α-MAXMIN EXPECTED UTILITY

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Abstract

We axiomatize preferences that admit a representation by α-maxmin expected utility, that is a weighted sum of maxmin and maximax expected utilities. The weight α in this representation is interpretable as the decision maker's ambiguity attitude index. The attitude can range from aversion when $\alpha = 1$ (as in the Gilboa–Schmeidler multiple priors model) to affinity when $\alpha = 0$. The axiomatization captures a *cognitive procedure* used by the decision maker to extend her preference from a subdomain where expected utility prevails to a ranking of all acts.

1 INTRODUCTION

Multiple Priors Model

Subjective expected utility axiomatized by Savage [10] and by Anscombe and Aumann [1], rules out situations like the Ellsberg Paradox where the decision maker is unwilling to assign sharp probabilities to all relevant events. To accommodate such situations, Gilboa and Schmeidler [6] generalize the expected utility paradigm. They propose the *multiple priors model* where preference is represented by maxmin expected utility with a non-singleton subjective set $\mathcal{M}$ of probability measures:

$$U(f) = \min_{m \in \mathcal{M}} \int u(f(s))dm.$$ [1]

According to this representation, the decision maker can be interpreted, roughly, as having a set $\mathcal{M}$ of possible "scenarios" and evaluating every uncertain prospect f via the least favorable "scenario" from this set. An intuitive criticism of maxmin expected utility stems from this interpretation that portrays the decision maker as excessively pessimistic.

A more appropriate target of criticism in the Gilboa–Schmeidler model is not the functional form for utility that they obtain but rather one of their axioms (Ambiguity Aversion). The axiom requires that if the decision maker is indifferent between two prospects, then she weakly prefers "mixing" them and, thereby, smoothing payoffs across states of the world. The following example describes a situation where Ambiguity Aversion is problematic.

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[1] The formal notation that we use in the introduction is defined later.

Motivating Example

Consider an Ellsberg urn containing balls of three possible colors R (red), G (green) and B (blue). No information about the composition of the urn is available to the decision maker — a situation of *complete ignorance.* A ball is drawn randomly from the urn and monetary payoffs are paid contingent on the color of the ball. Assume that the decision maker is risk neutral with respect to the monetary payoffs, or alternatively, that payoffs are in utils.

Symmetry suggests the ranking

$$\begin{bmatrix} 20 \text{ if } R \\ 0 \text{ if } G \\ 0 \text{ if } B \end{bmatrix} \sim \begin{bmatrix} 0 \text{ if } R \\ 20 \text{ if } G \\ 0 \text{ if } B \end{bmatrix} \sim \begin{bmatrix} 0 \text{ if } R \\ 0 \text{ if } G \\ 20 \text{ if } B \end{bmatrix}.$$

Ambiguity Aversion implies

$$\frac{1}{2}\begin{bmatrix} 0 \text{ if } R \\ 0 \text{ if } G \\ 20 \text{ if } B \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 \text{ if } R \\ 20 \text{ if } G \\ 0 \text{ if } B \end{bmatrix} \succeq \begin{bmatrix} 20 \text{ if } R \\ 0 \text{ if } G \\ 0 \text{ if } B \end{bmatrix}$$

or equivalently,

$$\begin{bmatrix} 0 \text{ if } \ R \\ 10 \text{ if not} \end{bmatrix} \succeq \begin{bmatrix} 20 \text{ if } \ R \\ 0 \text{ if not} \end{bmatrix}.$$

However, in the absence of any information about the composition of the urn, the decision maker may view R and "not R" as similarly attractive events upon which to bet. Intuitively, she may ignore the fact that "not R" contains two distinct states of nature because this fact is uninformative. After all, the event R might also be expressed as a union of more elementary states by introducing more aspects of the world to be described. Consequently, the decision maker might contradict Ambiguity Aversion and have the strict preference

$$\begin{bmatrix} 0 \text{ if } \ R \\ 10 \text{ if not} \end{bmatrix} \prec \begin{bmatrix} 20 \text{ if } \ R \\ 0 \text{ if not} \end{bmatrix}.$$

Thus, Ambiguity Aversion though natural in many situations, is problematic in others.

New Utility Representation

Motivated by such criticism, we axiomatize α-maxmin expected utility that uses a subjective set of priors $\mathcal{M}$ in a way different from the Gilboa–Schmeidler model. The utility of each prospect is evaluated by a weighted average of the most and the least favorable "scenarios"

$$U(f) = \alpha \min_{m \in \mathcal{M}} \int u(f(s))\, dm + (1-\alpha) \max_{m \in \mathcal{M}} \int u(f(s))\, dm.$$

The unique weight α is independent of the evaluated act and is interpretable as the decision maker's ambiguity attitude index. The attitude can range from aversion when $\alpha = 1$ (as in Gilboa and Schmeidler) to affinity when $\alpha = 0$.

To characterize α-maxmin expected utility, we assume that the decision maker conforms to expected utility on a suitable subdomain of *risky* (or unambiguous) acts but not on the entire domain. Then we replace Ambiguity Aversion with an axiom that reflects partial ignorance outside this subdomain. Roughly speaking, the Partial Ignorance axiom captures a *cognitive procedure* used by the decision maker to extend her preference over risky acts to a ranking of all acts. This axiom also implies additional structure for the set $\mathcal{M}$ beyond the technical conditions in the Gilboa–Schmeidler model. Two examples are (i) $\mathcal{M}$ consists of all extensions of a probability measure given on a class of unambiguous events; (ii) $\mathcal{M}$ consists of all "extensions" of conditional probabilities given on a class of unambiguous pairs of events.

Related Literature

The α-maxmin expected utility functional form appears first in Hurwicz [8] in the context of statistical decision problems where $\mathcal{M}$ is the statistician's *a priori* class of probability distributions. Luce and Raiffa [9] discuss Hurwicz's α-criterion for individual decision making with the emphasis on complete ignorance when $\mathcal{M}$ is the universal set of probability measures and

$$U(f) = \alpha \min_{s\in S} u(f(s)) + (1-\alpha) \max_{s\in S} u(f(s)).$$

Arrow and Hurwicz [2] axiomatize existence of such a utility representation with α being a function of the worst and best outcomes.

In independent research, Ghirardato, Maccheroni and Marinacci [5] axiomatize α-maxmin expected utility with the set $\mathcal{M}$ restricted only by technical conditions as in Gilboa and Schmeidler. Thus their model is more general than ours. There is also a difference in approaches. While the approach here begins with a subdomain of "risky acts," Ghirardato *et al.* begin with an "unambiguous subrelation."

Zhang [13] models situations when probabilities of some events are precisely known to the decision maker. Zhang proposes Approximation from Below axiom that reflects a pessimistic approach to evaluating acts on the basis of the known probabilities. Approximation from Below leads to a special case of Choquet expected utility. Adopting the Anscombe–Aumann setup, we eliminate some of the limitations of Zhang's approach as illustrated by his own "counterexample" (see section 3.3 below).

Outline

The paper is organized as follows. First, we briefly describe the Anscombe–Aumann setup and the Gilboa–Schmeidler multiple priors model. In sections 3.1 and 3.2 we describe the cognitive procedure underlying the α-maxmin expected utility representation and formulate our main result (Theorem 3.2). In sections 3.3 and 3.4 we illustrate the result with several examples and characterize a special case of maxmin expected utility suggested by our procedural approach (Theorem 3.3). Finally, we axiomatize a fully endogenous α-maxmin expected utility representation (Theorem 3.4).

2 SETUP

Anscombe–Aumann's setup is employed throughout. Given are a set of deterministic outcomes X and a set of states of the world $S = \{s, \dots\}$ equipped with an algebra of events Σ. $\Delta(S, \Sigma)$ denotes the set of finitely additive probability measures on the measurable space (S, Σ). The set $\Delta(S, \Sigma)$ is endowed with the weak* topology, that is the weakest topology such that for every bounded and Σ-measurable function $b : S \to \mathbb{R}$, the integration $\int b\, dm$ is a continuous mapping from $\Delta(S, \Sigma)$ into $\mathbb{R}$.

Probability distributions on X having finite support are called *lotteries*; the set of lotteries is written as $\mathcal{L} = \{l, \dots\}$. Prospects are modelled as *acts* — Σ-measurable functions mapping S into $\mathcal{L}$ and having finite range. [2] The set of acts is written as $\mathcal{H} = \{f, g, h, \dots\}$; the collection of constant acts is identified with $\mathcal{L}$.

Convex combinations, also called *mixtures*, are well-defined in $\mathcal{H}$. The act $\tau f + (1 - \tau)g$ is defined for $f, g \in \mathcal{H}$ and $\tau \in [0, 1]$ by

$$[\tau f + (1 - \tau)g](s) = \tau f(s) + (1 - \tau)g(s)$$

for every $s \in S$.

The decision maker's weak preference relation $\succeq$ over $\mathcal{H}$ is taken as primitive along with X, S and Σ. The following axioms are borrowed from Gilboa and Schmeidler [6].

Axiom (Weak Order). *$\succeq$ is complete and transitive.*

Axiom (Non-Degeneracy). *There exist $f, g \in \mathcal{H}$ such that $f \succ g$.*

Act f *weakly dominates* act g, written $f \geqq g$, if $f(s) \succeq g(s)$ for all $s \in S$.

Axiom (Monotonicity). *For all $f, g \in \mathcal{H}$, if $f \geqq g$, then $f \succeq g$.*

Monotonicity embodies a form of state independence for the decision maker's preference over lotteries.

Axiom (Mixture Continuity). *For all $f, g, h \in \mathcal{H}$, the sets $\{\tau : \tau f + (1 - \tau)g \succeq h\}$ and $\{\tau : \tau f + (1 - \tau)g \preceq h\}$ are closed in $[0, 1]$.*

Axiom (Certainty Independence). *For all $f, g \in \mathcal{H}$, $l \in \mathcal{L}$ and $\tau \in (0, 1)$,*

$$f \succeq g \Leftrightarrow \tau f + (1 - \tau)l \succeq \tau g + (1 - \tau)l.$$

Thereby the preference $\succeq$ is assumed unaffected by mixtures with constant acts but not necessarily by mixtures with non-constant acts. Note that Certainty Independence is weaker than the standard independence axiom.

