi-\gamma)Z} - 1 \right] = 0. \end{aligned} \tag{B.13}$$

To derive the ordinary differential equations (35) and (36), note that

$$G'(\lambda_t) = \int_0^\infty \exp\left\{ a_\phi(s) + b_\phi(s)\lambda_t \right\} b_\phi(s)\, ds \tag{B.14}$$

$$G''(\lambda_t) = \int_0^\infty \exp\left\{ a_\phi(s) + b_\phi(s)\lambda_t \right\} b_\phi(s)^2\, ds. \tag{B.15}$$

The solution must satisfy the boundary conditions $a_\phi(0) = b_\phi(0) = 0$ and

$$\lim_{t\to\infty} \exp\left\{ a_\phi(t) + b_\phi(t)\lambda_t \right\} = 0 \quad \forall \lambda_t > 0.$$

It follows from the boundary conditions and integration by parts that[8]

$$-\int_0^\infty \exp\left\{a_\phi(s) + b_\phi(s)\lambda_t\right\}\left(a'_\phi(s) + b'_\phi(s)\lambda_t\right) ds = 1. \quad \text{(B.16)}$$

It follows from (34), (B.14), (B.15) and (B.16) that each term in (B.13) takes the form of an integral of $\exp\left\{a_\phi(s) + b_\phi(s)\right\}$ multiplied by an expression that is linear in λ_t. Equation (34) can therefore be solved by setting the terms within the integral equal to zero, or equivalently, solving

$$\begin{aligned}\mu_Y + b_\phi(s)\kappa(\bar{\lambda} - \lambda_t) + \frac{1}{2}b_\phi^2(s)\sigma_\lambda^2\lambda_t - \mu - \beta + \gamma\sigma^2 - \lambda_t E_\nu\left[e^{-\gamma Z}(e^Z - 1)\right] \\ - \lambda_t E_\nu\left[e^{-\gamma Z} - 1\right] - a'_\phi(s) - b'_\phi(s)\lambda_t - \gamma\sigma^2\phi + b_\phi(s)b\sigma_\lambda^2\lambda_t \\ + \lambda_t E_\nu\left[e^{(\phi-\gamma)Z} - 1\right] = 0. \quad \text{(B.17)}\end{aligned}$$

Matching constant terms implies (35) and (36). Thus the price of the dividend claim is characterized by (33), where G is given by (34), and a_ϕ and b_ϕ are characterized by (35) and (36) respectively, with $a_\phi(0) = b_\phi(0) = 0$.

[8]The details of this calculation are as follows. Let $h_1(s) = \exp\left\{a_\phi(s) + b_\phi(s)\lambda_t\right\}$ and $h_2(s) = 1$. Integration by parts implies

$$\int_0^\infty h_1(s)h'_2(s)\, ds = \lim_{s\to\infty}(h_1(s)h_2(s)) - h_1(0)h_2(0) - \int_0^\infty h'_1(s)h_2(s)\, ds,$$

where the the left hand side of the above is identically zero.

References

Abel, Andrew, 1999, Risk premia and term premia in general equilibrium, *Journal of Monetary Economics* 43, 3–33.

Bansal, Ravi, and Amir Yaron, 2004, Risks for the long-run: A potential resolution of asset pricing puzzles, *Journal of Finance* 59, 1481–1509.

Barro, Robert J., 2006, Rare disasters and asset markets in the twentieth century, *The Quarterly Journal of Economics* pp. 823–866.

Barro, Robert J., 2007, Rare disasters, asset prices, and welfare costs, forthcoming, *American Economic Review.*

Campbell, John Y., 2003, Consumption-based asset pricing, in *Handbook of the Economics of Finance Volume IB* (North-Holland, Amsterdam).

Campbell, John Y., and Robert J. Shiller, 1988, The dividend-price ratio and expectations of future dividends and discount factors, *Review of Financial Studies* 1, 195–228.

Cochrane, John H., 1992, Explaining the variance of price-dividend ratios, *Review of Financial Studies* 5, 243–280.

