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The Local Recoverability of Risk Aversion and Intertemporal Substitution*

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Call an economy revealing if underlying preferences are recoverable from asset prices, specifically from the prices of aggregate equity or a one-period discount bond. In a Lucas [*Econometrica* **46** (1978), 1429–1444] model, with Kreps–Porteus [*Econometrica* **46** (1978), 185–200] nonexpected utility and Markov output growth rate process, local recoverability is shown to be a generic property. In this sense it is generically true that asset pricing models based on the more general Kreps–Porteus utility have more explanatory power than the usual expected utility, and both the risk aversion and intertemporal aspects of the Kreps–Porteus utility can be recovered from a single dynamic equilibrium. *Journal of Economic Literature* Classification Numbers: D11, G12, C60, D50, D80.

1. INTRODUCTION

Spurred by laboratory evidence contradicting expected utility theory, a number of generalizations have been studied. In addition, some attention has been devoted to the question of whether these generalized theories are useful in addressing standard problems in economics and in explaining market, as opposed to laboratory, data. In particular, Epstein and Zin [7, 8] demonstrate the theoretical and empirical gains from generalizing the standard intertemporally additive, expected utility model of preferences for the study of consumption and asset pricing. They formulate and use the following nonexpected utility function, which is based on earlier work by Kreps and Porteus [12],

$$U_t = [C_t^\rho + \beta(E_t U_{t+1}^\alpha)^{\rho/\alpha}]^{1/\rho}, \qquad t \geq 0, \tag{1.1}$$

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where C_t is the consumption, α a risk aversion parameter, β the discount factor, and $\sigma \equiv 1/(1-\rho)$ the elasticity of intertemporal substitution. It includes the following standard homothetic expected utility function as a special case when $\alpha = \rho$:

$$U_t^\alpha = E_t \sum_{s=0}^{\infty} \beta^s C_{t+s}^\alpha. \tag{1.2}$$

In (1.2), α represents both the risk aversion and the elasticity of intertemporal substitution. An advantage of the generalization of (1.2) with $\alpha = \rho$ to (1.1) with $\alpha \neq \rho$ is the *separation of the risk aversion parameter from the elasticity of intertemporal substitution.* Such separation would seem to be desirable since attitudes towards risk and towards intertemporal variations are conceptually distinct.

One important question concerning the Kreps–Porteus utility function is whether one could recover both σ and α from a *single* dynamic equilibrium. In particular, could an observer of equilibrium consumption and asset prices in a *given* economy distinguish between the intertemporal expected utility case ($\alpha = \rho$) and the more general case ($\alpha \neq \rho$)? If not, then the more general model does not provide any additional power for explaining time series from a single economy. Kocherlakota [11] has shown that if consumption growth rates are i.i.d., then indeed none of the Kreps–Porteus parameters α, β, and ρ is uniquely determined and the Kreps–Porteus model is observationally equivalent to the standard model. The intuition underlying this result is clear. Asset prices at time t reflect marginal rates of substitution at the conditional consumption program faced by the agent at t. If that program does not vary sufficiently across states, as in the i.i.d. case, then marginal rates of substitution will be delivered only on a limited domain. Indeed, the i.i.d. case is analogous to the situation in demand theory where a single price/quantity data point cannot be used to pin down the underlying utility function. This intuition suggests that Kocherlakota's observational equivalence result should apply only to a "small" set of endowment processes.

The objective of this paper is to confirm the above intuition by conducting a more general analysis of the recoverability question. The framework adopted is a Lucas [14] general equilibrium endowment model, modified so that the representative agent has Kreps–Porteus utility and specialized so that the output growth rate follows a first-order Markov process with a fixed and finite number of states. The assets considered are aggregate equity and a one-period discount bond. I show that generically in the space of growth rate processes, *the asset prices are locally determinate (unique) and the underlying preferences are locally recoverable*, where a property is *generically* true if it fails only on a closed zero-measure subset of economies. In this sense, for most economies, both risk aversion and intertemporal aspects of the Kreps–Porteus utility function can be recovered from a single dynamic equilibrium.

Prior to considering recoverability, I examine the determinacy of equilibrium asset prices. In an economy with complete markets and finitely many consumers and commodities, Debreu [2] shows that the economy is generically determinate. For the present model, Epstein [6] shows that

$$P_e^*(h_0) = \rho\beta Y_0^{1-\rho}\{E[U^\alpha(Y_1, Y_2, ...) \mid h_0]\}^{\rho/\alpha},$$

where $\{Y_t\}_{t=0}^{\infty}$ is the output process, h_0 the history, and P_e^* the equity price. This implies that if the utility solution U from (1.1) is unique then so is the equity price. Thus the potential cause of nonuniqueness of asset prices in our model is that the recursive relation (1.1) may have nonunique solutions (Epstein and Zin [7, pp. 963] offer a uniqueness result when $\alpha\rho > 0$). This gives rise to the problem of determinacy of equilibria. In other words, determinacy is an issue because I take the recursive relation (1.1) as given rather than a utility function. For this reason, the problem of determinacy is closely related to the analysis in Lucas and Stokey [15], in which a unique utility solution to the general recursive equation in a deterministic world is proven.

With complete markets, Mas-Colell [17] shows that if the economic analyst can observe an excess demand function $Z^*(P)$, $\forall P$, then the agent's function can be uniquely recovered. Roughly speaking, I assume the observability of the inverse demand function. The main difficulty here is that observability is assumed only for a single endowment process, and thus recoverability will only be possible if the conditional distribution function describing future growth rates varies sufficiently across states. For example, one nonrevealing case is the i.i.d. growth rate model in Kocherlakota [11].

This paper proceeds as follows: Section 2 describes the model and the principal results, Theorems 1–3, on determinacy and recoverability. Section 3 mentions some extensions. Proofs are collected in appendices.

2. Description of the Model and the Results

This section consists of three subsections. I will first describe briefly the definition of utility, then give the definitions of environment, economic model, and equilibrium, and then give the definition of a revealing economy. Finally, the main results of this paper, Theorems 1–3, are presented and discussed.

2.1. *The Utility Function*[1]

Consider an infinitely lived individual who gains utility from the consumption of a single good in each period. The formulation of the individual's intertemporal utility is based on two key assumptions. The first assumption is that,[2] in period t, with random utility U_{t+1} from the period $t+1$ onward, the individual computes a certainty equivalent μ_t of the random future utility

$$\mu_t \equiv (E_t U_{t+1}^{\alpha})^{1/\alpha}, \qquad 0 \neq \alpha < 1,$$

where α is the risk aversion parameter. Second, the individual is assumed to combine the certainty equivalent μ_t with current consumption C_t via an aggregator W of CES form

$$W(c, z) \equiv (c^{\rho} + \beta z^{\rho})^{1/\rho}, \qquad c, z \geqslant 0,\ 0 \neq \rho < 1,\ 0 < \beta < 1,$$

where ρ is the elasticity parameter and β the discount factor. Thus the intertemporal utility U is defined recursively by the following:

$$U_t = [C_t^{\rho} + \beta(E_t U_{t+1}^{\alpha})^{\rho/\alpha}]^{1/\rho}, \qquad t \geqslant 0. \tag{2.1}$$

A solution $\{U_t\}_{t=0}^{\infty}$ of this equation is a sequence of Borel measurable mappings $U_t\colon \mathbb{R}_{+}^{\infty} \to \mathbb{R}_{+}$, $t \geqslant 0$, satisfying (2.1). Note that the cases of $\alpha = 0$ or $\rho = 0$ could be similarly handled but are ignored for simplicity here. Following Epstein and Zin [7], α is interpreted as a risk aversion parameter with the degree of risk aversion increasing as α falls, and $\sigma \equiv 1/(1-\rho)$ is the elasticity of intertemporal substitution. Thus ρ is interpreted as reflecting intertemporal substitution. When $\alpha = \rho$, the above formulation of utility becomes the common homothetic expected utility formulation:

$$U_t = \left[E_t \left(\sum_{i=t}^{\infty} \beta^{i-t} C_i^{\alpha} \right) \right]^{1/\alpha}.$$

Of immediate concern are the circumstances under which there exist intertemporal utility functions U satisfying (2.1). Epstein and Zin [7, Theorem 3.1 and p. 963] offer an existence result and, for the case where

[1] See Epstein and Zin [7] for a rigorous and detailed presentation.

[2] In this paper, all random variables are real valued and are defined on a given probability space $(\Omega, \mathcal{F}, P)$. Given also is an increasing family $\{\mathcal{F}_t\}_{t=0}^{\infty}$ of sub-σ-algebras corresponding to the available information, time-by-time. I use E_t to denote a version of $\mathcal{F}_t$-conditional expectation, and I suppress "almost surely" for simplicity, taking equality between random variables to mean almost sure equality. Unless otherwise indicated, any random variables denoted with a t-subscript, as in "X_t," are taken to be $\mathcal{F}_t$-measurable.

$\alpha\rho > 0$, a uniqueness result. See also Chew and Epstein [1] for an axiomatic analysis. Epstein [6] shows that if U is unique then so is the equilibrium price of equity. Therefore, nonuniqueness of equilibrium asset prices can only result from nonuniqueness of U. This relates to the question of determinacy of equilibria explained in the Introduction.

2.2. *The Economy*

Consider a closed economy with one representative individual. There is one consumption good and two assets: equity and a one-period discount bond. In period t, the individual consumes C_t units of good, and holds Z_{et} shares of equity and Z_{bt} shares of bond.

The environment. The output level in period t is Y_t, which is a positive random variable. The (gross) growth rate process $\{X_t\}_{t=0}^{\infty}$, defined by $X_{t+1} \equiv Y_{t+1}/Y_t$ for $t \geq 0$ with given $Y_0 > 0$, is assumed to follow a first-order Markov process, i.e., there is a function $F(\cdot|\cdot)\colon \mathbb{R}^2 \to [0, 1]$ such that, at any time $t \geq 0$, X_{t+1} has $F(\cdot|X_t)$ as its conditional distribution function. In addition, for a given constant $B > 0$, assume that X_t has support in $[0, B]$, for all $t \geq 1$.

The bond. The one-period discount bond pays 1 unit of consumption good in the following period. Total supply of the bond is 0. There is a Borel measurable function $P_b\colon \mathbb{R}^2 \to \mathbb{R}_{++}$ such that, at any time t, the bond price is $P_b(X_t, Y_t)$.

The equity. The equity pays a sequence of dividends $\{Y_t\}_{t=0}^{\infty}$. Total supply of equity is 1. There is also a Borel measurable function $P_e\colon \mathbb{R}^2 \to \mathbb{R}_{++}$ such that, at any time t, the equity price is $P_e(X_t, Y_t)$.