[2] Such acts are usually called simple to emphasize the fact that they have finite range. In this terminology, all acts that we consider are simple.

Axiom (Independence). *For all* $f, g, h \in \mathcal{H}$ *and* $\tau \in (0,1)$,

$$f \succeq g \Leftrightarrow \tau f + (1-\tau)h \succeq \tau g + (1-\tau)h.$$

We say that $\succeq$ is *regular* if $\succeq$ satisfies Weak Order, Non-Degeneracy, Monotonicity, Mixture Continuity and Certainty Independence.

Gilboa and Schmeidler obtain a maxmin expected utility representation for regular preference by imposing an additional condition.

Axiom (Ambiguity Aversion). *For all* $f, g \in \mathcal{H}$ *and* $\tau \in (0,1)$, *if* $f \sim g$, *then* $\tau f + (1-\tau)g \succeq f$.

Intuitively, good and bad outcomes of f and g cancel out at least partially in the mixture $\tau f + (1-\tau)g$. Hence, the ambiguity averse decision maker is modelled as having the weak preference $\tau f + (1-\tau)g \succeq f$ whenever she is indifferent between f and g.

Gilboa–Schmeidler Theorem. $\succeq$ *is regular and satisfies Ambiguity Aversion if and only if* $\succeq$ *is represented by*

$$U(f) = \min_{m \in \mathcal{M}} \int u(f(s))\, dm \tag{2.1}$$

where $u : \mathcal{L} \to \mathbb{R}$ *is a non-constant affine function and* $\mathcal{M} \subset \Delta(S, \Sigma)$ *is non-empty, convex and closed. Moreover,* u *is unique up to a positive linear transformation and* $\mathcal{M}$ *is unique.*

If Certainty Independence is replaced by Independence, then $\mathcal{M}$ is singleton above and representation (2.1) reduces to expected utility.

To ensure uniqueness in the utility representations to follow, we employ

Axiom (Non-Independence). $\succeq$ *does not satisfy Independence.*

3 REPRESENTATION RESULTS

If the decision maker does not have comprehensive knowledge about the uncertain environment (for example, does not know the precise composition of the urn in Ellsberg-type experiments), then she may be unwilling to assign probabilities to all events and accordingly, to assign expected utility to all acts. However, she still assigns expected utility to some acts; we call such acts *risky.* In our model the decision maker identifies and ranks risky acts as a *first step* of her ranking the entire domain $\mathcal{H}$. We describe the second step later, after we obtain an expected utility representation over risky acts.

3.1 Exogenous Risk

We take the subdomain $\mathcal{R} = \{r, \dots\}$ of risky acts as exogenously given (in section 3.5, we define $\mathcal{R}$ endogenously). We interpret $\mathcal{R}$ as the domain where the decision maker assigns expected utility before ranking the entire $\mathcal{H}$. This informal interpretation and affinity of expected utility motivate the following formal conditions that we impose on $\mathcal{R}$:

(R1) $\mathcal{L} \subset \mathcal{R}$

(R2) if $r, r' \in \mathcal{R}$ and $\tau \in [0,1]$, then $\tau r + (1-\tau)r' \in \mathcal{R}$

(R3) if $r \in \mathcal{R}$, $r' \in \mathcal{H}$, $\tau \in (0,1)$ and $\tau r + (1-\tau)r' \in \mathcal{R}$, then $r' \in \mathcal{R}$.

In other words, the modelled decision maker assigns expected utility to lotteries; also, when possible she assigns expected utility via affinity.

Denote by $\succeq_{\mathcal{R}}$ the preference relation over $\mathcal{R}$. Given conditions $R1$–3, all the axioms formulated on the entire $\mathcal{H}$ are meaningful on the subdomain $\mathcal{R}$ as well. The following expected utility representation result holds for $\succeq_{\mathcal{R}}$.

Lemma 3.1. *If $\mathcal{R}$ satisfies conditions R1–3, then the following two statements are equivalent.*

1. *$\succeq_{\mathcal{R}}$ satisfies Weak Order, Non-Degeneracy, Monotonicity, Mixture Continuity and Independence.*

2. *$\succeq_{\mathcal{R}}$ is represented by*

$$U_{\mathcal{R}}(r) = \int u(r(s))\, dm \tag{3.1}$$

where $m \in \Delta(S, \Sigma)$ and the utility index $u : \mathcal{L} \to \mathbb{R}$ is non-constant and affine.

Moreover, u is unique up to a positive linear transformation and the probability measures m that permit representation (3.1) *form a non-empty, closed and convex set, written $\mathcal{M}_{\mathcal{R}}$.*

This lemma provides an axiomatic model of expected utility on the subdomain $\mathcal{R}$. If $\mathcal{R}$ satisfies $R1$–3 and if either of the equivalent statements of the lemma holds for $\succeq_{\mathcal{R}}$, then the pair $(\mathcal{R}, \succeq_{\mathcal{R}})$ is called a *risk profile*.

It merits emphasis that the probability measure that supports expected utility for the risk profile $(\mathcal{R}, \succeq_{\mathcal{R}})$ is not necessarily unique. For example, if $\mathcal{R} = \mathcal{L}$ and $\succeq_{\mathcal{L}}$ conforms to expected utility, then any probability measure $m \in \Delta(S, \Sigma)$ permits representation (3.1), and $\mathcal{M}_{\mathcal{R}} = \Delta(S, \Sigma)$. Of course, if $\mathcal{R} = \mathcal{H}$, then the measure in representation (3.1) is unique and Lemma 3.1 reduces to the Anscombe–Aumann Theorem.

3.2 Partial Ignorance Axiom

Suppose that after obtaining the risk profile $(\mathcal{R}, \succeq_{\mathcal{R}})$, the decision maker *extends* $\succeq_{\mathcal{R}}$ to preference $\succeq$ on $\mathcal{H}$ (that is, $\succeq$ is required to coincide with $\succeq_{\mathcal{R}}$ on $\mathcal{R}$). In this two-step cognitive procedure, the risk profile $(\mathcal{R}, \succeq_{\mathcal{R}})$ serves as a formal description of the decision maker's a priori knowledge about the uncertain environment and of her risk attitude.

Suppose further that $\succeq$ is regular. Recall that regularity includes axioms of Monotonicity and Certainty Independence. We now describe how the decision maker might go about using these axioms and the risk profile $(\mathcal{R}, \succeq_{\mathcal{R}})$ to arrive at the regular ranking $\succeq$ on the domain $\mathcal{H}$ of all acts.

The lottery space $\mathcal{L}$ provides a natural scale to evaluate an arbitrary act f. Using Monotonicity, the decision maker can conclude that f is strictly better than a lottery l_* if $f \geqq r_* \succ_{\mathcal{R}} l_*$ for some risky act r_*.[3] Conversely, she can conclude that f is strictly worse than l^* if $f \leqq r^* \prec_{\mathcal{R}} l^*$ for some $r^* \in \mathcal{R}$. Combining Monotonicity and Certainty Independence, the decision maker can arrive at more complex conclusions. We will say that the risk profile $(\mathcal{R}, \succeq_{\mathcal{R}})$

(A) *locates* a lottery l^* *above* the act f if

$$\tau f + (1-\tau)l \leqq r^* \prec_{\mathcal{R}} \tau l^* + (1-\tau)l$$

for some $\tau \in (0,1]$, $l \in \mathcal{L}$ and $r^* \in \mathcal{R}$;

(B) *locates* a lottery l_* *below* the act f if

$$\tau f + (1-\tau)l \geqq r_* \succ_{\mathcal{R}} \tau l_* + (1-\tau)l$$

for some $\tau \in (0,1]$, $l \in \mathcal{L}$ and $r_* \in \mathcal{R}$.

There are no other situations when a strict preference between an act and a lottery is implied by the risk profile $(\mathcal{R}, \succeq_{\mathcal{R}})$, Monotonicity and Certainty Independence.

The "locating" observations may coincide for some acts f and g. We will say that the risk profile $(\mathcal{R}, \succeq_{\mathcal{R}})$ is *uninformative* for the comparison of f and g if for all $l_*, l^* \in \mathcal{L}$,

$$\begin{array}{lcl} (\mathcal{R}, \succeq_{\mathcal{R}}) \text{ locates } l^* \text{ above } f & \Leftrightarrow & (\mathcal{R}, \succeq_{\mathcal{R}}) \text{ locates } l^* \text{ above } g \\ (\mathcal{R}, \succeq_{\mathcal{R}}) \text{ locates } l_* \text{ below } f & \Leftrightarrow & (\mathcal{R}, \succeq_{\mathcal{R}}) \text{ locates } l_* \text{ below } g. \end{array}$$

The following axiom characterizes the comparison of acts when $(\mathcal{R}, \succeq_{\mathcal{R}})$ is uninformative.

Axiom (Partial Ignorance outside $(\mathcal{R}, \succeq_{\mathcal{R}})$). *If the risk profile $(\mathcal{R}, \succeq_{\mathcal{R}})$ is uninformative for the comparison of f and g, then $f \sim g$.*

The axiom portrays the decision maker as conditioning her global preference on her ranking of the subdomain $\mathcal{R}$ where she assigns expected utility. Arriving at $\succeq$, she focuses exclusively on the implications of Monotonicity, Certainty Independence and the risk profile $(\mathcal{R}, \succeq_{\mathcal{R}})$. If there are no implications for the comparison of f and g, then she is indifferent between the two. Such a procedure underlies a representation by α-maxmin expected utility as described in our main result.