Cochrane, John H., 1994, Permanent and transitory components of GDP and stock prices, *Quarterly Journal of Economics* 109, 241–265.

Cox, John C., Jonathan C. Ingersoll, and Stephen A. Ross, 1985, A theory of the term structure of interest rates, *Econometrica* 53, 385–408.

Drechsler, Itamar, and Amir Yaron, 2008, What's vol got to do with it, Working paper, University of Pennsylvania.

Duffie, Darrell, 2001, *Dynamic Asset Pricing Theory.* (Princeton University Press Princeton, NJ) 3 edn.

Duffie, Darrell, and Larry G Epstein, 1992, Asset pricing with stochastic differential utility, *Review of Financial Studies* 5, 411–436.

Duffie, Darrell, Jun Pan, and Kenneth Singleton, 2000, Transform analysis and asset pricing for affine jump-diffusions, *Econometrica* 68, 1343–1376.

Duffie, Darrell, and Costis Skiadas, 1994, Continuous-time asset pricing: A utility gradient approach, *Journal of Mathematical Economics* 23, 107–132.

Epstein, Larry, and Stan Zin, 1989, Substitution, risk aversion and the temporal behavior of consumption and asset returns: A theoretical framework, *Econometrica* 57, 937–969.

Fama, Eugene F., and Kenneth R. French, 1989, Business conditions and expected returns on stocks and bonds, *Journal of Financial Economics* 25, 23–49.

Gabaix, Xavier, 2007, A simple, unified, exactly solved framework for ten puzzles in macro-finance, Working paper, New York University.

Gourio, Francois, 2007, Disasters and recoveries, Working paper, Boston University.

Hall, Robert E., 1988, Intertemporal substitution in consumption, *Journal of Political Economy* 96, 221–273.

Hodrick, Robert J., 1992, Dividend yields and expected stock returns: Alternative procedures for inference and measurement, *Review of Financial Studies* 5, 357–386.

Keim, Donald B., and Robert F. Stambaugh, 1986, Predicting returns in the stock and bond markets, *Journal of Financial Economics* 17, 357–390.

Kreps, D., and E. Porteus, 1978, Temporal resolution of uncertainty and dynamic choice theory, *Econometrica* 46, 185–200.

LeRoy, Stephen F., and Richard D. Porter, 1981, The present-value relation: Tests based on implied variance bounds, *Econometrica* 49, 555–574.

Lettau, Martin, and Sydney C. Ludvigson, 2001, Consumption, aggregate wealth and expected stock returns, *Journal of Finance* 56, 815–849.

Lettau, Martin, and Jessica A. Wachter, 2007, Why is long-horizon equity less risky? A duration-based explanation of the value premium, *Journal of Finance* 62, 55–92.

Liu, Jun, Jun Pan, and Tan Wang, 2005, An equilibrium model of rare-event premia and its implication for option smirks, *Review of Financial Studies* 18, 131–164.

Lord, Roger, Remmert Koekkoek, and Dick van Dijk, 2006, A comparison of biased simulation schemes for stochastic volatility models, Tinbergen Institute Discussion Paper 06-046/4.

Martin, Ian, 2007, Consumption-based asset pricing with higher cumulants, Working paper, Harvard University.

Mehra, Rajnish, and Edward Prescott, 1985, The equity premium puzzle, *Journal of Monetary Economics* 15, 145–161.

Naik, Vasanttilak, and Moon Lee, 1990, General equilibrium pricing of options on the market portfolio with discontinuous returns, *Review of Financial Studies* 3, 493–521.

Reitz, Thomas A., 1988, The equity risk premium: A solution, *Journal of Monetary Economics* 22, 117–131.

Santa-Clara, Pedro, and Shu Yan, 2006, Crashes, volatility, and the equity premium: Lessons from S&P 500 options, Working paper, UCLA and University of Arizona.

