Disasters Implied by Equity Index Options

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University of Glasgow | March 4, 2010

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The idea

- Disasters are infrequent $\Rightarrow$ hard to estimate their distribution
- Idea: infer from option prices (market prices of bets on disasters)
- We find:
  - disasters apparent in options data
  - the mechanism generating disasters is more modest than what is assumed based on macro data

Entropy

Entropy

- Hans-Otto Georgii (quoted by Hansen and Sargent):

> *When Shannon had invented his quantity and consulted von Neumann on what to call it, von Neumann replied: "Call it entropy. It is already in use under that name and, besides, it will give you a great edge in debates because nobody knows what entropy is anyway."*

Outline

- Preliminaries: entropy, Alvarez-Jermann bound, cumulants
- **Disasters in macroeconomic data**
- Risk-neutral probabilities
- **Disasters in options data**
- **Compare the implications of the two approaches**
- Extensions and related work

Alvarez-Jermann bound

- Pricing relation

$$E_t \left(m_{t+1} r^j_{t+1} \right) \quad = \quad 1$$

- Entropy: for any $x > 0$

$$L(x) \quad \equiv \quad \log Ex - E \log x \quad \geq \quad 0$$

- AJ bound (i.i.d. case)

$$L(m) \quad \geq \quad E \left(\log r^j - \log r^1 \right)$$

Cumulants

- Cumulant generating function

$$k(s;x) = \log Ee^{sx} = \sum_{j=1}^{\infty} \kappa_j(x)s^j/j!$$

- Cumulants are almost moments

$$\begin{aligned} \text{mean} &= \kappa_1(x) \\ \text{variance} &= \kappa_2(x) \\ \text{skewness} &= \kappa_3(x)/\kappa_2^{3/2}(x) \\ \text{(excess) kurtosis} &= \kappa_4(x)/\kappa_2^2(x) \end{aligned}$$

- If x is normal, $\kappa_j(x) = 0$ for $j > 2$

Entropy and cumulants

Entropy and cumulants

- Entropy of pricing kernel

$$L(m) \quad = \quad \log E e^{\log m} - E \log m \quad = \quad k(1, \log m) - \kappa_1(\log m)$$

Entropy and cumulants

- Entropy of pricing kernel

$$L(m) \;=\; \log E e^{\log m} - E \log m \;=\; k(1, \log m) - \kappa_1(\log m)$$

- Zin's "never a dull moment" conjecture

$$L(m) \;=\; \underbrace{\kappa_2(\log m)/2!}_{\text{(log)normal term}} + \underbrace{\kappa_3(\log m)/3! + \kappa_4(\log m)/4! + \cdots}_{\text{high-order cumulants (incl disasters)}}$$

Alvarez-Jerman bound vs. Hansen-Jagannathan bound

- Entropy: for $x > 0$

$$L(x) \equiv \log Ex - E\log x \geq 0$$

- AJ bound

$$L(m) \geq E\left(\log r^j - \log r^1\right)$$

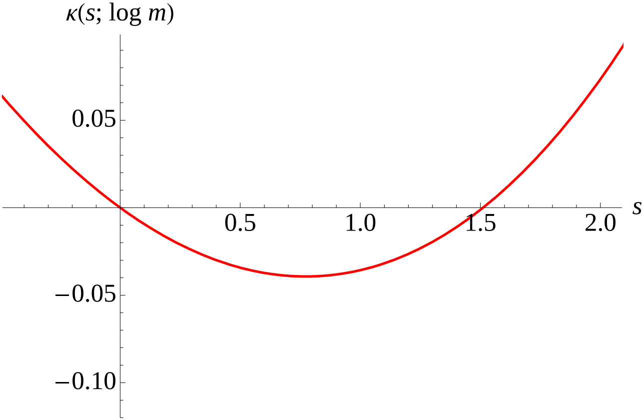
- HJ: for $x > 0$

$$HJ(x) \equiv \frac{\sigma(x)}{Ex} \geq 0$$

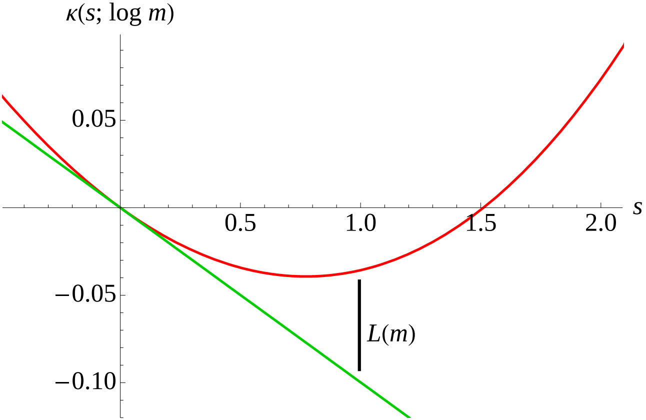
- HJ bound

$$HJ(m) \geq \frac{Er^j - r^1}{\sigma(r^j - r^1)}$$

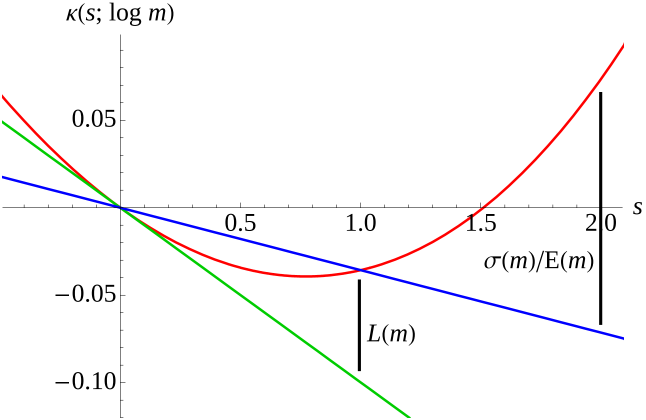
Alvarez-Jermann bound vs. Hansen-Jagannathan bound



Alvarez-Jermann bound vs. Hansen-Jagannathan bound



Alvarez-Jermann bound vs. Hansen-Jagannathan bound



Disasters based on macro fundamentals

Macro disasters: Model

- Consumption growth iid

$$\begin{aligned}
\log g_{t+1} &= w_{t+1} + z_{t+1} \\
w_{t+1} &\sim \mathcal{N}(\mu, \sigma^2) \\
z_{t+1} | j &\sim \mathcal{N}(j\theta, j\delta^2) \\
j &\geq 0 \text{ has probability } e^{-\omega}\omega^j / j!
\end{aligned}$$