Theorem 3.2. *Given a binary relation $\succeq$ and a risk profile $(\mathcal{R}, \succeq_{\mathcal{R}})$, the following two statements are equivalent.*

1. *$\succeq$ is regular and satisfies Partial Ignorance outside $(\mathcal{R}, \succeq_{\mathcal{R}})$.*

[3] As $\mathcal{L} \subset \mathcal{R}$, the weak dominance relation $\geqq$ is well-defined given $\succeq_{\mathcal{R}}$.

2. $\succeq$ *is represented by*

$$U(f) = \alpha \min_{m \in \mathcal{M}_{\mathcal{R}}} \int u(f(s))\, dm + (1-\alpha) \max_{m \in \mathcal{M}_{\mathcal{R}}} \int u(f(s))\, dm \tag{3.2}$$

where u *and* $\mathcal{M}_{\mathcal{R}}$ *are delivered by Lemma 3.1.*

Moreover, given Non-Independence, α *is unique.*

The theorem shows that Partial Ignorance outside $(\mathcal{R}, \succeq_{\mathcal{R}})$ leads to α-maxmin expected utility representation 3.2 for regular preference $\succeq$. In this representation, both the set of measures $\mathcal{M}_{\mathcal{R}}$ and the utility index u are determined by the exogenously given $(\mathcal{R}, \succeq_{\mathcal{R}})$. The set $\mathcal{M}_{\mathcal{R}}$ consists of all probability measures that support expected utility for $(\mathcal{R}, \succeq_{\mathcal{R}})$ (Lemma 3.1 asserts that $\mathcal{M}_{\mathcal{R}}$ is non-empty, convex and closed). The affine utility index u represents preference over $\mathcal{L} \subset \mathcal{R}$.

3.3 Applications

Next we illustrate α-maxmin expected utility representation (3.2) via specializations of the risk profile $(\mathcal{R}, \succeq_{\mathcal{R}})$.

Complete Ignorance

Let $\mathcal{R} = \mathcal{L}$. If $\succeq$ is regular, then $(\mathcal{L}, \succeq_{\mathcal{L}})$ is a risk profile and $\mathcal{M}_{\mathcal{L}} = \Delta(S, \Sigma)$. Theorem 3.2 asserts that $\succeq$ satisfies Partial Ignorance outside $(\mathcal{L}, \succeq_{\mathcal{L}})$ if and only if $\succeq$ can be represented by Hurwicz's α-maxmin utility

$$U(f) = \alpha \min_{s \in S} u(f(s)) + (1-\alpha) \max_{s \in S} u(f(s)),$$

where $\alpha \in [0,1]$ and the utility index u is non-constant and affine.

The decision maker whose preference satisfies Partial Ignorance outside $(\mathcal{L}, \succeq_{\mathcal{L}})$ can be interpreted as being completely ignorant about the uncertain environment, hence, assigning expected utility only to constant acts. In the resulting α-maxmin expected utility representation, complete ignorance is reflected by the use of the universal set $\Delta(S, \Sigma)$ of "probabilistic scenarios."

Sharp Probabilities on a Subalgebra

Let $\Sigma_0 \subset \Sigma$ be a subalgebra endowed with a probability measure $m_0 : \Sigma_0 \to [0,1]$ describing all probabilities that are precisely known to the decision maker. Let $\mathcal{R} \subset \mathcal{H}$ be the set of Σ_0-measurable acts. It is intuitive that the decision maker assigns expected utility on $\mathcal{R}$, this expected utility being $U_{\mathcal{R}}(r) = \int u(r(s))\, dm_0$, with non-constant and affine utility index $u : \mathcal{L} \to \mathbb{R}$. Then $(\mathcal{R}, \succeq_{\mathcal{R}})$ is a risk profile and by construction,

$$\mathcal{M}_{\mathcal{R}} = \mathcal{E}(m_0) = \{m \in \Delta(S, \Sigma) : m(E) = m_0(E) \text{ for all } E \in \Sigma_0\}.$$

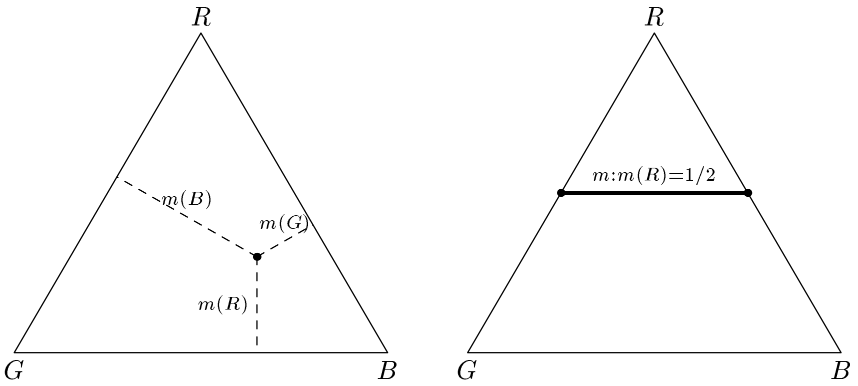

Figure 1: $\mathcal{M}_{\mathcal{R}}$ is determined by the known probability $m(R) = 1/2$

In other words, $\mathcal{M}_{\mathcal{R}}$ consists of all probability measures that extend m_0 from Σ_0 to Σ.

Theorem 3.2 asserts that, given Σ_0, m_0 and u, the following two statements are equivalent for the corresponding $(\mathcal{R}, \succeq_{\mathcal{R}})$ and $\mathcal{E}(m_0)$.

1. $\succeq_{\mathcal{R}}$ is represented by $U_{\mathcal{R}}$, $\succeq$ is regular and satisfies Partial Ignorance outside $(\mathcal{R}, \succeq_{\mathcal{R}})$.
2. $\succeq$ is represented by

$$U(f) = \alpha \min_{m \in \mathcal{E}(m_0)} \int u(f(s))\, dm + (1 - \alpha) \max_{m \in \mathcal{E}(m_0)} \int u(f(s))\, dm$$

   for some $\alpha \in [0, 1]$.

For example, consider an Ellsberg urn with three possible colors R, G and B. Suppose that the decision maker is told only that the total number of balls in the urn is 100 and that 50 of them are red. Then $\Sigma_0 = \{S, \emptyset, \{R\}, \{G, B\}\}$ and $m_0(\{R\}) = m_0(\{G, B\}) = \frac{1}{2}$. The set $\mathcal{E}(m_0)$ of probability measures that extend m_0 is illustrated by Figure 1.

Sharp Probabilities: General Case

In some situations the family of events where probabilities are known is not an algebra (Zhang [13]). Consider an Ellsberg urn with four possible colors R, G, B and W (white). Suppose the decision maker is told only that the total number of balls in the urn is 100, and that $R + B = G + B = 30$, that is the combined number of red and blue balls or alternatively, of green and blue balls is 30. Then the domain where probabilities are known is

$$\Gamma = \{S, \emptyset, \{R, B\}, \{G, B\}, \{G, W\}, \{R, W\}\}.$$

Γ is not an algebra because, for instance, $\{R, B\} \cap \{G, B\} = \{B\} \notin \Gamma$.

To accommodate such situations, let $\Gamma \subset \Sigma$ be a non-empty family of events and let a function $m_0 : \Gamma \to [0, 1]$ describe all probabilities known to the decision maker. Suppose that m_0 can be extended to a probability measure $m_* \in \Delta(S, \Sigma)$. Denote by $\mathcal{B}$ the set of all binary acts $b(s) = \begin{bmatrix} l \text{ if } s \in E \\ l' \text{ if } s \notin E \end{bmatrix}$ for $E \in \Gamma$. It is intuitive that the decision maker assigns expected utility to every $b \in \mathcal{B}$, $U_{\mathcal{R}}(b) = m_0(E)u(l) + (1 - m_0(E))u(l')$, where u is non-constant and affine. Let $\mathcal{R}$ be the minimal subdomain that satisfies conditions $R1$–3 and contains $\mathcal{B}$. Note that $r \in \mathcal{R}$ if and only if

$$\tau_0 r + (1 - \tau_0)(\tau_1 b_1 + \cdots + \tau_n b_n) = \tau_1' b_1' + \cdots + \tau_k' b_k'$$

for some binary acts $b_i, b_j' \in \mathcal{B}$ and weights $\tau_i, \tau_j' \in [0, 1]$ such that $\tau_0 > 0$, $\sum_{i=1}^{n} \tau_i = 1$ and $\sum_{j=1}^{k} \tau_j' = 1$.[4] Hence, the decision maker can assign expected utility to every $r \in \mathcal{R}$ as

$$U_{\mathcal{R}}(r) = \frac{1}{\tau_0}(\sum_{j=1}^{k} \tau_j' U_{\mathcal{R}}(b_j') - \sum_{i=1}^{n} \tau_i U_{\mathcal{R}}(b_i)).$$

Then m_* supports expected utility on $(\mathcal{R}, \succeq_{\mathcal{R}})$ and $(\mathcal{R}, \succeq_{\mathcal{R}})$ is a risk profile. By construction,

$$\mathcal{M}_{\mathcal{R}} = \mathcal{E}(\Gamma, m_0) = \{m \in \Delta(S, \Sigma) : m(E) = m_0(E) \text{ for all } E \in \Gamma\}.$$

In other words, $\mathcal{M}_{\mathcal{R}}$ consists of all probability measures that extend m_0 from Γ to Σ.

Theorem 3.2 asserts that, given Γ, m_0 and u, the following two statements are equivalent for the corresponding $(\mathcal{R}, \succeq_{\mathcal{R}})$ and $\mathcal{E}(\Gamma, m_0)$.

1. $\succeq_{\mathcal{R}}$ is represented by $U_{\mathcal{R}}$, $\succeq$ is regular and satisfies Partial Ignorance outside $(\mathcal{R}, \succeq_{\mathcal{R}})$.

2. $\succeq$ is represented by

$$U(f) = \alpha \min_{m \in \mathcal{E}(\Gamma, m_0)} \int u(f(s))\, dm + (1 - \alpha) \max_{m \in \mathcal{E}(\Gamma, m_0)} \int u(f(s))\, dm$$

for some $\alpha \in [0, 1]$.