Shiller, Robert J., 1981, Do stock prices move too much to be justified by subsequent changes in dividends?, *American Economic Review* 71, 421–436.

Shiller, Robert J., 1989, *Market Volatility.* (MIT Press Cambridge, MA).

Tauchen, George, 2005, Stochastic volatility in general equilibrium, Working paper, Duke University.

Veronesi, Pietro, 2004, The Peso problem hypothesis and stock market returns, *Journal of Economic Dynamics and Control* pp. 707–725.

Weil, Philippe, 1990, Nonexpected utilitiy in macroeconomics, *Quarterly Journal of Economics* 105, 29–42.

Table 1: Parameters for the time-varying disaster risk model

Panel A: Cash flow parameters	
Average growth in consumption (normal times) μ	0.0252
Volatility of consumption growth (normal times) σ	0.020
Leverage ϕ	3.0
Average probability of a rare disaster $\bar{\lambda}$	0.017
Mean reversion κ	0.142
Volatility parameter σ_λ	0.07
Probability of default given disaster q	0.40
Panel B: Preference parameters	
Rate of time preference β	0.02
Relative risk aversion γ	3.0
Elasticity of intertemporal substitution ψ	1.0

Notes: The table shows parameter values for the time-varying disaster risk model. The process for the intensity of a disaster is given by

$$d\lambda_t = \kappa(\bar{\lambda} - \lambda_t)\, dt + \sigma_\lambda \sqrt{\lambda_t}\, dB_{\lambda,t}.$$

The consumption (endowment) process is given by

$$dC_t = \mu C_t\, dt + \sigma C_t\, dB_t + (e^{Z_t} - 1)C_{t^-}\, dN_t,$$

where N_t is a Poisson process with intensity λ_t, and Z_t is calibrated to the distribution of large declines in GDP in the data. The dividend Y_t equals C_t^ϕ. The representative agent has recursive utility defined by $V_t = E_t \int_t^\infty f(C_s, V_s)\, ds$, with normalized aggregator

$$f(C, V) = \beta(1-\gamma)V \left[\log C - \frac{1}{1-\gamma} \log((1-\gamma)V)\right].$$

Parameter values are in annual terms.

Table 2: Population moments from simulated data and sample moments from the historical time series

	Model		U.S. Data
	Population	Conditional	
$E[R^b]$	2.53	2.75	2.01
$\sigma(R^b)$	3.17	1.68	5.91
$E[R^e - R^b]$	5.58	6.49	5.97
$\sigma(R^e)$	17.38	15.05	18.48
Sharpe Ratio	0.33	0.42	0.32
$\sigma(\Delta c)$	5.89	1.98	3.56
$\sigma(\Delta y)$	17.68	5.96	11.51

Notes: The model is simulated at a monthly frequency and simulated data are aggregated to an annual frequency. Moments are calculated using the annual data and (except for the Sharpe ratio) expressed in percentage terms. The second column reports population moments from simulated data. The third column reports moments from simulated data that are calculated over years in which a disaster did not occur. The last column reports sample moments from annual U.S. data from 1890 to 2004. R^b denotes the gross return on the government bond, R^e the gross equity return, Δc growth in log consumption and Δy growth in log dividends.

Table 3: Long-horizon regressions: Excess returns

	Horizon in years					
	1	2	4	6	8	10
	Panel A: Model – Population moments					
β_1	-0.14	-0.26	-0.45	-0.59	-0.68	-0.77
R^2	0.03	0.05	0.07	0.09	0.09	0.09
	Panel B: Model – Conditional moments					
β_1	-0.21	-0.39	-0.69	-0.91	-1.08	-1.20
R^2	0.13	0.23	0.37	0.46	0.50	0.50
	Panel B: U.S. Data					
β_1	-0.09	-0.14	-0.36	-0.48	-0.73	-0.98
t-stat	-2.04	-1.84	-2.91	-2.69	-3.22	-3.75
R^2	0.04	0.04	0.12	0.12	0.17	0.23

