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# Economic Growth with Intergenerational Altruism

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We consider the properties of equilibrium behaviour in an aggregative growth model with intergenerational altruism. Various positive properties such as the cyclicity of equilibrium programs, and the convergence of equilibrium stocks to a steady state, are analyzed. Among other normative properties, it is established that under certain natural conditions, Nash equilibrium programs are efficient and "modified Pareto optimal", in a sense made clear in the paper, but never Pareto optimal in the traditional sense.

## 1. INTRODUCTION

In this paper, we study the properties of equilibria in an aggregative growth model with intergenerational altruism.[1] In this model, each generation is active for a single period. At the beginning of this period, it receives an endowment of a single homogeneous good which is the output from a "bequest investment" made by the previous generation. It divides the endowment between consumption and investment. The return from this investment constitutes the endowment of the next generation. Each generation derives utility from its own consumption and that of its immediate successor. However, since altruism is *limited*, in the sense that no generation cares about later successors, the interests of distinct agents come into conflict.

Models of this type have been used to analyze a number of issues concerning intergenerational altruism. One line of research, pursued by Arrow (1973) and Dasgupta (1974*a*) elucidates the implications of Rawls' principle of just savings. Others, beginning with Phelps and Pollak (1968), have addressed the question of how an "altruistic growth economy" might actually evolve over time. Topics of subsequent investigation have included the efficiency and optimality of equilibrium programs, and the implications of intergenerational altruism for the distribution of wealth.

Unfortunately, the positive features of equilibrium programs have received little attention from previous authors. Aside from a few comments by Kohlberg (1976), virtually nothing is known about the asymptotic behaviour of capital stocks in these models. In particular, will the long-run capital stock which arises from intergenerational conflict be higher or lower than the "turnpike" associated with the solution to the optimal planning problem? On *a priori* grounds, the answer is not clear. Agents who take only a limited interest in the future will tend to bequeath less than those who are far-sighted. However, since each generation views its children's bequest as pure waste, it must bequeath a larger sum to obtain the same consumption value.

In this paper, we obtain steady-state results for equilibrium capital stocks completely analogous to the well-known optimal planning results. By comparing "steady-states", we show that no limit point of equilibrium capital stocks can exceed the planning turnpike. Under slightly more restrictive conditions, we show that the equilibrium capital stock never exceeds the planning stock in any period.

We also address a number of normative issues which have been raised by other authors. In particular, Dasgupta (1974*b*) has shown that, for a specific parameterization of the general model, equilibrium programs are never Pareto optimal. Nevertheless, Lane and Mitra (1981) have established (under appropriate conditions) the existence of equilibrium programs which are Pareto optimal in a modified sense. Unfortunately, their study employs a restrictive notion of equilibrium, which typically implies a certain amount of myopic behaviour. In general, the equilibria which they consider correspond to subgame perfect equilibria only under very specific parametrizations of the model. In this paper, we establish that, under very general conditions, the set of Markov perfect equilibrium[2] programs for this class of altruistic growth economies are always efficient and modified Pareto optimal, but never Pareto optimal in the traditional sense. These results do not depend upon parametric forms.

The current paper is organized as follows. Section 2 displays the model, basic assumptions and definitions of equilibria, and reviews some important results which appear in Bernheim and Ray (1983) and Leininger (1983). Positive aspects of equilibrium programs are considered in Section 3; normative aspects are discussed in Section 4. All proofs are deferred to Section 5.

## 2. THE MODEL

The model is closely related to those of Arrow (1973), Dasgupta (1974a, b), Kohlberg (1976), Lane and Mitra (1981), and Leininger (1983), and is a special case of the stationary altruistic growth economy described in Bernheim and Ray (1983). There is one commodity, which may be consumed or invested. The transformation of capital stock into output takes one period, and is represented by a *production function* $f$. In the following sections, certain results require only one weak assumption about $f$:

*Assumption* 1. $f: R_+ \to R_+$ is increasing, continuous and $f(0) = 0$.

To establish other results, we strengthen this assumption by adding combinations of the following additional restrictions:

*Assumption* 2. $f$ is continuously differentiable, and $\lim_{k\to 0} f'(k) > 1$.

*Assumption* 3.1. $f$ is concave.

*Assumption* 3.2. $f$ is strictly concave.

In each time period, decisions concerning production and consumption are made by a fresh generation. Thus, generation $t$ is endowed with some initial output $(y_t)$, which it divides between consumption $(c_t)$ and investment $(k_t = y_t - c_t)$. Each generation derives utility from its own consumption, and the consumption of the generation immediately succeeding it. Preferences are represented by a common utility function, $w$. We assume that $w$ satisfies certain relatively weak conditions:[3]

*Assumption* 4. $w: R_+^2 \to R$ is increasing and continuous. Moreover, given $(c_t', c_t'', c_{t+1}) \geqq 0$ with $c_t' > c_t''$, if, for some $(\varepsilon, \eta) \gg 0$, $w(c_t', c_{t+1}) \leqq w(c_t' + \varepsilon, c_{t+1} - \eta)$, then $w(c_t'', c_{t+1}) < w(c_t'' + \varepsilon, c_{t+1} - \eta)$.

In most of the relevant literature, a stronger version of Assumption 4 is employed. At some points, we adopt this more restrictive formulation, in part for technical reasons, and in part to facilitate a comparison of equilibrium and planning programs. Specifically,

*Assumption* 5. There exists an increasing, continuously differentiable, strictly concave function $v: R_+ \to R$ with $v(c) \to \infty$ as $c \to \infty$ and a discount factor $\delta > 0$ such that $w(c_t, c_{t+1}) = v(c_t) + \delta v(c_{t+1})$ for $(c_t, c_{t+1}) \in R_+^2$.

For certain results (particularly those concerning comparisons between equilibrium and planning programs) it will be convenient to assume that agents discount the future at some positive rate. In these cases, we will impose one of the following restrictions on $\delta$.

*Assumption* 5.1. $\delta \in (0, 1]$.

*Assumption* 5.2. $\delta \in (0, 1)$.

Finally, to prove certain results, we employ the following assumption concerning the relationship between production and utility.

*Assumption* 6. $\lim_{k\to\infty} f'(k) < \delta^{-1}$.