Zhang [13] provides an alternative model of preference based on known probabilities m_0. More precisely, Zhang employs the Savage setup and axiomatizes Choquet expected utility with the capacity being the inner measure induced on Σ by the function m_0. In the four-color example, his model predicts the indifference between \$0 paid for sure and a hundred-dollar bet that pays \$100 if $s = W$ and \$0 otherwise. The indifference is implied by the fact that the inner measure of event $\{W\}$ is zero. As Zhang points out, this indifference is problematic because the decision maker knows that there are at least 40 white balls in the urn.

Our utility representation eliminates this limitation of Zhang's approach. Consider α-maxmin expected utility representation with

$$\mathcal{E}(\Gamma, m_0) = \{m \in \Delta(S, \Sigma) : m(R, B) = m(B, G) = 0.3\}.$$

Let $u(\$0) = 0$. Then the utility of a hundred-dollar bet on W lies between $0.4u(\$100)$ (when $\alpha = 1$) and $0.7u(\$100)$ (when $\alpha = 0$) and is definitely larger than zero. This improvement becomes possible in the Anscombe–Aumann setup. In this setup, the decision maker can use affinity to assign expected utility to acts that are not necessarily Γ-measurable, hence, making more accurate comparisons on the entire domain $\mathcal{H}$.

[4]The proof of this statement is simple and is omitted.

Sharp Conditional Probabilities

The decision maker's a priori knowledge about the uncertain environment may take a more complex form. For example, some *conditional* probabilities may be known to her. To illustrate, consider an Ellsberg urn with three possible colors R, G and B. Suppose that the decision maker is told only that $B = G$. Then the probability of B conditional on $\{G, B\}$ is known to be $\frac{1}{2}$. At the same time all probabilities except the trivial $m(S) = 1$ and $m(\emptyset) = 0$ are imprecisely known.

To accommodate such situations, let $\Theta \subset \Sigma \times \Sigma$ be an arbitrary collection of pairs of events and let a function $\mu : \Theta \to [0, 1]$ describe all the conditional probabilities known to the decision maker. Suppose that $\mu : \Theta \to [0, 1]$ agrees with a probability measure $m_* \in \Delta(S, \Sigma)$, that is for all $(E, C) \in \Theta$,

$$m_*(E \cap C) = \mu(E, C) \cdot m_*(C).$$

Denote by $\mathcal{T}$ the collection of all ternary acts $t(s) = \begin{bmatrix} l \text{ if } s \in E \cap C \\ l' \text{ if } s \in C \setminus E \\ l'' \text{ if } \quad s \notin C \end{bmatrix}$, where $(E, C) \in \Theta$ and $\mu(E, C)l + (1 - \mu(E, C))l' = l''$. In other words, every t yields l'' if $s \notin C$ and l'' on average if $s \in C$. Here, the average is based on the known conditional probability $\mu(E, C)$. It is intuitive that the decision maker assigns expected utility $u(t) = u(l'')$ to every $t \in \mathcal{T}$. Let $\mathcal{R}$ be the minimal subdomain that satisfies conditions $R1$–3 and contains $\mathcal{T}$. Then $r \in \mathcal{R}$ if and only if

$$\tau_0 r + (1 - \tau_0)(\tau_1 t_1 + \cdots + \tau_n t_n) = \tau'_1 t'_1 + \cdots + \tau'_k t'_k$$

for some ternary acts $t_i, t'_j \in \mathcal{B}$ and weights $\tau_i, \tau'_j \in [0, 1]$ such that $\tau_0 > 0$, $\sum_{i=1}^n \tau_i = 1$ and $\sum_{j=1}^k \tau'_j = 1$. Therefore, the decision maker can assign expected utility over $\mathcal{R}$ as

$$U_{\mathcal{R}}(r) = \frac{1}{\tau_0}(\sum_{j=1}^k \tau'_j U_{\mathcal{R}}(t'_j) - \sum_{i=1}^n \tau_i U_{\mathcal{R}}(t_i)).$$

Then m_* supports expected utility on $(\mathcal{R}, \succeq_{\mathcal{R}})$ and $(\mathcal{R}, \succeq_{\mathcal{R}})$ is a risk profile. By construction

$$\mathcal{M}_{\mathcal{R}} = \mathcal{E}(\Theta, \mu) = \{m \in \Delta(S, \Sigma) : m(E \cap C) = \mu(E, C)m(C) \text{ for all } (E, C) \in \Theta\}.$$

In other words, $\mathcal{E}(\Theta, \mu)$ consists of all probability measures that agree with conditional probabilities $\mu(\cdot)$ on Θ.

Theorem 3.2 asserts that, given Θ, μ and u, the following two statements are equivalent for the corresponding $(\mathcal{R}, \succeq_{\mathcal{R}})$ and $\mathcal{E}(\Theta, \mu)$.

1. $\succeq_{\mathcal{R}}$ is represented by $U_{\mathcal{R}}$, $\succeq$ is regular and satisfies Partial Ignorance outside $(\mathcal{R}, \succeq_{\mathcal{R}})$.

2. $\succeq$ is represented by

$$U(f) = \alpha \min_{m \in \mathcal{E}(\Theta, \mu)} \int u(f(s))\, dm + (1 - \alpha) \max_{m \in \mathcal{E}(\Theta, \mu)} \int u(f(s))\, dm$$

for some $\alpha \in [0, 1]$.

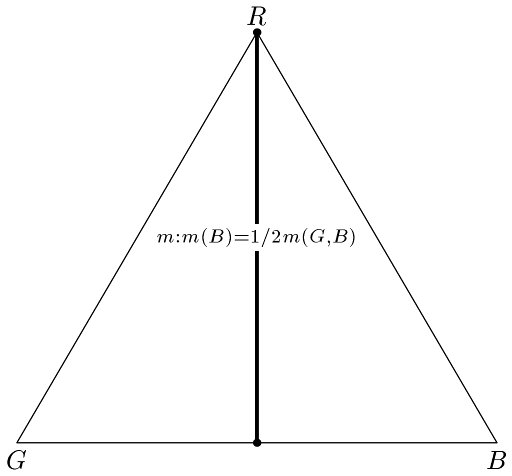

Figure 2: $\mathcal{M}_{\mathcal{R}}$ is determined by the probability of $\{B\}$ conditional on $\{G, B\}$.

In the three-color example, where the decision maker is told only that $R = B$, let $\Theta = \{(\{G\}, \{G, B\})\}$ and $\mu(\{G\}, \{G, B\}) = \frac{1}{2}$. Figure 2 illustrates the set of measures that agree with $\mu(\cdot)$ on Θ.

3.4 Back to Maxmin Expected Utility

A version of the Partial Ignorance axiom describes a more conservative approach that the decision maker might take to extend her preference from risky acts to the entire domain. If taking this approach, she is unwilling to express a strict preference in a broader class of situations than prescribed by Partial Ignorance.

We say that the risk profile $(\mathcal{R}, \succeq_{\mathcal{R}})$ is *semi-uninformative* for the comparison of acts f and g if for all $l_* \in \mathcal{L}$,

$$(\mathcal{R}, \succeq_{\mathcal{R}}) \text{ locates } l_* \text{ below } f \quad \Leftrightarrow \quad (\mathcal{R}, \succeq_{\mathcal{R}}) \text{ locates } l_* \text{ below } g.$$

Of course, if the risk profile $(\mathcal{R}, \succeq_{\mathcal{R}})$ is uninformative for the comparison of f and g, then $(\mathcal{R}, \succeq_{\mathcal{R}})$ is also semi-uninformative for that comparison.

Axiom (Pessimism outside $(\mathcal{R}, \succeq_{\mathcal{R}})$). *If the risk profile $(\mathcal{R}, \succeq_{\mathcal{R}})$ is semi-uninformative for the comparison of f and g, then $f \sim g$.*

The Pessimism axiom strengthens Partial Ignorance. The Pessimism axiom models a decision maker who focuses on "locating from below" observations and attaches no significance to "locating from above." The Pessimism axiom underlies a representation by a special case of maxmin expected utility.

Theorem 3.3. *Given a binary relation $\succeq$ and a risk profile $(\mathcal{R}, \succeq_{\mathcal{R}})$, the following two statements are equivalent.*

1. *$\succeq$ is regular and satisfies Pessimism outside $(\mathcal{R}, \succeq_{\mathcal{R}})$.*

2. $\succeq$ *is represented by*

$$U(f) = \min_{m \in \mathcal{M}_{\mathcal{R}}} \int u(f(s))\, dm \tag{3.3}$$

where u *and* $\mathcal{M}_{\mathcal{R}}$ *are delivered by Lemma 3.1.*

Thus, if $\succeq$ is regular, then Pessimism outside $(\mathcal{R}, \succeq_{\mathcal{R}})$ strengthens Ambiguity Aversion. The set $\mathcal{M}_{\mathcal{R}}$ in the resulting maxmin expected utility representation is determined by the risk profile $(\mathcal{R}, \succeq_{\mathcal{R}})$. Two examples are: i) $\mathcal{M} = \mathcal{E}(\Gamma, m_0)$ consists of all extensions of probabilities $m_0(\cdot)$ given on a class $\Gamma \subset \Sigma$; (ii) $\mathcal{M} = \mathcal{E}(\Theta, \mu)$ consists of all "extensions" of conditional probabilities $\mu(\cdot)$ given on a class $\Theta \subset \Sigma \times \Sigma$.

3.5 Endogenized Risk

Theorem 3.2 takes the risk profile $(\mathcal{R}, \succeq_{\mathcal{R}})$ as given. The exogenous formulation is natural in applications where $(\mathcal{R}, \succeq_{\mathcal{R}})$ is derived from objectively given information but not in situations where the decision maker assigns expected utility subjectively. In this section we show how the risk profile underlying α-maxmin expected utility representation (3.2) can be obtained endogenously.