Notes: Excess returns are regressed on the lagged price-dividend ratio in data simulated from the model and in annual data from 1890 to 2004. Specifically, the table reports coefficients β_1, R^2 statistics and, for the sample, Newey-West t-statistics for regressions

$$\sum_{i=1}^{h} \log(R^e_{t+i}) - \log(R^b_{t+i}) = \beta_0 + \beta_1(p_t - y_t) + \epsilon_t,$$

where R^e_{t+i} and R^b_{t+i} are, respectively, the return on the aggregate market and the return on the government bill between $t+i-1$ and $t+i$ and $p_t - y_t$ is the log price-dividend ratio on the aggregated market. The model is simulated at a monthly frequency and simulated data are aggregated to an annual frequency. Panel A reports population moments from simulated data. Panel B reports moments from simulated data that are calculated over years in which a disaster does not take place (for a horizon of 2, for example, all 2-year periods in which a disaster takes place are eliminated). Panel C reports sample moments.

Table 4: Long-horizon regressions: Consumption growth

	Horizon in years					
	1	2	4	6	8	10
Panel A: Model – Population moments						
β_1	0.03	0.05	0.09	0.11	0.14	0.16
R^2	0.01	0.02	0.03	0.03	0.04	0.04
Panel B: U.S. Data						
β_1	0.003	-0.001	-0.007	0.002	0.006	0.040
t-stat	0.45	-0.06	-0.38	0.08	0.16	0.82
R^2	0.0011	0.0000	0.0013	0.0001	0.0004	0.0129

Notes: Growth in aggregate consumption is regressed on the lagged price-dividend ratio in data simulated from the model and in annual data from 1890 to 2004. Specifically, the table reports coefficients β_1, R^2 statistics and, for the sample, Newey-West t-statistics for regressions

$$\sum_{i=1}^{h} \Delta c_{t+i} = \beta_0 + \beta_1(p_t - y_t) + \epsilon_t,$$

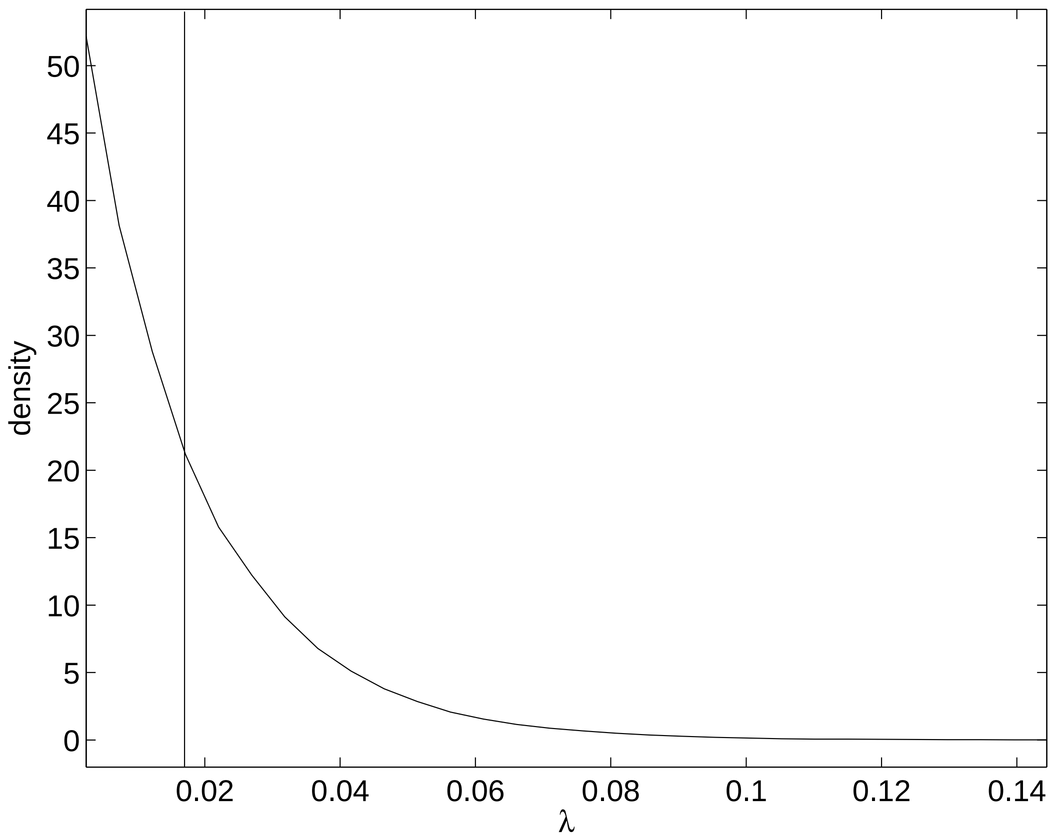
where Δc_{t+i} is log growth in aggregate consumption between periods $t+i-1$ and $t+i$ and $p_t - y_t$ is the log price-dividend ratio on the aggregated market. The model is simulated at a monthly frequency and simulated data are aggregated to an annual frequency. Panel A reports population moments from simulated data. Panel B reports sample moments. The conditional moments, namely the slope coefficient and the R^2 calculated over periods in the simulation without disasters, are equal to zero.