- Parameter values
  - Match mean and variance of log consumption growth
  - Average number of disasters ($\omega = 0.01$), mean ($\theta = -0.3$) and variance ($\delta^2 = 0.15^2$)
  - Similar to Barro, Nakamura, Steinsson, and Ursua (2009)

Macro disasters: Deviations from normality

- Pricing kernel

$$
\begin{aligned}
\log m_{t+1} &= \log \beta - \alpha \log g_{t+1} \\
L(m) &= \log E e^{\log m} - E \log m = k(-\alpha; \log g) + \alpha \kappa_1(\log g)
\end{aligned}
$$

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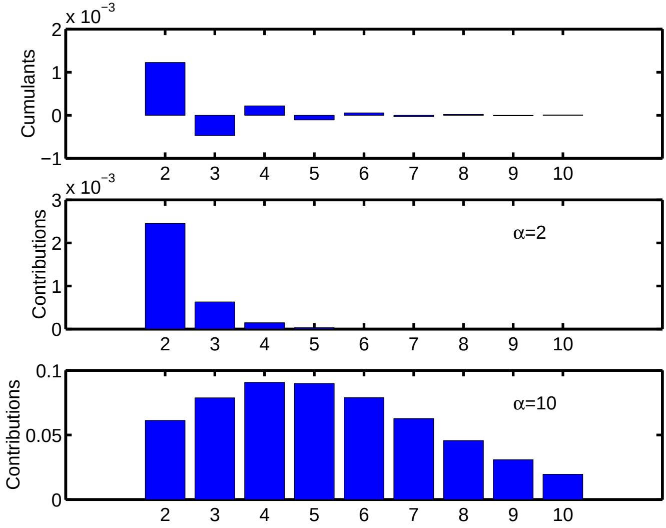
- Yaron's "bazooka"

$$\kappa_j(\log m)/j! = \kappa_j(\log g)(-\alpha)^j/j!$$

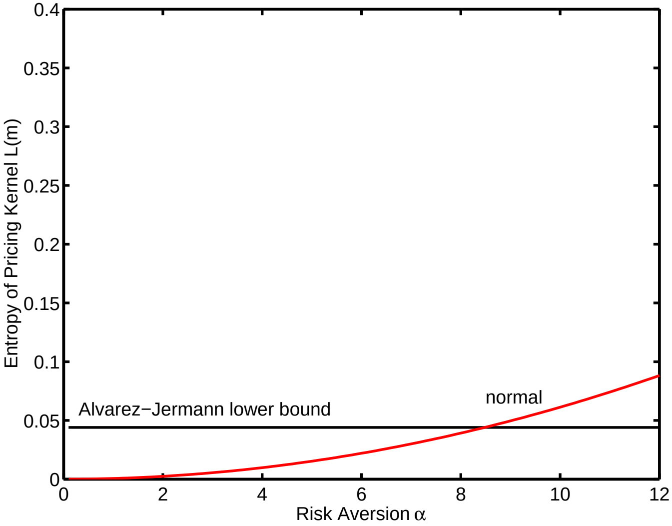
- The contribution of higher-order cumulants peaks at $j=\alpha$

$$\frac{\alpha^j}{j!} = \frac{\alpha}{1}\cdot\frac{\alpha}{2}\cdot\ldots\cdot\frac{\alpha}{j-1}\cdot\frac{\alpha}{j}$$

Macro disasters: Cumulants



Macro disasters: Entropy



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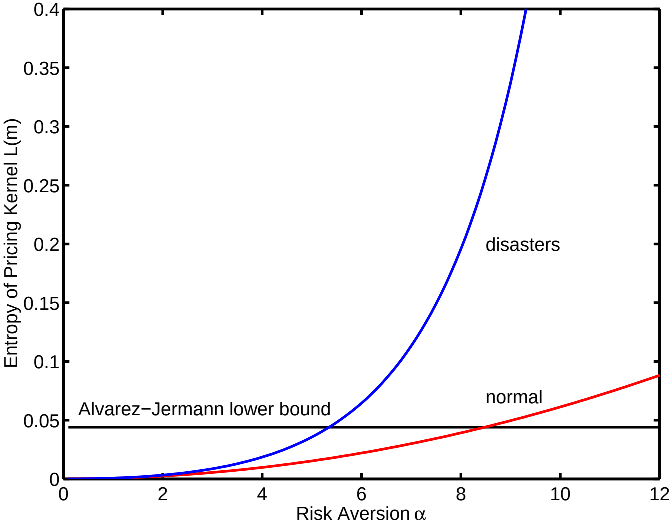

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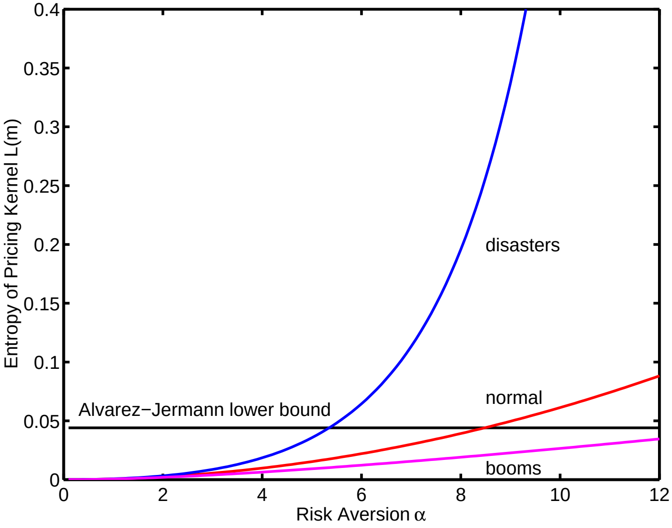

Macro-finance and risk-neutral pricing

Macro-finance and risk-neutral pricing

- Pricing relation

$$q^1 E_t^* \left(r_{t+1}^j \right) = 1,$$

  - where q^1 is a price of a one-period riskless bond

- Translating between preferences and risk-neutral probabilities

$$\begin{aligned} p(x)m(x) &= q^1 p^*(x) \\ p^*(x) &= p(x)m(x)/q^1 \end{aligned}$$