An act $r \in \mathcal{H}$ is called *endogenously risky* if for all $f, g \in \mathcal{H}$ and $\tau \in (0,1)$,

$$f \succeq g \quad \Leftrightarrow \quad \tau f + (1-\tau) r \succeq \tau g + (1-\tau) r.$$

In other words, r is endogenously risky if the invariance required by the Independence Axiom holds for mixtures with r. For example, Certainty Independence requires that constant acts be endogenously risky. Denote by $\mathcal{R}_e$ the collection of all endogenously risky acts.

Theorem 3.4. *If* $\succeq$ *is regular, then* $(\mathcal{R}_e, \succeq_{\mathcal{R}_e})$ *is a risk profile and the following statements are equivalent.*

1. $\succeq$ *satisfies Partial Ignorance outside* $(\mathcal{R}, \succeq_{\mathcal{R}})$ *for some risk profile* $(\mathcal{R}, \succeq_{\mathcal{R}})$.

2. $\succeq$ *satisfies Partial Ignorance outside* $(\mathcal{R}_e, \succeq_{\mathcal{R}_e})$.

3. $\succeq$ *is represented by*

$$U(f) = \alpha \min_{m \in \mathcal{M}_{R_e}} \int u(f(s))\, dm + (1-\alpha) \max_{m \in \mathcal{M}_{R_e}} \int u(f(s))\, dm \tag{3.4}$$

where u *and* $\mathcal{M}_{\mathcal{R}_e}$ *are delivered by Lemma 3.1.*

Moreover, $\mathcal{R} \subset \mathcal{R}_e$ *whenever Partial Ignorance outside* $(\mathcal{R}, \succeq_{\mathcal{R}})$ *holds. Given Non-Independence, the weight* α *in utility representation* (3.2) *is independent of the underlying risk profile* $(\mathcal{R}, \succeq_{\mathcal{R}})$.

Theorem 3.4 shows that if there is at least one risk profile $(\mathcal{R}, \succeq_{\mathcal{R}})$ for which Partial Ignorance holds (and generally, there may be many), then Partial Ignorance must hold also for the *endogenous* risk profile $(\mathcal{R}_e, \succeq_{\mathcal{R}_e})$. Moreover, the endogenous risk profile is the largest one for which this axiom holds. In other words, according to the two-step cognitive procedure described by Partial Ignorance, $\mathcal{R}_e$ is the largest subdomain where the decision maker can assign expected utility in the first step.

Theorem 3.4 also establishes that even though the modeler usually has freedom in choosing the risk profile $(\mathcal{R}, \succeq_{\mathcal{R}})$ to obtain an α-maxmin expected utility representation, the weight α in the representation is independent of $(\mathcal{R}, \succeq_{\mathcal{R}})$. Therefore, α is uniquely determined by the preference $\succeq$ and can be interpreted as the decision maker's ambiguity index. However, this interpretation is based only on the α-maxmin expected utility that represents $\succeq$. Whether the index α has any direct behavioral implications is an open question.

4 CONCLUDING REMARKS

We axiomatize a new model of utility that uses a set of priors $\mathcal{M}$ and evaluates each act by a mixture of the most and the least favorable elements in $\mathcal{M}$. The α-maxmin expected utility representation reflects a two-step cognitive procedure: the first step is ranking risky acts via expected utility; the second step is described by the Partial Ignorance axiom. According to Partial Ignorance, the decision maker focuses exclusively on the implications of Monotonicity, Certainty Independence and her initial risk profile $(\mathcal{R}, \succeq_{\mathcal{R}})$ to arrive at her global preference. The set of priors in the resulting α-maxmin expected utility representation is tightly associated with the underlying risk profile. Two examples are: i) $\mathcal{M} = \mathcal{E}(\Gamma, m_0)$ consists of all extensions of probabilities $m_0(\cdot)$ given on a class $\Gamma \subset \Sigma$; (ii) $\mathcal{M} = \mathcal{E}(\Theta, \mu)$ consists of all "extensions" of conditional probabilities $\mu(\cdot)$ given on a class $\Theta \subset \Sigma \times \Sigma$. The modeler may have freedom in choosing the risk profile $(\mathcal{R}, \succeq_{\mathcal{R}})$ to interpret preference $\succeq$ via Partial Ignorance. We show that the largest such $(\mathcal{R}, \succeq_{\mathcal{R}})$ has a simple subjective definition.

Our model fails to accommodate some situations when information is given by inequalities. Consider an Ellsberg urn with two possible colors R and G. Suppose that the decision maker is told only that there are 100 balls in the urn and *at least* 60 of them are red. Unless the decision maker is willing to assign sharp subjective probabilities, she can assign expected utility only to lotteries. Partial Ignorance outside $(\mathcal{L}, \succeq_{\mathcal{L}})$ leads to the complete ignorance α-maxmin representation. This representation ignores the fact that there are strictly more red than green balls in the urn. Roughly, our model fails because the decision maker is not ignorant outside the domain where she assigns expected utility.

A APPENDIX: PROOFS

Extension Lemma

Denote by $B_s = \{a, b, \ldots\}$ the linear space of Σ-measurable functions $b : S \to \mathbb{R}$ having finite range. For every $E \in \Sigma$, denote by I_E the indicator function

$$I_E(s) = \begin{bmatrix} 1 \text{ if } s \in E \\ 0 \text{ if } s \notin E \end{bmatrix}.$$

In particular, $I_S(s) = 1$ for all $s \in S$. For $a, b \in B_s$, write $a \leqq b$ if $a(s) \leq b(s)$ for all $s \in S$. A function V is called *monotonic* if $a \leqq b$ implies $V(a) \leq V(b)$.

Function $V : B_s \to \mathbb{R}$ is linear, monotonic and satisfies $V(I_S) = 1$ if and only if there is a probability measure $m \in \Delta(S, \Sigma)$ such that $V(b) = \int b\, dm$ for all $b \in B_s$. Such m is uniquely defined as $m(E) = V(I_E)$ for all $E \in \Sigma$.

Lemma. *Let B_0 be a linear subspace of B_s such that $I_S \in B_0$. Let $V_0 : B_0 \to R$ be a monotonic linear function satisfying $V_0(I_S) = 1$. Then the set of probability measures*

$$\mathcal{M}_0 = \{m \in \Delta(S, \Sigma) : \int b_0\, dm = V_0(b_0) \textit{ for all } b_0 \in B_0\}$$

is non-empty, convex and closed. Moreover, for all $a \in B_s$,

$$\begin{aligned} \min_{m \in \mathcal{M}_0} \int a\, dm &= \sup_{b_0 \in B_0 : b \leqq a} V_0(b_0) \\ \max_{m \in \mathcal{M}_0} \int a\, dm &= \inf_{b_0 \in B_0 : b_0 \geqq a} V_0(b_0). \end{aligned} \tag{A.1}$$

Proof. Fix $a \notin B_0$. As V_0 is monotonic and $V_0(I_S) = 1$, for all $b_0 \in B_0$,

$$\begin{aligned} b_0 \leqq a &\Rightarrow V_0(b_0) \leq \max_{s \in S} a(s) \\ b_0 \geqq a &\Rightarrow V_0(b_0) \geq \min_{s \in S} a(s). \end{aligned}$$

Therefore, the values $\underline{V}(a) = \sup_{b_0 \in B_0 : b \leqq a} V_0(b_0)$ and $\overline{V}(a) = \inf_{b_0 \in B_0 : b \geqq a} V_0(b_0)$ are well-defined and $\underline{V}(a) \leq \overline{V}(a)$.

Denote by B_a the linear space of functions $\{b_0 + \gamma a \ : b_0 \in B_0, \gamma \in \mathcal{R}\}$. Fix an arbitrary value $v_a \in [\underline{V}(a), \overline{V}(a)]$ and define V_a on B_a as

$$V_a(b_0 + \gamma a) = V_0(b_0) + \gamma v_a$$

for all b_0 and γ. Such V_a is linear. The following argument shows that V_a is also monotonic.

Suppose that $b_0 + \gamma a \geqq b'_0 + \gamma' a$ for some $b_0, b'_0 \in B_0$ and $\gamma, \gamma' \in \mathbb{R}$. There are three possible cases:

- $\gamma = \gamma'$, implying $b_0 \geqq b'_0$. Then $V_0(b_0) \geq V_0(b'_0)$ and $V_0(b_0) + \gamma v_a \geq V_0(b'_0) + \gamma' v_a$.
- $\gamma > \gamma'$, implying $a \geqq \frac{b'_0 - b_0}{\gamma - \gamma'} \in B_0$.
  Then $v_a \geq \underline{V}(a) \geq V_0\left(\frac{b'_0 - b_0}{\gamma - \gamma'}\right)$ and $V_0(b_0) + \gamma v_a \geq V_0(b'_0) + \gamma' v_a$.
- $\gamma < \gamma'$, implying $a \leqq \frac{b_0 - b'_0}{\gamma' - \gamma} \in B_0$.
  Then $v_a \leq \overline{V}(a) \leq V_0\left(\frac{b_0 - b'_0}{\gamma' - \gamma}\right)$ and $V_0(b_0) + \gamma v_a \geq V_0(b'_0) + \gamma' v_a$.

Thus, $V_a(b_0 + \gamma a) \geq V_a(b'_0 + \gamma' a)$ and V_a is monotonic on B_a.

By Zorn's Lemma, there exists a linear monotonic extension V of V_0 to a maximal linear subspace in B_s. We have shown that this maximal subspace cannot be a proper subset of B_s. Hence, V is an extension to the entire B_s. As $V(I_S) = V_0(I_S) = 1$, there exists $m \in \Delta(S, \Sigma)$ such that $V(b) = \int b\, dm$ for all $b \in B_s$. Therefore, $\mathcal{M}_0$ is not empty. Convexity of $\mathcal{M}_0$ is obvious. Finally $\mathcal{M}_0$ is closed as an intersection of closed sets of probability measures: $\mathcal{M}_0 = \cap_{b_0 \in B_0} \{m \in \Delta(S, \Sigma) : \int b_0\, dm = V_0(b_0)\}$.

