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UNCERTAINTY AVERSION, RISK AVERSION, AND THE OPTIMAL CHOICE OF PORTFOLIO

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1. INTRODUCTION

IN THIS PAPER we describe some implications for economic analysis of a model of decision making under uncertainty which generalizes the expected-utility model accepted by most economists as a representation of rational behavior. The model we use is the model of expected utility under a nonadditive probability measure, which seeks to distinguish between quantifiable "risks" and unknown "uncertainties." An axiomatic treatment of the model may be found in Schmeidler (1982, 1989), Gilboa (1987), and Gilboa and Schmeidler (1989).

The focus of this paper is the problem of optimal investment decisions. Under the standard theory of expected utility, an agent who must allocate his or her wealth between a safe and a risky asset will buy some of the asset if the price is less than the expected (present) value. Conversely the agent will sell the asset short when the price is greater than the expected value. Our main theorem is a generalization of this result to the case of uncertainty. We also provide a definition of an increase in perceived uncertainty, and analyze the effects of an increase on the investment decision.

The problem of making decisions under uncertainty has been of central importance to economics and statistics throughout the development of these disciplines. The expected-utility theory, which owes its axiomatic development to von Neumann and Morgenstern (1947), initiates from the work of Bernoulli (1730). Savage (1954) has made a persuasive case that rational behavior necessarily entails actions represented by such a utility function and by a prior subjective probability distribution over possible events. For example, an agent gambling on the toss of a coin about which he knows nothing may behave qualitatively differently from when he knows whether the coin is biased and if so by how much. According to Savage, this distinction would be unreasonable: in the first case the agent should behave exactly as if he knew that the bias was equal to some value (of course, this value need not be the "true" value since the agent does not have sufficient information).

Nevertheless, for both theoretical and empirical reasons economists have developed models which generalize the expected-utility model. One group of these models is based on a distinction between risk and uncertainty: the idea was proposed by Knight (1921) and has been further explored by Ellsberg (1961) and Bewley (1986) among others. In the series of papers referred to above, Schmeidler and Gilboa have given an axiomatic development of a model which incorporates this distinction. Based on a weakening of the independence axiom, the model entails maximizing expected utility with a nonadditive probability measure. With a nonadditive probability measure, the "probability" that either of two mutually exclusive events will occur is not necessarily equal to the sum of their two "probabilities." If it is less than the sum, then expected-utility calculations using this probability measure will reflect uncertainty aversion as well as (possibly) risk aversion. The reader may be disturbed by "probabilities" that do not sum to one. It should be stressed that the probabilities, together with the utility function, provide a representation of behavior. They are not objective probabilities.

Although the expected-utility model has been questioned, there is one factor which is strongly in its favor. While the theory of consumer behavior under certainty has only the most pedestrian empirical implications (homogeneity of degree zero and continuity of the demand function, and symmetry and negative semi-definiteness of the Slutsky matrix

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where demand is differentiable), the theory of expected utility yields some strong predictions, in particular the results on local risk neutrality and on complete insurance with actuarially fair policies. A generalization of the theory which eliminated the independence axiom completely would also lead to the loss of these useful predictions. The purpose of this paper is to show that the model of expected-utility maximization with nonadditive probabilities reflecting uncertainty aversion preserves strong results which are analogous to these. We focus on the local risk-neutrality theorem (Arrow (1965)).

According to this result, an agent who starts from a position of certainty will invest in an asset if, and only if, the expected value of the asset exceeds the price. The amount of the asset that is bought depends on the agent's attitude to risk. This result holds in the absence of transactions costs whenever it is possible to buy small quantities of an asset. Conversely, if the expected value is lower than the price of the asset the agent will wish to sell the asset short. Consequently an agent's demand for an asset should be positive below a certain price, negative above that price, and zero at exactly that price. In case there are many risky assets, this price will not necessarily be the expected value.

With a nonadditive subjective probability distribution over returns on the asset, we show that this result has a straightforward analog which is intuitively plausible and is compatible with observed investment behavior. There is an interval of prices within which the agent neither buys nor sells short the asset. At prices below the lower limit of this interval, the agent is willing to buy this asset. At prices above the upper end of this interval, the agent is willing to sell the asset short. The highest price at which the agent will buy the asset is the expected value of the asset under the nonadditive probability measure. The lowest price at which the agent sells the asset is the expected value of selling the asset short. This reservation price is larger than the other one if the beliefs reflect uncertainty aversion: with a nonadditive probability measure, the expectation of a random variable is less than the negative of the expectation of the negative of the random variable. The computation of expected values with nonadditive probability measures is explained below.