*Remark.* A sufficient condition for Assumption 6 under either Assumption 5.1 or Assumption 5.2 is that the production function eventually cross (and stay below) the 45° line. This assumption rules out the case, $\lim_{k\to\infty} f'(k) > \delta^{-1}$, but for most of the results presented here, our techniques are readily applicable to that situation.[4] See Kohlberg (1976) for a partial analysis of the "utility-productive" case when $f$ is linear.[5]

We take the historically given initial output at time zero, $y$, to lie in some compact interval $[0, Y]$, $Y > 0$. A program $\langle y_t, c_t, k_t \rangle_0^\infty$ is *feasible* from $y \in [0, Y]$ if

$$\begin{aligned} &y_0 = y \\ &y_t = c_t + k_t, \qquad t \geqq 0 \\ &y_{t+1} = f(k_t), \qquad t \geqq 0 \\ &(y_t, c_t, k_t) \geqq 0, \quad t \geqq 0. \end{aligned}$$

Denote by $\langle c_t \rangle_0^\infty$ the corresponding *feasible consumption program.* The *pure accumulation program* is a sequence $\langle \bar{y}_t, \bar{c}_t, \bar{k}_t \rangle_0^\infty$ with $\bar{c}_t = 0$ for all $t \geqq 0$, $\bar{y}_t = \bar{k}_t$ for all $t \geqq 0$, $\bar{y}_{t+1} = f(\bar{k}_t)$ for all $t \geqq 0$, and $\bar{y}_0 = y$.

Define $S_t$ as the set of *savings functions* $s: [0, \bar{y}_t] \to [0, \bar{y}_t]$, with $s(y) \leqq y$ for all $y \in [0, \bar{y}_t]$. Define $W(y, k; s_{t+1}) = w(y - k, f(k) - s_{t+1}(f(k)))$ for all $s_{t+1} \in S_{t+1}$, and $(y, k) \geqq 0$ with $k \leqq y \leqq \bar{y}_t$.

We will impose the behavioural assumption that all generations select subgame perfect Nash strategies (see Selten (1975)). Formally,

*Definition.* The sequence $\langle s_t^* \rangle_0^\infty$, $s_t^* \in S_t$, $t \geqq 0$, is a bequest equilibrium (or simply, equilibrium) if for all $t \geqq 0$ and $y \in [0, \bar{y}_t]$,

$$s_t^*(y) \in \arg\max_{0 \leqq k \leqq y} W(y, k; s_{t+1}^*)$$

A bequest equilibrium is *stationary* if the equilibrium savings functions $\langle s_t^* \rangle_0^\infty$ satisfy $s_t^*(y) = s_{t+1}^*(y)$ for all $y \in [0, \bar{y}_t]$, $t \geqq 0$.

The reader should be aware that although we have restricted attention to the class of strategies for which consumption depends only upon initial endowment (Markov perfect equilibria), our bequest equilibria continue to be perfect equilibria when all restrictions on strategic choice are removed. See Bernheim and Ray (1983) for a more complete discussion.

Under Assumptions 1 and 4, existence of a stationary equilibrium is guaranteed within the class of *monotonic* (non-decreasing) lower semicontinuous savings functions; furthermore, equilibrium policy functions are always monotonic, regardless of whether they are associated with stationary or nonstationary equilibria (see Bernheim and Ray (1983) and Leininger (1983)). Since we use this monotonicity property extensively throughout the current paper, we restate the relevant theorem here, without proof.

**Theorem 2.1.** *Suppose that for some savings function $s_{t+1} \in S_{t+1}$, and optimal savings function for generation $t$, $s_t \in S_t$, given by*

$$s_t(y) \in \arg\max_{0 \leqq k \leqq y} W(y, k; s_{t+1}), \qquad y \in [0, \bar{y}_t]$$

*is well defined. Then under Assumptions 1 and 4, $s_t$ is monotonic.*

*Remark.* This theorem establishes that $t$'s best response to any policy function for $t$'s successor is monotonic, regardless of whether $t$'s successor employs a monotonic function. Monotonicity of equilibrium policy functions follows as an immediate corollary.

Finally, define, for each $s_t \in S_t$, $\bar{y}_t \geqq y \geqq 0$ and $\varepsilon > 0$,

$$\lambda(s_t, y, \varepsilon) \equiv \inf\left\{ \frac{s_t(y^2) - s_t(y^1)}{y^2 - y^1} \,\middle|\, (y^1, y^2) \geqq 0 \text{ and } y - \varepsilon \leqq y^1 < y^2 \leqq y + \varepsilon \right\}.$$

Clearly, if $s_t$ is an equilibrium savings function, $\lambda(s_t, y, \varepsilon) \geqq 0$ for all $y \in [0, \bar{y}_t]$ and $\varepsilon > 0$. For one result, we will wish to rule out the case of equality.

*Definition.* Suppose, for some $s_t \in S_t$ and $y \in [0, \bar{y}_t]$, there is $\varepsilon$ such that $\lambda(s_t, y, \varepsilon) > 0$. Then we say that $s_t$ is *strictly monotonic* at $y$.

Surprisingly, it is difficult to guarantee that equilibrium policy functions are strictly monotonic at all $y$. Specifically, Leininger (1983) has established that if any equilibrium savings schedule is discontinuous, then all previous savings schedules have intervals over which they are not strictly monotonic. This observation is important, since no one has yet been able to establish existence within the class of continuous savings schedules. Fortunately, strict monotonicity plays a small role in the following sections.

## 3. POSITIVE BEHAVIOUR

In intertemporal optimal planning models, an important characteristic of optimum capital stocks and consumption levels is that these converge, over time, to some stationary input-output-consumption configuration. In this section, we establish some analogous results for the limiting behaviour of capital stocks under a bequest equilibrium.

In stationary models, *stationary* equilibria always exist (Bernheim and Ray (1983), Leininger (1983)). Of course, this does not preclude the existence of non-stationary equilibria in such models. Of particular interest for asymptotic stock behaviour are *periodic* non-stationary equilibria.

*Definition.* An equilibrium $\langle s_t^* \rangle_0^\infty$ is *periodic* if there exists an integer $T$ and $T$ functions $(\bar{s}_1, \ldots, \bar{s}_T)$ such that $s_{t+Tn}^* = \bar{s}_t$, $n = 0, 1, 2, \ldots$; $t = 1, \ldots, T$. The integer $T$ is the *period* of the equilibrium.