To show equalities (A.1), note that for any $a \in B_s$, v_a can be taken equal to $\underline{V}(a)$ and V taken to be an extension of V_a so that $V(a) = \underline{V}(a)$. On the other hand, for all $m \in \mathcal{M}_0$ and $b_0 \in B_0$ such that $b_0 \leqq a$, $\int a\, dm \geq \int b_0\, dm = V_0(b_0)$. Thus, $\min_{m \in \mathcal{M}_0} \int a\, dm \geq \underline{V}(a)$. The second equality is proven analogously. □

A.1 Proof of Lemma 3.1

The Anscombe–Aumann Theorem implies the necessity part in the lemma.

Suppose that $\mathcal{R} \subset \mathcal{H}$ satisfies conditions $R1$–3 and that that $\succeq_{\mathcal{R}}$ satisfies Weak Order, Non-Degeneracy, Monotonicity, Mixture Continuity and Independence. Denote by $u : \mathcal{L} \to R$ a non-constant, affine utility function representing $\succeq_{\mathcal{L}}$ (by the von Neumann - Morgenstern Theorem, such u exists and is unique up to a positive linear transformation). Without loss of generality, u is fixed throughout so that its range contains the interval $[-1, 1]$. Also, fix a lottery l_0 such that $u(l_0) = 0$.

Denote by $e_{\mathcal{R}} : \mathcal{R} \to \mathcal{L}$ a function attaching a *certainty equivalent* $e_{\mathcal{R}}(r) \sim r$ to every $r \in \mathcal{R}$ (Weak Order, Monotonicity and Mixture Continuity are sufficient for existence of certainty equivalents). Then $\succeq_{\mathcal{R}}$ is represented by

$$U_{\mathcal{R}}(r) = u(e_{\mathcal{R}}(r).)$$

Affinity of u and Independence imply affinity of $U_{\mathcal{R}}$.

Denote by $B_{\mathcal{R}}$ the set of $u-$images of elements of $\mathcal{R}$:

$$B_{\mathcal{R}} = \{b \in B_s : \; b = u \circ r_b, \; r_b \in \mathcal{R}\},$$

and by $L(B_{\mathcal{R}})$ the linear hull of $B_{\mathcal{R}}$ in B_s:

$$L(B_{\mathcal{R}}) = \{b \in B_s : b = \sum_{i=1}^{n} \gamma_i b_i, \quad i = 1 \ldots n, \; \gamma_i \in \mathbb{R}, \; b_i \in B_{\mathcal{R}}\}.$$

The utility function $U_{\mathcal{R}}$ extends to a unique linear function $V_{\mathcal{R}}$ on $L(B_{\mathcal{R}})$.

Lemma A.1. *There exists a unique linear function $V_{\mathcal{R}}$ on $L(B_{\mathcal{R}})$ such that $V_{\mathcal{R}}(u \circ r) = U_{\mathcal{R}}(r)$ for all $r \in \mathcal{R}$. Moreover, $V_{\mathcal{R}}$ is monotonic and satisfies $V_{\mathcal{R}}(I_S) = 1$.*

Proof. Fix $b \in L(B_{\mathcal{R}})$ and fix corresponding $\gamma_i \in \mathbb{R}$, $b_i \in B_{\mathcal{R}}$ and $r_i \in \mathcal{R}$ such that $b = \sum_{i=1}^{n} \gamma_i b_i$ and $b_i = u \circ r_i$ for all $i = 1 \ldots n$.

Suppose that $b \geqq 0$. Then $\sum_{i=1}^{n} \gamma_i b_i \geqq 0$ can be rewritten as

$$\sum_{i=1}^{n} \gamma_i^+ b_i \geqq \sum_{i=1}^{n} \gamma_i^- b_i,$$

where $\gamma_i^+ = \max(\gamma_i, 0)$ and $\gamma_i^- = \max(-\gamma_i, 0)$. Choose $\gamma > 0$ such that $|\gamma\gamma_i| \leq 1$ for all i. Then

$$\sum_{i=1}^{n} \frac{1}{n}(\gamma\gamma_i^+ b_i) \geqq \sum_{i=1}^{n} \frac{1}{n}(\gamma\gamma_i^- b_i).$$

Let $r_i^+ = \gamma\gamma_i^+ r_i + (1 - \gamma\gamma_i^+) l_0$ and $r_i^- = \gamma\gamma_i^- r_i + (1 - \gamma\gamma_i^-) l_0$ for all i. Let $r^+ = \frac{1}{n} r_i^+$ and $r^- = \frac{1}{n} r_i^-$. Condition $R2$ implies that $r_i^+, r_i^-, r^+, r^- \in \mathcal{R}$. Then $u(r^+(s)) \geq u(r^-(s))$ for all s implying $r^+ \geqq r^-$, $r^+ \succeq_{\mathcal{R}} r^-$ and $U_{\mathcal{R}}(r^+) \geq U_{\mathcal{R}}(r^-)$. By affinity of $U_{\mathcal{R}}$,

$$\sum_{i=1}^{n} \frac{1}{n} \gamma\gamma_i^+ U_{\mathcal{R}}(r_i) \geq \sum_{i=1}^{n} \frac{1}{n} \gamma\gamma_i^- U_{\mathcal{R}}(r_i).$$

Note that $\gamma_i = \gamma_i^+ - \gamma_i^-$ for all i. Therefore, $\sum_{i=1}^{n} \gamma_i U_{\mathcal{R}}(r_i) \geq 0$.

Define

$$V_{\mathcal{R}}(b) = \sum_{i=1}^{n} \gamma_i U_{\mathcal{R}}(r_i), \tag{A.2}$$

Such $V_{\mathcal{R}}$ is well-defined. To see this, suppose that b is represented as another linear combination $b = \sum_{i=1}^{n'} \gamma_i' b_i'$, where $b_i' = u \circ r_i'$ for $r_i' \in \mathcal{R}$. Then

$$\sum_{i=1}^{n} \gamma_i b_i - \sum_{i=1}^{n'} \gamma_i' b_i' \geqq (\leqq) 0$$

implies that $\sum_{i=1}^{n} \gamma_i U_{\mathcal{R}}(r_i) - \sum_{i=1}^{n'} \gamma_i' U(r_i') \geq (\leq) 0$. Therefore,

$$\sum_{i=1}^{n} \gamma_i b_i = \sum_{i=1}^{n'} \gamma_i' b_i'.$$

Next, $V_{\mathcal{R}}$ is linear, monotonic, and $V_{\mathcal{R}}(I_S) = U_{\mathcal{R}}(l) = 1$, where $l \in \mathcal{L}$ is such that $u(l) = 1$. Finally, definition of $V_{\mathcal{R}}$ implies that $V_{\mathcal{R}}$ is the unique possible linear function on $L(B_{\mathcal{R}})$ that extends $U_{\mathcal{R}}$ from $B_{\mathcal{R}}$. □

Ranking $\succeq_{\mathcal{R}}$ is represented by $\int u \circ r \, dm$ if and only if $\int u \circ r \, dm = U_{\mathcal{R}}(r)$ for all $r \in \mathcal{R}$ (because $U_{\mathcal{R}}$ is the only utility representation of $\succeq_{\mathcal{R}}$ that agrees with u). This is if and only if $\int b \, dm = V_{\mathcal{R}}(b)$ for all $b \in L(B_{\mathcal{R}})$ because $\int b \, dm$ is linear and, by Lemma A.1, $V_{\mathcal{R}}$ is the unique linear extension of $U_{\mathcal{R}}$ to $L(B_{\mathcal{R}})$. By the Extension Lemma, measures m that satisfy $\int b \, dm = V_{\mathcal{R}}(b)$ for all $b \in L(B_{\mathcal{R}})$ form a non-empty, closed and convex set $\mathcal{M}_{\mathcal{R}} \subset \Delta(S, \Sigma)$.

A.2 Proofs of Theorems 3.2 and 3.3

Let $(\mathcal{R}, \succeq_{\mathcal{R}})$ be a risk profile. Define functions $\underline{U}$ and $\overline{U}$ on $\mathcal{H}$ as

$$\underline{U}(f) = \sup_{b \in L(B_{\mathcal{R}}): b \leqq u \circ f} V_{\mathcal{R}}(b)$$
$$\overline{U}(f) = \inf_{b \in L(B_{\mathcal{R}}): b \geqq u \circ f} V_{\mathcal{R}}(b).$$

The Extension Lemma asserts that for all $f \in \mathcal{H}$,

$$\begin{aligned} \underline{U}(f) &= \min_{m \in \mathcal{M}_{\mathcal{R}}} \int (u \circ f)\ dm \\ \overline{U}(f) &= \max_{m \in \mathcal{M}_{\mathcal{R}}} \int (u \circ f)\ dm. \end{aligned} \tag{A.3}$$

To prove α-maxmin expected utility representation (3.2), we show that $\succeq$ is represented

$$U(f) = \alpha \underline{U}(f) + (1-\alpha)\overline{U}(f).$$

Define functions $\underline{l} : \mathcal{H} \to \mathcal{L}$ and $\overline{l} : \mathcal{H} \to \mathcal{L}$ such that $\underline{U}(f) = u(\underline{l}(f))$ and $\overline{U}(f) = u(\overline{l}(f))$ for all $f \in \mathcal{H}$. For example, $\underline{l}(f)$ and $\overline{l}(f)$ can be taken to be mixtures of the worst and best lotteries l_{min} and l_{max}in the range of f (obviously, $u(l_{min}) \leq \underline{U}(f) \leq \overline{U}(f) \leq u(l_{max})$).

The functions $\underline{U}$ and $\overline{U}$ satisfy the following properties.