Table 5: Moments implied by the constant disaster risk model

	Population	Conditional
Panel A: $\phi = 1.5$, $\beta = .03$, $\gamma = 4$, $\psi = 1/4$		
$E[R^b]$	3.64	3.85
$\sigma(R^b)$	2.75	0
$E[R^e - R^b]$	5.57	6.08
$\sigma(R^e)$	6.76	3.30
Sharpe Ratio	0.98	1.84
$\sigma(\Delta y)$	8.67	3.00
Panel B: $\phi = 1.5$, $\beta = .03$, $\gamma = 4$, $\psi = 1$		
$E[R^b]$	1.45	1.66
$\sigma(R^b)$	2.69	0
$E[R^e - R^b]$	5.46	5.95
$\sigma(R^e)$	6.62	3.23
Sharpe Ratio	0.98	1.84
$\sigma(\Delta y)$	8.67	3.00
Panel C: $\phi = 3$, $\beta = .03$, $\gamma = 4$, $\psi = 1$		
$E[R^b]$	1.45	1.66
$\sigma(R^b)$	2.69	0
$E[R^e - R^b]$	8.25	9.16
$\sigma(R^e)$	11.04	6.65
Sharpe Ratio	0.83	1.38
$\sigma(\Delta y)$	17.34	6.00

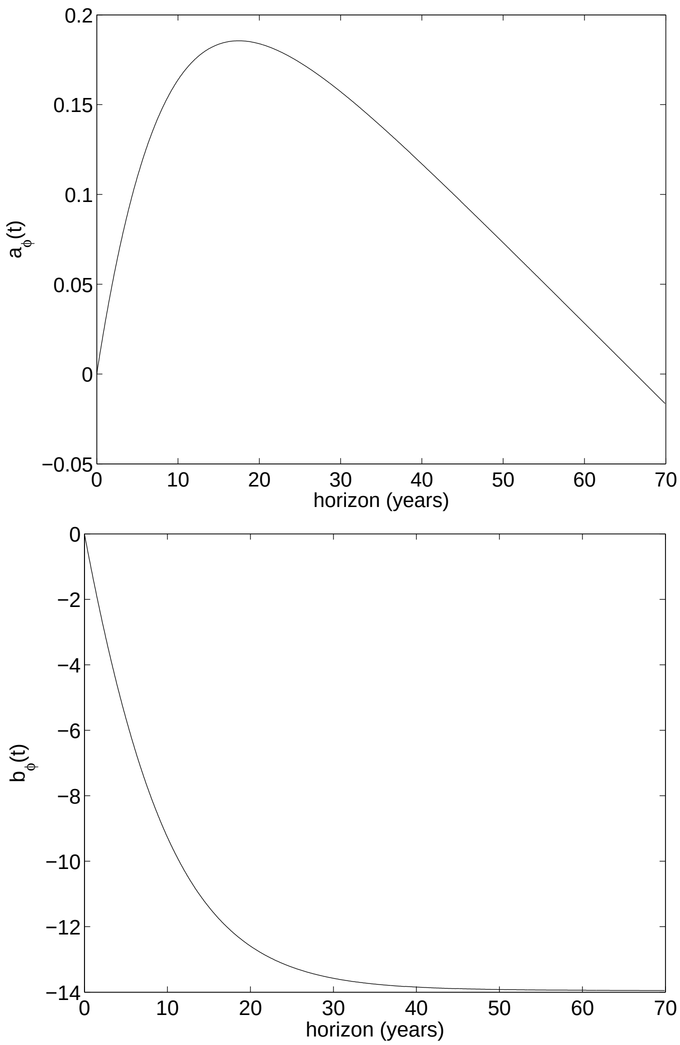
Notes: All three panels assume that the endowment growth is independent and identically distributed across time, namely that $\sigma_\lambda = 0$ and λ_t is equal to $\bar{\lambda}$. Unless otherwise noted, parameter values are as in Table 1. For all panels, the volatility of log consumption growth equals 5.67% per year in population and 2% per year conditional on no disasters. See Table 2 for further details.