These two reservation prices, then, depend only on the beliefs and aversion to uncertainty incorporated in the agent's prior, and not on attitudes to risk. This result is the nonadditive analog of the local risk-neutrality result.

The local risk-neutrality result has a counterpart in the analysis of insurance. An agent who can buy actuarially fair insurance in any amount will choose to be fully insured. It follows from the results presented here that there will be a range of premium costs at which the agent buys full insurance (the model, like Savage's model, has no objective probabilities and hence actuarial fairness is not defined).

We suggest that a reasonable person may not act according to Savage's model. Maximizing utility with a nonadditive prior may be a reasonable model of rational behavior in some circumstances. However, we do not argue that this model is the only way, nor necessarily the best way, to represent genuine uncertainty. What we show here is that it provides a tractable framework for economic analysis of the types of problems for which expected-utility theory itself is useful.

In terms of empirical implications of the Schmeidler-Gilboa model, broadly similar types of behavior could be caused by transactions costs or asymmetric information, or by the preferences in Bewley's (1986) model. The main difference is that in each of those three cases there is a tendency not to trade, whereas in Schmeidler-Gilboa there is a tendency not to hold a position. In other words the agent's frame of reference here is the safe allocation, rather than the *status quo*.

In this paper we have set out the simplest investment decision to analyze, namely where there is only one uncertain asset. In case there are several assets the analysis becomes more complex because one must consider statistical dependence of the risks and uncertainties of the different asset returns. We hope to pursue this issue in the future. We have also refrained here from describing equilibrium interaction among many uncertainty averse traders. Dow, Madrigal, and Werlang (1989) discuss this in relation to the no-trade theorem.

The organization of the rest of the paper is as follows. In Section 2 we present a simple example which illustrates the basic features of the model. In Section 3 we present a definition of an increase in uncertainty aversion and results on expectation of a random variable with a nonadditive distribution. In Section 4 we give our main theorem on asset choice under uncertainty. The Appendix contains mathematical results for reference. Several of the proofs are omitted for brevity, and are available on request from the authors.

2. AN EXAMPLE

In this section we present an example which illustrates the portfolio decisions of an agent whose preferences are represented by a nonadditive probability measure. The example is based on a risk-neutral agent and an asset which can take only two possible values. The agent has wealth W and the (present) value of the asset will be either high, H, or low, L. The probabilities of these two outcomes are π and π' respectively. If $\pi + \pi' < 1$ the agent's decisions reflect uncertainty aversion. We stress that the nonadditive prior represents both the presence of uncertainty and the agent's aversion to it. For instance, in this example we could have $\pi = \pi' = 1/2$, which does not necessarily mean that the agent "knows" the risk with certainty. It could be that the agent thinks both outcomes are equally likely and is not averse to uncertainty.

Consider the expected return from buying one unit of the asset at price p. The value will be at worst $(L-p)$ net of the price, but with probability π it will be $(H-p)$, that is, an improvement of $(H-L)$ over the worst outcome. The assessment of this possible improvement reflects its uncertainty: the expected payoff from buying one unit of the asset is $[L+\pi(H-L)]-p$. If the price p is less than $[L+\pi(H-L)]$, a risk neutral investor will buy the asset.

Now consider the return from selling one unit of the asset short. The payoff will be $(p-H)$ if the asset is worth H, which is the worst outcome. With probability π' it will

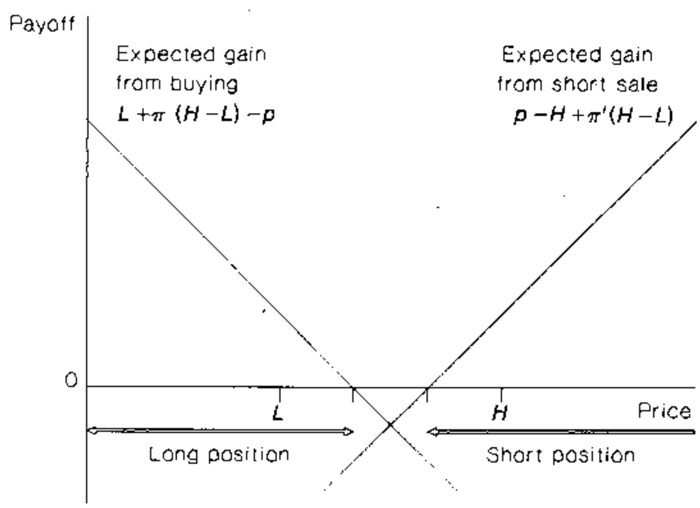

FIGURE 1.—Expected gains from buying and selling short one unit of the asset.

increase to $(p - L)$. The expected payoff is therefore $p - H + \pi'(H - L)$. Thus if p exceeds $H - \pi'(H - L)$, the investor will sell the asset short. Because $\pi + \pi' < 1$, $H - \pi'(H - L) > L + \pi(H - L)$. At prices in between these two numbers the investor will not hold the asset. Figure 1 shows the expected payoff from buying and selling the asset as a function of p.