When equilibria are non-stationary, the intertemporal behaviour of stocks is governed by a non-stationary process, even though the underlying model is stationary. In these situations, while limiting stocks may not exhibit convergence, a bound on their oscillatory behaviour may be obtained.

**Theorem 3.1.** *Suppose that $\langle s_t^* \rangle_0^\infty$ is a periodic equilibrium (with period $T$). Then under Assumptions* 1 *and* 4 *the sequence of equilibrium stocks has at most $T$ limit points in $R \cup \{+\infty\}$.*

Theorem 3.1 immediately yields a steady state result for stationary equilibria.[6]

**Corollary 3.1** (Steady State Theorem for Stationary Bequest Equilibria). *Suppose that $\langle s_t^* \rangle_0^\infty$ is a stationary equilibrium with equilibrium stocks $\langle k_t^* \rangle_0^\infty$. Then under Assumptions* 1 *and* 4 *capital stocks are monotonic in time, and* $\lim_{t\to\infty} k_t^* \equiv k^*$ *exists in $R \cup \{+\infty\}$.*

*Remark.* The steady state theorem has been obtained *without* assuming separability or concavity of the utility functions, or convexity of the technology. In these respects, compare the results to that obtained by Mitra and Ray (1984) for planning models.

We now turn to a comparison of limiting capital stocks for bequest equilibrium with turnpike levels obtained in aggregative planning models. An omniscient planner who takes into account the infinite stream of utilities of all generations is clearly acting more farsighted than a single generation which only cares about the consumption of its successor. On that score, one would expect a larger stock to be generated in the long-run, under planning. However, while each generation cares only about its successor, it *recognizes* that its successor will do the same, and, in anticipating bequests to be made by the successor, may compensate by bequeathing a larger amount. This tends to *increase* the limiting stock under a bequest equilibrium. The question of which steady state is larger, is, therefore, non-trivial.

To facilitate comparison, assume that Assumption 5 holds. In the corresponding planned economy version of the model, a "planner" seeks a feasible consumption program $\langle \hat{c}_t \rangle_0^\infty$ such that for all feasible consumption programs $\langle c_t \rangle_0^\infty$,

$$\liminf_{T\to\infty} \sum_{t=0}^{T} \delta^t [v(\hat{c}_t) - v(c_t)] \geqq 0, \tag{3.1}$$

or, if all feasible utility sums converge, the planner maximizes, subject to feasibility constraints,

$$\sum_{t=0}^{\infty} \delta^t v(c_t). \tag{3.2}$$

Call such a maximizing program an *optimal* program.

That this maximization process adequately represents the corresponding planned economy may be rationalized in two ways. First, we may simply envisage a *formal* comparison between two economies, identical in technology and one-period utilities; the one governed by two-period bequest motives, the other by an omniscient planner whose social welfare function is expressible as (3.2), or the form implicit in (3.1). Secondly, we can imagine all consumption choices in the altruistic growth economy being left to the planner who has the same discount factor $\delta$ as each generation. In this case, the planner replaces the maximization of (3.2) by[7]

$$\max_{(c_t)_0^\infty \text{ feasible}} v(c_0)+\sum_{t=0}^{\infty}[v(c_t)+\delta v(c_{t+1})]\delta^t \qquad (3.3)$$

(the inclusion of $v(c_0)$ separately signifies that the planner also cares for the utility of generation $-1$). But this is simply a scalar multiple of (3.2).

We now state, without proof, a well-known turnpike theorem for the planning problem ((3.1) or (3.2)).[8]

**Theorem 3.2** (Turnpike Theorem Under Optimal Planning). *Under Assumptions* 1, 2, 3.2, 5, 5.1, *and* 6, *an optimal program with stocks* $\langle \hat{k}_t\rangle_0^\infty$ *exists. The sequence of stocks* $\langle \hat{k}_t\rangle_0^\infty$ *converges, as* $t\to\infty$, *to a limit stock* $\hat{k}\in[0,\infty)$. *If* $\hat{k}>0$, *it solves the equation* $\delta f'(\hat{k})=1$. *If* $\delta \lim_{k\downarrow 0} f'(k)>1$, *then* $\hat{k}>0$.

Theorem 3.3 establishes a general result on the relative asymptotic behaviour of $\langle k_t^*\rangle_0^\infty$ and $\langle \hat{k}_t\rangle_0^\infty$. For stationary bequest equilibria, the comparison between $\hat{k}$ and $k^*$ is then obtained as an immediate corollary.

**Theorem 3.3.** *Under Assumptions* 1, 2, 3.2, 5, 5.1, *and* 6, *suppose that* $\langle s_t^*\rangle_0^\infty$ *is a bequest equilibrium with stocks* $\langle k_t^*\rangle_0^\infty$. *Then* $\limsup_{t\to\infty} k_t^*\leqq \hat{k}$, *the planning turnpike.*

**Corollary 3.2.** *Under Assumptions* 1, 2, 3.2, 5, 5.1, *and* 6, *a stationary bequest equilibrium with limiting capital stock* $k^*$ *has the property* $k^*\leqq\hat{k}$.

Theorem 3.3 establishes that, *in the limit*, a planned economy must accumulate at least as much capital as an altruistic growth economy (whether it accumulates strictly more remains an open question). If we assume strict discounting, then we obtain a much stronger result: bequest equilibrium capital stocks do not exceed optimal planning stocks in any period, given the same initial output.[9]

**Theorem 3.4.** *Under Assumptions* 1, 2, 3.2, 5, 5.2 *and* 6, *let* $\langle y_t^*, k_t^*, c_t^*\rangle_0^\infty$ *be a program originating from* $y_0\in(0, Y]$, *generated by the bequest equilibrium* $\langle s_t^*\rangle_0^\infty$. *Then, for all* $t\geqq 0$, $k_t^*\leqq\hat{k}_t$, *where* $\langle\hat{k}_t\rangle_0^\infty$ *is the sequence of optimal planning stocks.*

## 4. NORMATIVE BEHAVIOUR

Following the existing literature, we consider three normative notions: efficiency, Pareto optimality, and modified Pareto optimality. Formal definitions follow:

*Definition.* A feasible program $\langle y_t, k_t, c_t\rangle_0^\infty$ from $y_0\in(0, Y]$ is *efficient* if there does not exist a feasible program $\langle y_t', k_t', c_t'\rangle_0^\infty$ with $c_t'\geqq c_t$ for all $t\geqq 0$, and $c_s'>c_s$ for some $s\geqq 0$.