Given the bound B of the output growth rate, let

$$\mathscr{C} \equiv \{\{C_t\}_{t=0}^{\infty} \mid C_t \text{ is a } \mathscr{F}_t\text{-measurable random variable, } 0 \leq C_t \leq MB^t, \text{ for all } t \geq 0 \text{ and some constant } M > 0\},$$

and, given a constant $A > 1$, let

$$\Lambda \equiv \{\{\lambda_t\}_{t=0}^{\infty} \mid \lambda_t \text{ is a } \mathscr{F}_t\text{-measurable random variable, } |\lambda_t| \leq A, \text{ for all } t \geq 0\}.$$

Let $U_t\colon \mathbb{R}_+^{\infty} \to \mathbb{R}$ be a solution of (2.1). Then the consumer's problem is

$$J(X_0, Y_0, W_0) \equiv \max_{\{C_t\} \in \mathscr{C},\, \{\lambda_t\} \in \Lambda} U_0(C_0, C_1, C_2, \ldots)$$
$$\text{s.t. } W_{t+1} = \lambda_t (W_t - C_t) R_{e\,t+1} + (1 - \lambda_t)(W_t - C_t) R_{b\,t+1}$$
$$\text{given } X_0, Y_0, W_0 \geq 0, \tag{2.2}$$

where W_t is the wealth; λ_t is the proportion of investment in equity; $R_{e\,t+1}$ and $R_{b\,t+1}$ are, respectively, the (gross) returns for equity and the bond,

$$R_{e\,t+1} \equiv \frac{Y_{t+1} + P_e(X_{t+1}, Y_{t+1})}{P_e(X_t, Y_t)}, \qquad R_{b\,t+1} \equiv \frac{1}{P_b(X_t, Y_t)};$$

and $Z_{e\,t+1}$ and $Z_{b\,t+1}$ are, respectively, the shares of holdings in equity and the bond,

$$Z_{e\,t+1} \equiv \frac{\lambda_t(W_t - C_t)}{P_e(X_t, Y_t)}, \qquad Z_{b\,t+1} \equiv \frac{(1-\lambda_t)(W_t - C_t)}{P_b(X_t, Y_t)}.$$

As required by Epstein and Zin [7], I will assume that the returns are bounded: there exist constants $\underline{R}$ and $\bar{R}$, $0 = \underline{R} < \bar{R} < \infty$, such that $R_{e\,t}$ and $R_{b\,t}$ as random variables have supports in $[\underline{R}, \bar{R}]$, for all $t \geqslant 1$. Epstein and Zin [7, Theorem 5.1] offer an existence result of a maximum for the consumer's problem (2.2). Since all the functions involved in the maximization problem (2.2) are Borel measurable, J is also Borel measurable. By the recursive definition (2.1) of U_t, any solution of (2.2) is a solution of the "Bellman equation" (Ma [16] offers a proof)

$$J(X_t, Y_t, W_t) = \max_{0 \leqslant C_t \leqslant MB^t,\, |\lambda_t| \leqslant A} \{C_t^\rho + \beta[E_t J^\alpha(X_{t+1}, Y_{t+1}, W_{t+1})]^{\rho/\alpha}\}^{1/\rho}$$

$$\text{s.t.} \quad W_{t+1} = \lambda_t(W_t - C_t) R_{e\,t+1} + (1-\lambda_t)(W_t - C_t) R_{b\,t+1}$$
$$\text{given } X_t, Y_t, W_t, \text{ for } t \geqslant 0 \tag{2.3}$$

for some constant $M > 0$.

For solutions $\{C_t^*\}$, $\{Z_{et}^*\}$, and $\{Z_{bt}^*\}$ of (2.3), equilibrium conditions are

$$C_t^* = Y_t, \qquad Z_{e\,t+1}^* = 1, \qquad Z_{b\,t+1}^* = 0, \qquad \text{for } t \geqslant 0. \tag{2.4}$$

Following Lucas [14],

DEFINITION. An equilibrium consists of three functions P_e, P_b, and J satisfying (2.3) and (2.4).

A dynamic programming analysis would provide circumstances under which the solutions of (2.3) are solutions of (2.2). Such a Verification Theorem is, however, not needed for this paper. Equilibria defined by (2.2) and (2.4) are included in equilibria defined by (2.3) and (2.4). If local uniqueness (determinacy) and recoverability of utility from equilibrium asset prices are true for the latter set of equilibria, they must also be true for the former set of equilibria. That is, my principal results regarding determinacy and recoverability are made *stronger* by my use of a *weak* notion of equilibrium.

By Formula (6.6) in Epstein and Zin [7], the Euler equations for this dynamic problem are

$$\delta = E_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{\alpha-\gamma} R^{\gamma-1}_{m\,t+1} R_{e\,t+1}\right] = E_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{\alpha-\gamma} R^{\gamma-1}_{m\,t+1} R_{b\,t+1}\right], \quad (2.5)$$

where $R_{m\,t+1} \equiv \lambda_t R_{e\,t+1} + (1-\lambda)\, R_{b\,t+1}$ is the market return, $\gamma \equiv \alpha/\rho$, and $\delta \equiv \beta^{-\gamma}$. Note that by the boundedness of the consumption growth rate and asset returns the integrability of the expectations in (2.5) are guaranteed. Conditions (2.4) immediately imply

$$\lambda_t = 1, \qquad R_{m\,t+1} = R_{e\,t+1}, \qquad W_t = Y_t + P_e(X_t, Y_t), \qquad \text{for} \quad t \geqslant 0.$$

Note that the budget constraint is satisfied in the equilibrium. By (2.5), we thus have the following *pricing equations*:

$$\begin{aligned} \delta &= E_t\left[X^{\alpha-\gamma}_{t+1}\left(\frac{Y_{t+1} + P_e(X_{t+1}, Y_{t+1})}{P_e(X_t, Y_t)}\right)^{\gamma}\right] \\ \delta &= E_t\left[X^{\alpha-\gamma}_{t+1}\left(\frac{Y_{t+1} + P_e(X_{t+1}, Y_{t+1})}{P_e(X_t, Y_t)}\right)^{\gamma-1} \frac{1}{P_b(X_t, Y_t)}\right]. \end{aligned} \quad (2.6)$$

Since $Y_{t+1} = X_{t+1} Y_t$, Eqs. (2.6) imply that there are Borel measurable functions $Q_e, Q_b \colon \mathbb{R} \to \mathbb{R}_{++}$ such that

$$P_e(X_t, Y_t) = Q_e(X_t)\, Y_t, \qquad P_b(X_t, Y_t) = Q_b(X_t),$$

where Q_e and Q_b satisfy

$$\delta = E_t\left[X^{\alpha}_{t+1}\left(\frac{1 + Q_e(X_{t+1})}{Q_e(X_t)}\right)^{\gamma}\right] = E_t\left[X^{\alpha-1}_{t+1}\left(\frac{1 + Q_e(X_{t+1})}{Q_e(X_t)}\right)^{\gamma-1} \frac{1}{Q_b(X_t)}\right].$$

The boundedness of the asset returns will be satisfied if there exist constants $\underline{Q}$ and $\bar{Q}$, $0 < \underline{Q} < \bar{Q} < \infty$, such that $\underline{Q} \leqslant Q_b(x)$, $Q_e(x) \leqslant \bar{Q}$, for $0 \leqslant x \leqslant B$. The boundedness of Q_b and Q_e also ensures the integrability of the above expectations. Since $X_{t+1} \sim F(\cdot \mid X_t)$, the above set of equations becomes

$$\begin{aligned} \delta Q^{\gamma}_e(X_t) &= \int z^{\alpha}[1 + Q_e(z)]^{\gamma}\, dF(z \mid X_t) \\ \delta Q_b(X_t)\, Q^{\gamma-1}_e(X_t) &= \int z^{\alpha-1}[1 + Q_e(z)]^{\gamma-1}\, dF(z \mid X_t). \end{aligned} \quad (2.7)$$

Following Mehra and Prescott [19], to ensure tractability, I assume that[3]

Assumption. The (gross) growth rate X_t takes only m distinct positive values $x_1, x_2, ..., x_m$, for all $t \geq 0$.

Let $q_{ei} \equiv Q_e(x_i)$, $q_{bi} \equiv Q_b(x_i)$, and $\pi_{ij} \equiv P(X_t = x_j \mid X_t = x_i)$. Then (2.7) becomes

$$\begin{aligned} \delta q_{ei}^{\gamma} &= \sum_{j=1}^{m} \pi_{ij} x_j^{\alpha} (1 + q_{ej})^{\gamma} \\ \delta q_{bi} q_{ei}^{\gamma - 1} &= \sum_{j=1}^{m} \pi_{ij} x_j^{\alpha - 1} (1 + q_{ej})^{\gamma - 1} \end{aligned} \tag{2.8}$$

for $i = 1, 2, ..., m$. Let

$$\Pi \equiv \begin{pmatrix} \pi_{11} & \pi_{12} & \cdots & \pi_{1m} \\ \pi_{21} & \pi_{22} & \cdots & \pi_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \pi_{m1} & \pi_{m2} & \cdots & \pi_{mm} \end{pmatrix}$$

$$x \equiv (x_1, x_2, ..., x_m), \qquad q_e \equiv (q_{e1}, q_{e2}, ..., q_{em}), \qquad q_b \equiv (q_{b1}, q_{b2}, ..., q_{bm}),$$

and

$$\theta \equiv (\alpha, \gamma, \delta), \qquad \mathscr{E} \equiv (x, \Pi).$$

Then (2.8) becomes

$$G(\theta, q_e, q_b, \mathscr{E}) \equiv \begin{pmatrix} \Pi & \\ & \Pi \end{pmatrix} \begin{pmatrix} x_1^{\alpha}(1 + q_{e1})^{\gamma} \\ \vdots \\ x_m^{\alpha}(1 + q_{em})^{\gamma} \\ x_1^{\alpha - 1}(1 + q_{e1})^{\gamma - 1} \\ \vdots \\ x_m^{\alpha - 1}(1 + q_{em})^{\gamma - 1} \end{pmatrix} - \delta \begin{pmatrix} q_{e1}^{\gamma} \\ \vdots \\ q_{em}^{\gamma} \\ q_{b1} q_{e1}^{\gamma - 1} \\ \vdots \\ q_{bm} q_{em}^{\gamma - 1} \end{pmatrix} = 0. \tag{2.9}$$

In particular, q_e is determined by

$$H(\theta, q_e, \mathscr{E}) \equiv \Pi \begin{pmatrix} x_1^{\alpha}(1 + q_{e1})^{\gamma} \\ x_2^{\alpha}(1 + q_{e2})^{\gamma} \\ \vdots \\ x_m^{\alpha}(1 + q_{em})^{\gamma} \end{pmatrix} - \delta \begin{pmatrix} q_{e1}^{\gamma} \\ q_{e2}^{\gamma} \\ \vdots \\ q_{em}^{\gamma} \end{pmatrix} = 0. \tag{2.10}$$

[3] The boundedness of the output growth rate and asset returns is automatically satisfied given this assumption.

Let the space of parameter vectors (α, β, ρ) be

$$\Theta_0 \equiv \{(\alpha, \beta, \rho) \in \mathbb{R}^3 \mid \alpha \neq 0, \rho \neq 0, 1, \beta > 0\}.$$

The cases of $\alpha = 0$ or $\rho = 0$ could also be considered. Here I ignore them for simplicity. The results regarding these cases are mentioned in the concluding remarks of Section 4. For our model to make economic sense and to guarantee the transversality condition (see Epstein and Zin [7] for details), we should restrict further: $\alpha, \beta, \rho < 1$. But these restrictions are not required by the mathematical proofs in this paper. And if conclusions in this paper are true for all the parameter vectors in Θ_0, they are of course true for all parameter vectors in a subset set of Θ_0. For convenience, I will use (α, γ, δ) instead of (α, β, ρ) as preference parameters. Let

$$\Theta \equiv \{(\alpha, \gamma, \delta) \in \mathbb{R}^3 \mid \alpha \neq 0, \alpha \neq \gamma, \delta > 0\}.$$

By directly solving transformation $\alpha = \alpha$, $\gamma = \alpha/\rho$, $\delta = \beta^{-\gamma}$ for (α, β, ρ) given (α, γ, δ), we know that these two parameter vectors are related by a smooth (differentiable of any order) one-to-one relation defined between Θ_0 and Θ. Therefore, given the parameter spaces, (α, γ, δ) and (α, β, ρ) are equivalent for our discussions.

Let

$$X \equiv \{(x_1, x_2, ..., x_m) \in \mathbb{R}^m_{++} \mid x_1, x_2, ..., x_m \text{ are distinct}\}$$

and

$$\Delta \equiv \left\{ \Pi \in (0, 1)^{m(m-1)} \,\middle|\, \pi_{im} = 1 - \sum_{j=1}^{m-1} \pi_{ij} > 0, \forall i \right\}.$$

The space of all economies is then defined as

$$\mathbb{E} \equiv \left\{ (x, \Pi) \in X \times \Delta \,\middle|\, \delta^{1/\gamma} > \max_i \left(\sum_j \pi_{ij} x_j^\alpha \right)^{1/\gamma} \right\},$$

with the induced Euclidean topology and Lebesgue measure.

Remark 2.1. The condition $\delta^{1/\gamma} > \max_i(\sum \pi_{ij} x_j^\alpha)^{1/\gamma}$ or $\beta < \min_i(\sum \pi_{ij} x_j^\alpha)^{-\rho/\alpha}$ means that the discount factor β is relatively small. I will show in the proposition in Section 3.2 that this is sufficient for the existence of asset prices and thus an equilibrium.[4] It can be shown that it is necessary when the economy is close to certainty (i.e., when π_{ii}'s are close to 1). Under expected utility $(\alpha = \rho)$, if we let $a_{ij} \equiv \beta \pi_{ij} x_j^\alpha$ and $A \equiv (a_{ij})_{m \times m}$,

[4] Since $[\max_i\{x_i\}]^{-\rho} \leqslant \min_i(\sum \pi_{ij} x_j^\alpha)^{-\rho/\alpha}$ when $\rho > 0$, the assumption relating β, ρ, and consumption growth rates in Epstein and Zin [7, Theorem 3.1] is stronger than my restriction.