This example illustrates how the expected value is computed under a nonadditive distribution. In this case, $E(X) = L + \pi(H - L)$ (the details are given in the Appendix). It should be clear from the discussion that adding a constant to a random variable or multiplying it by a positive constant has the same effect on its expectation. On the other hand, this property does not hold for negative constants: $-E(-X) = H + \pi'(H - L)$, so that $-E(-X) > E(X)$. It is this inequality which gives rise to the interval of prices with no asset holdings.

A closely related representation of decisions is to suppose that the agent evaluates expected utility for a set of prior (additive) probability distributions and acts to maximize the minimum of expected utility over these priors (see Gilboa and Schmeidler (1989)). At one extreme, the agent considers only one prior—a "known" distribution—and acts according to the standard theory of expected utility. At the other extreme, if all prior distributions over outcomes are considered, the agent considers only the worst possible outcome. In the above example we would consider a set of additive priors where the chance of a high return lies between π (at least) and $1 - \pi'$ (at most). The payoff from buying a unit of the asset is then

$$\operatorname{Min}\{L + \lambda(H - L) - p \mid \lambda \in [\pi, 1 - \pi']\} = L + \pi(H - L) - p,$$

and from selling it short,

$$\operatorname{Min}\{p - H + \lambda(H - L) \mid \lambda \in [\pi', 1 - \pi]\} = p - H + \pi'(H - L).$$

3. UNCERTAINTY AVERSION

We define a measure of uncertainty aversion, following an idea of Schmeidler (1989) for the case of two states of nature. The reader should refer as necessary to the Appendix for the notation, the definition of nonadditive probabilities, and a summary of their mathematical properties.

3.1. DEFINITION: Let P be a probability and $A \subset \Omega$ an event. The *uncertainty aversion of P at A* is defined by

$$c(P, A) = 1 - P(A) - P(A^c).$$

This number measures the amount of probability "lost" by the presence of uncertainty aversion. It gives the deviation of P from additivity at A. Notice that $c(P, A) = c(P, A^c)$, which is natural.

3.2. LEMMA: *$c(P, A) = 0$ for all events $A \subset \Omega$ if, and only if, P is additive.*

The proof is omitted.

3.3. EXAMPLE: *Constant Uncertainty Aversion*. Let Ω be finite with n elements and let the event space be the power set of Ω, 2^{Ω}. For all $\omega \in \Omega$, set $P(\{\omega\}) = (1 - c)/n$, where $c \in [0, 1]$. For $A \subset \Omega$, $A \neq \Omega$, define $P(A) = \Sigma_{\omega \in A} P(\{\omega\})$. It is easy to verify that $c(P, A) = c$, $\forall A \neq \Omega, \varnothing$. In other words this is a distribution with constant uncertainty aversion. In general a nonadditive probability need not be so simple.

3.4. EXAMPLE: *Maximin Behavior*. A person with extreme uncertainty aversion who is completely uninformed maximizes the payoff of the worst possible outcome. Suppose $c(P,A)=1$ for all events $A \neq \Omega, \varnothing$. Then $P(A)=0$ for all $A \neq \Omega$. Let $u: \mathbb{R} \to \mathbb{R}_+$ be the utility function of the agent. Then:

$$Eu = \int_\Omega u\,dp = \int_0^\infty P(u \geqslant \alpha)\,d\alpha.$$

Let $\underline{u} = \inf_{x \in \mathbb{R}} u(x)$. Then $P(u \geqslant \underline{u}) = 1$ and $P(u \geqslant \underline{u} + \varepsilon) = 0\ \forall \varepsilon > 0$. Therefore

$$Eu = \int_0^{\underline{u}} 1\,d\alpha = \underline{u} = \inf_{x \in \mathbb{R}} u(x).$$

This "maximin" rule was proposed by Wald (1950) for situations of complete uncertainty, and Ellsberg (1961) and Rawls (1971) also suggest that this rule should be considered in such circumstances. Simonsen (1986) is a recent application to the theory of inflationary inertia.

We now proceed to extend this "local" measure of uncertainty aversion to the whole range of two nonadditive probabilities.

