Disasters Implied by Equity Index Options

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Summary

The idea

- Problem: disasters infrequent $\Rightarrow$ hard to estimate their distribution
- Solution: infer from option prices

What we find

- Disasters apparent in options data
- More modest than disasters in macro data

Why this is harder than we thought

- Barro data gives us "true" distribution of consumption growth
- Option prices give us "risk-neutral" distribution of returns

Outline

Preliminaries: entropy, cumulants, plan

Disasters in macroeconomic models

Digression: risk-neutral probabilities

Disasters in option models

Comparing models

Entropy

Hans-Otto Georgii (quoted by Hansen and Sargent):

> *When Shannon had invented his quantity and consulted von Neumann on what to call it, von Neumann replied: "Call it entropy. It is already in use under that name and, besides, it will give you a great edge in debates because nobody knows what entropy is anyway."*

Entropy bound

Entropy is a measure of dispersion: for $x > 0$

$$L(x) \equiv \log Ex - E \log x \geq 0$$

Pricing relation: there exists $m > 0$ such that

$$E_t \left(m_{t+1} r_{t+1} \right) = 1$$

Entropy bound

$$L(m) \geq E \left(\log r - \log r^1 \right)$$

Cumulants

Cumulant generating function

$$k(s;x) \;=\; \log E e^{sx} \;=\; \sum_{j=1}^{\infty} \kappa_j(x) s^j / j!$$

Cumulants are almost moments

$$\begin{aligned}
\text{mean} &= \kappa_1 \\
\text{variance} &= \kappa_2 \\
\text{skewness} &= \kappa_3 / \kappa_2^{3/2} \\
\text{(excess) kurtosis} &= \kappa_4 / \kappa_2^2
\end{aligned}$$

Entropy and cumulants

Entropy of pricing kernel

$$\begin{aligned} L(m) &= \log E e^{\log m} - E \log m \\ &= k(1; \log m) - E \log m \;=\; \sum_{j=2}^{\infty} \kappa_j(\log m)/j! \end{aligned}$$

Zin's "never a dull moment" conjecture

$$L(m) = \underbrace{\kappa_2(\log m)/2!}_{\text{(log)normal term}} + \underbrace{\kappa_3(\log m)/3! + \kappa_4(\log m)/4! + \cdots}_{\text{high-order cumulants (incl disasters)}}$$

Plan of attack

Modeling assumptions

- iid
- Tight link between consumption growth and equity returns
- Representative agent with power utility [if needed]

Parameter choices

- Match mean and variance of log consumption growth
- Ditto log equity return
- Base "disasters" on Barro's macroeconomic evidence
- Or on equity index options

Compare macro- and option-based examples

Macro disasters: environment

Consumption growth and "equity" return are iid

$$
\begin{aligned}
g_{t+1} &= c_{t+1}/c_t \\
d_t &= c_t^{\lambda} \\
\log r^e_{t+1} &= \text{constant} + \lambda \log g_{t+1}
\end{aligned}
$$

Power utility

$$
\log m_{t+1} = \log \beta - \alpha \log g_{t+1}
$$

Macro disasters: the bazooka

Cumulant generating functions

$$k(s; \log m) \quad = \quad k(-\alpha s; \log g)$$

Yaron's "bazooka"

$$\kappa_j(\log m)/j! \quad = \quad \kappa_j(\log g)(-\alpha)^j/j!$$

Macro disasters: Poisson-normal mixture

Consumption growth

$$\begin{aligned} \log g_{t+1} &= w_{t+1} + z_{t+1} \\ w_{t+1} &\sim \mathcal{N}(\mu, \sigma^2) \\ z_{t+1}|j &\sim \mathcal{N}(j\theta, j\delta^2) \\ j &\geq 0 \text{ has probability } e^{-\omega}\omega^j/j! \end{aligned}$$

Parameter values

- Match mean and variance of log consumption growth
- Jump probability ($\omega = 0.01$), mean ($\theta = -0.3$), and variance ($\delta^2 = 0.15^2$) [similar to Barro, Nakamura, Steinsson, and Ursua]