Mehra and Prescott [19] show that the condition "$A^n \to 0$ as $n \to \infty$" is necessary and sufficient for the existence of equilibrium. My condition reduces in this expected utility case to a stronger condition "$\sum_j a_{ij} < 1, \forall i$." Note that I use Brouwer's fixed-point theorem rather than the Contraction Mapping Theorem to prove existence. The Contraction Mapping Theorem would provide both existence and global uniqueness, while the Brouwer Theorem only offers existence. However, the latter requires weaker conditions than the former.

Remark 2.2. In the definition of the space of economies, the following set could also be included:

$$\mathcal{M}_0 \equiv \{(x, \Pi) \in \mathbb{R}^{m^2} \mid \pi_{ij} = 0 \text{ or } 1 \text{ for some } i, j\}.$$

This set can be easily shown to be a closed null set in $\mathbb{R}^{m^2}$, where a set is called a *null set* if it has zero Lebesgue measure in the appropriate Euclidean space. Therefore, my results are unaffected if this set is included in $\mathbb{E}$. This set is excluded from $\mathbb{E}$ for convenience.

2.3. *Determinacy and Recoverability*

The main results of this paper are the theorems stated in this subsection.

A property on the space of economies is *generic* if it fails only on a closed null subset of $\mathbb{E}$. Since a null set necessarily has empty interior, my definition of genericity is stronger than the usual one, implying that the set of "bad economies" under my definition is generally smaller than the one under the usual definition.

Given $\theta \in \Theta$, $\mathscr{E} \in \mathbb{E}$, equilibrium asset prices are said to be *determinate* if they are locally unique and smooth: for any q_e, $q_b \in \mathbb{R}^m_{++}$ satisfying $G(\theta, q_e, q_b, \mathscr{E}) = 0$, there exist a neighborhood $N_{(\theta, \mathscr{E})}$ of $(\theta, \mathscr{E})$, neighborhoods N_e and N_b of q_e and q_b, respectively, and unique and smooth price functions $q_e^*: N_{(\theta, \mathscr{E})} \to N_e$ and $q_b^*: N_{(\theta, \mathscr{E})} \to N_b$, such that

$$q_e \neq q_e^*(\theta, \mathscr{E}), \qquad q_b = q_b^*(\theta, \mathscr{E})$$

and

$$G[\theta', q_e^*(\theta', \mathscr{E}'), q_b^*(\theta', \mathscr{E}'), \mathscr{E}'] = 0, \qquad \forall (\theta', \mathscr{E}') \in N_{(\theta, \mathscr{E})}.$$

THEOREM 1 (Determinacy). *For $m \geqslant 1$, for any parameter vector $\theta \in \Theta$, there exists a closed null subset $\mathscr{A}_1$ of $\mathbb{E}$ such that equilibrium asset prices in any economy $\mathscr{E} \in \mathbb{E} \backslash \mathscr{A}_1$ are determinate. In other words, asset prices are generically determinate, for all $\theta \in \Theta$.*

This theorem is technically a necessary step towards recoverability analysis. Notice that the smoothness of q_e^* and q_b^* is with respect to the vector $(\theta, \mathscr{E})$, which has nothing to do with the smoothness with respect to the state of an economy; the latter is obviously satisfied since there are only a finite number of states. By Mas-Colell [18, pp. 316–317], persistent equilibria are those that do not depend too precisely on environmental variables; more precisely, *persistence* means that the equilibrium moves smoothly as system parameters shift smoothly. The smoothness of the asset prices directly implies the persistence of the equilibrium. Theorem 1 serves as both a lemma for Theorems 2 and 3 and an extension of the literature on determinacy to a model with nonexpected utility preferences.

Given a parameter vector $\theta \in \Theta$ and an economy $\mathscr{E} \in \mathbb{E}$, the resulting asset price $q(\theta, \mathscr{E})$ ($q = q_e$ or q_b) and the economy $\mathscr{E}$ are said to be (locally) *revealing* if there exists a neighborhood N_θ of θ such that, $\forall \theta', \theta'' \in N_\theta$,

$$\theta' \neq \theta'' \qquad \text{implies} \quad q(\theta', \mathscr{E}) \neq q(\theta'', \mathscr{E}).$$

That is, there is a neighborhood of θ in which different parameter vectors give different asset prices in the economy $\mathscr{E}$. A revealing asset price theoretically gives modelers the possibility of identifying the underlying preferences from the asset price.

If the price of equity is observable, we can try to recover underlying preferences from it.

THEOREM 2 (Recoverability). *For $m \geq 3$, for any parameter vector $\theta \in \Theta$, there exists a closed null set $\mathscr{A}_2$ in $\mathbb{E}$ such that all economies in $\mathbb{E} \setminus \mathscr{A}_2$ are revealing in terms of equity prices. In other words, economies are generically revealing in terms of equity prices.*

Note that since there are three parameters, it is obvious that we need at least three equity price values to identify them, which means that we have to have at least three states ($m \geq 3$).

A difficulty with this result is that equity here is an aggregate asset which may include many nontradeable components like human-capital, etc., and its price may not be observable. The bond on the other hand is more readily viewed as a tradeable asset. For this reason, recoverability in terms of the bond price is also provided.

THEOREM 3 (Recoverability). *For[5] $m > 8$, for any parameter vector $\theta \in \Theta$, there exists a closed zero-measure set $\mathscr{A}_3$ in $\mathbb{E}$ such that all the economies in $\mathbb{E} \setminus \mathscr{A}_3$ are revealing in terms of bond prices. In other words, economies are generically revealing in terms of bond prices.*

Theorems 1–3 together tell us that, given any $\theta \in \Theta$, it is generically true that the asset prices are locally unique, revealing of preference parameters, and smooth with respect to the economy and the parameters.

[5] I suspect (but have not proven) that $m \geq 3$ is sufficient.

Proving recoverability is, unfortunately, quite complicated. We need eight lemmas in Appendix 1 for the proofs of Theorems 1 and 2. The existence of the asset prices is guaranteed by a proposition in Appendix 2. The proofs of Theorems 1 and 2 are presented in Appendices 3 and 4, respectively. The proof for Theorem 3, which may be found in Wang [20], is similar in structure to that of Theorem 2 and is omitted in the interest of brevity.

3. Extensions

The cases $\alpha = 0$ or $\rho = 0$ can be similarly handled. Determinacy is still true for all these cases. Recoverability is, however, only true for the case $\rho \neq 0$. Intuitively, recoverability fails when $\rho = 0$ because in that case, the agent becomes myopic and risk aversion α drops out. In fact, q^* only depends on β, and thus we are unable to identify α from q^*.

When the growth rate follows a finite-order Markov process with a finite but unspecified number of states, determinacy and recoverability still hold and the proofs are similar.

The extension to general $F(\cdot|x_t)$ is currently being pursued. This extension offers us the possibility of dealing with general (nonparametric) recursive utility functions, but it poses serious technical difficulties. A further extension of Sard's Theorem to Banach spaces may be needed. See, for example, Kehoe *et al.* [10].

I also have a global recoverability result in Wang [20]. I show that the general Kreps–Porteus specification is observationally distinguishable from any nonparametric expected additive intertemporal utility function satisfying specified regularity conditions, without restricting the Kreps–Porteus and expected utility functions to be close to one another in any sense, e.g., in parameter space.

APPENDIX 1: Lemmas

This appendix contains lemmas needed for the proofs of the theorems. Proofs not provided here may be found in Wang [20].

First, the following two lemmas will be used to show that the "bad" set taken away from $\mathbb{E}$ is a closed set.

LEMMA 1. *For sets $Z \subset \mathbb{R}^n$, $\Lambda \subset \mathbb{R}^m$ and continuous function $f: Z \times \Lambda \to \mathbb{R}^k$, set*

$$\mathcal{M} \equiv \{\mathscr{E} \in Z \mid f(\mathscr{E}, \lambda) = 0 \text{ for some } \lambda \in \Lambda\}$$

is closed in Z *if the following condition is satisfied: for any convergent sequence* $\{\mathscr{E}_n\} \subset \mathscr{M}$, $\mathscr{E}_n \to \mathscr{E}_0 \in Z$, *the corresponding sequence* $\{\lambda_n\} \subset \Lambda$ *satisfying* $f(\mathscr{E}_n, \lambda_n) = 0$ *has a convergent sub-sequence converging to a point in* Λ.

LEMMA 2. *Let* X *be a topological space and* $X_1 \subset X$ *a sub-space of* X *equipped with the induced topology from* X. *Then*

(1) $A \subset X_1$ *is open in* $X_1 \Rightarrow A$ *is open in* X *if* X_1 *is open in* X.

(2) $A \subset X_1$ *is closed in* $X_1 \Rightarrow A$ *is closed in* X *if* X_1 *is closed in* X.

(3) $A \subset X$ *is closed in* X *and* $B \subset X \setminus A$ *is closed in* $X \setminus A \Rightarrow C \equiv A \cup B$ *is closed in* X.

The following lemma is an analogue to Sard's Theorem. It deals with the small size of a subset in the domain of a mapping rather than in the range of a mapping. The Inverse Function Theorem (see Mas-Colell [18]) is invoked in its proof. The lemma will be used to show that the "bad" set that I take away from $\mathbb{E}$ is a null set. In this paper, the only condition that really needs to be checked is the condition "rank $> k$," i.e., the rank of the Jacobian matrix is greater than the number of free parameters. The critical conditions are the "C^1" and "rank $> k$." It can be shown that both conditions are necessary.

LEMMA 3. *Let* $f: X \times \Lambda \to \mathbb{R}^m$ *be* C^1, *let* $X \subset \mathbb{R}^n$ *and* $\Lambda \subset \mathbb{R}^k$ *be two open sets, and*

$$\mathscr{M} \equiv \{x \in X \mid f(x, \lambda) = 0 \text{ for some } \lambda \in \Lambda\}.$$

If $D_x f(x, \lambda) \equiv \partial(f_1, ..., f_m)/\partial(x_1, ..., x_n)$ *has* rank $> k$ *for all* $(x, \lambda) \in X \times \Lambda$ *satisfying* $f(x, \lambda) = 0$, *then* $\mathscr{M}$ *has zero measure.*

In order to use Lemma 1 for the special sets that we encounter in the proofs of the theorems, we need the following two lemmas.

LEMMA 4. *Given* $\theta \in \Theta$, *for any* $\mathscr{E}_n$, $\mathscr{E}_0 \in \mathbb{E}$, $\mathscr{E}_n \to \mathscr{E}_0$, *and* $q^n \in \mathbb{R}^m_{++}$ *satisfying* $H(\theta, q^n, \mathscr{E}_n) = 0$, *there exist two positive numbers* d *and* D *such that*

$$d \leqslant q_i^n \leqslant D, \qquad \textit{for all } i, n.$$

Proof. This proof proceeds in three steps. The first two steps show the boundedness of $q(\theta, \mathscr{E})$, where the boundaries depend on $(\theta, \mathscr{E})$. The last step shows the boundedness of $\{q(\theta, \mathscr{E})\}$, where the boundaries depend on $\{\mathscr{E}_n\}$, $\mathscr{E}_0$, and θ.

Step 1. For all $\mathscr{E} \in \mathbb{E}$ and $q \in \mathbb{R}^m_{++}$ satisfying $H(\theta, q, \mathscr{E}) = 0$, we have

$$q_i \geqslant \min_i \left(\frac{1}{\delta} \sum_j \pi_{ij} x_j^\alpha \right)^{1/\gamma}, \qquad \text{for all } i. \tag{A.1.1}$$

For all γ ($\gamma \neq 0$), since $q(\theta, \mathscr{E}) > 0$, we have

$$\delta^{1/\gamma} q_i = \left[\sum_j \pi_{ij} x_j^\alpha (1 + q_j)^\gamma \right]^{1/\gamma} \geqslant \left(\sum_j \pi_{ij} x_j^\alpha \right)^{1/\gamma} \geqslant \min_i \left(\sum_j \pi_{ij} x_j^\alpha \right)^{1/\gamma}.$$

This implies (A.1.1).