3.5. DEFINITION: Given two nonadditive probabilities P and Q defined on the same space of events, we say that *P is at least as uncertainty averse as Q* if for all events $A \subset \Omega$, $c(P,A) \geqslant c(Q,A)$.

The terminology is clumsy, but shorter than alternatives such as "P reflects at least as much perceived uncertainty as Q," etc. This definition allows us to formalize the statement that the gap between buying and selling prices increases as the uncertainty aversion increases.

3.6. THEOREM: *The following statements are equivalent:*
(i) *P is at least as uncertainty averse as Q.*
(ii) *For all random variables X for which the integrals are finite,*

$$-E_P(-X) - E_P X \geqslant -E_Q(-X) - E_Q X.$$

PROOF: (i) $\Rightarrow$ (ii): Let $A(\alpha) = \{\omega \in \Omega \mid X(\omega) \geqslant \alpha\}$. Then

$$E_P X = \int_{-\infty}^0 [P(A(\alpha)) - 1]\,d\alpha + \int_0^\infty P(A(\alpha))\,d\alpha.$$

Notice that $\{\omega \in \Omega \mid -X(\omega) > \alpha\} = A(-\alpha)^c$. Thus

$$E_P(-X) = \int_{-\infty}^0 \left[P\left(A(-\alpha)^c\right) - 1\right] d\alpha + \int_0^\infty P\left(A(-\alpha)^c\right) d\alpha$$

$$= \int_0^\infty \left[P\left(A(\alpha)^c\right) - 1\right] d\alpha + \int_{-\infty}^0 P\left(A(\alpha)^c\right) d\alpha.$$

Hence

$$-E_P(-X) - E_P(X) = \int_{-\infty}^\infty \left[1 - P(A(\alpha)) - P\left(A(\alpha)^c\right)\right] d\alpha.$$

By the same argument,

$$-E_Q(-X) - E_Q(X) = \int_{-\infty}^{\infty} \left[1 - Q(A(\alpha)) - Q(A(\alpha)^c)\right] d\alpha.$$

Since P is at least as uncertainty averse as Q, the result follows immediately.

(ii) $\Rightarrow$ (i): For all events $A \in \Sigma$, define the random variable $X = 1_A$ (the characteristic function of the set A). Then $E_P X = P(A)$, $E_P(-X) = P(A^c) - 1$, $E_Q X = Q(A)$, and $E_Q(-X) = Q(A^c) - 1$. Applying (ii) to X, we get (i). *Q.E.D.*

The next example illustrates the effect of uncertainty aversion on the difference between $-E(-X)$ and $E(X)$.

3.7. Example: Let X be a random variable with $\underline{X} = \inf_{\omega \in \Omega} X(\omega) \geqslant 0$ and $\overline{X} = \sup_{\omega \in \Omega} X(\omega) \leqslant \infty$. Let P be an additive probability, and fix $c \in [0,1]$. We define a nonadditive probability which is obtained by uniformly increasing the uncertainty aversion from P: let $P_c(\Omega) = 1$, and $P_c(A) = (1-c)P(A)$ for $A \neq \Omega$. It is easy to verify that $c(P_c, A) = c$ for all $A \neq \Omega, \varnothing$, and that

$$E_{P_c} X = c\underline{X} + (1-c) E_P X \quad \text{and}$$

$$-E_{P_c}(-X) = c\overline{X} + (1-c) E_P X.$$

Thus $-E_{P_c}(-X) - E_{P_c} X = c(\overline{X} - \underline{X})$, which is increasing in the uncertainty aversion c in accordance with Theorem 3.6. Here we have taken an additive distribution and squeezed it uniformly. A risk-neutral agent whose behavior is represented by this distribution will maximize a weighted average of the worst possible outcome and the expectation of the additive distribution. Ellsberg (1961) suggested this as an *ad hoc* decision rule; this example provides some rationale for the rule.

4. PORTFOLIO CHOICE

In this section we derive our main result, namely that there will be a range of prices, from $E(X)$ to $-E(-X)$, at which the investor has no position in the asset. At prices below these, the investor holds a positive amount of the asset, and at higher prices he holds a short position. Notice that this range of prices depends only on the beliefs and attitudes to uncertainty incorporated in the agent's prior, and not on the attitudes towards risk captured by the utility function.

Let $W > 0$ be the investors' initial wealth, $u \geqslant 0$ the utility function, and X a random variable with nonadditive distribution P. We assume that u is C^2, $u' > 0$, and $u'' \leqslant 0$.