Macro disasters: entropy

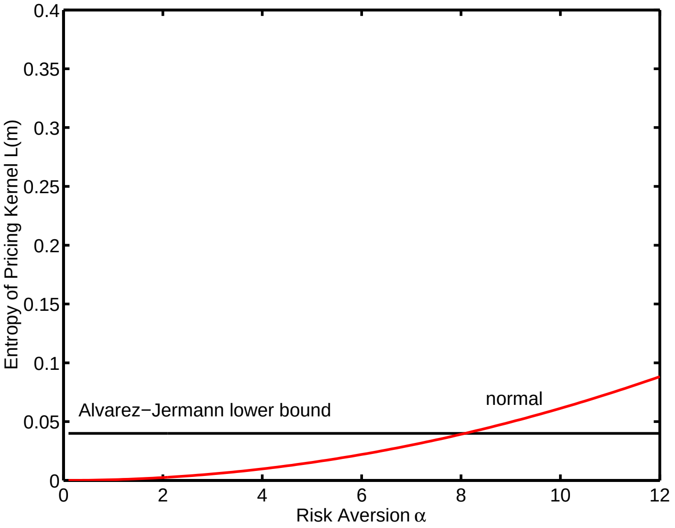
Cumulant generating functions

$$\begin{aligned} k(s;\log g) &\equiv \log E e^{s\log g} = k(s;w) + k(s;z) \\ k(s;w) &\equiv \log E e^{sw} = s\mu + (s\sigma)^2/2 \\ k(s;z) &\equiv \log E e^{sz} = \omega\left(e^{s\theta+(s\delta)^2/2} - 1\right) \end{aligned}$$

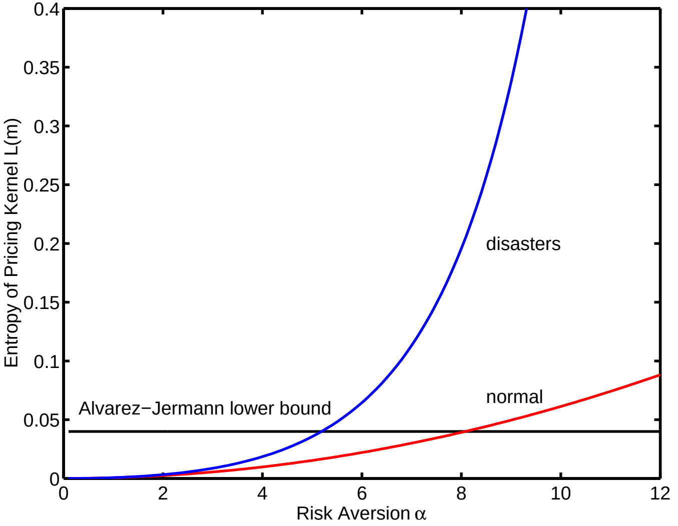
Entropy

$$L(m) = (-\alpha\sigma)^2/2 + \omega\left(e^{-\alpha\theta+(\alpha\delta)^2/2} - 1\right) + \alpha\omega\theta,$$