*Definition.* A feasible program $\langle y_t, k_t, c_t\rangle_0^\infty$ from $y_0\in(0, Y]$ is *Pareto-optimal* if there does not exist a feasible program $\langle y_t', k_t', c_t'\rangle_0^\infty$ with $w(c_t', c_{t+1}')\geqq w(c_t, c_{t+1})$ for all $t\geqq 0$, and $w(c_s', c_{s+1}')>w(c_s, c_{s+1})$ for some $s\geqq 0$.

*Definition.* A feasible program $\langle y_t, k_t, c_t\rangle_0^\infty$ from $y_0\in(0, Y]$ is *modified-Pareto-optimal* if there does not exist a feasible program $\langle y_t', k_t', c_t'\rangle_0^\infty$ with $w(c_t', c_{t+1}')\geqq w(c_t, c_{t+1})$ for all $t\geqq 0$, $w(c_s', c_{s+1}')>w(c_s, c_{s+1})$ for some $s\geqq 0$, and $c_0'\geqq c_0$.

The definitions of efficiency and Pareto optimality are standard. The notion of modified Pareto optimality is due to Lane and Mitra (1981). The restriction that $c_0 \geqq c'_0$ for any comparison program $\langle y'_t, k'_t, c'_t\rangle_0^\infty$ reflects the recognition that time 0 is *not* the beginning of all mankind, and therefore, in considering Pareto dominance, the utility of generation $-1$ (which depends on $c_0$) must not be tampered with, or at least must not be reduced.

Given Theorem 3.3, it is possible to establish the efficiency of equilibrium programs by applying known results.

**Theorem 4.1.** *Under Assumptions* 1, 2, 3.2, 5, 5.2, *and* 6, *if* $\langle y_t, c_t, k_t\rangle_0^\infty$ *is a feasible program from* $y_0$ *generated by some bequest equilibrium* $\langle s_t^*\rangle_0^\infty$, *then it is efficient.*

Since the utility of each generation depends on its own consumption as well as that of its successor, efficiency in consumption does not guarantee Pareto optimality. In fact, as long as the marginal propensity to consume of generation 1 is less than unity, a transfer of consumption from generation 0 to generation 1 always yields a Pareto dominating allocation. In this way, we establish

**Theorem 4.2.** *Under Assumptions* 1, 2, *and* 5, *assume* $\langle y_t, k_t, c_t\rangle_0^\infty$ *is a feasible program from* $y_0$ *generated by a bequest equilibrium* $\langle s_t^*\rangle_0^\infty$. *Then if* $s_1^*$ *is strictly monotonic at* $y_1$, *and if* $k_0 > 0$, $c_0 > 0$, $\langle y_t, k_t, c_t\rangle_0^\infty$ *is not Pareto optimal.*

As mentioned earlier, $s_1^*$ may not be strictly monotonic over certain intervals; therefore, we cannot generally guarantee that the equilibrium program is Pareto sub-optimal for all initial conditions. However, it is important to note that, while strict monotonicity at $y_1$ is a sufficient condition for suboptimality, it is not a necessary condition. Indeed, with substantial work, one can obtain a stronger result: if $s_t^*$ is strictly monotonic at $y_t$ for *any odd t*, then, under the other conditions stated above, the equilibrium is not Pareto optimal. We believe (but have not proven) that this weaker condition is necessary as well as sufficient for domination to be possible.

Of course, a scheme for dominating the equilibrium program by lowering $c_0$ leaves generation $-1$ strictly worse off. If we rule out alternatives which are damaging to this pre-historic generation, it becomes impossible to dominate efficient equilibrium programs. The efficiency of these programs alone is sufficient to guarantee modified Pareto optimality. This is stated in

**Theorem 4.3.** *Let* $\langle y_t, k_t, c_t\rangle_0^\infty$ *be a feasible program from* $y_0$ *associated with some bequest equilibrium* $\langle s_t^*\rangle_0^\infty$. *Under Assumptions* 1, 2, 3.1, *and* 5, $\langle y_t, k_t, c_t\rangle_0^\infty$ *is modified Pareto optimal if and only if it is efficient.*

Coupling Theorems 4.1 and 4.3, we obtain as an immediate corollary:

**Corollary 4.1.** *Let* $\langle y_t, k_t, c_t\rangle_0^\infty$ *be a feasible program* $y_0$ *associated with some bequest equilibrium* $\langle s_t^*\rangle_0^\infty$. *Under Assumptions* 1, 2, 3.2, 5, 5.2, *and* 6, $\langle y_t, k_t, c_t\rangle_0^\infty$ *is modified Pareto optimal.*

## 5. PROOFS

**Lemma 5.1.** *Suppose that* $\langle s_t^*\rangle_0^\infty$ *is an equilibrium, and let* $y_0$, $y'_0$ *be two initial output levels. Let* $\langle k_t\rangle_0^\infty$, $\langle k'_t\rangle_0^\infty$ *be the corresponding sequence of capital stocks. Then, if* $y_0 \leqq y'_0$ (or $k_0 \leqq k'_0$), $k_t \leqq k'_t$ *for all* $t > 0$.

*Proof.* Since $y_0 \leqq y'_0$, $k_0 = s^*_0(y_0) \leqq s^*_0(y'_0) = k'_0$, by Theorem 2.1. Now proceed by induction. Let $k_T \leqq k'_T$ for some $T \geqq 0$. Then, since $f_T$ is increasing, $y_{T+1} = f(k_T) \leqq f(k'_T) = y'_{T+1}$. Using Theorem 2.1 again, $k_{t=1} = s^*_{t+1}(y_{T+1}) \leqq s^*_{T+1}) = k'_{T+1}$. This establishes the lemma. ‖

*Remark.* Lemma 5.1 establishes an analogue of the Brock "monotonicity" result (Brock (1971)) when initial stocks are changed.