Figure 1: Distribution of λ_t



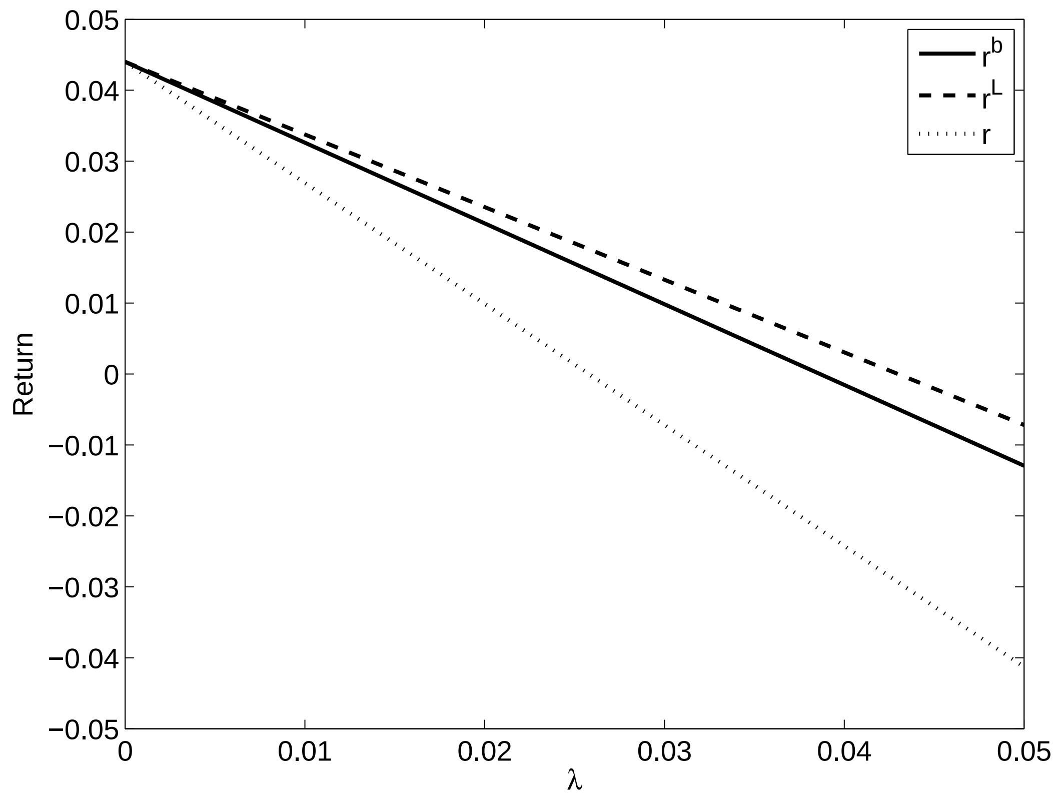
Notes: This figure shows the probability density function for λ_t, the time-varying intensity (per year) of a disaster. Parameter values are given in Table 1. The solid line is located at the unconditional mean $\bar{\lambda} = 0.017$.