Property 1. For all $r \in \mathcal{R}$, $\underline{U}(r) = \overline{U}(r) = U_{\mathcal{R}}(r)$ because $\int u \circ r\, dm = U_{\mathcal{R}}(r)$ for all $m \in \mathcal{M}_{\mathcal{R}}$. In particular, for all $l \in \mathcal{L}$, $\underline{U}(l) = \overline{U}(l) = u(l)$.

Property 2. For all $f, g \in \mathcal{H}$ and $\tau \in [0,1]$,

$$\begin{aligned} \tau\underline{U}(f) + (1-\tau)\underline{U}(g) \leq \underline{U}(\tau f + (1-\tau)g) \leq \tau\underline{U}(f) + (1-\tau)\overline{U}(g) \leq \\ \overline{U}(\tau f + (1-\tau)g) \leq \tau\overline{U}(f) + (1-\tau)\overline{U}(g). \end{aligned} \tag{A.4}$$

For example, the first inequality follows from

$$\min_{m \in \mathcal{M}_{\mathcal{R}}} \int \big(\tau(u \circ f) + (1-\tau)(u \circ g)\big)\ dm \geq \tau \min_{m \in \mathcal{M}_{\mathcal{R}}} \int (u \circ f)\, dm + (1-\tau) \min_{m \in \mathcal{M}_{\mathcal{R}}} \int (u \circ g)\, dm.$$

In particular, for all $f \in \mathcal{H}$, $r \in \mathcal{R}$ and $\tau \in [0,1]$,

$$\begin{aligned} \underline{U}(\tau f + (1-\tau)r) &= \tau\underline{U}(f) + (1-\tau)\underline{U}(r) = \tau\underline{U}(f) + (1-\tau)U(r) \\ \overline{U}(\tau f + (1-\tau)r) &= \tau\overline{U}(f) + (1-\tau)\overline{U}(r) = \tau\overline{U}(f) + (1-\tau)U(r). \end{aligned} \tag{A.5}$$

Property 3. For all $f, g \in \mathcal{H}$, the functions $\underline{U}(\tau f + (1-\tau)g)$ and $\overline{U}(\tau f + (1-\tau)g)$ are continuous in the weight $\tau \in [0,1]$.

Property 4. $U(f)=\alpha\underline{U}(f)+(1-\alpha)\overline{U}(f)$ represents a regular preference.

The following technical lemma employs condition $R3$ that we impose on $\mathcal{R}$.

Lemma A.2.

1. *For every $b\in B_{\mathcal{R}}$ there exist $b'\in B_{\mathcal{R}}$ and $\tau\in(0,1)$ such that $\tau b+(1-\tau)b'=0$.*

2. *For every $b\in L(B_{\mathcal{R}})$ there exists $\varepsilon\in(0,1)$ such that $\varepsilon b\in B_{\mathcal{R}}$.*

Proof. Fix $b\in B_{\mathcal{R}}$ together with $r_b\in\mathcal{R}$ such that $b=u\circ r_b$. Act r_b can be written as $r_b=[l_i \text{ if } s\in E_i]_{i=1}^n$, where $\{E_1,\dots,E_n\}$ is a partition of the state space S. Let $r'=[\sum_{j\neq i}\frac{1}{n-1}l_j \text{ if } s\in E_i]_{i=1}^n$. Then

$$\frac{1}{n}r_b+\frac{n-1}{n}r'=\sum_i\frac{1}{n}l_i.$$

By condition $R3$, $r'\in\mathcal{R}$. As the range $u(\mathcal{L})$ is taken to contain the interval $[-1,1]$, there exist $l\in\mathcal{L}$ and $\gamma\in(0,1)$ such that $\gamma u\left(\sum_{i=1}^n\frac{1}{n}l_i\right)+(1-\gamma)u(l)=0$. It follows that $\tau b+(1-\tau)b'=0$, where $\tau=\frac{\gamma}{n}$ and $b'=u\circ\left(\frac{\gamma(n-1)}{n-\gamma}r'+\frac{n(1-\gamma)}{n-\gamma}l\right)$ belongs to $B_{\mathcal{R}}$.

Next, fix $b\in L(B_{\mathcal{R}})$ and fix $\gamma_i\in R$ and $b_i\in B_{\mathcal{R}}$ such that $b=\sum_{i=1}^n\gamma_i b_i$. Without loss of generality γ_i's can be assumed non-negative for all i: if $\gamma_i<0$, then the member $\gamma_i b_i$ can be replaced by $-\gamma_i\frac{1-\tau}{\tau}b_i'$, where $b_i'\in B_{\mathcal{R}}$ and $\tau\in(0,1)$ satisfy $\tau b_i+(1-\tau)b_i'=0$. Choose $\varepsilon>0$ small enough so that $\varepsilon n\gamma_i<1$ for all i. Then $\varepsilon b=\sum_{i=1}^n\frac{1}{n}(\varepsilon n\gamma_i b_i)$. Thus,

$$\varepsilon b=u\circ\left(\sum_{i=1}^n(\varepsilon n\gamma_i r_{b_i}+(1-\varepsilon n\gamma_i)l_0)\right)$$

implying $\varepsilon b\in B_{\mathcal{R}}$. □

The following lemma characterizes "locating observations" in terms of $\underline{U}$ and $\overline{U}$.

Lemma A.3.

1. *$(\mathcal{R},\succeq_{\mathcal{R}})$ locates l_* below f if and only if $u(l_*)<\underline{U}(f)$.*

2. *$(\mathcal{R},\succeq_{\mathcal{R}})$ locates l^* above f if and only if $u(l^*)>\overline{U}(f)$.*

Proof. Suppose that $(\mathcal{R},\succeq_{\mathcal{R}})$ locates l_* below f, that is

$$\tau f+(1-\tau)l\geqq r_*\succ_{\mathcal{R}}\tau l_*+(1-\tau)l$$

for some $\tau\in(0,1]$, $l\in\mathcal{L}$ and $r_*\in\mathcal{R}$. Ranking $r_*\succ_{\mathcal{R}}\tau l_*+(1-\tau)l$ implies

$$U_{\mathcal{R}}(r_*)>U_{\mathcal{R}}(\tau l_*+(1-\tau)l)$$
$$V_{\mathcal{R}}(u\circ r_*)>V_{\mathcal{R}}(u\circ(\tau l_*+(1-\tau)l)$$
$$V_{\mathcal{R}}\left(\frac{1}{\tau}((u\circ r_*)-(1-\tau)(u\circ l))\right)>u(l_*).$$

Then the weak dominance $\tau f + (1-\tau)r_*$ implies

$$\begin{aligned} u \circ (\tau f + (1-\tau)l) &\geqq u \circ r_* \\ u \circ f &\geqq \frac{1}{\tau}\big((u \circ r_*) - (1-\tau)(u \circ l)\big) = b \in L(B_{\mathcal{R}}), \\ \underline{U}(f) &\geq V_{\mathcal{R}}(b) = V_{\mathcal{R}}\left(\frac{1}{\tau}((u \circ r_*) - (1-\tau)(u \circ l))\right) > u(l_*). \end{aligned}$$

Conversely, suppose that $u(l_*) < \underline{U}(f)$, that is $u(l_*) = V_{\mathcal{R}}(u \circ l_*) < V_{\mathcal{R}}(b)$ for some $b \in L(B_{\mathcal{R}})$ satisfying $b \leqq u \circ f$. Choose $\tau \in (0,1)$ so that $\tau b \in B_{\mathcal{R}}$ and take $r_{\tau b} \in \mathcal{R}$ such that $\tau b = u \circ r_{\tau b}$. Note that $b \leqq u \circ f$ implies $\tau b \leqq u \circ (\tau f + (1-\tau)l_0)$ and $r_{\tau b} \leqq \tau f + (1-\tau)l_0$. Therefore,

$$\begin{aligned} V_{\mathcal{R}}(u \circ (\tau l_* + (1-\tau)l_0) &< V_{\mathcal{R}}(\tau b) \\ U_{\mathcal{R}}(\tau l_* + (1-\tau)l_0) &< U_{\mathcal{R}}(r_{\tau b}) \\ \tau l_* + (1-\tau)l_0 &\prec_{\mathcal{R}} r_{\tau b} \leqq \tau f + (1-\tau)l_0. \end{aligned}$$

Thus, $(\mathcal{R}, \succeq_{\mathcal{R}})$ locates l_* below f.

The second statement of the lemma is proven analogously. □

The lemma implies that the risk profile $(\mathcal{R}, \succeq_{\mathcal{R}})$ is uninformative for the comparison of acts f and if and only if both $\underline{U}(f) = \underline{U}(g)$ and $\overline{U}(f) = \overline{U}(g)$. In particular, if $\succeq$ is represented by $U(f) = \alpha\underline{U}(f) + (1-\alpha)\overline{U}(f)$, then $\succeq$ is regular and satisfies Partial Ignorance outside $(\mathcal{R}, \succeq_{\mathcal{R}})$.

Conversely, suppose that $\succeq$ is regular and satisfies Partial Ignorance outside $(\mathcal{R}, \succeq_{\mathcal{R}})$. Let $e : \mathcal{H} \to \mathcal{L}$ attach a certainty equivalent $e(f) \sim f$ to every $f \in \mathcal{H}$. Let $U(f) = u(e(f))$ for all $f \in \mathcal{H}$. Then U represents $\succeq$, and for all $f \in \mathcal{H}$, $l \in \mathcal{L}$ and $\tau \in [0,1]$,

$$U(\tau f + (1-\tau)l) = u(e(\tau f + (1-\tau)l)) = u(\tau e(f) + (1-\tau)l) = \tau U(f) + (1-\tau)u(l).$$

For all $f \in \mathcal{H}$,

$$\underline{U}(f) \leq U(f) \leq \overline{U}(f).$$

To prove these bounding inequalities, suppose that $U(f) < \underline{U}(f)$. Then $(\mathcal{R}, \succeq_{\mathcal{R}})$ locates $e(f)$ below f, that is

$$\tau f + (1-\tau)l \geqq r_* \succ_{\mathcal{R}} \tau e(f) + (1-\tau)l$$

for some $\tau \in (0,1]$, $l \in \mathcal{L}$ and $r_* \in \mathcal{R}$. Monotonicity and Certainty Independence imply $f \succ e(f)$ which contradicts $f \sim e(f)$.