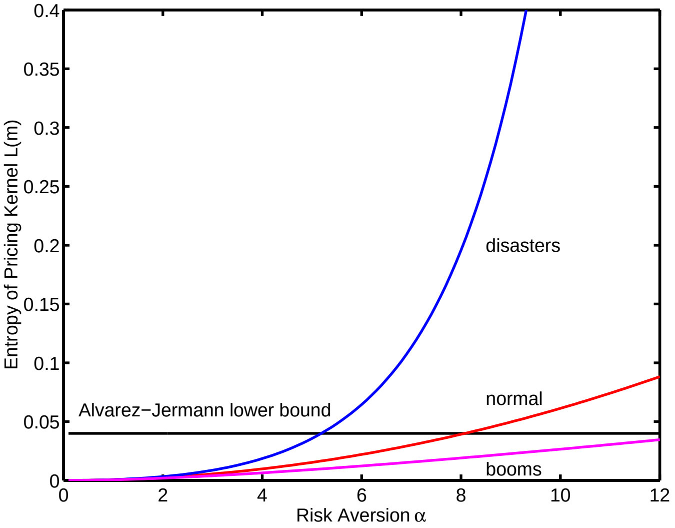
Macro disasters: entropy



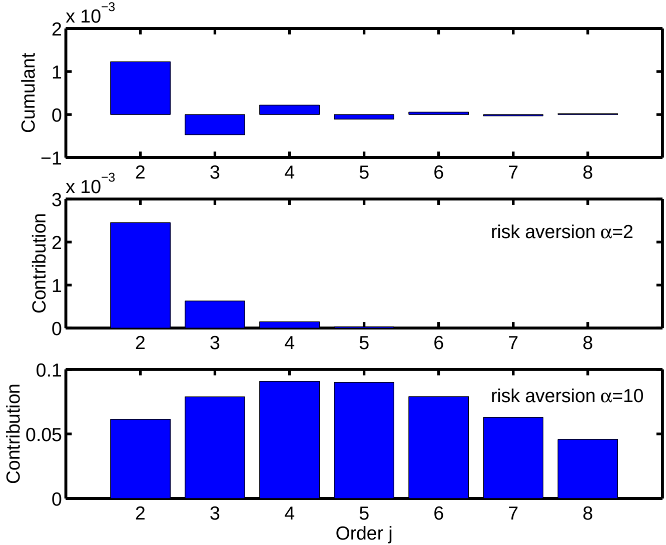
Macro disasters: entropy



Macro disasters: entropy



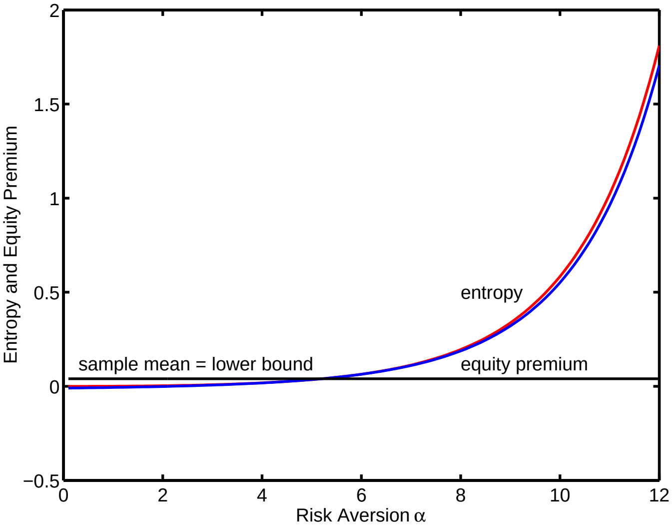
Macro disasters: cumulants



Macro disasters: cumulants

			High-Order Cumulants	
Model ($\alpha = 10$)	**Entropy**	Variance/2	Odd	Even
Normal	**0.0613**	0.0613	0	0
Poisson disaster	**0.5837**	0.0613	0.2786	0.2439
Poisson boom	**0.0266**	0.0613	**–0.2786**	0.2439

Macro disasters: equity premium



Digression: risk-neutral probabilities

Notation: states x have (true) probabilities $p(x)$

Risk-neutral probabilities p^*

$$\begin{aligned} p^*(x) &= p(x)m(x)/q^1 \\ m(x) &= q^1 p^*(x)/p(x) \\ q^1 &= Em \quad \text{(1-period bond price)} \end{aligned}$$

Entropy (aka "relative entropy" or "Kullback-Leibler divergence")

$$L(m) = L(p^*/p) = E \log(p/p^*)$$

Risk-neutral probabilities: power utility

Normal log consumption growth

- If $\log g \sim \mathcal{N}(\mu, \sigma^2)$ (true distribution)
- Then risk-neutral distribution also lognormal with $\mu^* = \mu - \alpha\sigma^2, \sigma^* = \sigma$

Poisson log consumption growth

- Jumps have probability ω and distribution $\mathcal{N}(\theta, \delta^2)$
- Risk-neutral distribution has same form with $\omega^* = \omega \exp[-\alpha\theta + (\alpha\delta)^2/2], \theta^* = \theta - \alpha\delta^2, \delta^* = \delta$

Option disasters: overview

Options an obvious source of information ...