Macro-finance and risk-neutral pricing: Examples

- Normal log consumption growth
  - If $\log g \sim \mathcal{N}(\mu, \sigma^2)$ (true distribution)
  - Then risk-neutral distribution also lognormal with $\mu^* = \mu - \alpha\sigma^2, \sigma^* = \sigma$
- Poisson log consumption growth
  - If disasters have probability ω and distribution $\mathcal{N}(\theta, \delta^2)$
  - Then risk-neutral distribution has same form with $\omega^* = \omega \exp(-\alpha\theta + (\alpha\delta)^2/2), \theta^* = \theta - \alpha\delta^2, \delta^* = \delta$

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- Entropy

$$L(m) \;=\; L(p^*/p) \;=\; -E \log(p^*/p)$$

Disasters in options

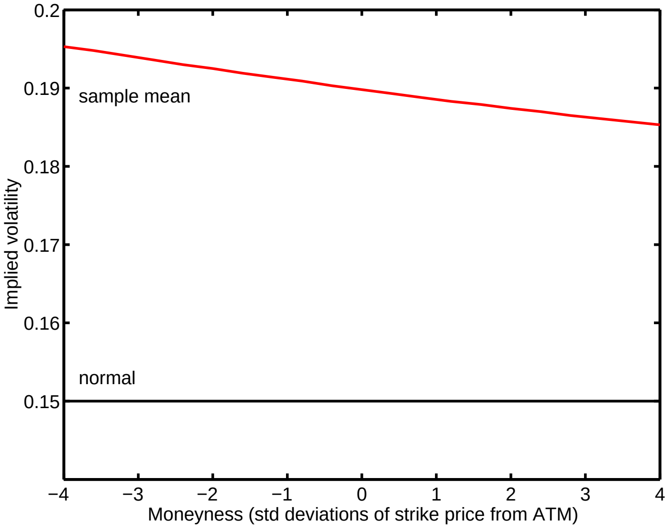
Disasters in options

- Put option (bet on low returns)

$$q_t^p \quad = \quad q^1 E_t^*(b - r_{t+1}^e)^+$$

- Estimate p^* by varying strike price b (cross section) (Breeden and Litzenberger, 1978)
- Black-Scholes-Merton benchmark
  - Quote prices as implied volatilities [high price $\Leftrightarrow$ high vol]
  - Horizontal line if (log)normal
  - “Skew” suggests disasters

Disasters in options: Data vs normal benchmark



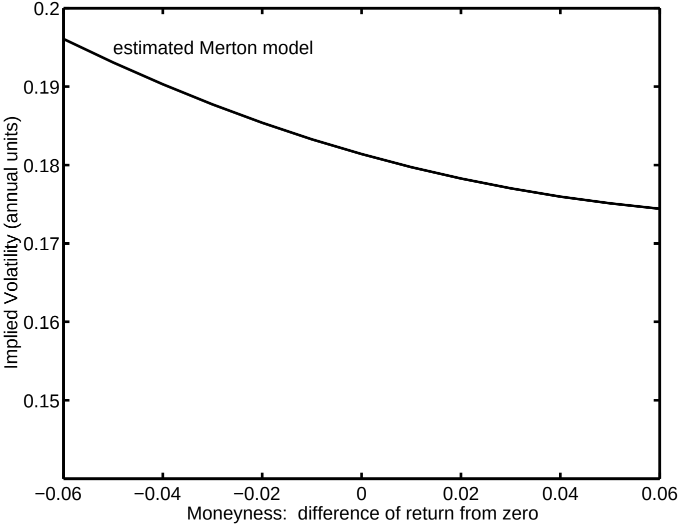
Disasters in options: Merton model

- Equity returns iid

$$\begin{aligned} \log r^e_{t+1} - \log r^1 &= w_{t+1} + z_{t+1} \\ w_{t+1} &\sim \mathcal{N}(\mu, \sigma^2) \\ z_{t+1}|j &\sim \mathcal{N}(j\theta, j\delta^2) \\ j &\geq 0 \text{ has probability } e^{-\omega}\omega^j/j! \end{aligned}$$

- Risk-neutral distribution: the same with *s
- Parameter values
  - Choose risk-neutral parameters to match option prices
  - Average number of disasters: $\omega = \omega^* = 1.5$,
    mean: $\theta = -0.03$, $\theta^* = -0.05$,
    variance: $\delta^2 = 0.04^2$, $\delta^{*2} = 0.10^2$
  - Calibration is based on Broadie, Chernov, and Johannes (2007)

Calibrating option parameters



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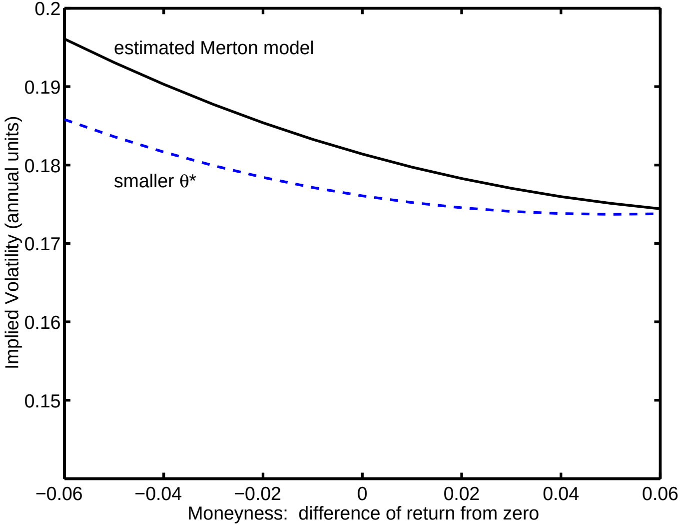

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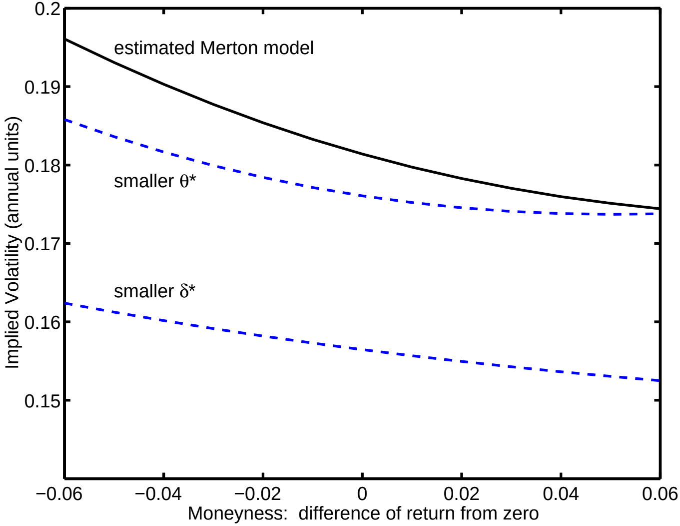

