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Notes on Option Pricing*

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The idea is to review the origins of the Black-Scholes-Merton formula for option pricing and a related formula for the so-called Merton model.

Options

Options. An options gives the owner right to buy or sell an asset at a preset price within a given time period. For example, movie producers might have a 5-year option to produce a movie based on a book. Or a property developer might have a 6-month option to purchase a piece of property. After that time, the option expires and the seller can do what she wants with the asset. In financial markets you see options on lots of things: stocks, bonds, foreign currencies, pork belly futures, and so on. They're the classic "financial derivative" and show up all over the place. Options are useful to buyers because give them flexibility. In return for this flexibility, the seller collects a fee — the price of the option.

A typical option is defined by these features: (i) The underlying: that asset on which the option is based. (ii) The strike price: the price at which you can buy or sell the underlying. (iii) The term or maturity: the period of time over which the option can be exercised. (iv) Call or put: whether it's an option to buy (a call) or sell (a put). (v) "European" or "American": most options can be exercised any time up to maturity (American), but it's mathematically simpler to work with options that can be exercised only at maturity (European).

Option cash flows. With a small investment in notation, we can express an option's cash flows mathematically. Let us say we have a τ-period option. If the price of some underlying asset at time t is s_t (s for "stock"), then a τ-period call option at strike price k generates a cash flow at time $t+\tau$ of $(s_{t+\tau} - k)^+$, where $x^+ \equiv \max(x, 0)$. Why? Because you would only exercise the option if the price is above k. Otherwise you let it expire. Similarly, a τ-period put option at strike price k generates the cash flow $(k - s_{t+\tau})^+$.

Put-call parity. There's a useful connection between European put and call prices that tells us, in practice, that if you know the price of one you can easily compute the price of the other. With a call option, you buy the asset if its price is above k, and with a put you sell it if the price is below k. So buying a call and selling a put leads you to own the stock in all cases. The price is k, paid at $t+\tau$, so we discount it using the price q_t^τ (the price at t of one dollar paid at $t+\tau$). That gives us two ways to buy the stock — with options or directly — and they should have the same price:

$$\underbrace{q_t^c}_{\text{buy call}} - \underbrace{q_t^p}_{\text{sell put}} + \underbrace{q_t^\tau k}_{\text{present value of strike}} = \underbrace{s_t}_{\text{buy stock}} .$$

*Working notes, no guarantee of accuracy or sense.

This is known as "put-call parity." A fine point: this is for an asset that doesn't pay a dividend between t and $t + \tau$. If it does, we need to work that cash flow into the equation.

Useful links: Wikipedia on options and put-call parity.

Black-Scholes-Merton formula

The idea behind option pricing — indeed asset pricing in general — is that the future price of the underlying asset is random: we don't know what it will be, although we might have a good idea of the probabilities of different outcomes. The BSM formula is based on a normal (or gaussian) distribution. We won't use that in any significant way yet, but we do need a quick introduction in order to understand the formula.

Normal random variables. We say a "random variable" x is normal with mean zero and variance one (the so-called "standard normal") if its probability density function (pdf) is

$$p(x) \;=\; (2\pi)^{-1/2} \exp(-x^2/2).$$

This is the equation for the well-known bell-shaped curve. [You might want to graph it. Try a range of x from -3 to 3.] It has two properties shared by all pdfs: (i) $p(x) \geq 0$ (probabilities can't be negative) and (ii) $\int_{-\infty}^{\infty} p(x)dx = 1$ (total probability is one). It has another property that's less general: (iii) $p(x) = p(-x)$ (symmetry).

Suppose we were interested in the probability that x is less than some specific value x^*. We could write this

$$\text{Prob}(x \leq x^*) \;=\; \int_{-\infty}^{x^*} p(x)dx \;=\; N(x^*).$$

The last is common notation (or sometimes people use Φ instead of N). There's no simple antiderivative of p, it comes up enough that we give the integral the letter N. It's also a common function in lots of software. The function N corresponds to `normcdf` in Matlab or Octave and `NORMSDIST` in Excel or OpenOffice.

Black-Scholes-Merton formula. This will be a little mysterious, but there's a popular formula for option prices that's taught in pretty much every finance course you'll find. We can do better, but it's a good start. The formula for the price of a call option is

$$q_t^c \;=\; s_t N(d_1) - q_t^\tau k N(d_2) \tag{1}$$

where

$$\begin{aligned} d_1 &= \frac{\log(s_t/q_t^\tau k) + \tau\sigma^2/2}{\tau^{1/2}\sigma} \\ d_2 &= d_1 - \tau^{1/2}\sigma. \end{aligned}$$

Here σ is a parameter ("volatility") that reflects the uncertainty in the future price.

Practice.
1. Using put-call parity and the symmetry of the normal distribution, show that the BSM put price is

$$q_t^p \quad = \quad b_t^\tau k N(-d_2) - s_t N(-d_1) \tag{2}$$

2. Compute call and put prices using these parameters: $s_t = 100$, $\tau = 1/2$, $\sigma = 0.20$, and $k = 90, 95, 100, 105, 110$. Plot call and put prices against $\log(k/s_t)$.
3. Get option prices from the WSJ online for the S&P 500 futures contract. This is the so-called "mini" contract which us popular because it comes in smaller sizes. I'm hoping this link works: `http://online.wsj.com/mdc/public/page/2_3028.html?category=Index&subcategory=U.S.&contract=SP%2520500%2520-%2520Mini%2520-%2520cme&catandsubcat=Index|U.S.&contractset=SP%2520500%2520-%2520Mini%2520-%2520cme` Click on the option button next to the Dec '10 contract and write down the option prices corresponding to any strike prices that are available. Since December is about 6 months away, that tells us $\tau = 1/2$. Also write down the last price of the underlying, labelled "Last." That gives you s_t. And take my word for it that $q_t^\tau = 0.985$.

Or try this: `http://online.wsj.com/mdc`

You mission is to take these prices. For each one, find the value of σ that generates the observed price. This is called the "implied volatility." Graph them against $\log(k/s_t)$.

Useful links: Wikipedia on the normal distribution and the Black-Scholes-Merton formula.