Figure 2: Model solution



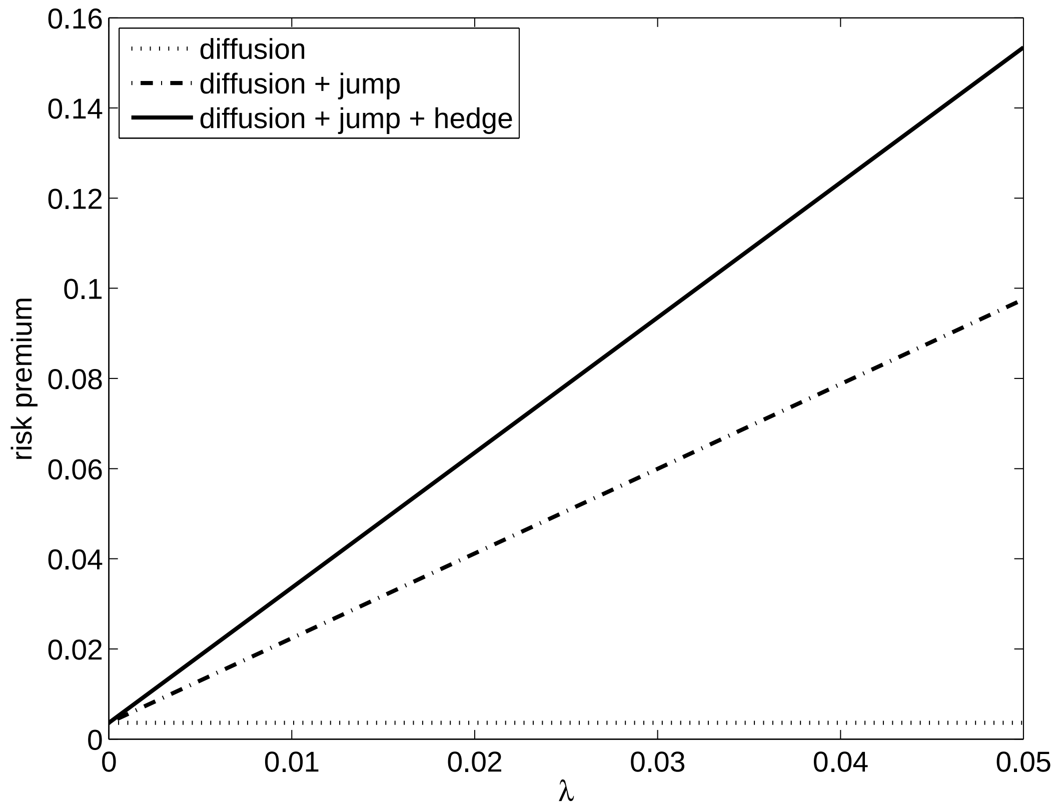
Notes: The upper and lower panels show the model's solution for $a_\phi(t)$ and $b_\phi(t)$, components of the solution for the price-dividend ratio when the risk of disaster is time varying. Parameter values are given in Table 1.

Figure 3: Government bill return



Notes: This figure shows r^b, the instantaneous expected return on a government bill; r^L, the instantaneous expected return on the bill conditional on no default and r, the rate of return on a default-free security as functions of the disaster intensity λ. Returns are annual.

Figure 4: Decomposition of the equity premium



Notes: This figure shows the instantaneous equity premium relative to the government bond. The solid line shows the full equity premium, the dashed line shows the equity premium without the hedging term that accounts for dynamic effects, and the dotted line shows the equity premium assuming there is no disaster risk.