*Proof of Theorem* 3.1. We establish that the $T$ subsequences $\langle k^*_{t+nT}\rangle^\infty_{n=0}$, $t = 0, \ldots, T-1$ are each monotone in $n$. Suppose that $k^*_T \geqq k^*_0$. Given a period of $T$, we can invoke Lemma 5.1 to claim that $k^*_{t+T} \geqq k^*_t$ for all $t \geqq 0$. (A similar argument applies if $k^*_T \leqq k^*_0$.) This immediately yields monotonicity of the relevant subsequences, and proves the theorem. ‖

*Remark.* We have established a stronger result: that the $T$ subsequences $\langle k^*_{t+nT}\rangle^\infty_{n=0}$, $t = 0, \ldots, T-1$, are either *all* monotone non-increasing in $n$, or *all* monotone non-decreasing in $n$.

In proving Theorem 3.3, we consider two cases. *In the first case*, $\lim_t \sup k^*_t \equiv k^* < \infty$. We will take the pure accumulation program $\langle \bar{y}_t\rangle^\infty_0$ in this case to be unbounded (the analysis of $\lim_t \bar{y}_t < \infty$ is similar and easier to handle, and is omitted). In case 1, $\langle k^*_t\rangle^\infty_0$ is a bounded sequence. Choose $\varepsilon > 0$, and define $\hat{y} \equiv f(\lim_t \sup y^*_t) + \varepsilon$.

Define $\boldsymbol{M}$ as the set of all monotonic (non-decreasing) functions $s:[0, \hat{y}] \to [0, \hat{y}]$. As is well-known, $s \in \boldsymbol{M}$ has at most a countable number of discontinuities. We endow $\boldsymbol{M}$ with the topology of weak convergence.

Let $T$ be an integer such that $\bar{y}_t \geqq \hat{y}$ for all $t \geqq T$. For $t \geqq T$, define $\hat{s}_t:[0, \hat{y}] \to [0, \hat{y}]$ by $\hat{s}_t(y) = s^*_t(y)$, $y \in [0, \hat{y}]$.

Below, Lemmas 5.2–5.6 hold under the assumptions of Theorem 3.3. We assume throughout that $k^* > 0$, otherwise there is nothing to prove. First, we establish

**Lemma 5.2.** *When $k^* < \infty$, there exists a subsequence $\langle t_q\rangle^\infty_{q=0}$ with $t_0 \geqq T$, such that*

(i) $k^*_{t_q} \to k^*$ *as* $q \to \infty$

(ii) $k^*_{t_q-1} \to \bar{k} \leqq k^*$ *as* $q \to \infty$

(iii) $k^*_{t_q+1} \to \underline{k}$ *as* $q \to \infty$, *with* $\underline{k} \leqq k^*$

(iv) $(\hat{s}_{t_q+1}) \to s^* \in \boldsymbol{M}$, *for some lower semicontinuous* $s^*$.

*Proof.* By definition of $k^*$, there is a subsequence $\langle t_m\rangle^\infty_{m=0}$ such that $k^*_{t_m} \to k^*$ as $m \to \infty$. Since we are in Case 1, $k^*_{t_m-1}$ and $k^*_{t_m+1}$ are bounded sequences. Hence there is a subsequence $\langle t_n\rangle^\infty_{n=0}$ of $\langle t_m\rangle^\infty_{m=0}$ such that $k^*_{t_n} \to k^*$, $k^*_{t_n-1} \to \bar{k}$, $k^*_{t_n+1} \to \underline{k}$ as $n \to \infty$, with $\bar{k} \leqq k^* \geqq \underline{k}$.

The sequence $\{\hat{s}_{t_n+1}\}^\infty_{n=0}$ is in $\boldsymbol{M}$ for all $t_n \geqq T$ (Theorem 2.1). By Helly's selection theorem, $\boldsymbol{M}$ endowed with the weak topology is sequentially compact (see Billingsley (1968), p. 227). Thus, we may take a subsequence $\langle t_q\rangle^\infty_{q=0}$ with $t_0 \geqq T$, such that $\hat{s}_{t_q+1} \to s^* \in \boldsymbol{M}$. Since the weak topology ignores points of discontinuity, we may take $s^*$ to be continuous from the left. Since $s^*$ is monotonic, it must therefore be lower semicontinuous. Now the subsequence $t_q$ has all the desired properties. ‖

Next, define $\bar{y} = f(\bar{k})$. Our immediate objective is to demonstrate that if an agent is endowed with $\bar{y}$ and if his successor's policy is $s^*$, then $k^*$ maximizes his well-being (Lemma 5.4). This requires a preliminary result.

**Lemma 5.3.** *Consider sequences* $\langle s^n \rangle_{n=0}^{\infty}$ *in* $M$ *and* $\langle x^n \rangle_{n=0}^{\infty}$ *in* $[0, \hat{y}]$. *Suppose* $s^n \to s$ *where* $s$ *is lower semicontinuous and* $x^n \to x$. *Then*

(i) $\lim_{n\to\infty} s^n(x^n) \geqq s(x)$.

*Furthermore, if* $s$ *is continuous at* $x$, *then*

(ii) $\lim_{n\to\infty} s^n(x^n) = s(x)$.

*Proof.* Suppose that (i) is false. Then for some $\varepsilon > 0$ there exists $N$ such that for all $n > N$,

$$s^n(x^n) < s(x) - 2\varepsilon$$

and

$$|x^n - x| < \delta,$$

where $\delta > 0$ is chosen such that

$$s(x-\delta) > s(x) - \varepsilon,$$

and such that $s$ is continuous at $x - \delta$ (this is possible since $s$ is continuous from the left, and has only a countable number of discontinuities). Since $s^n$ is monotonic,

$$s^n(x-\delta) \leqq s^n(x^n).$$

Combining these statements, we see that

$$s^n(x-\delta) < s(x-\delta) - \varepsilon$$

for all $n > N$. But this contradicts the fact that $s^n$ must converge to $s$ at $x - \delta$, a point of continuity.