- ... about risk-neutral distribution of equity returns

Critical ingredients

- Option prices
- Merton model
- Estimated parameters
- Implied volatilities

Option disasters: information in option prices

Put option (bet on low returns)

$$q_t^p \quad = \quad q_t^1 E_t^*(b - r_{t+1}^e)^+$$

Strategy

- Estimate p^* by varying strike price b (cross section)

Black-Scholes-Merton benchmark

- Quote prices as implied volatilities (high price $\Leftrightarrow$ high vol)
- Horizontal line if lognormal
- "Skew" suggests disasters

Option disasters: Merton model

Equity returns iid

$$\begin{aligned} \log r^e_{t+1} &= \log r^1 + w_{t+1} + z_{t+1} \\ w_{t+1} &\sim \mathcal{N}(\mu, \sigma^2) \\ z_{t+1}|j &\sim \mathcal{N}(j\theta, j\delta^2) \\ j &\geq 0 \text{ has probability } e^{-\omega}\omega^j/j! \end{aligned}$$

Risk-neutral distribution: ditto with *s

Option disasters: parameter values

Set $(\omega^*, \theta^*, \delta^*)$ to match option prices

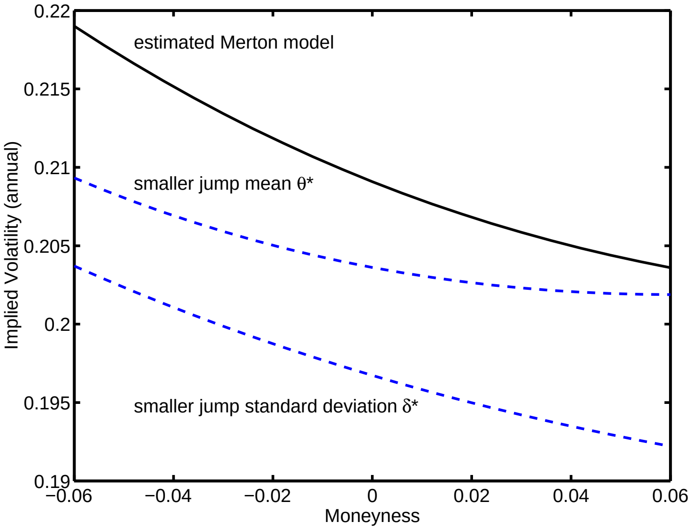
- Jumps: $\omega^* = \omega$, $\theta^* = -0.0482$, $\delta^* = 0.0981$
- Set $\sigma^* = \sigma$
- Set μ^* to satisfy pricing relation $(q^1 E^* r^e = 1)$

Later: choose $(\mu, \sigma, \omega, \theta, \delta)$ to match distribution of equity returns

- Jumps: $\omega = 1.512$, $\theta = -0.0259$, $\delta = 0.0229$
- Equity premium: $\mu + \omega\theta$
- Variance of equity returns: $\sigma^2 + \omega(\theta^2 + \delta^2)$

All from Broadie, Chernov, and Johannes (JF, 2007)