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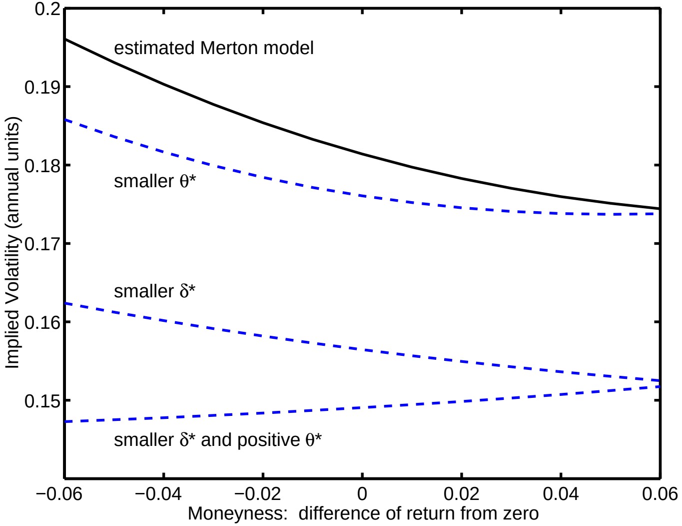

Comparing macro- and option-based models

- Entropy and cumulants of pricing kernel
  - Result: option-based entropy is large
- Consumption growth implied by option prices
  - Option-based $p^* +$ power utility $\Rightarrow p$
  - Result: more modest skewness and kurtosis, tail probabilities
- Option prices implied by consumption growth
  - Macro-based $p +$ power utility $\Rightarrow p^*$
  - Compute option prices
  - Result: steeper volatility smile
- Risk aversion implied by options
  - Result: Risk aversion declines with increase in returns

Comparing models: components of entropy

			High-Order Cumulants	
Model	Entropy	Variance/2	Odd	Even
Macro ($\alpha = 5.38$)	**0.0449**	0.0177	0.0173	0.0099
		39%	39%	22%
Options	**0.7647**	0.4699	0.1130	0.1819
		61%	15%	24%

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Comparing models: one more step

- "Levered equity"
  - Claim to c^{λ}
  - Log return $\log r^e = \lambda \log g + \text{constant}$
- Calibrate λ to match volatility of returns

$$\lambda = \underbrace{0.15}_{vol(\log r^e)} \Big/ \underbrace{0.035}_{vol(\log g)} = 4.3$$

Comparing models: consumption implied by options

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	Calibration	Implied
α	5.38	10.07
ω	0.0100	1.3864
θ	–0.3000	–0.0060
δ	0.1500	0.0229

Comparing models: consumption implied by options

	Calibration	Implied	
α	5.38	10.07	
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δ	0.1500	0.0229	
Skew	–11.02	–0.31	– 0.35
Excess Kurt	145.06	0.87	1.10

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Tail prob (≤ -3 st dev)	0.0090	0.0086	Great Depression

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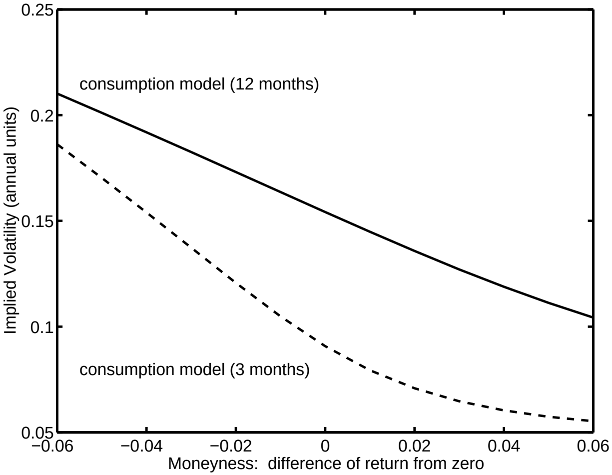
Tail prob (≤ -3 st dev)	0.0090	0.0086	Great Depression
Tail prob (≤ -5 st dev)	0.0079	0.0002	

Comparing models: options implied by macro model



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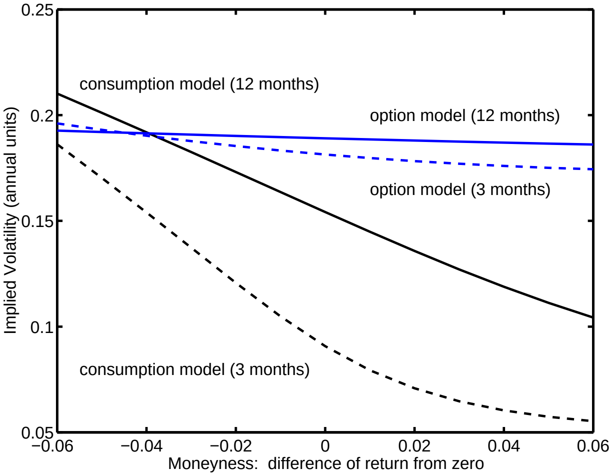

Comparing models: risk aversion

- In option model, implicit risk aversion accounts for
  - Equity premium
  - Prices of options (high entropy)
- Form differs from power utility
  - Not constant
  - Parameters imply greater aversion to adverse risks

Comparing models: risk aversion computation

- Math

$$\text{RA} = -\frac{\partial \log m}{\partial \log g} = -\frac{\partial \log(p^*/p)}{\partial \log r^e} \cdot \underbrace{\frac{\partial \log r^e}{\partial \log g}}_{\lambda}$$

Comparing models: risk aversion computation

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$$\text{RA} = -\frac{\partial \log m}{\partial \log g} = -\frac{\partial \log(p^*/p)}{\partial \log r^e} \cdot \underbrace{\frac{\partial \log r^e}{\partial \log g}}_{\lambda}$$

- Example: $\log r^e \sim \mathcal{N}(\mu, \sigma^2)$ and $\mathcal{N}(\mu^*, \sigma^{*2})$

$$\text{RA} = [(\sigma^2/\sigma^{*2} - 1)\log r^e + \mu - (\sigma^2/\sigma^{*2})\mu^*]/\sigma^2 \cdot \lambda$$