Step 2. For all $\mathscr{E} \in \mathbb{E}$ and $q \in \mathbb{R}^m_{++}$ satisfying $H(\theta, q, \mathscr{E}) = 0$, we have

$$q_i \leqslant \frac{\max_i (\sum_j \pi_{ij} x_j^\alpha)^{1/\gamma}}{\delta^{1/\gamma} - \max_i (\sum_j \pi_{ij} x_j^\alpha)^{1/\gamma}}, \qquad \text{for all } i. \tag{A.1.2}$$

For all γ ($\gamma \neq 0$), for $q = q(\theta, \mathscr{E})$ and $q_{\max} \equiv \max_i \{q_i\}$, we have

$$\begin{aligned}\delta^{1/\gamma} q_i = \left[\sum_j \pi_{ij} x_j^\alpha (1 + q_j)^\gamma \right]^{1/\gamma} &\leqslant (1 + q_{\max}) \left(\sum_j \pi_{ij} x_j^\alpha \right)^{1/\gamma} \\ &\leqslant (1 + q_{\max}) \max_i \left(\sum_j \pi_{ij} x_j^\alpha \right)^{1/\gamma}\end{aligned}$$

for all i. Therefore,

$$\delta^{1/\gamma} q_{\max} \leqslant (1 + q_{\max}) \max_i \left(\sum_j \pi_{ij} x_j^\alpha \right)^{1/\gamma}.$$

This immediately implies (A.1.2).

Step 3. $\{q_{in}\}$ is bounded for any i.

For any $\mathscr{E}_n \to \mathscr{E}_0$, $\mathscr{E}_n$, $\mathscr{E}_0 \in Z$, and $q^n \in \mathbb{R}^m_{++}$ satisfying $H(\theta, q^n, \mathscr{E}_n) = 0$, define two sequences $\{d(\mathscr{E}_n)\}$ and $\{D(\mathscr{E}_n)\}$ by

$$d(\mathscr{E}_n) \equiv \min_i \left(\frac{1}{\delta} \sum_j \pi^n_{ij} x^\alpha_{jn} \right)^{1/\gamma} \qquad \text{and} \qquad D(\mathscr{E}_n) \equiv \frac{\max_i (\sum_j \pi^n_{ij} x^\alpha_{jn})^{1/\gamma}}{\delta^{1/\gamma} - \max_i (\sum_j \pi^n_{ij} x^\alpha_{ij})^{1/\gamma}}.$$

Then, by Steps 1 and 2,

$$d(\mathscr{E}_n) \leqslant q_{ij} \leqslant D(\mathscr{E}_n) \qquad \text{for all } i, n.$$

Since $\mathscr{E}_n \to \mathscr{E}_0$, $d(\mathscr{E}_n)$ and $D(\mathscr{E}_n)$ converge to positive numbers $d(\mathscr{E}_0)$ and $D(\mathscr{E}_0)$, respectively, as $n \to \infty$. Thus, there exists an integer N such that, when $n > N$,

$$\tfrac{1}{2} d(\mathscr{E}_0) \leqslant q_{in} \leqslant 2D(\mathscr{E}_0), \qquad \text{for all } i.$$

Therefore, there exist positive numbers

$$d \equiv \min\{\tfrac{1}{2}d(\mathscr{E}_0), q_{i1}, q_{i2}, ..., q_{iN}, i=1, 2, ..., m\}$$

and

$$D \equiv \max\{2D(\mathscr{E}_0), q_{i1}, q_{i2}, ..., q_{iN}, i=1, 2, ..., m\}$$

such that

$$d \leqslant q_{in} \leqslant D, \qquad \text{for all } i, n. \quad \blacksquare$$

LEMMA 5. *Let*

$$\mathscr{M}_3 \equiv \{(x, \Pi) \in \mathbb{E} \mid |\Pi| = 0\}.$$

For any sequence $\{\mathscr{E}_n\} \subset \mathbb{E}\backslash\mathscr{M}_3$, $\mathscr{E}_n \to \mathscr{E}_0 \in \mathbb{E}\backslash\mathscr{M}_3$, *and equity prices* $\{q^n\}$ *satisfying* $H(\theta, q^n, \mathscr{E}_n) = 0$, *any sequence of solutions* $\{(\lambda_n, \mu_n)\}$ *from the following equation set*

$$\lambda_n \Pi^n \begin{pmatrix} b_{1n} \ln x_{1n} \\ b_{2n} \ln x_{2n} \\ \vdots \\ b_{mn} \ln x_{mn} \end{pmatrix} + \Pi^n \begin{pmatrix} b_{1n} \ln(1+q_{1n}) \\ b_{2n} \ln(1+q_{2n}) \\ \vdots \\ b_{mn} \ln(1+q_{mn}) \end{pmatrix} + \mu_n \Pi^n \begin{pmatrix} b_{1n} \\ b_{2n} \\ \vdots \\ b_{mn} \end{pmatrix} = \delta \begin{pmatrix} q_{1n}^{\gamma} \ln q_{1n} \\ q_{2n}^{\gamma} \ln q_{2n} \\ \vdots \\ q_{mn}^{\gamma} \ln q_{mn} \end{pmatrix} \tag{A.1.3}$$

is bounded, where $b_i \equiv x_i^{\alpha}(1+q_i)^{\gamma}$.

Proof. Equation (A.1.3) can become

$$\lambda_n \begin{pmatrix} \ln x_{1n} \\ \ln x_{2n} \\ \vdots \\ \ln x_{mn} \end{pmatrix} + \begin{pmatrix} \ln(1+q_{1n}) \\ \ln(1+q_{2n}) \\ \vdots \\ \ln(1+q_{mn}) \end{pmatrix} + \mu_n \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \delta \begin{pmatrix} e_{1n} \\ e_{2n} \\ \vdots \\ e_{mn} \end{pmatrix}, \tag{A.1.4}$$

where

$$\begin{pmatrix} e_{1n} \\ e_{2n} \\ \vdots \\ e_{mn} \end{pmatrix} \equiv \delta \begin{pmatrix} b_{1n}^{-1} & & & \\ & b_{2n}^{-1} & & \\ & & \ddots & \\ & & & b_{mn}^{-1} \end{pmatrix} (\Pi^n)^{-1} \begin{pmatrix} q_{1n}^{\gamma_n} \ln q_{1n} \\ q_{2n}^{\gamma_n} \ln q_{2n} \\ \vdots \\ q_{mn}^{\gamma_n} \ln q_{mn} \end{pmatrix}.$$

We know that

$$\Pi^{-1} = \frac{1}{|\Pi|}\begin{pmatrix} A_{11} & A_{21} & \cdots & A_{m1} \\ A_{12} & A_{22} & \cdots & A_{m2} \\ \vdots & \vdots & & \vdots \\ A_{1m} & A_{2m} & \cdots & A_{mm} \end{pmatrix},$$

where $A_{ij} = (-1)^{i+j} M_{ij}$, and M_{ij} is the minor of element π_{ij} in Π. Since $\mathscr{E}_n \in \mathbb{E} \backslash \mathscr{M}_3$, Π^n has an inverse. Because $\mathscr{E}_n \to \mathscr{E}_0 \in \mathbb{E} \backslash \mathscr{M}_3$, we know that $\{(\Pi^n)^{-1}\}$ is bounded. By Lemma 4, $\{q_n\}$ is bounded above and bounded from zero. Besides that, it is obvious that $\{\mathscr{E}_n\}$ is bounded. Therefore, $\{e^n\} \equiv \{(e_{1n}, e_{2n}, ..., e_{mn})\}$ is bounded.

Equation (A.1.4) can be expressed in the following form:

$$\lambda_n \ln x_{in} + \mu_n + \ln(1 + q_{in}) = e_{in}, \qquad i = 1, 2, .., m.$$

This implies that

$$\lambda_n \ln \frac{x_{2n}}{x_{1n}} + \ln \frac{1 + q_{2n}}{1 + q_{1n}} = e_{2n} - e_{1n}.$$

And thus,

$$\lambda_n = \frac{1}{\ln(x_{2n}/x_{1n})}\left[e_{2n} - e_{1n} - \ln \frac{1 + q_{2n}}{1 + q_{1n}}\right]$$

$$\mu_n = e_{1n} - \lambda_n \ln x_{1n} - \ln(1 + q_{1n}).$$

We know that $\ln(x_{2n}/x_{1n}) \to \ln(x_{20}/x_{10}) \neq 0$, and that, by Lemma 4, $\{(1 + q_{2n})/(1 + q_{1n})\}$ is bounded from zero and from above. Therefore, $\{\lambda_n\}$ and thus $\{\mu_n\}$ are bounded. ∎

The following two lemmas show that, except for a null set, most economies result in equity prices which very across states, and similarly for the b_i's.

LEMMA 6. *Given $\theta \in \Theta$, the following set*

$$\mathscr{M}_4 \equiv \{\mathscr{E} \in \mathbb{E} \backslash \mathscr{M}_3 \mid H(\theta, q, \mathscr{E}) = 0,\ q_i = q_m \text{ for some } i \neq m \text{ and } q \in \mathbb{R}^m_{++}\}$$

is a closed null set in $\mathbb{E} \backslash \mathscr{M}_3$.

Proof. Given $i \in \{1, 2, ..., (m-1)\}$, for

$$\mathscr{M}_4^i \equiv \{\mathscr{E} \in \mathbb{E} \backslash \mathscr{M}_3 \mid H(\theta, q, \mathscr{E}) = 0 \text{ and } q_i = q_m \text{ for some } q \in \mathbb{R}^m_{++}\}$$

we have

$$\left|\frac{\partial H}{\partial x}\right| = \alpha^m \frac{b_1 b_2 \cdots b_m}{x_1 x_2 \cdots x_m} \, |\Pi| \neq 0$$

for any $\mathscr{E} \in \mathbb{E} \backslash \mathscr{M}_3$ and $q \in \mathbb{R}^{m-1}_{++}$. That is, $\partial H / \partial \mathscr{E}$ has full rank m, with only $(m-1)$ free parameters. Thus, by Lemma 3, $\mathscr{M}_4^i$ has zero measure for $i = 1, 2, ..., (m-1)$. Therefore $\mathscr{M}_4 = \sum_{i=1}^{m-1} \mathscr{M}_4^i$ has zero measure.

By Lemmas 1 and 4, $\mathscr{M}_4$ is closed in $\mathbb{E} \backslash \mathscr{M}_3$. ∎

LEMMA 7. *Given $\theta \in \Theta$, the following set*

$$\mathscr{M}_5 \equiv \{\mathscr{E} \in \mathbb{E} \mid H(\theta, q, \mathscr{E}) = 0, b_i = b_m \text{ for some } i \neq m \text{ and } q \in \mathbb{R}^m_{++}\}$$

is a closed null set in $\mathbb{E}$, where $b_i \equiv x_i^\alpha (1 + q_i)^\gamma$.

Proof. Given $i \in \{1, 2, ..., (m-1)\}$, define

$$Q_i(\theta, q, \mathscr{E}) \equiv \begin{pmatrix} H(\theta, q, \mathscr{E}) \\ b_i - b_m \end{pmatrix}.$$

For any $\mathscr{E} \in \mathbb{E}$ and $q \in \mathbb{R}^m_{++}$ satisfying $Q_i(\theta, q, \mathscr{E}) = 0$, I need to show that $\partial Q_i / \partial \mathscr{E}$ has full rank.

Let me first show that, for any $q \in \mathbb{R}^m_{++}$ and $\mathscr{E} \in \mathbb{E}$ satisfying $H(\theta, q, \mathscr{E}) = 0$, not all the b_i's are equal. Suppose not, that is, $b_1 = b_2 = \cdots = b_m$. Then $H(\theta, q, \mathscr{E}) = 0$ implies that $b_1 = \delta q_i^\gamma$ for all i. That is, $q_1 = q_2 = \cdots = q_m$. By substituting this into $b_1 = b_2 = \cdots = b_m$, we get $x_1 = x_2 = \cdots = x_m$. This is a contradiction.