We establish (ii) through a completely analogous argument, using the fact that $s$ is continuous from the right. ‖

**Lemma 5.4.** $k^*$ *maximizes* $V(\bar{y}, k) \equiv v(\bar{y} - k) + \delta v(f(k) - s^*(f(k)))$.

*Proof.* Note that there exists $T' \geqq T$ such that for all $t_q \geqq T'$, $y^*_{t_q} \leqq f^{-1}(\hat{y})$. Therefore it is meaningful to write that for all $t_q \geqq T'$, $k^*_{t_q}$ maximizes

$$v(y^*_{t_q} - k) + \delta v(f(k) - \hat{s}_{t_q+1}(f(k)))$$

over $k$ in $[0, y^*_{t_q}]$. As $q \to \infty$, $\hat{s}_{t_q} \to s^*$, and $(y^*_{t_q}, k^*_{t_q}) \to (\bar{y}, k^*)$. Thus, by Lemma 5.3(i),

$$\lim_{q\to\infty} [v(y^*_{t_q} - k^*_{t_q}) + \delta v(f(k^*_{t_q}) - \hat{s}_{t_q+1}(f(k^*_{t_q})))] \leqq v(\bar{y} - k^*) + \delta v(f(k^*) - s^*(f(k^*))).$$

Now suppose that the lemma is false. Then there exists $k'$ and $\varepsilon > 0$ such that

$$V(\bar{y}, k') > V(\bar{y}, k^*) + 3\varepsilon.$$

If $s^*$ is continuous at $f(k')$, define $k'' = k'$. If $s^*$ is discontinuous at $f(k')$, we can choose $k'' < k'$ such that

$$V(\bar{y}, k') - V(\bar{y}, k'') < \varepsilon$$

(since $s^*$ is continuous from the left), and such that $s^*$ is continuous at $f(k'')$ (since $s^*$ has at most a countable number of discontinuities). In either case, we have

$$V(\bar{y}, k'') > V(\bar{y}, k^*) + 2\varepsilon,$$

and $s^*$ continuous at $f(k'')$.

Now define $k''_{t_q} = \min(y^*_{t_q}, k'')$. Clearly, $f(k''_{t_q}) \to f(k'')$. By Lemma 5.3(ii), $\hat{s}_{t_q+1}(f(k''_{t_q})) \to s^*(f(k''))$. Thus, for $q$ sufficiently large, we have

$$\begin{aligned} v(y^*_{t_q} - k''_{t_q}) + \delta v(f(k''_{t_q}) - \hat{s}_{t_q+1}(f(k''_{t_q}))) &> v(\bar{y} - k'') + \delta v(f(k'') - s^*(f(k''))) - \varepsilon \\ &> v(\bar{y} - k^*) + \delta v(f(k^*) - s^*(f(k^*)) + \varepsilon \\ &> v(y^*_{t_q} - k^*_{t_q}) + \delta v(f(k^*_{t_q}) - \hat{s}_{t_q}(f(k^*_{t_q}))), \end{aligned}$$

which is a contradiction. ‖

**Lemma 5.5.** *If* $\lim_t \sup k^*_t = k^* < \infty$, *then* $k^* \leqq \hat{k}$.

*Proof.* Consider any sequence $k^n \uparrow k^*$, $k^n < k^*$. Now

$$\begin{aligned} V(\bar{y}, k^n) - V(\bar{y}, k^*) = [v(\bar{y} - k^n) - v(\bar{y} - k^*)] + \delta[v(f(k^n) - s^*(f(k^n)) \\ - v(f(k^*) - s^*(f(k^*)))]. \end{aligned} \tag{5.1}$$

By the mean value theorem, there is a $\alpha^n \in [\bar{y} - k^*, \bar{y} - k^n]$ and

$$\begin{aligned} \beta^n \in [\min\{f(k^n) - s^*(f(k^n)), f(k^*) - s^*(f(k^*))\}, \\ \max\{f(k^n) - s^*(f(k^n)), f(k^*) - s^*(f(k^*))\}] \end{aligned}$$

such that for all $n$,

$$\begin{aligned} V(\bar{y}, k^n) - V(\bar{y}, k^*) = v'(\alpha^n)(k^* - k^n) + \delta v'(\beta^n)[f(k^n) - s^*(f(k^n)) \\ - f(k^*) + s^*(f(k^*))]. \end{aligned} \tag{5.2}$$

Define

$$A^n \equiv \frac{\delta[f(k^*) - f(k^n) + s^*(f(k^n)) - s^*(f(k^*))]}{k^* - k^n}. \tag{5.3}$$

Then

$$V(\bar{y}, k^n) - V(\bar{y}, k^*) = [v'(\alpha^n) - A^n v'(\beta^n)](k^* - k^n). \tag{5.4}$$

Since $v(\cdot)$ is continuously differentiable, $\alpha^n \to \bar{y} - k^*$ as $n \to \infty$. Also, since $s^*$ is continuous from the left and $f(k^n) < f(k^*)$ for all $n$, $\beta_n \to f(k^*) - s^*(f(k^*))$ as $n \to \infty$.

*Actual* consumption along the sequence $t_q + 1$, $c^*_{t_q+1} \to f(k^*) - \underline{k}$ as $q \to \infty$ (Lemma 5.2). Using lower semicontinuity of $s^*$, and Lemma 5.3(i),

$$f(k^*) - s^*(f(k^*)) \geqq f(k^*) - \underline{k} \geqq f(\bar{k}) - k^* = \bar{y} - k^*. \tag{5.5}$$

As a result,

$$\lim_{n\to\infty} v'(\alpha^n) \geqslant \lim_{n\to\infty} v'(\beta^n). \tag{5.6}$$

By Lemma 5.4, $V(\bar{y}, k^n) - V(\bar{y}, k^*) \leqq 0$ for all $n$. Therefore, $\lim_{n\to\infty} \inf A^n \geqq 1$, and so, by (5.3),

$$\lim_{n\to\infty} \inf \frac{\delta[f(k^*) - f(k^n)]}{k^* - k^n} - \frac{\delta[s^*(f(k^*)) - s^*(f(k^n))]}{k^* - k^n} \geqq 1. \tag{5.7}$$

Using the fact that $s^*$ is monotonic

$$\lim_{n\to\infty}\inf \frac{\delta[f(k^*)-f(k^n)]}{k^*-k^n} \geqq 1 \tag{5.8}$$

and so, since $f$ is differentiable,

$$\delta f'(k^*) \geqq 1. \tag{5.9}$$

Combining this with $k^*>0$, Assumption 3.2, and Theorem 3.2, $k^* \leqq \hat{k}$. ‖

*In the second case*, we have the possibility that $\lim_t \sup k_t^* \equiv k^* = \infty$. This is ruled out in