Comparing models: risk aversion computation

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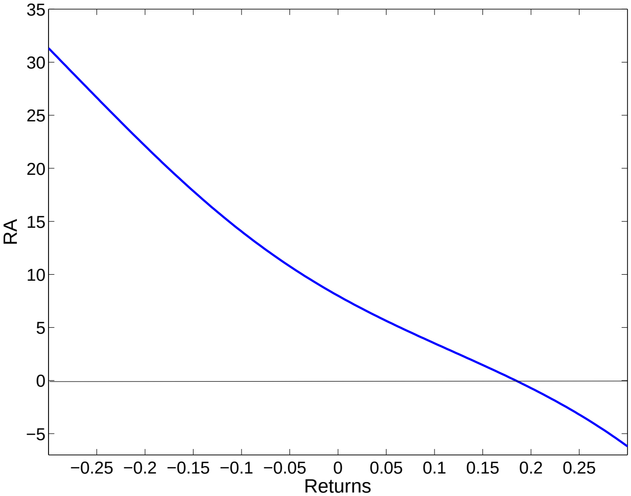
$$\text{RA} \;=\; -\frac{\partial \log m}{\partial \log g} \;=\; -\frac{\partial \log(p^*/p)}{\partial \log r^e} \cdot \underbrace{\frac{\partial \log r^e}{\partial \log g}}_{\lambda}$$

- Example: $\log r^e \sim \mathcal{N}(\mu, \sigma^2)$ and $\mathcal{N}(\mu^*, \sigma^{*2})$

$$\text{RA} = [(\sigma^2/\sigma^{*2} - 1)\log r^e + \mu - (\sigma^2/\sigma^{*2})\mu^*]/\sigma^2 \cdot \lambda$$

- Interpretation
  - If $\sigma = \sigma^*$, RA is constant
  - If $\sigma < \sigma^*$, RA decreases with $\log r^e$

Comparing models: risk aversion variation



Bottom line

- Barro (2006), Longstaff and Piazzesi (2004), and Rietz (1988)
  - Disasters account for equity premium
  - Evident in macro data
- We look at options
  - Disasters evident in option prices
  - More modest than in macro data
  - Suggest high average risk aversion, greater aversion to bad outcomes
  - Imply higher entropy than equity premium

Open questions

- Consumption and dividends
  - Examples: Bansal and Yaron (2004), Gabaix (2009) , Longstaff and Piazzesi (2004)
- Source of apparent risk aversion
  - Exotic preferences
  - Heterogeneous agents
  - Examples: Alvarez, Atkeson, and Kehoe (2009); Bates (2008); Chan and Kogan (2002); Lustig and Van Nieuwerburgh (2005)
- Time dependence
  - Short rate, predictable returns, stochastic volatility
  - Examples: Drechsler and Yaron (2008), Wachter (2008)

Consumption and Dividends

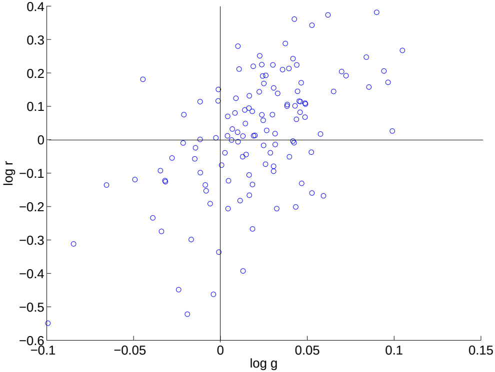
- So, far they were perfectly correlated:

$$\log g^d = \lambda \log g \Rightarrow \log r^e = \lambda \log g + \text{constant}$$

- Can we relax this and is it important?

$$\log r^e = \lambda \log g + \text{constant} + \text{noise}$$

Returns and consumption growth

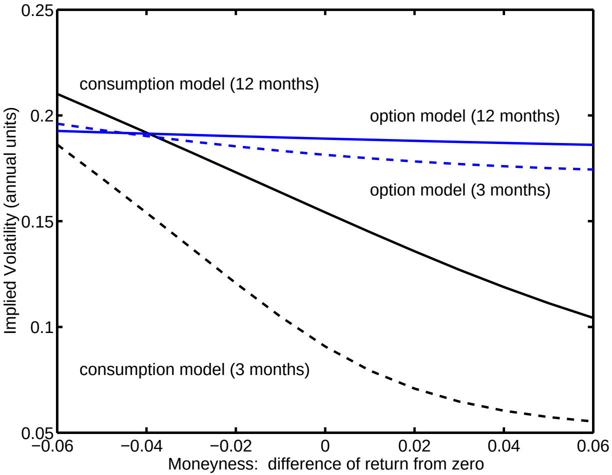


- OLS (using Shiller's data)

$$\log r^e = 3 \cdot \log g + \text{constant} + \text{noise}$$

- What is the nature of the noise term?

Using information in options to model noise



Consumption and Dividends

- Add noise:

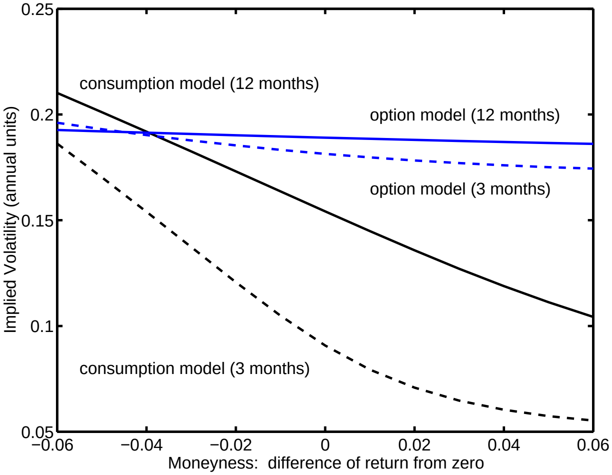
$$\begin{aligned} \log g^d &= \lambda \log g + w_t^d \Rightarrow \log r^e = \lambda \log g + \text{constant} + w_t^d \\ w_t^d &\sim \mathcal{N}(0, \sigma^{d2}) \end{aligned}$$

- Calibration

$$\begin{aligned} \lambda &= 3 \\ \sigma^d &= \sqrt{0.15^2 - (3 \cdot 0.035)^2} = 0.1071 \end{aligned}$$

- As a result, $\alpha = 5.79$ (instead of 5.38)