Therefore, since $Q_i(\theta, q, \mathscr{E}) = 0$ implies $b_i = b_m$, there is a j, $j \neq i$, such that $b_j \neq b_m$. We then have

$$\left|\frac{\partial Q_i}{\partial(\pi_{1j}, \pi_{2j}, ..., \pi_{mj}, x_i)}\right| = \frac{\alpha}{x_i} b_i (b_j - b_m)^m \neq 0.$$

That is, $\partial Q_i / \partial \mathscr{E}$ has full rank. Therefore, by Lemma 3,

$$\mathscr{M}_5^i \equiv \{\mathscr{E} \in \mathbb{E} \mid H(\theta, q, \mathscr{E}) = 0 \text{ and } b_i = b_m \text{ for some } q \in \mathbb{R}^m_{++}\}$$

has zero measure, for any i. Then, $\mathscr{M}_5 = \sum_{i=1}^{m-1} \mathscr{M}_5^i$ has zero measure.

By Lemmas 1 and 4, $\mathscr{M}_5$ is closed in $\mathbb{E}$. ∎

Finally, the following lemma tells us how small the "bad" set that we encounter in the proof of Theorem 2 is.

LEMMA 8. *Given $\theta \in \Theta$, let $f(\cdot \mid \theta) : \mathbb{R}^5 \to \mathbb{R}$,*

$$f(z, x, a, b, \lambda \mid \theta) \equiv x^{-\alpha} e^{-\gamma z} + a(\lambda \ln x + z) + b.$$

Then there exist smooth functions $\varphi(\cdot\,|\,\theta)$: $D(\varphi)\to\mathbb{R}$, $\psi(\cdot\,|\,\theta)$: $D(\psi)\to\mathbb{R}$, *and* $\phi(\cdot\,|\,\theta)$: $\mathbb{R}^4\to\mathbb{R}$ *such that*

$$\begin{aligned} f^{-1}(0\,|\,\theta)\subset &\{(z,x,a,b,\lambda)\mid \varphi(x,a,b,\lambda\,|\,\theta)=z\}\\ &\cup\{(z,x,a,b,\lambda)\mid \psi(x,a,b,\lambda\,|\,\theta)=z\}\\ &\cup\{(z,x,a,b,\lambda)\mid \phi(x,a,b,\lambda\,|\,\theta)=z\}, \end{aligned} \tag{A.15}$$

where $D(\varphi)$ *and* $D(\psi)$ *are some open sets in* $\mathbb{R}^4$.

In short, Lemma 8 shows that the solutions of the equation $f(z,x,a,b,\lambda\,|\,\theta)=0$ for (z,x,a,b,λ) can be covered by three smooth functions φ, ψ, and ϕ.

Proof of Lemma 8. We have

$$f_z=a-\gamma x^{-\alpha}e^{-\gamma z} \qquad\text{and}\qquad f_{zz}=\gamma^2x^{-\alpha}e^{-\gamma z}>0.$$

Since f is convex with respect to z, given (x,a,b,λ), $f=0$ will have at most two solutions $\bar{z}$ and $\check{z}$. By this, I can define two functions φ and ψ. For any $(x,a,b,\lambda)\in\mathbb{R}^4$, if there is a solution $\bar{z}$ from $f=0$ such that $f_z<0$, then I define

$$\bar{z}=\varphi(x,a,b,\lambda\,|\,\theta).$$

φ is defined on a set $D(\varphi)\subset\mathbb{R}^4$ in which such $\bar{z}$ exists. Since f is convex with respect to z, there is at most one such $\bar{z}$. This means that φ is well defined on the set $D(\varphi)$.

I can also similarly define a function ψ. For any $(x,a,b,\lambda)\in\mathbb{R}^4$, if there is a solution $\check{z}$ from $f=0$ such that $f_z>0$, then I define

$$\check{z}=\psi(x,a,b,\lambda\,|\,\theta).$$

ψ is defined on a set $D(\psi)\subset\mathbb{R}^4$. By the same argument as above, ψ is well defined.

Another possible case is that at a solution point $\hat{z}$ we have $f_z=0$. This time we can easily find this $\hat{z}$. If this $\hat{z}$ exists, we can easily know that the a cannot be zero and that from $f_z=0$ we can solve for this $\hat{z}$:

$$\hat{z}=-\frac{1}{\gamma}\ln\left(\frac{a}{\gamma}x^{\alpha}\right)\equiv\phi(x,a,b,\lambda\,|\,\theta).$$

We see that this kind of $\hat{z}$ is unique, given any (x,a,b,λ).

The above derivation immediately implies (A.1.5). It is obvious that ϕ is smooth. I now need to show that φ and ψ are smooth, and $D(\varphi)$ and $D(\psi)$ are open.

Given any point $(z_0, x_0, a_0, b_0, \lambda_0)$ satisfying $z_0 = \varphi(x_0, a_0, b_0, \lambda_0 \mid \theta)$, as $f_z \neq 0$ at this point and f is smooth, by the Implicit Function Theorem, there are two open neighborhoods V and N of z_0 and $(x_0, a_0, b_0, \lambda_0)$, respectively, and a smooth function $\bar{\varphi}: N \to V$ such that $z = \bar{\varphi}(x, a, b, \lambda \mid \theta)$ is the unique solution for any $(x, a, b, \lambda) \in N$. We know that, by definition,

$$f_z(z_0, x_0, a_0, b_0, \lambda_0 \mid \theta) < 0.$$

This implies that there exists an open neighborhood W of $(x_0, a_0, b_0, \lambda_0)$, $W \subset N$, such that

$$f_z[\bar{\varphi}(x, a, b, \lambda \mid \theta), x, a, b, \lambda \mid \theta] < 0.$$

As I have argued before, the number of solutions z satisfying $f_z < 0$ cannot exceed one. Therefore, on W, φ and $\bar{\varphi}$ are identical. This first implies that φ is smooth on $D(\varphi)$. Second, since for any $(x_0, a_0, b_0, \lambda_0) \in D(\varphi)$ I find an open neighborhood W such that $W \subset D(\varphi)$, $D(\varphi)$ must be open.

Similarly, I can show that ψ is smooth and $D(\psi)$ is open. ∎

APPENDIX 2: Proposition on Existence

This appendix provides a proposition on existence of equilibrium.

Proposition (Existence of Asset Prices). *For any $\theta \in \Theta$ and $\mathscr{E} \in \mathbb{E}$, there exist asset prices $q_e, q_b \in \mathbb{R}^m_{++}$ satisfying $G(\theta, q_e, q_b, \mathscr{E}) = 0$.*

Proof. For $\theta \in \Theta$, $\mathscr{E} \in \mathbb{E}$, and $b \in \mathbb{R}$, define a mapping $T: [0, b]^m \to \mathbb{R}^m$ by

$$T^i(q) \equiv \left[\frac{1}{\delta} \sum_{j=1}^{m} \pi_{ij} x_j^{\alpha} (1 + q_j)^{\gamma}\right]^{1/\gamma}, \qquad i = 1, 2, ..., m.$$

$T(\cdot)$ is obviously a continuous function. For any $\theta \in \Theta$ and $\mathscr{E} \equiv (x, \Pi) \in \mathbb{E}$, we want to find a positive number b so that $T(\cdot)$ has a fixed point in $[0, b]^m$.

For $\theta \neq \Theta$ and $\mathscr{E} \in \mathbb{E}$, we have $\delta^{1/\gamma} > \max_i (\sum_j \pi_{ij} x_j^{\alpha})^{1/\gamma}$. Since $(1 + b)/b \to 1$ as $b \to \infty$, we can find a sufficiently large $b > 0$ such that

$$\frac{1+b}{b} \delta^{-1/\gamma} \max_i \left(\sum_j \pi_{ij} x_j^{\alpha}\right)^{1/\gamma} \leqslant 1.$$

This implies that, for this b and any $q \in [0, b]^m$,

$$T^i(q) \leqslant (1+b)\left(\frac{1}{\delta}\sum_j \pi_{ij} x_j^\alpha\right)^{1/\gamma} \leqslant b, \qquad i = 1, 2, ..., m.$$

Therefore, T: $[0, b]^m \to [0, b]^m$. Notice that here γ can be negative. By the Brouwer Theorem, $T(\cdot)$ has a fixed-point in $[0, b]^m$. This means that an equity price q_e exists for $(\theta, \mathscr{E})$. From (2.9), q_b can be explicitly solved, given q_e. Therefore, the asset prices exist. ▌

APPENDIX 3: PROOF OF THEOREM 1

When $m = 1$, it is the certainty case. The proof in this case is obvious. I thus only need to prove it for $m \geqslant 2$.

The price equations set (2.9) can be separated into two parts: $H(\theta, q_e, \mathscr{E}) = 0$ and

$$q_{bi} = \frac{1}{\delta}\sum_{j=1}^{m} \pi_{ij} x_j^{\alpha - 1}\left(\frac{1 + q_{ej}}{q_{ei}}\right)^{\gamma - 1}, \qquad i = 1, 2, ..., m.$$

Thus, if I can show that the q_e determined by $H = 0$ is generically determinate, then Theorem 1 is proven.

We want to prove that, for any $\theta \in \Theta$ and almost any $\mathscr{E} \in \mathbb{E}$, the equity price[6] $q(\theta, \mathscr{E})$ satisfying $H(\theta, q, \mathscr{E}) = 0$ is locally unique and smooth with respect to variables $(\theta, \mathscr{E})$. This can be done by the Implicit Function Theorem, if we can prove that the *Jacobian* determinant of $H(\theta, q, \mathscr{E})$ with respect to q valued at $(\theta, \mathscr{E})$ and $q(\theta, \mathscr{E})$ is nonzero. The *Jacobian* determinant is

$$|J_q(\theta, q, \mathscr{E})| \equiv \left|\frac{\partial H(\theta, q, \mathscr{E})}{\partial q}\right|$$
$$= \gamma^m (q_1, q_2, ..., q_m)^{\gamma - 1} \left| \Pi \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_m \end{pmatrix} - \delta I \right|,$$

where $a_i \equiv x_i^\alpha (1 + 1/q_i)^{\gamma - 1}$. Unfortunately, we cannot rule out the possibility that $|J_q(\theta, q, \mathscr{E})|$ may be zero for some $(\theta, q, \mathscr{E})$ satisfying $H(\theta, q, \mathscr{E}) = 0$. My strategy is to show that, given any $\theta \in \Theta$, the set of the economies satisfying both $|J_q(\theta, q, \mathscr{E})| = 0$ and $H(\theta, q, \mathscr{E}) = 0$ for some $q \in \mathbb{R}^m_{++}$ is a closed null set.

[6] The existence of such q is guaranteed by the proposition.

$|J_q(\theta, q, \mathscr{E})| = 0$ is equivalent to: $\exists \lambda \equiv (\lambda_1, \lambda_2, ..., \lambda_m)$, $\lambda \neq 0$, such that

$$Q(\theta, q, \lambda, \mathscr{E}) \equiv \Pi \begin{pmatrix} \lambda_1 a_1 \\ \lambda_2 a_2 \\ \vdots \\ \lambda_m a_m \end{pmatrix} - \delta \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_m \end{pmatrix} = 0.$$

Without loss of generality, I will henceforth assume that $\lambda_m = 1$. This means that $\lambda \in \mathbb{R}^{m-1}$. Let

$$K(\theta, q, \lambda, \mathscr{E}) \equiv \begin{pmatrix} H(\theta, q, \mathscr{E}) \\ Q(\theta, q, \lambda, \mathscr{E}) \end{pmatrix}$$

$$\mathscr{M}_1 \equiv \{\mathscr{E} \in \mathbb{E} \mid |J_q(\theta, q, \mathscr{E})| = 0 \text{ and } H(\theta, q, \mathscr{E}) = 0 \text{ for some } q \in \mathbb{R}^m_{++}\}$$

and

$$\mathscr{M}_6 \equiv \{\mathscr{E} \in \mathbb{E} \backslash \mathscr{M}_3 \backslash \mathscr{M}_5 \mid K(\theta, q, \lambda, \mathscr{E}) = 0 \text{ for some } (q, \lambda) \in \mathbb{R}^m_{++} \times \mathbb{R}^{m-1}\},$$

where $\mathscr{M}_3$ and $\mathscr{M}_5$ are defined in Lemmas 5 and 7, respectively. By the definition, we see that

$$\mathscr{M}_1 \subset \mathscr{M}_3 \cup \mathscr{M}_5 \cup \mathscr{M}_6.$$

By Lemmas 1 and 4, $\mathscr{M}_1$ is closed in $\mathbb{E}$. By Lemma 7, $\mathscr{M}_5$ has zero measure. By Sard's Theorem, it is very easy to see that $\mathscr{M}_3$ has zero measure too. Therefore, if I can prove that $\mathscr{M}_6$ has zero measure, then $\mathscr{M}_1$ is a closed null set.