**Lemma 5.6.** *It is impossible for* $\lim_t \sup k_t^* \equiv k^*$ *to equal* $+\infty$.

*Proof.* Suppose, on the contrary, that $k^*=\infty$. Then we claim that there exists $T$ such that $k_T^* > \hat{k}$ and $c_{T+1}^* > c_T^*$. Suppose not. Then for all $T$ with $k_T^* > \hat{k}$ (such $T$ exist since $\hat{k}=0$ or $\delta f'(\hat{k})=1$, and Assumption 6 holds), $c_{T+1}^* \leqq c_T^*$. For all $t$ with $k_t^* \leqq \hat{k}$, $c_{t+1} \leqq f(\hat{k})$. It follows, therefore, that for all $t \geqq 0$, $c_t^* \leqq B < \infty$. Consider a sequence $\langle T_n \rangle$ with $k_{T_n}^* \to \infty$. Observe that for each $n$, $c_{T_n+1}^*$ maximizes, over $c$ in $[0, f(k_{T_n}^*)]$,

$$Z(f(k_{T_n}^*), c) \equiv v(c) + \delta v[f(f(k_{T_n}^*)-c) - s_{T_n-2}^*(f(f(k_{T_n}^*)-c))]. \tag{5.10}$$

But $Z(f(k_{T_n}^*), c_{T_n+1}^*) \leqq v(B)(1+\delta)$, since $c_t^* \leqq B$ for all $t$.

Since $k_{T_n}^* \to \infty$, so does $f(k_{T_n}^*)$. By Assumption 5, there exists $n$ such that

$$v(f(k_{T_n}^*)) + \delta v(0) > v(B)(1+\delta). \tag{5.11}$$

For such $n$, using (5.10) and (5.11),

$$Z(f(k_{T_n}^*), f(k_{T_n}^*)) > Z(f(k_{T_n}^*), c_{T_n+1}^*),$$

a contradiction. So our claim is true, and there exists $T$ with

$$k_T^* > k \quad \text{and} \quad c_{T+1}^* > c_T^*. \tag{5.12}$$

Further, $k_T^*$ maximizes

$$V(y_T^*, k) \equiv v(y_T^* - k) + \delta v(f(k) - s_{T+1}^*(f(k)))$$

and clearly, $f(k_T^*) - s_{T+1}^*(f(k^*)) = c_{T+1}^*$.

Now we simply retrace the steps in the proof of Lemma 5.5, substituting $s_{T+1}^*$ for $s^*$, $y_T^*$ for $\bar{y}$, and $k^*$ for $k_T^*$. The only difference is that the inequality $f(k_T^*) - s_{T+1}^*(f(k_T^*)) \geqq y_T^* - k_T^*$ (analogous to (5.5)) is now verified simply by noting that $c_{T+1}^* > c_T^*$ (by choice of $T$). Following these steps yields, finally,

$$\delta f'(k_T^*) \geqq 1 \tag{5.13}$$

which, together with Assumption 3.2 and Theorem 3.2, contradicts our construction $k_T^* > \hat{k}$. ‖

*Proof of Theorem* 3.3. Combine Lemmas 5.5 and 5.6. ‖

Theorem 3.4 is proved below, following Lemma 5.7.

*Proof of Theorem* 4.1. By Theorem 3.3, $\lim_{t\to\infty} \sup k_t \leqq \hat{k}$, where $\hat{k}$ is the planning turnpike. If $\hat{k}>0$, it solves $\delta f'(\hat{k})=1$, by Theorem 3.2. In this case, $\lim_{t\to\infty} \inf f'(k_t) \geqq f'(\hat{k}) = 1/\delta > 1$, by Assumption 5.2. If $\hat{k}=0$, $\lim_{t\to\infty} k_t = 0$, and so again, $\lim_{t\to\infty} \inf f'(k_t) = f'(0) > 1$, by Assumption 2. Define a sequence $\langle p_t \rangle$ by $p_0 = 1/f'(f^{-1}(y))$, and $p_{t+1} = p_t / f'(k_t)$,

$t \geqq 0$. Then it is easily verified that $\lim_{t\to\infty} p_t k_t = 0$, so by a well-known criterion for efficiency (see, for example, Mitra (1979, Corollary 1)), $\langle y_t, c_t, k_t \rangle_0^\infty$ is efficient. ||

**Lemma 5.7.** *Under Assumptions* 1, 2, *and* 5, *suppose that* $\langle s_t^* \rangle_0^\infty$ *is a bequest equilibrium. Let* $\langle y_t, c_t, k_t \rangle_0^\infty$ *be an equilibrium path generated by* $\langle s_t^* \rangle_0^\infty$ *from* $y \in (0, Y]$. *Then, if* $k_t > 0$,

$$v'(c_t) \leqq \delta f'(k_t) v'(c_{t+1}), \tag{5.11}$$

*If, in addition, for any* $t \geqq 1$, $s_t^*$ *is strictly monotonic at* $y_t$, *then, for such* $t$, *if* $k_{t-1} > 0$,

$$v'(c_{t-1}) < \delta f'(k_{t-1}) v'(c_t). \tag{5.12}$$

*Proof.* First observe that $k_t > 0$ implies $c_{t+1} > 0$. If not, this easily contradicts the fact that $k_t$ is an equilibrium action for generation $t$.

Now suppose, contrary to (5.11), that there is some $t \geqq 0$, with $v'(c_t) > \delta f'(k_t) v'(c_{t+1})$. Then for $\eta > 0$ sufficiently small (using the definition of derivatives),

$$v(c_t + \eta) - v(c_t) > \delta v(c_{t+1}) - \delta v(c_{t+1} - f(k_t) + f(k_t - \eta)). \tag{5.13}$$

Suppose generation $t$ consumes $c_t + \eta$ instead of $c_t$. Then, by (5.13)

$$\begin{aligned} v(c_t) + \delta v(c_{t+1}) &< v(c_t + \eta) + \delta v(c_{t+1} - f(k_t) + f(k_t - \eta)) \\ &\leqq v(c_t + \eta) + \delta v(f(k_t - \eta) - s_{t+1}^*(f(k_t - \eta))) \end{aligned}$$

which contradicts the optimality of $t$'s decision (the weak inequality above follows from the fact that $s_{t+1}^*(\cdot)$ is monotonic). This establishes (5.11).