4.1. Lemma: *Suppose $EX < \infty$ and $-E(-X^2) < \infty$. For $\lambda \in \mathbb{R}$ define $f(\lambda) = Eu(W + \lambda X)$. Then:* (i) *$f$ is right-differentiable at $\lambda = 0$;* (ii) *$f'_+(0) = u'(W)EX$.*

The proof is omitted. We now proceed to the main result, namely the behavior of the risk-averse or risk-neutral agent under uncertainty aversion. Suppose the investor is faced with the problem of choosing the sum of money S he will invest in an asset. The present value of one unit of the asset next period is a random amount X with nonadditive probability distribution P. We characterize the demand for the asset as a function of the price.

4.2. Theorem: *A risk averse or risk neutral investor with certain wealth W, who is faced with an asset which yields X per unit, whose price is $p > 0$ per unit, will buy the asset if $p < EX$ and only if $p \leqslant EX$. He will sell the asset if $p > -E(-X)$ and only if $p \geqslant -E(-X)$.*

PROOF: By Jensen's inequality (see the Appendix):

$$Eu(W-S+(S/p)X) \leqslant u(E[W-S+(S/p)X]).$$

If $EX \leqslant p$, then $E[W-S+(S/p)X] \leqslant W$ (by property (iv) of the integral in the Appendix). Thus the investor is at least as well off not holding the asset, giving expected utility $u(W)$, as buying any positive amount. Similarly if $EX < p$, no holding is strictly better than investing in the asset.

We now show that if $p < EX$ the investor will buy some of the asset. The investor's objective is to maximize $g(S) = Eu(W - S + (S/p)X)$. By Lemma 4.1,

$$g'_+(0) = u'(W)E[(X-p)/p] > 0,$$

since $EX > p$. Thus the investor will buy a strictly positive amount of the asset.

Similar arguments give the corresponding results for short sales. *Q.E.D.*

Notice that if u is not differentiable at some point W, then there is a range of prices with no trade even with an additive measure (if u is concave, the set of such points has measure zero).

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APPENDIX

The mathematical treatment of nonadditive probabilities may be found in Schmeidler (1982, 1986, 1989), Choquet (1955), Dellacherie (1970), Gilboa (1987), Gilboa and Schmeidler (1989), Shafer (1976), and Dempster (1967). The reader is referred to these sources. In particular, Schmeidler (1986) contains only material related to the mathematical aspects of the theory.

Let Ω be a set, and Σ an algebra, i.e. a set of subsets of Ω such that (i) $\Omega \in \Sigma$, (ii) $A, B \in \Sigma \Rightarrow A \cup B \in \Sigma$, and (iii) $A \in \Sigma \Rightarrow A^c \in \Sigma$ (here A^c means the set of elements of Ω not in A). Ω is the set of *states of nature* and the elements of Σ are called *events*. A function P: $\Sigma \to [0,1]$ is a *nonadditive probability* if (i) $P(\varnothing) = 0$, (ii) $P(\Omega) = 1$, and (iii) $P(A) \leqslant P(B)$ if $A \subset B$. We impose an additional restriction (see Gilboa and Schmeidler (1989), Schmeidler (1986), and Shafer (1976)): (iv) $\forall A, B \in \Omega$, $P(A \cup B) + P(A \cap B) \geqslant P(A) + P(B)$. In Section 3 of the paper we show that this corresponds to uncertainty aversion.

A real valued function X: $\Omega \to \mathbb{R}$ is said to be a *random variable* if for all open sets O of $\mathbb{R}$, $X^{-1}(O) \in \Sigma$. The *expected value* of a random variable X is defined as:

$$EX = \int_\Omega X\,dp = \int_{-\infty}^0 (P(X \geqslant \alpha) - 1)\,d\alpha + \int_0^\infty P(X \geqslant \alpha)\,d\alpha,$$

whenever these integrals exist (in the improper Riemann sense) and are finite. Notice that since $P(X \geqslant \alpha) = P(X > \alpha)$ a.e., the expression for the expected value may also be written with strict inequalities. When it is necessary to distinguish between P and other distributions, we write $E_P X$.

The following properties of the integral are either proved in the papers referred to above, or else can be proved immediately:

(i) $X \geqslant Y \Rightarrow EX \geqslant EY$;

(ii) $E(X+Y) \geqslant EX + EY$;

(iii) $-E(-X) \geqslant EX$;

(iv) $\forall a \geqslant 0$ and $b \in \mathbb{R}$, $E(aX+b) = aEX + b$;

(v) For all concave functions u: $\mathbb{R} \to \mathbb{R}$, $Eu(X) \leqslant u(EX)$ (Jensen's inequality).

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