To complete the proof of Theorem 3.2, consider two cases. First, suppose that for all $f \in \mathcal{H}$, $\underline{U}(f) = \overline{U}(f)$. Then $\underline{U}(f) = U(f) = \overline{U}(f)$ and $U(f) = \alpha\underline{U}(f) + (1-\alpha)\overline{U}(f)$ for any $\alpha \in [0,1]$. In this case, $\succeq$ satisfies Independence because U is affine. For all $f, g \in \mathcal{H}$ and $\tau \in [0,1]$, inequalities (A.4) imply

$$\tau U(f) + (1-\tau)U(g) = \tau\underline{U}(f) + (1-\tau)\underline{U}(g) \leq \underline{U}(\tau f + (1-\tau)g) \leq \\ \tau\underline{U}(f) + (1-\tau)\overline{U}(g) = \tau U(f) + (1-\tau)U(g),$$

and $U(\tau f+(1-\tau)g)=\underline{U}(\tau f+(1-\tau)g)=\tau U(f)+(1-\tau)U(g)$. Therefore, Non-Independence rules out this case.

Given Non-Independence, fix $f\in\mathcal{H}$ such that $\underline{U}(f)<\overline{U}(f)$. Define

$$\alpha=\frac{\overline{U}(f)-U(f)}{\overline{U}(f)-\underline{U}(f)}.$$

Fix an arbitrary $g\in\mathcal{H}$. Choose $\tau\in[0,1)$ so that $\tau\overline{U}(f)+(1-\tau)\underline{U}(g)=\tau\underline{U}(f)+(1-\tau)\overline{U}(g)$. Such τ is uniquely determined by $\frac{\tau}{1-\tau}=\frac{\overline{U}(g)-\underline{U}(g)}{\overline{U}(f)-\underline{U}(f)}$. Let $h=\tau f+(1-\tau)\underline{l}(g)$ and $h'=\tau\underline{l}(f)+(1-\tau)g$, where $u(\underline{l}(f))=\underline{U}(f)$ and $u(\overline{l}(f))=\overline{U}(f)$. Then $(\mathcal{R},\succeq_{\mathcal{R}})$ is uninformative for the comparison of h and h' because

$$\underline{U}(h)=\tau\underline{U}(f)+(1-\tau)\underline{U}(g)=\underline{U}(h')$$
$$\overline{U}(h)=\tau\overline{U}(f)+(1-\tau)\underline{U}(g)=\tau\underline{U}(f)+(1-\tau)\overline{U}(g)=\overline{U}(h').$$

Partial Ignorance outside $(\mathcal{R},\succeq_{\mathcal{R}})$ implies that $h\sim h'$ and consequently, $U(h)=U(h')$. It follows that

$$\tau U(f)+(1-\tau)\underline{U}(g)=\tau\underline{U}(f)+(1-\tau)U(g)$$
$$U(g)=\frac{\tau}{1-\tau}(U(f)-\underline{U}(f))+\underline{U}(g)=(1-\alpha)(\overline{U}(g)-\underline{U}(g))+\underline{U}(g)$$
$$U(g)=\alpha\underline{U}(g)+(1-\alpha)\overline{U}(g).$$

In the resulting α-maxmin expected utility representation, α is uniquely determined by the equality $U(f)=\alpha\underline{U}(f)+(1-\alpha)\overline{U}(f)$.

The proof of Theorem 3.2 is complete.

When Pessimism outside $(\mathcal{R},\succeq_{\mathcal{R}})$ holds, like in Theorem 3.3, Lemma A.3 implies that $f\sim g$ whenever $\underline{U}(f)\sim\underline{U}(g)$. It follows that

$$f\succeq g\Leftrightarrow e(f)\succeq e(g)\Leftrightarrow\underline{U}(e(f))\geq\underline{U}(e(g))\Leftrightarrow\underline{U}(f)\geq\underline{U}(g)$$

that is $\underline{U}$ represents $\succeq$.

A.3 Proof of Theorem 3.4

First, we prove that if $\succeq$ is regular, than $(\mathcal{R}_e,\succeq_{\mathcal{R}_e})$ is a risk profile. $\mathcal{R}_e$ satisfies $R1$ because Certainty Independence implies that $\mathcal{L}\subset\mathcal{R}_e$. Second, suppose that $r\in\mathcal{R}_e$ and $r'\in\mathcal{R}_e$. Fix arbitrary $\tau,\xi\in(0,1)$. Let $\beta=1-(1-\xi)\tau$ and $\gamma=\frac{\xi}{1-(1-\xi)\tau}=\frac{1-(1-\xi)}{1-(1-\xi)\tau}$. Both β and γ belong to $(0,1)$. Then

$$f\succeq g\Leftrightarrow\gamma f+(1-\gamma)r'\succeq\gamma g+(1-\gamma)r'\Leftrightarrow$$
$$\beta(\gamma f+(1-\gamma)r')+(1-\beta)r\succeq\beta(\gamma g+(1-\gamma)r')+(1-\beta)r\Leftrightarrow$$
$$\beta\gamma f+(1-\beta\gamma)(\frac{1-\beta}{1-\beta\gamma}r+\frac{\beta-\beta\gamma}{1-\beta\gamma}r')\succeq\beta\gamma g+(1-\beta\gamma)(\frac{1-\beta}{1-\beta\gamma}r+\frac{\beta-\beta\gamma}{1-\beta\gamma}r')\Leftrightarrow$$
$$\xi f+(1-\xi)(\tau r+(1-\tau)r')\succeq\xi g+(1-\xi)(\tau r+(1-\tau)r').$$

It follows that $\mathcal{R}_e$ satisfies $R2$.

Suppose that $\tau r + (1-\tau)r' \in \mathcal{R}_e$, where $r \in \mathcal{R}_e$, $r' \in \mathcal{H}$ and $\tau \in (0,1)$. Fix arbitrary $\gamma \in (0,1)$. Let $\beta = \frac{1-\tau}{1-\tau\gamma}$ and $\xi = \frac{(1-\tau)\gamma}{1-\tau\gamma}$. Then

$$f \succeq g \Leftrightarrow \xi f + (1-\xi)(\tau r + (1-\tau)r') \succeq \xi g + (1-\xi)(\tau r + (1-\tau)r') \Leftrightarrow$$
$$\beta(\gamma f + (1-\gamma)r') + (1-\beta)r \succeq \beta(\gamma g + (1-\gamma)r') + (1-\beta)r \Leftrightarrow$$
$$\gamma f + (1-\gamma)r' \succeq \gamma g + (1-\gamma)r'.$$

Thus, $r' \in \mathcal{R}_e$ and $\mathcal{R}_e$ satisfies $R3$.

The preference $\succeq_{\mathcal{R}}$ inherits Weak Order, Mixture Continuity and Monotonicity from $\succeq$, and satisfies Independence by definition of endogenously risky acts.

Next, we prove that if Partial Ignorance outside $(\mathcal{R}, \succeq_{\mathcal{R}})$ holds, then $\mathcal{R} \subset \mathcal{R}_e$. Utility representation (3.2) and equation (A.5) imply for all $f \in \mathcal{H}$,$r \in \mathcal{R}$ and $\tau \in (0,1]$,

$$U(\tau f + (1-\tau)r) = \alpha \underline{U}(\tau f + (1-\tau)r) + (1-\alpha)\overline{U}(\tau f + (1-\tau)r) =$$
$$\tau(\alpha \underline{U}(f) + (1-\alpha)\overline{U}(f)) + (1-\tau)U(r) = \tau U(f) + (1-\tau)U(r).$$

If Partial Ignorance holds for the smaller profile $(\mathcal{R}, \succeq_{\mathcal{R}})$, then it also holds for the larger profile $(\mathcal{R}_e, \succeq_{\mathcal{R}_e})$. The equivalence of all three statements in Theorem 3.4 follows.

Finally, we prove that given Non-Independence, α in representations (3.2) is independent of $(\mathcal{R}, \succeq_{\mathcal{R}})$. Fix $(\mathcal{R}, \succeq_{\mathcal{R}})$ such that Partial Ignorance outside $(\mathcal{R}, \succeq_{\mathcal{R}})$ holds and let α be the unique ambiguity index in corresponding representation (3.2). Let α_e be the unique ambiguity index in representation (3.4) based on the endogenous risk profile $(\mathcal{R}_e, \succeq_{\mathcal{R}_e})$. We prove that $\alpha = \alpha_e$: it will follow that α is independent of $(\mathcal{R}, \succeq_{\mathcal{R}})$.

Suppose first that $\alpha \neq \frac{1}{2}$. Then $r \in \mathcal{R}_e$ if and only if $\underline{U}(r) = \overline{U}(r)$. Therefore $\int r\, dm$ represents $\succeq_{\mathcal{R}}$ if and only if it represents $\succeq_{\mathcal{R}_e}$. It follows that $\mathcal{M}_e = \mathcal{M}_{\mathcal{R}_e}$ and $\alpha = \alpha_e$.

Suppose that $\alpha = \frac{1}{2}$. Fix arbitrary $f \in \mathcal{H}$ and choose $f' \in \mathcal{H}$ and $\tau \in (0,1)$ so that $u \circ (\tau f + (1-\tau)f') = 0$. Then $a = \frac{1}{2}$ implies $\tau U(f) + (1-\tau)U(f') = 0$. It follows that $\alpha_e = \frac{1}{2}$ as well because otherwise $\tau U(f) + (1-\tau)U(f') \neq 0$ whenever $\underline{U}(f) < \overline{U}(f)$.

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