Option disasters: implied volatility



Comparing macro and option models

Approach 1: compare pricing kernels

- Required: estimated p from daily data on equity returns

Approach 2: compare consumption growth distributions

- Required: connections between g and r^e, p and p^*

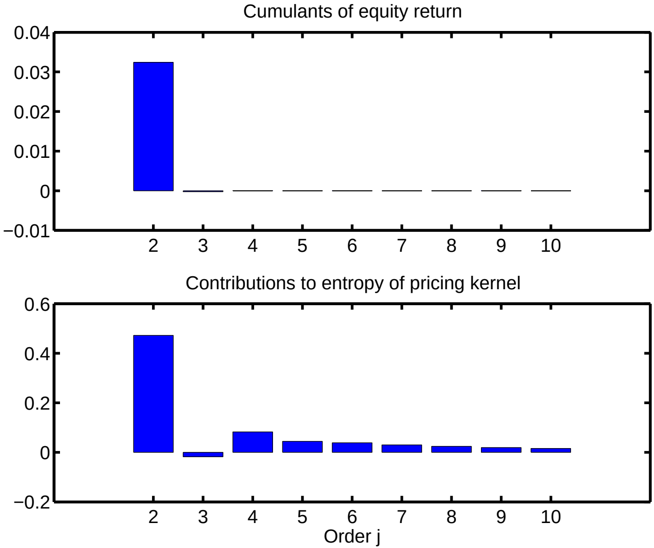
Approach 3: compare option prices

- Required: connections between g and r^e, p and p^*

Comparing pricing kernels: components of entropy

			High-Order Cumulants	
Model	Entropy	Variance/2	Odd	Even
Consumption-based models				
Normal ($\alpha = 10$)	0.0613	0.0613	0	0
Poisson ($\alpha = 10$)	0.5837	0.0613	0.2786	0.2439
Poisson ($\alpha = 5.38$)	0.0449	0.0177	0.0173	0.0099
Option-based model				
Option model	**0.7647**	0.4699	**0.1130**	**0.1819**

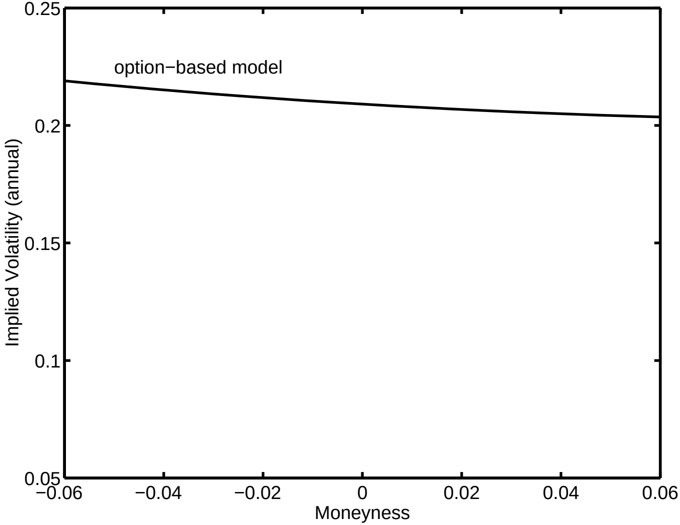
Comparing pricing kernels: cumulants



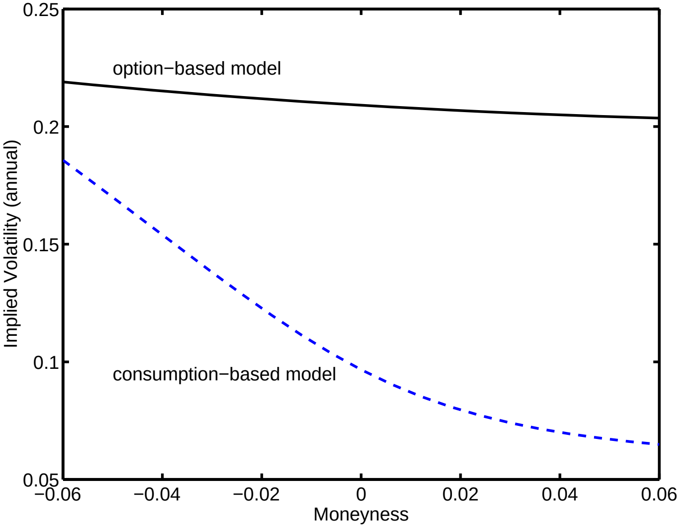
Comparing consumption growth

	Consumption Process Based on	
	Cons Growth	Option Prices
α	5.38	10.07
ω	0.0100	1.3864
θ	–0.3000	–0.0060
δ	0.1500	0.0229
Skewness	–11.02	–0.31
Excess Kurtosis	145.06	0.87
Tail prob (≤ -3 st dev)	**0.0090**	**0.0086**
Tail prob (≤ -5 st dev)	**0.0079**	**0.0002**

Comparing option prices



Comparing option prices



Comparing models

All of these comparisons point in the same direction

- Macro disasters more pronounced than option disasters

Reconsidering ...