Implied volatilities revisited



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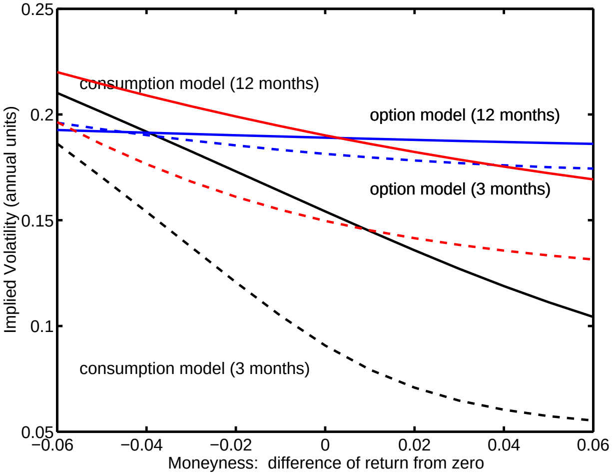

Comparing models: consumption implied by options

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α	5.38	10.07	11.92
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δ	0.1500	0.0229	0.0320

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- Initial conclusions are robust to imperfect correlation between consumption and dividends
- Returns and consumption seem to share a common jump component

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Sources of Entropy in Dynamic Representative Agent Models

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Understanding dynamic models

- Are dynamic features important for the disaster story?
- More generally, how does one discern critical features of modern dynamic models?
  - The size of equity premium is no longer an overidentifying restriction

Market-adjusted excess returns

Asset Class	Value	Momentum
US stocks	4.3%	6.1%
UK stocks	2.7%	10.8%
Euro stocks	4.2%	10.9%
Jpn stocks	11.3%	4.2%
FX	4.9%	2.7%
Bonds	0.3%	0.3%
Commodities	6.4%	8.8%

- Annualized alphas relative to the MSCI world equity index in excess of the US Treasury Bill rate
- Source: Asness, Moskowitz, and Pedersen (2009)

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Understanding dynamic models

- Are dynamic features important for the disaster story?
- More generally, how does one discern critical features of modern dynamic models?
  - The size of equity premium is no longer an overidentifying restriction
  - The models are built up from different state variables
  - Which pieces are most important quantitatively?
- We start by thinking about how risk is priced in these models
  - What is the source of the evident high entropy in the data?
  - We use ACE to characterize this

AJ bound, non-i.i.d. case

- AJ bound

$$L(m) \geq E\left(\log r^j - \log r^1\right) + \underbrace{L(q^1)}_{\text{non-i.i.d. piece}}$$

AJ bound, non-i.i.d. case

- AJ bound

$$L(m) \geq E\left(\log r^j - \log r^1\right) + \underbrace{L(q^1)}_{\text{non-i.i.d. piece}}$$

- Conditional entropy:

$$L_t(m_{t+1}) = \log E_t m_{t+1} - E_t \log m_{t+1}$$

- Average conditional entropy (ACE)

$$\begin{aligned} L(m) &= EL_t(m_{t+1}) + L(E_t(m_{t+1})) = EL_t(m_{t+1}) + L(q^1) \\ EL_t(m_{t+1}) &\geq E\left(\log r^j - \log r^1\right) \end{aligned}$$

Advantages of average conditional entropy (ACE)

- Transparent lower bound: expected excess return (in logs)
- Alternatively, ACE measures the highest risk premium in the economy
- Conditional entropy is easy to compute; to compute ACE evaluate conditional entropy at steady-state values
- ACE is comparable across different models with different state variables, preferences, etc.

Key models

- **External habit**
- **Recursive preferences**
- Heterogeneous preferences
- ....

A change in notation

- α is replaced by $1-\alpha$
- Example: CRRA preferences; RA$=5$
- Old $\alpha = 5$
- New $\alpha = -4$

External habit

- Equations (Abel/Campbell-Cochrane/Chan-Kogan/Heaton)

$$\begin{aligned} U_t &= \sum_{j=0}^{\infty} \beta^j u(c_{t+j}, x_{t+j}), \\ u(c_t, x_t) &= \left(f(c_t, x_t)^{\alpha} - 1\right)/\alpha. \end{aligned}$$

- Habit is a function of past consumption: $x_t = h(c^{t-1})$, e.g., Abel: $x_t = c_{t-1}$.
- Dependence on habit
  - Abel: $f(c_t, x_t) = c_t / x_t$
  - Campbell-Cochrane: $f(c_t, x_t) = c_t - x_t$
- Pricing kernel:

$$m_{t+1} = \beta \frac{u_c(c_{t+1}, x_{t+1})}{u_c(c_t, x_t)} = \beta \left(\frac{f(c_{t+1}, x_{t+1})}{f(c_t, x_t)} \right)^{\alpha - 1} \left(\frac{f_c(c_{t+1}, x_{t+1})}{f_c(c_t, x_t)} \right)$$

Example 1: Abel (1990) + Chan and Kogan (2002)

- Preferences: $f(c_t, x_t) = c_t / x_t$
- Chan and Kogan have extended the habit formulation:

$$\log x_{t+1} = (1-\phi)\sum_{i=0}^{\infty}\phi^i \log c_{t-i} = \phi \log x_t + (1-\phi)\log c_t$$

- Relative (log) consumption

$$\log s_t \equiv \log(c_t/x_t) = \phi \log s_{t-1} + \log g_t$$

- Pricing kernel:

$$\begin{aligned}\log m_{t+1} &= \log\beta + (\alpha-1)\log g_{t+1} - \alpha\log(x_{t+1}/x_t) \\ &= \log\beta + (\alpha-1)\log g_{t+1} - \alpha(1-\phi)\log s_t\end{aligned}$$

ACE: Abel-Chan-Kogan

- Pricing kernel:

$$\log m_{t+1} = \log\beta + (\alpha-1)\log g_{t+1} - \alpha(1-\phi)\log s_t$$

- Conditional entropy: $L_t(m_{t+1}) = \log E_t e^{\log m_{t+1}} - E_t \log m_{t+1}$

$$\begin{aligned}
\log E_t e^{\log m_{t+1}} &= \log\beta + k(\alpha-1;\log g) - \alpha(1-\phi)\log s_t (= -\log r^1) \\
E_t \log m_{t+1} &= \log\beta + (\alpha-1)\kappa_1(\log g) - \alpha(1-\phi)\log s_t \\
L_t(m_{t+1}) &= k(\alpha-1;\log g) - (\alpha-1)\kappa_1(\log g)
\end{aligned}$$