I will proceed in two steps to finish this proof. Step 1 shows that $\mathscr{M}_6$ has zero measure; Step 2 then uses the Implicit Function Theorem to explain that $\mathscr{M}_1$ is the set that we want for the theorem.

Step 1. $\mathscr{M}_6$ (and thus $\mathscr{M}_1$) has zero measure.

Suppose that there exist some $(q, \lambda) \in \mathbb{R}^m_{++} \times \mathbb{R}^{m-1}$ and $\mathscr{E} \in \mathbb{E} \backslash \mathscr{M}_3 \backslash \mathscr{M}_5$ satisfying $K(\theta, q, \lambda, \mathscr{E}) = 0$ so that $\partial K / \partial \mathscr{E}$ does not have full rank. I am going to find a contradiction from this.

For these $(q, \lambda) \in \mathbb{R}^m_{++} \times \mathbb{R}^{m-1}$ and $\mathscr{E} \in \mathbb{E}$, all the 6×6 sub-matrices in $\partial K / \partial \mathscr{E}$ have to be degenerate. The following one is one of the simple ones that I consider. Let $\Pi^{1i} \equiv (\pi_{11}, \pi_{1i}, \pi_{21}, \pi_{2i}, ..., \pi_{m1}, \pi_{mi})$. Then, for $i = 2, 3, ..., (m-1)$, we have

$$\left|\frac{\partial K}{\partial \Pi^{1i}}\right| = \pm \begin{vmatrix} b_1 - b_m & b_i - b_m \\ \lambda_1 a_1 - a_m & \lambda_i a_i - a_m \end{vmatrix}^m = \pm \begin{vmatrix} b_1 & b_i & b_m \\ \lambda_1 a_1 & \lambda_i a_i & a_m \\ 1 & 1 & 1 \end{vmatrix}^m = 0. \quad \text{(A.3.1)}$$

The following is another simple one:

$$\left|\frac{\partial K}{\partial(x, \pi_{11}, \pi_{21}, ..., \pi_{m1})}\right|$$
$$= (-1)^m (b_1 - b_m)^m \, |\Pi| \, \alpha^m \frac{b_1 b_2 \cdots b_m}{x_1 x_2 \cdots x_m} \prod_{i=1}^{m} \left(\frac{\lambda_i a_i}{b_i} - \frac{\lambda_1 a_1 - a_m}{b_1 - b_m}\right). \qquad \text{(A.3.2)}$$

Notice that in $\mathbb{E}\backslash\mathcal{M}_3\backslash\mathcal{M}_5$, $|\Pi| \neq 0$ and $b_i \neq b_m$ for $i = 1, 2, ..., (m-1)$.

I am now going to find a contradiction if the above two types of special $(2m) \times (2m)$ sub-determinants of $|\partial K/\partial \mathscr{E}|$ are all zero.

By (A.3.1), $|\partial K/\partial \Pi^{1i}|$ implies that

$$\frac{\lambda_1 a_1 - a_m}{b_1 - b_m} = \frac{\lambda_i a_i - a_m}{b_i - b_m}$$

for $i = 2, 3, ..., (m-1)$. By (A.3.2), this in turn implies that

$$\left|\frac{\partial K}{\partial(x, \pi_{11}, \pi_{21}, ..., \pi_{m1})}\right|$$
$$= \pm (b_1 - b_m)^m \, |\Pi| \, \alpha^m \frac{b_1 b_2 \cdots b_m}{x_1 x_2 \cdots x_m} \left(\frac{a_m}{b_m} - \frac{a_m - \lambda_1 a_1}{b_m - b_1}\right)$$
$$\times \prod_{i=1}^{m-1} \left(\frac{\lambda_i a_i}{b_i} - \frac{\lambda_i a_i - a_m}{b_i - b_m}\right).$$

Thus, $|\partial K/\partial(x, \pi_{11}, \pi_{21}, ..., \pi_{m1})| = 0$ implies that $(\lambda_i a_i/b_i) - ((\lambda_i a_i - a_m)/(b_i - b_m)) = 0$ for some $i \in \{1, 2, ..., (m-1)\}$, or $(a_m/b_m) - ((a_m - \lambda_1 a_1)/(b_m - b_1)) = 0$. Since all these equalities are symmetric, without loss of generality, we can assume that

$$\frac{\lambda_1 a_1}{b_1} - \frac{\lambda_1 a_1 - a_m}{b_1 - b_m} = 0. \qquad \text{(A.3.3)}$$

By (A.3.1), $|\partial K/\partial \Pi^{12}| = 0$ also implies that: $\exists a, b, c \in \mathbb{R}$, $(a, b, c) \neq 0$, such that

$$a \begin{pmatrix} b_1 \\ b_2 \\ b_m \end{pmatrix} + b \begin{pmatrix} \lambda_1 a_1 \\ \lambda_2 a_2 \\ a_m \end{pmatrix} + c \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0 \qquad \text{(A.3.4)}$$

$\Rightarrow$

$$\begin{aligned} a(b_1 - b_m) + b(\lambda_1 a_1 - a_m) &= 0 \\ a(b_2 - b_m) + b(\lambda_2 a_2 - a_m) &= 0. \end{aligned} \qquad \text{(A.3.5)}$$

By the first equations in (A.3.5) and (A.3.3) we then have (evidently, $b=0 \Rightarrow a=c=0$; so $b \neq 0$):

$$\frac{\lambda_1 a_1}{b_1} = -\frac{a}{b}.$$

Using the first equation in (A.3.4), this implies that

$$0 = ab_1 + b\lambda_1 a_1 + c = bb_1\left(\frac{a}{b} + \lambda_1 \frac{a_1}{b_1}\right) + c = c.$$

Then, equation set (A.3.4) implies that: $\exists \mu \in \mathbb{R}$, $\mu \neq 0$, such that

$$\begin{pmatrix} b_1 \\ b_2 \\ b_m \end{pmatrix} = \mu \begin{pmatrix} \lambda_1 a_1 \\ \lambda_2 a_2 \\ a_m \end{pmatrix}.$$

Similarly, for any $i \in \{3, 4, ..., (m-1)\}$, $\exists \tilde{\mu} \in \mathbb{R}$, $\tilde{\mu} \neq 0$, such that

$$\begin{pmatrix} b_1 \\ b_i \\ b_m \end{pmatrix} = \tilde{\mu} \begin{pmatrix} \lambda_1 a_1 \\ \lambda_i a_i \\ a_m \end{pmatrix}.$$

By the fact that $b_m = \mu a_m$ and $b_m = \tilde{\mu} a_m$, we know that $\tilde{\mu} = \mu$. Therefore, $\exists \mu \in \mathbb{R}$, $\mu \neq 0$, such that

$$b_i = \mu \lambda_i a_i, \qquad i = 1, 2, ..., m. \tag{A.3.6}$$

Note that here $\lambda_m = 1$. By $K = 0$, we thus have

$$q_i^\gamma = \mu \lambda_i, \qquad i = 1, 2, ..., m.$$

Substitute this back into (A.3.6) to get

$$b_i = q_i^\gamma a_i, \qquad i = 1, 2, ..., m.$$

By the definition of a_i's and b_i's, we then have

$$\frac{q_i}{1 + q_i} = 1, \qquad i = 1, 2, ..., m.$$

This is a contradiction.

Therefore, I prove that $\partial K / \partial \mathscr{E}$ has full rank for any $(q, \lambda) \in \mathbb{R}^m_{++} \times \mathbb{R}^{m-1}$ and $\mathscr{E} \in \mathbb{E} \setminus \mathscr{M}_3 \setminus \mathscr{M}_5$ satisfying $K(\theta, q, \lambda, \mathscr{E}) = 0$, given $\theta \in \Theta$. Since $\mathscr{M}_3$ and $\mathscr{M}_5$ are closed in $\mathbb{E}$, $\mathbb{E} \setminus \mathscr{M}_3 \setminus \mathscr{M}_5$ is open in $\mathbb{E}$. By Lemma 2(1), since $\mathbb{E}$ is open in $\mathbb{R}^{m^2}$, $\mathbb{E} \setminus \mathscr{M}_3 \setminus \mathscr{M}_5$ is open in $\mathbb{R}^{m^2}$. Therefore, the mapping $K(\theta, \cdot, \cdot, \cdot)$:

$\mathbb{R}^m_{++} \times \mathbb{R}^{m-1} \times (\mathbb{E} \backslash \mathcal{M}_3 \backslash \mathcal{M}_5) \rightarrow \mathbb{R}^{2m}$ is well defined on an open set, which means that Lemma 3 is applicable to mapping K. Since the rank is $2m$ and there are only $(2m-1)$ free parameters (q, λ), by Lemma 3, $\mathcal{M}_6$ has zero measure.

Step 2. The property of the economies in $\mathbb{E} \backslash \mathcal{M}_1$.

Given $\theta \in \Theta$, for any $\mathscr{E} \in \mathbb{E} \backslash \mathcal{M}_1$ and $q \in \mathbb{R}^m_{++}$ satisfying $H(\theta, q, \mathscr{E}) = 0$, since $\mathscr{E} \notin \mathcal{M}_1$, we have

$$|J_q(\theta, q, \mathscr{E})| \equiv \left| \frac{\partial H(\theta, q, \mathscr{E})}{\partial q} \right| \neq 0.$$

By the Implicit Function Theorem, there then exist two open sets $N_{(\theta, \mathscr{E})}$ in $\mathbb{R}^{m^2+3}$ and N_q in $\mathbb{R}^m$ with $(\theta, \mathscr{E}) \in N_{(\theta, \mathscr{E})}$ and $q \in N_q$ and a unique smooth function $q^*: N_{(\theta, \mathscr{E})} \rightarrow N_q$ such that $q = q^*(\theta, \mathscr{E})$ and

$$H[\theta', q^*(\theta', \mathscr{E}'), \mathscr{E}'] = 0, \qquad \forall (\theta', \mathscr{E}') \in N_{(\theta, \mathscr{E})}.$$

This finishes the proof. ▮

APPENDIX 4: PROOF OF THEOREM 2

For the general case with $m \geqslant 3$, the proof is much more complicated. I will only present here the proof for $m \geqslant 5$. The procedure of the proof is the same as the general case $m \geqslant 3$. I will mention in this proof where the $m \leqslant 4$ case needs more work.

I will use the Implicit Function Theorem to prove this theorem. To do this, I need to show that the *Jacobian* matrix with respect to the parameters has full rank at $[\theta, q(\theta, \mathscr{E}), \mathscr{E}]$,

$$J_\theta(\theta, q, \mathscr{E}) \equiv \frac{\partial H(\theta, q, \mathscr{E})}{\partial \theta}$$

$$= \Pi \begin{pmatrix} b_1 \ln x_1 & b_1 \ln(1+q_1) & -\frac{1}{\delta} b_1 \\ b_2 \ln x_2 & b_2 \ln(1+q_2) & -\frac{1}{\delta} b_2 \\ \vdots & \vdots & \vdots \\ b_m \ln x_m & b_m \ln(1+q_m) & -\frac{1}{\delta} b_m \end{pmatrix} - \begin{pmatrix} 0 & \delta q_1^\gamma \ln q_1 & 0 \\ 0 & \delta q_2^\gamma \ln q_2 & 0 \\ \vdots & \vdots & \vdots \\ 0 & \delta q_m^\gamma \ln q_m & 0 \end{pmatrix},$$

where $H(\theta, q, \mathscr{E}) = 0$ is used. Unfortunately, we cannot rule out the possibility that $J_\theta(\theta, q, \mathscr{E})$ may not have full rank for some economies. My strategy is to show that, given any $\theta \in \Theta$, $J_\theta(\theta, q(\theta, \mathscr{E}), \mathscr{E})$ has full rank for almost all the economies in $\mathbb{E}$.