Our models are based on

- iid
- Tight link between consumption growth and equity returns
- Representative agent with power utility

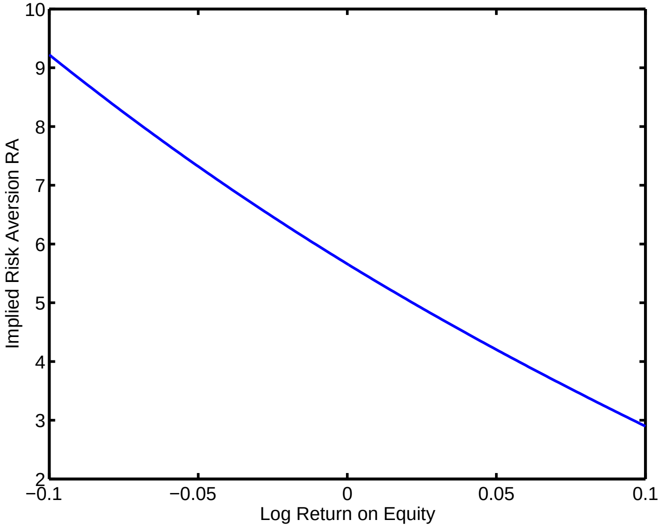
Let's take a close look at the last two

Reconsidering power utility

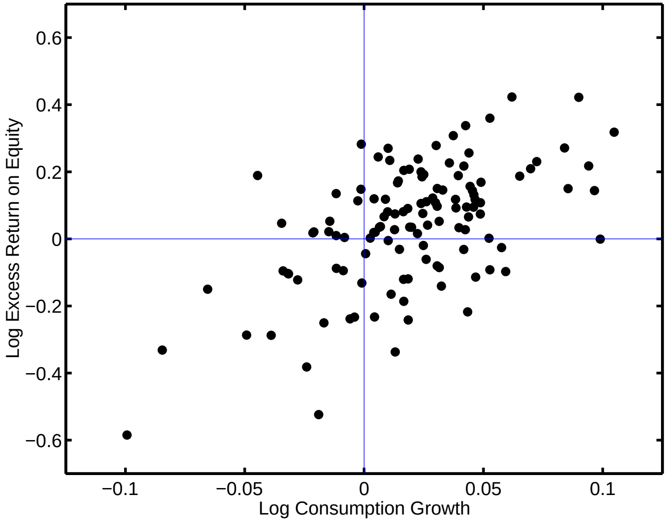
"Risk aversion" implied by arbitrary pricing kernel

$$\text{RA} \equiv -\frac{\partial \log m}{\partial \log g} = -\frac{\partial \log(p^*/p)}{\partial \log r^e} \cdot \frac{\partial \log r^e}{\partial \log g}$$

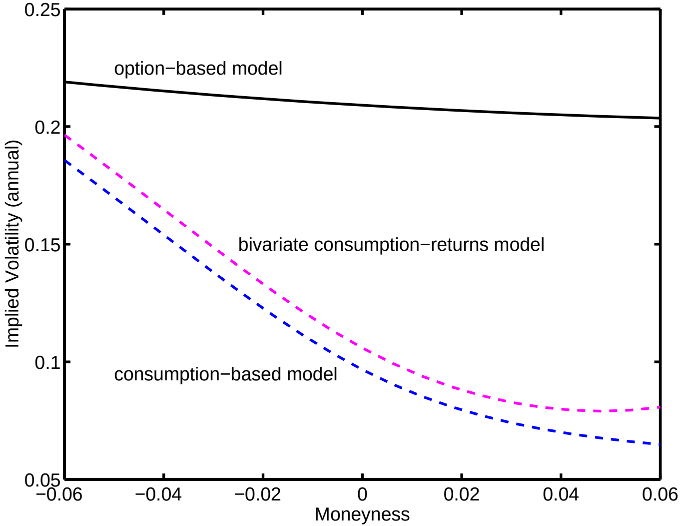
Reconsidering power utility



Reconsidering the link between g and r^e



Reconsidering the link between g and r^e



Recapitulation

Barro, Longstaff & Piazzesi, Rietz

- Disasters contribute to equity premium [entropy]
- Evident in macro data
- Range of opinion on magnitude

We look at options

- Smile/smirk suggests something like disasters
- Prices available even for outcomes that don't occur in sample
- Implied disasters less severe than macro data
- High entropy from options suggests it's not enough to match equity premium