- ACE: $EL_t(m_{t+1}) = k(\alpha-1;\log g) - (\alpha-1)\kappa_1(\log g)$

ACE: Abel-Chan-Kogan

- Pricing kernel:

$$\log m_{t+1} = \log \beta + (\alpha - 1) \log g_{t+1} - \alpha(1-\phi) \log s_t$$

- Conditional entropy: $L_t(m_{t+1}) = \log E_t e^{\log m_{t+1}} - E_t \log m_{t+1}$

$$\begin{aligned} \log E_t e^{\log m_{t+1}} &= \log \beta + k(\alpha - 1; \log g) - \alpha(1-\phi) \log s_t (= -\log r^1) \\ E_t \log m_{t+1} &= \log \beta + (\alpha - 1)\kappa_1(\log g) - \alpha(1-\phi) \log s_t \\ L_t(m_{t+1}) &= k(\alpha - 1; \log g) - (\alpha - 1)\kappa_1(\log g) \end{aligned}$$

- ACE: $EL_t(m_{t+1}) = k(\alpha - 1; \log g) - (\alpha - 1)\kappa_1(\log g)$
- It is exactly the same as in the CRRA case

Example 2: Campbell and Cochrane (1999)

- Preferences: $f(c_t, x_t) = c_t - x_t$
- Campbell and Cochrane specify (log) surplus consumption ratio directly:

$$
\begin{aligned}
\log s_t &= \log[(c_t - x_t)/c_t] \\
\log s_t &= \phi(\log s_{t-1} - \log \bar{s}) + \lambda(\log s_{t-1})(\log g_t - \kappa_1(\log g)).
\end{aligned}
$$

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- Compare to relative (log) consumption in Chan and Kogan

$$\log s_t \equiv \log(c_t/x_t) = \phi \log s_{t-1} + \log g_t$$

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- Pricing kernel:

$$\begin{aligned} \log m_{t+1} &= \log \beta + (\alpha - 1)\log g_{t+1} + (\alpha - 1)\log(s_{t+1}/s_t) \\ &= \log \beta - (\alpha - 1)\lambda(\log s_t)\kappa_1(\log g) \\ &+ (\alpha - 1)(1 + \lambda(\log s_t))\log g_{t+1} \\ &+ (\alpha - 1)(\phi - 1)(\log s_t - \log \bar{s}) \end{aligned}$$

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&+ (\alpha-1)(1+\lambda(\log s_t))\log g_{t+1} \\
&+ (\alpha-1)(\phi-1)(\log s_t - \log\bar{s})
\end{aligned}
$$

- Conditional entropy:

$$
\begin{aligned}
L_t(m_{t+1}) &= k((\alpha-1)(1+\lambda(\log s_t)); \log g) \\
&- (\alpha-1)(1+\lambda(\log s_t))\kappa_1(\log g)
\end{aligned}
$$

Additional assumptions

- To compute ACE, we have to specify λ and $\log g$
- Conditional volatility of the consumption surplus ratio

$$\lambda(\log s_t) = \frac{1}{\sigma}\sqrt{\frac{1-\phi-b/(1-\alpha)}{1-\alpha}}\sqrt{1-2(\log s_t - \log \bar{s})} - 1$$

  - In discrete time, there is an upper bound on $\log s_t$ to ensure positivity of λ
  - In continuous time, this bound never binds so we will ignore it
  - In Campbell and Cochrane, $b = 0$ to ensure a constant $\log r$[1]
- Consumption growth is i.i.d.
  - Case 1. $\log g_{t+1} = w_{t+1}$, $w_{t+1} \sim \mathcal{N}(\mu, \sigma^2)$
  - Case 2. $\log g_{t+1} = w_{t+1} - z_{t+1}$, $z_{t+1}|j \sim Gamma(j, \theta^{-1}), \bar{j} = \omega$

ACE: Campbell and Cochrane, Case 1

- Conditional entropy:

$$L_t(m_{t+1}) = ((\alpha - 1)(\phi - 1) - b)/2 + b(\log s_t - \log \bar{s})$$

- ACE: $EL_t(m_{t+1}) = ((\alpha - 1)(\phi - 1) - b)/2$

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- All authors use $\alpha = -1$
- ACE for different calibrations (quarterly)

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  - Wachter (2006): $\phi = 0.97$, $b = 0.011$;
    $EL_t(m_{t+1}) = 0.0245$ (0.098 annual)
  - Verdelhan (2009): $\phi = 0.99$, $b = -0.011$;
    $EL_t(m_{t+1}) = 0.0155$ (0.062 annual)

ACE: Campbell and Cochrane, Case 2

- Conditional entropy:

$$\begin{aligned} L_t(m_{t+1}) &= (\alpha-1)(1+\lambda(\log s_t))\omega\theta \\ &+ ((1+(\alpha-1)(1+\lambda(\log s_t))\theta)^{-1}-1)\omega \\ &+ ((\alpha-1)(\phi-1)-b)/2+b(\log s_t-\log\bar{s}) \end{aligned}$$

- ACE: use log-linearization around $\log\bar{s}$

$$\begin{aligned} EL_t(m_{t+1}) &= \omega d^2/(1+d)+((\alpha-1)(\phi-1)-b)/2 \\ d &= \frac{\theta}{\sigma}\sqrt{(\alpha-1)(\phi-1)-b} \end{aligned}$$

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- Calibration as above + vol of $\log g$ + jump parameters:
  - $\sigma^2=(0.035)^2/4-\omega\theta^2$
  - BNSU: $\omega=0.01/4$, $\theta=0.15$
  - BCM: $\omega=1.3864/4$, $\theta=0.0229$

ACE: Campbell and Cochrane, Case 2

Calibration	ACE	ACE (case 1)	ACE jumps
CC + BNSU	0.0341	0.0300	0.0041
W + BNSU	0.0281	0.0245	0.0036
V + BNSU	0.0181	0.0155	0.0026
CC + BCM	0.0883	0.0300	0.0583
W + BCM	0.0737	0.0245	0.0492
V + BCM	0.0487	0.0155	0.0332

Time dependence via external habit

- No time-dependence in consumption growth
- Nevertheless: habit with varying volatility may have a substantial impact on the entropy of the pricing kernel
- Could be relevant for option prices (Du, 2008)