$J_\theta(\theta, q, \mathscr{E})$ not having full rank means that: $\exists \lambda,\ \mu \in \mathbb{R}$, such that

$$Q(\theta, q, \lambda, \mu, \mathscr{E}) \equiv \Pi \begin{pmatrix} b_1 d_1 \\ b_2 d_2 \\ \vdots \\ b_m d_m \end{pmatrix} - \delta \begin{pmatrix} q_1^\gamma \ln q_1 \\ q_2^\gamma \ln q_2 \\ \vdots \\ q_m^\gamma \ln q_m \end{pmatrix} = 0,$$

where $d_i \equiv \lambda \ln x_i + \ln(1 + q_i) + \mu$, $\forall i$. Given $\theta \in \Theta$, let

$$F(\theta, q, \lambda, \mu, \mathscr{E}) \equiv \begin{pmatrix} Q(\theta, q, \lambda, \mu, \mathscr{E}) \\ H(\theta, q, \mathscr{E}) \end{pmatrix},$$

$$\mathscr{M}_7 \equiv \left\{ \mathscr{E} \in \mathbb{E} \setminus \mathscr{M}_3 \mid F(\theta, q, \lambda, \mu, \mathscr{E}) = 0, \text{ and } \frac{\partial F(\theta, q, \lambda, \mu, \mathscr{E})}{\partial \mathscr{E}} \right.$$
$$\left. \text{not having full rank, for some } (q, \lambda, \mu) \in \mathbb{R}^m_{++} \times \mathbb{R}^2 \right\}$$

and

$$\mathscr{M}_8 \equiv \{ \mathscr{E} \in \mathbb{E} \setminus \mathscr{M}_3 \setminus \mathscr{M}_7 \mid H(\theta, q, \mathscr{E}) = 0 \text{ and } J_\theta(\theta, q, \mathscr{E})$$
$$\text{not having full rank, for some } q \in \mathbb{R}^m_{++} \},$$

where $\mathscr{M}_3$ is defined in Lemma 5. $J_\theta(\theta, q, \mathscr{E})$ has full rank except in $\mathscr{M}_3 \cup \mathscr{M}_7 \cup \mathscr{M}_8$. Therefore, if I can show that $\mathscr{M}_3 \cup \mathscr{M}_7 \cup \mathscr{M}_8$ is a closed null set, then it will be the set that I need to take away from $\mathbb{E}$.

By Lemmas 4 and 5, for any $\{\mathscr{E}_n\} \subset \mathscr{M}_7$, $\mathscr{E}_n \to \mathscr{E}_0 \in \mathbb{E} \setminus \mathscr{M}_3$, the $\{q^n, \lambda_n, \mu_n\}$ satisfying $F(\theta, q^n, \lambda_n, \lambda_n, \mathscr{E}_n) = 0$ is bounded. Then, by Lemma 1, $\mathscr{M}_7$ is closed in $\mathbb{E} \setminus \mathscr{M}_3$. By Lemmas 1 and 4, using an argument similar to that used for $\mathscr{M}_7$, $\mathscr{M}_8$ is closed in $\mathbb{E} \setminus \mathscr{M}_3 \setminus \mathscr{M}_7$. By Lemma 2(3), $\mathscr{M}_7 \cup \mathscr{M}_8$ is then closed in $\mathbb{E} \setminus \mathscr{M}_3$, and $\mathscr{M}_3 \cup \mathscr{M}_7 \cup \mathscr{M}_8$ is closed in $\mathbb{E}$. By Sard's Theorem, it is very easy to see that $\mathscr{M}_3$ has zero measure. What is left is to show that $\mathscr{M}_7$ and $\mathscr{M}_8$ have zero measure.

In the following first two steps, I will show that $\mathscr{M}_7$ has zero measure, and then in Step 3 I show that $\mathscr{M}_8$ has zero measure too. Step 4 finishes off this proof.

I will first define a simple set $\mathscr{M}_9$ that contains $\mathscr{M}_7$ in Step 1, and then show that $\mathscr{M}_9$ has zero measure in Step 2. In this way, $\mathscr{M}_7$ is thus proved to be of zero measure.

Step 1. Define a subset $\mathscr{M}_9$ in $\mathbb{E}$ such that $\mathscr{M}_9 \supset \mathscr{M}_7$.

I will look at one type of simple $(2m) \times (2m)$ sub-matrices in $\partial F / \partial \mathscr{E}$. $\mathscr{M}_9$ will be defined by these sub-matrices. Notice that for the $m \leqslant 4$ case, I need to consider more types of sub-matrices in $\partial F/\partial \mathscr{E}$ in order to get two more constraints on the economy. The assumption $m \geq 5$ gives us two extra constraints from $H(\theta, q, \mathscr{E})=0$ which makes the construction of $\mathscr{M}_9$ much easier.

I will discuss the rank of $\partial F/\partial \mathscr{E}$ in $\mathbb{E}\backslash\mathscr{M}_3\backslash\mathscr{M}_4\backslash\mathscr{M}_5$, where $\mathscr{M}_4$ and $\mathscr{M}_5$ are defined in Lemmas 6 and 7, respectively. That is, we have

$$b_i \neq b_m \quad \text{and} \quad q_i \neq q_m$$

for $i=1, 2, ..., (m-1)$. I will then add $\mathscr{M}_3$, $\mathscr{M}_4$, and $\mathscr{M}_5$ into $\mathscr{M}_9$ to include all the economies in $\mathscr{M}_7$.

In $\mathscr{M}_7\backslash\mathscr{M}_4$, not all d_i's are equal. If not, by $K=0$, $d_1=d_2=\cdots=d_m$ first implies that

$$d_1 \Pi \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} = \delta \begin{pmatrix} q_1^{\gamma} \ln q_1 \\ q_2^{\gamma} \ln q_2 \\ \vdots \\ q_m^{\gamma} \ln q_m \end{pmatrix}$$

and then $d_1 = \ln q_i$, $\forall i$. That is, $q_1=q_2=\cdots=q_m$. This is a contradiction. I can thus assume that

$$d_1 \neq d_m. \tag{A.4.1}$$

The following one is one of the simple ones that I can easily handle. Let $\Pi^{1i} \equiv (\pi_{11}, \pi_{1i}, \pi_{21}, \pi_{2i}, ..., \pi_{m1}, \pi_{mi})$. Then

$$\left|\frac{\partial F}{\partial \Pi^{12}}\right| = \pm \begin{vmatrix} b_1 d_1 - b_m d_m & b_2 d_2 - b_m d_m \\ b_1 - b_m & b_2 - b_m \end{vmatrix}^m.$$

For this, we see that in general, for $i=2, 3, ..., (m-1)$, $|\partial F/\partial \Pi^{1i}|=0$ imply that

$$\begin{vmatrix} b_1 d_1 - b_m d_m & b_i d_i - b_m d_m \\ b_1 - b_m & b_i - b_m \end{vmatrix}$$

$$= \begin{vmatrix} x_1^{-\alpha}(1+q_1)^{-\gamma} & x_i^{-\alpha}(1+q_i)^{-\gamma} & x_m^{-\alpha}(1+q_m)^{-\gamma} \\ \lambda \ln x_1 + \ln(1+q_1) & \lambda \ln x_i + \ln(1+q_i) & \lambda \ln x_m + \ln(1+q_m) \\ 1 & 1 & 1 \end{vmatrix} = 0. \tag{A.4.2}$$

Notice that in the above determinants I choose 1 and m as two fixed indices because of the assumption in (A.4.1). This will matter later.

For any $\mathscr{E} \in \mathscr{M}_7$, $\exists(q, \lambda, \mu) \in \mathbb{R}^m_{++} \times \mathbb{R}^2$ such that the formula in (A.4.2) and $F(\theta, q, \lambda, \mu, \mathscr{E})$ are zero. Let us see in the following what this implies.

By (A.4.2), $|\partial F/\partial \Pi^{12}| = 0$ imply that: $\exists \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$, $(\lambda_1, \lambda_2, \lambda_3) \neq 0$, such that

$$\lambda_1 x_i^{-\alpha}(1+q_i)^{-\gamma} + \lambda_2[\lambda \ln x_i + \ln(1+q_i)] + \lambda_3 = 0, \qquad \text{for} \quad i = 1, 2, m. \tag{A.4.3}$$

Let me first show that $\lambda_1 \neq 0$. If not, then $\lambda_2 \neq 0$ and for some $b \in \mathbb{R}$

$$\lambda \ln x_i + \ln(1+q_i) + b = 0, \qquad \text{for} \quad i = 1, 2, m.$$

I can then solve for q_1, q_2, and q_m:

$$q_i = Bx_i^{-\lambda} - 1, \qquad \text{for} \quad i = 1, 2, m; \text{ and some } B \in \mathbb{R}_{++}.$$

Thus,

$$d_i \equiv \lambda \ln x + \ln(Bx_i^{-\lambda}) + \mu = \mu + \ln B, \qquad \text{for} \quad i = 1, 2, m.$$

This contradicts (A.4.1). Therefore, the λ_1 cannot be zero.

Since $\lambda_1 \neq 0$, (A.4.3) implies that there exists $(a, b) \in \mathbb{R}^2$ such that

$$x_i^{-\alpha}(1+q_i)^{-\gamma} + a[\lambda \ln x_i + \ln(1+q_i)] + b = 0, \qquad \text{for} \quad i = 1, 2, m.$$

Similarly, for any $j \in \{3, 4, ..., (m-1)\}$, $|\partial F/\partial \Pi^{1j}| = 0$ implies that there exists $(\tilde{a}, \tilde{b}) \in \mathbb{R}^2$ such that

$$x_i^{-\alpha}(1+q_i)^{-\gamma} + \tilde{a}[\lambda \ln x_i + \ln(1+q_i)] + \tilde{b} = 0, \qquad \text{for} \quad i = 1, j, m.$$

The above two equation sets imply that

$$(a-\tilde{a})[\lambda \ln x_1 + \ln(1+q_1)] + b - \tilde{b} = 0$$
$$(a-\tilde{a})[\lambda \ln x_m + \ln(1+q_m)] + b - \tilde{b} = 0.$$

By (A.4.1), I thus have $a = \tilde{a}$ and then $b = \tilde{b}$. Therefore, $\exists(a, b) \in \mathbb{R}^2$ such that

$$x_i^{-\alpha}(1+q_i)^{-\gamma} + a[\lambda \ln x_i + \ln(1+q_i)] + b = 0, \qquad \text{for} \quad i = 1, 2, ..., m. \tag{A.4.4}$$

Define $z_i \equiv \ln(1+q_i)$. Then, by Lemma 8, there exist smooth functions $\bar{q}(\cdot|\theta)\colon D(\bar{q}) \to \mathbb{R}$, $\check{q}(\cdot|\theta)\colon D(\check{q}) \to \mathbb{R}$, and $\hat{q}(\cdot|\theta)\colon \mathbb{R}^4 \to \mathbb{R}$ such that the equity price satisfying (A.4.4) can only be of the following three possible types,

$$q_i = \bar{q}(x_i, a, b, \lambda \mid \theta), \qquad q_i = \check{q}(x_i, a, b, \lambda \mid \theta), \qquad \text{or} \qquad q_i = \hat{q}(x_i, a, b, \lambda \mid \theta),$$

where $D(\bar{q})$ and $D(\check{q})$ are some open sets in $\mathbb{R}^4$.

Denote $\tilde{q}$ as $\bar{q}$, $\check{q}$, or $\hat{q}$, and $\tilde{q}_i \equiv \tilde{q}(x_i, a, b, \lambda \mid \theta)$. Substitute the $\tilde{q}_i$'s into function F to get

$$\tilde{F}(\theta, a, b, \lambda, \mu, \mathscr{E}) = F[\theta, \tilde{q}(x_i, a, b, \lambda \mid \theta), \lambda, \mu, \mathscr{E}].$$

By the above derivation, $\mathscr{E} \in \mathscr{M}_7 \setminus \mathscr{M}_4 \setminus \mathscr{M}_5$ implies that there exists $(a, b, \lambda, \mu) \in \mathbb{R}^4$ such that $\tilde{F}(\theta, a, b, \lambda, \mu, \mathscr{E}) = 0$. Denote

$$\mathscr{M}_{10} \equiv \{\mathscr{E} \in \mathbb{E} \setminus \mathscr{M}_3 \setminus \mathscr{M}_4 \setminus \mathscr{M}_5 \mid \tilde{F}(\theta, a, b, \lambda, \mu, \mathscr{E}) = 0 \text{ for some } (a, b, \lambda, \mu) \in \mathbb{R}^4\}.$$

Let

$$\mathscr{M}_9 \equiv \mathscr{M}_{10} \cup \mathscr{M}_3 \cup \mathscr{M}_4 \cup \mathscr{M}_5.$$

If I can prove that $\mathscr{M}_{10}$ has zero measure, as $\mathscr{M}_9 \supset \mathscr{M}_7$ by the definition, then $\mathscr{M}_7$ has zero measure (we already know that $\mathscr{M}_3$, $\mathscr{M}_4$, and $\mathscr{M}_5$ have zero measure).