Recursive preferences: traditional version

- Equations (Kreps-Porteus/Epstein-Zin/Weil)

$$
\begin{aligned}
U_t &= \left[(1-\beta)c_t^{\rho} + \beta\mu_t(U_{t+1})^{\rho}\right]^{1/\rho} \\
\mu_t(U_{t+1}) &= \left(E_t U_{t+1}^{\alpha}\right)^{1/\alpha} \\
IES &= 1/(1-\rho) \\
CRRA &= 1-\alpha \\
\alpha &= \rho \quad \Rightarrow \quad \text{additive preferences}
\end{aligned}
$$

Recursive preferences: pricing kernel

- Scale problem by c_t ($u_t = U_t/c_t$, $g_{t+1} = c_{t+1}/c_t$)

$$u_t \quad = \quad [(1-\beta) + \beta\mu_t(g_{t+1}u_{t+1})^{\rho}]^{1/\rho}$$

- Pricing kernel (mrs)

$$\begin{aligned} m_{t+1} &= \beta \left(\frac{c_{t+1}}{c_t}\right)^{\rho-1} \left(\frac{U_{t+1}}{\mu_t(U_{t+1})}\right)^{\alpha-\rho} \\ &= \beta\, g_{t+1}^{\rho-1} \left(\frac{g_{t+1}u_{t+1}}{\mu_t(g_{t+1}u_{t+1})}\right)^{\alpha-\rho} \end{aligned}$$

Loglinear approximation

- Loglinear approximation

$$\begin{aligned}\log u_t &= \rho^{-1}\log\left[(1-\beta)+\beta\mu_t(g_{t+1}u_{t+1})^{\rho}\right] \\ &= \rho^{-1}\log\left[(1-\beta)+\beta e^{\rho\log\mu_t(g_{t+1}u_{t+1})}\right] \\ &\approx b_0+b_1\log\mu_t(g_{t+1}u_{t+1}).\end{aligned}$$

- Exact if $\rho=0$: $b_0=0$, $b_1=\beta$

- Solve by guess and verify

Example 1: Bansal-Yaron

- Consumption growth

$$
\begin{aligned}
\log g_t &= g + \gamma(L) v_{t-1}^{1/2} w_{1t} \\
v_t &= v + \nu(L) w_{2t} \\
(w_{1t}, w_{2t}) &\sim \text{NID}(0, I)
\end{aligned}
$$

- Guess value function

$$
\log u_t = u + \omega_g(L) v_{t-1}^{1/2} w_{1t} + \omega_v(L) w_{2t}
$$

- Solution includes

$$
\begin{aligned}
\omega_{g0} + \gamma_0 &= \gamma(b_1) \equiv \sum_{i=0}^{\infty} b_1^i \gamma_i \\
\omega_{v0} &= b_1 (\alpha/2) \gamma(b_1)^2 \nu(b_1)
\end{aligned}
$$

ACE: Bansal-Yaron

- Pricing kernel

$$\begin{aligned}
\log m_{t+1} &= \log\beta + (\rho-1)g - (\alpha-\rho)(\alpha/2)\omega_{v0}^2 \\
&\quad + (\rho-1)[\gamma(L)/L]_+ v_{t-1}^{1/2} w_{1t} - (\alpha-\rho)(\alpha/2)\gamma(b_1)^2 v_t \\
&\quad + [(\rho-1)\gamma_0 + (\alpha-\rho)\gamma(b_1)] v_t^{1/2} w_{1t+1} \\
&\quad + (\alpha-\rho)\omega_{v0}^2 w_{2t+1}
\end{aligned}$$

- Conditional entropy

$$L_t(m_{t+1}) = [(\rho-1)\gamma_0 + (\alpha-\rho)\gamma(b_1)]^2 v_t/2 + (\alpha-\rho)^2\omega_{v0}^2/2$$

- ACE (Bansal, Kiku, Yaron, 2007; monthly)

$$\begin{aligned}
0.0218 &= 0.0065 + 0.0153 \\
0.0026 &= 0.0026 + 0.0000 \text{ if } \rho = \alpha
\end{aligned}$$

Example 2: Wachter

- Consumption growth

$$\begin{aligned}
\log g_t &= g + \sigma w_{1t} + z_t \\
\lambda_t &= (1-\varphi)\lambda + \varphi\lambda_{t-1} + \sigma_\lambda w_{2t} \\
(w_{1t}, w_{2t}) &\sim \text{NID}(0, I) \\
z_t | j &\sim \mathcal{N}(j\theta, j\delta^2) \\
j &\geq 0 \text{ has jump intensity } \lambda_{t-1}
\end{aligned}$$

- Guess value function

$$\log u_t = u + \omega_\lambda \lambda_t$$

- Solution includes

$$\omega_\lambda = (1 - b_1\varphi)^{-1} b_1 \left[e^{\alpha\theta + (\alpha\delta)^2/2} - 1 \right] / \alpha$$

ACE: Wachter

- Pricing kernel

$$\begin{aligned}\log m_{t+1} &= \log\beta + (\rho-1)x - (\alpha-\rho)(\alpha/2)[\sigma^2 + (\omega_\lambda\sigma_\lambda)^2] \\ &\quad - \lambda_t(e^{\alpha\theta+(\alpha\delta)^2/2} - 1)/\alpha \\ &\quad + (\alpha-1)(\sigma w_{1t+1} + z_{t+1}) + (\alpha-\rho)(\omega_\lambda\sigma_\lambda)w_{2t+1}\end{aligned}$$

- Conditional entropy (monthly)

$$\begin{aligned}L_t(m_{t+1}) &= (\alpha-1)^2\sigma^2/2 + (\alpha-\rho)^2(\omega_\lambda\sigma_\lambda)^2/2 \\ &\quad + \lambda_t\left\{[e^{(\alpha-1)\theta+(\alpha-1)^2\delta^2/2} - 1] - (\alpha-1)\theta\right\}\end{aligned}$$

- ACE (monthly)

$$\begin{aligned}0.0100 &= 0.0001 + 0.0087 + 0.0012 \\ 0.0013 &= 0.0001 + 0.0000 + 0.0012 \text{ if } \rho = \alpha\end{aligned}$$

Time dependence via recursive preferences

- Little time-dependence in pricing kernel
- Nevertheless: interaction of (modest) dynamics in consumption growth and recursive preferences can have a substantial impact on the entropy of the pricing kernel
- Not clear it's relevant to option prices, but it's a route to magnify the impact of disasters on excess returns