Note that, in the definition of $\mathscr{M}_9$, we should think of $\mathscr{M}_{10}$ as a union of several sets like $\mathscr{M}_{10}$ as $\tilde{q}$ takes three possible forms $\bar{q}$, $\check{q}$, and $\hat{q}$.

Step 2. $\mathscr{M}_{10}$ has zero measure.

Since $\tilde{F}$ has four free parameters (a, b, λ, μ), to prove that $\mathscr{M}_{10}$ has zero measure, I need to show that $\partial \tilde{F}/\partial \mathscr{E}$ has a rank $\geqslant 5$ for any $(a, b, \lambda, \mu) \in \mathbb{R}^4$ and $\mathscr{E} \in \mathbb{E} \setminus \mathscr{M}_3 \setminus \mathscr{M}_4 \setminus \mathscr{M}_5$ satisfying $\tilde{F}(\theta, a, b, \lambda, \mu, \mathscr{E}) = 0$.

Let the corresponding b_i's and d_i's for the equity price $\tilde{q}_i$ be denoted as $\tilde{b}_i$'s and $\tilde{d}_i$'s, respectively. Since $\tilde{b}_i$'s and $\tilde{d}_i$'s do not depend on Π, we have,

$$\frac{\partial \tilde{F}}{\partial(\pi_{11}, \pi_{21}, ..., \pi_{m1})} = \begin{pmatrix} (\tilde{b}_1 \tilde{d}_1 - \tilde{b}_m \tilde{d}_m)\, I_m \\ (\tilde{b}_1 - \tilde{b}_m)\, I_m \end{pmatrix},$$

where I_m is the $m \times m$ identity matrix. Since

$$|(\tilde{b}_1 - \tilde{b}_m)\, I_m| = (\tilde{b}_1 - \tilde{b}_m)^m \neq 0$$

for any $(a, b, \lambda, \mu) \in \mathbb{R}^4$ and $\mathscr{E} \in \mathbb{E} \setminus \mathscr{M}_3 \setminus \mathscr{M}_4 \setminus \mathscr{M}_5$ satisfying $\tilde{F}(\theta, a, b, \lambda, \mu, \mathscr{E}) = 0$, $\partial \tilde{F}/\partial(\pi_{11}, \pi_{21}, ..., \pi_{m1})$ thus has full rank $(= m > 4)$ for any parameters $(a, b, \lambda, \mu) \in \mathbb{R}^4$ and economy $\mathscr{E} \in \mathbb{E} \setminus \mathscr{M}_3 \setminus \mathscr{M}_4 \setminus \mathscr{M}_5$ satisfying $\tilde{F}(\theta, a, b, \lambda, \mu, \mathscr{E}) = 0$. Notice that in $\mathbb{E} \setminus \mathscr{M}_3 \setminus \mathscr{M}_4 \setminus \mathscr{M}_5$, we have $b_i \neq b_m$ for any equity price. By Lemma 6, $\mathscr{M}_4$ is closed in $\mathbb{E} \setminus \mathscr{M}_3$, and thus by Lemma 2(3) $\mathscr{M}_4 \cup \mathscr{M}_3$ is closed in $\mathbb{E}$. Therefore, also by Lemma 2, $\mathbb{E}\setminus\mathcal{M}_3\setminus\mathcal{M}_4\setminus\mathcal{M}_5$ is open in $\mathbb{R}^{m l^2}$. Thus, by Lemma 3, $\mathcal{M}_{10}$ has zero measure.

Since $\mathcal{M}_{10}$, $\mathcal{M}_3$, $\mathcal{M}_4$, and $\mathcal{M}_5$ have zero measure, $\mathcal{M}_9$ has zero measure too. As $\mathcal{M}_9 \supset \mathcal{M}_7$, $\mathcal{M}_7$ has zero measure.

Notice that for the $m \leqslant 4$ case, we need two more columns in $\partial \tilde{F}/\partial \mathscr{E}$. For this, we need to consider more types of simple sub-matrices in $\partial F/\partial \mathscr{E}$ such as those corresponding to the combinations of partial derivatives with respect to π_{ij}'s and x_i's in order to get more constraints on the economy. This extra work is very difficult. The assumption $m \geqslant 5$ gives us two extra constraints from $H(\theta, q, \mathscr{E}) = 0$ which makes this proof much easier.

Step 3. The inverse problem.

By the definition of $\mathcal{M}_7$, $\partial F/\partial \mathscr{E}$ has full rank for any $(q, \lambda, \mu) \in \mathbb{R}^m_{++} \times \mathbb{R}^2$ and $\mathscr{E} \in \mathcal{M}_7$ satisfying $F(\theta, q, \lambda, \mu, \mathscr{E}) = 0$. Since $\mathcal{M}_7$ is closed in $\mathbb{E}\setminus\mathcal{M}_3$, $\mathcal{M}_3 \cup \mathcal{M}_7$ is closed in $\mathbb{E}$, and thus $\mathbb{E}\setminus\mathcal{M}_3\setminus\mathcal{M}_7$ is open in $\mathbb{E}$. By Lemma 2(1), $\mathbb{E}\setminus\mathcal{M}_3\setminus\mathcal{M}_7$ is open in $\mathbb{R}^{m l^2}$. This means that Lemma 3 is applicable to the mapping $F(\theta, \cdot, \cdot, \cdot, \cdot)$: $\mathbb{R}^m_{++} \times \mathbb{R}^2 \times (\mathbb{E}\setminus\mathcal{M}_3\setminus\mathcal{M}_7) \to \mathbb{R}^{2m}$. By the definition, we have

$$\mathcal{M}_8 = \{\mathscr{E} \in \mathbb{E}\setminus\mathcal{M}_3\setminus\mathcal{M}_7 \mid F(\theta, q, \lambda, \mu, \mathscr{E}) = 0 \text{ for some } (q, \lambda, \mu) \in \mathbb{R}^m_{++} \times \mathbb{R}^2\}.$$

Because the full rank $2m$ of $\partial F/\partial \mathscr{E}$ is greater than the number $(m+2)$ of free parameters (q, a, b), by Lemma 3, $\mathcal{M}_8$ has zero measure.

I have finally proved that $\mathcal{M}_7$ and $\mathcal{M}_8$ are null sets. By this, I can now go back to the beginning of this proof.

Take

$$\mathcal{M}_2 \equiv \mathcal{M}_1 \cup \mathcal{M}_3 \cup \mathcal{M}_7 \cup \mathcal{M}_8,$$

where $\mathcal{M}_1$ is defined in Theorem 1. First, since $\mathcal{M}_1$, $\mathcal{M}_3$, $\mathcal{M}_7$, and $\mathcal{M}_8$ all have zero measure, $\mathcal{M}_2$ has zero measure too. Second, as explained at the beginning, $\mathcal{M}_2$ is a closed subset in $\mathbb{E}$. For all the economies in $\mathbb{E}\setminus\mathcal{M}_2$, I now discuss the inverse problem.

Given $\theta \in \Theta$, for any $\mathscr{E} \in \mathbb{E}\setminus\mathcal{M}_2$, by Proposition 1, we have a q satisfying $H(\theta, q, \mathscr{E}) = 0$. Since $\mathscr{E} \notin \mathcal{M}_3 \cup \mathcal{M}_7 \cup \mathcal{M}_8$, $J_\theta(\theta, q, \mathscr{E}) \equiv \partial H(\theta, q, \mathscr{E})/\partial \theta$ has full rank $(=3)$. This means that there is a 3×3 sub-matrix in $J_\theta(\theta, q, \mathscr{E})$ having nonzero determinant. For simplicity, assume that this sub-matrix consists of the first three rows in $J_\theta(\theta, q, \mathscr{E})$. Let

$$H^{123}(\theta, q, \mathscr{E}) \equiv \begin{pmatrix} H_1(\theta, q, \mathscr{E}) \\ H_2(\theta, q, \mathscr{E}) \\ H_3(\theta, q, \mathscr{E}) \end{pmatrix}.$$

We then have

$$\left|\frac{\partial H^{123}(\theta, q, \mathscr{E})}{\partial \theta}\right| \neq 0.$$

Then, by the Implicit Function Theorem, there exist two open sets $N_{(q,\mathscr{E})} \subset \mathbb{R}^{m(m+1)}$ and $N_\theta \subset \mathbb{R}^3$ with $(q, \mathscr{E}) \in N_{(q,\mathscr{E})}$ and $\theta \in N_\theta$ and a unique smooth function $\theta^*: N_{(q,\mathscr{E})} \to N_\theta$ such that $\theta = \theta^*(q, \mathscr{E})$ and

$$H^{123}[\theta^*(q', \mathscr{E}'), q', \mathscr{E}'] = 0, \qquad \forall (q', \mathscr{E}') \in N_{(q,\mathscr{E})}.$$

Step 4. Revealingness.

I am now going to show that all the economies in $\mathbb{E}\backslash\mathscr{M}_2$ are revealing.

For any $\mathscr{E} \in \mathbb{E}\backslash\mathscr{M}_2$, by Proposition 1, there is an equity price q satisfying $H(\theta, q, \mathscr{E}) = 0$. Since $\mathscr{E} \notin \mathscr{M}_1$, by Theorem 1, there exist open neighborhoods $N_\theta \subset \mathbb{R}^3$ and $N_q \subset \mathbb{R}^m$ of θ and q, respectively, and a unique smooth equity price function $q^*(\cdot \mid \mathscr{E}): N_\theta \to N_q$ such that $q = q^*(\theta \mid \mathscr{E})$, and

$$H[\theta', q^*(\theta' \mid \mathscr{E}), \mathscr{E}] = 0, \qquad \forall \theta' \in N_\theta. \tag{A.4.5}$$

Since $\mathscr{E} \notin \mathscr{M}_3 \cup \mathscr{M}_7 \cup \mathscr{M}_8$, by the last step, there exist open neighborhoods $V_q \subset \mathbb{R}^m$ and $V_\theta \subset \mathbb{R}^3$ of q and θ, respectively, and a unique smooth parameters function $\theta^*(\cdot \mid \mathscr{E}): V_q \to V_\theta$ such that $\theta = \theta^*(q \mid \mathscr{E})$, and

$$H^{123}(\theta', q', \mathscr{E}) = 0 \quad \text{iff} \quad \theta' = \theta^*(q' \mid \mathscr{E}) \tag{A.4.6}$$

for all $q' \in V_q$ and $\theta' \in V_\theta$.

Now take $\tilde{N}_\theta \equiv N_\theta \cap V_\theta \cap q^{*-1}(N_q \cap V_q)$. It is obvious that $\tilde{N}_\theta$ is an open neighborhood of θ. For any $\theta' \in \tilde{N}_\theta$, let $q' \equiv q^*(\theta' \mid \mathscr{E})$; since $\theta' \in N_\theta$, by (A.4.5), we have

$$H^{123}(\theta', q', \mathscr{E}) = 0.$$

Since $\theta' \in V_\theta$ and $q' \in V_q$, by (A.4.6), we have

$$\theta' = \theta^*(q' \mid \mathscr{E}).$$

That is, for all $\theta' \in \tilde{N}_\theta$, we have

$$\theta' = \theta^*[q^*(\theta' \mid \mathscr{E}) \mid \mathscr{E}].$$

From this, we immediately see that in the neighborhood $\tilde{N}_\theta$ of θ, different θ's give different q's. That is, the economy $\mathscr{E}$ is revealing. Therefore, all the economies in $\mathbb{E}\backslash\mathscr{M}_2$ are (locally) revealing. This finishes the proof. ∎

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