Now suppose, contrary to (5.12), that $s_t^*$ is strictly monotonic at $y_t$ but $v'(c_{t-1}) \geqq \delta f'(k_{t-1}) v'(c_t)$. Then, defining $\lambda_t \equiv 1 - \lambda(s_t^*, y_t, \varepsilon_t)$ with $\varepsilon_t$ chosen so that $\lambda_t < 1$, for sufficiently small $\eta$ we have

$$v(c_{t-1} + \eta) - v(c_{t-1}) > \delta v(c_t) - \delta v(c_t - \lambda_t [f(k_{t-1}) - f(k_{t-1} - \eta)]). \tag{5.14}$$

Now suppose that generation $t-1$ consumes $c_{t-1} + \eta$ instead of $c_{t-1}$. Then, by (5.14),

$$\begin{aligned} v(c_{t-1}) + \delta v(c_t) &< v(c_{t-1} + \eta) + \delta v(c_t - \lambda_t [f(k_{t-1}) - f(k_{t-1} - \eta)]) \\ &\leqq v(c_{t-1} + \eta) + \delta v(f(k_{t-1} - \eta) - s_t^*(f(k_{t-1} - \eta))) \end{aligned}$$

which contradicts the optimality of $t-1$'s choice (the weak inequality follows from the definition of $\lambda_t$). This establishes (5.12). ||

**Lemma 5.8.** *Under Assumptions* 1, 2, *and* 5, *let* $\langle \hat{y}_t, \hat{k}_t, \hat{c}_t \rangle_0^\infty$ *be the optimal planning program from* $y \in (0, Y]$. *Then if* $\hat{c}_t > 0$ *for any* $t \geqq 0$,

$$v'(\hat{c}_t) \geqq \delta f'(\hat{k}_t) v'(\hat{c}_{t+1}). \tag{5.15}$$

*Proof.* Suppose, on the contrary, that $v'(\hat{c}_s) < \delta f'(\hat{k}_s) v'(\hat{c}_{s+1})$, and $\hat{c}_s > 0$, for some $s \geqq 0$. Then, by an argument similar to that in Lemma 5.7, there is $\eta \in (0, \hat{c}_s)$, such that

$$v(\hat{c}_s) - v(\hat{c}_s - \eta) < \delta v(\hat{c}_{s+1} + f(\hat{k}_s + \eta) - f(\hat{k}_s)) - \delta v(\hat{c}_{s+1}). \tag{5.16}$$

Now define $\langle y'_t, c'_t, k'_t \rangle_0^\infty$ from $y \in (0, Y]$ by $y'_t = \hat{y}_t$, $t \neq s+1$, $y'_{s+1} = f(\hat{k}_s + \eta)$, $k'_t = \hat{k}_t$, $t \neq s$, $k'_s = \hat{k}_s + \eta$, and $c'_t = \hat{c}_t$, $t \neq s$, $s+1$, $c'_s = \hat{c}_s - \eta$, $c'_{s+1} = \hat{c}_s + f(\hat{k}_s + \eta) - f(\hat{k}_s)$. Clearly, this is feasible. Moreover, $v(c'_t) = v(\hat{c}_t)$, $t \neq s$, $s+1$, and $v(c'_t) + \delta v(c'_{t+1}) = v(\hat{c}_s - \eta) + \delta v(\hat{c}_{s+1} + f(\hat{k}_s + \eta) - f(\hat{k}_s)) > v(\hat{c}_s) + \delta v(\hat{c}_{s+1})$. So $\langle \hat{y}_t, \hat{k}_t, \hat{c}_t \rangle_0^\infty$ is not optimal, a contradiction. ||

*Proof of Theorem* 3.4. Suppose on the contrary, that $k_\tau^* > \hat{k}_\tau$ for some *first* time period $\tau \geqq 0$. Then $y_\tau^* \leqq \hat{y}_\tau$, where $\langle \hat{y}_t \rangle_0^\infty$ represents optimal output levels under planning.

Clearly, $\langle s_t^* \rangle_\tau^\infty$ is a bequest equilibrium, and $\langle y_t^*, k_t^*, c_t^* \rangle_\tau^\infty$ is a program generated by this equilibrium, from $y_\tau^*$. Let $\langle y_t', k_t', c_t' \rangle_\tau^\infty$ be the program generated by $\langle s_t^* \rangle_0^\infty$, from $\hat{y}_\tau$. Then, by Lemma 5.1, $k_t' \geqq k_t^*$, $t \geqq \tau$. So $k_\tau' > \hat{k}_\tau$. Hence $c_\tau' < \hat{c}_\tau$.

Now proceed by induction. Suppose that for some $t \geqq \tau$, $c_t' < \hat{c}_t$, and $\hat{k}_t < k_t'$. Then, using the strict concavity of $v$ (and/or $f$), Lemma 5.6 (noting that $k_t' > 0$), and Lemma 5.8 (noting that $\hat{c}_t > 0$),

$$\begin{aligned} \delta f'(k_t')v'(c_{t+1}') &\geqq v'(c_t') > v'(\hat{c}_t) \\ &\geqq \delta f'(\hat{k}_t)v'(\hat{c}_{t+1}) > \delta f'(k_t')v'(\hat{c}_{t+1}). \end{aligned} \tag{5.17}$$

Using (5.17), it follows that $v'(c_{t+1}') > v'(\hat{c}_{t+1})$, so $c_{t+1}' < \hat{c}_{t+1}$. Since $k_t' > \hat{k}_t$ (by hypothesis) and $f$ is increasing, $k_{t+1}' = f(k_t') - c_{t+1}' > f(\hat{k}_t) - \hat{c}_{t+1} = \hat{k}_{t+1}$. Hence $c_t' < \hat{c}_t$ for all $t \geqq \tau$. This establishes the inefficiency of $\langle y_t', k_t', c_t' \rangle_\tau^\infty$ from $\hat{y}_\tau$, which contradicts Theorem 4.1.

Hence $k_t^* \leqq \hat{k}_t$ for all $t \geqq 0$. ‖

*Proof of Theorem* 4.2. Since $s_1^*$ is strictly monotonic at $y_1$, and $k_0 > 0$, by Lemma 5.7, we know that for sufficiently small $\eta$,

$$v(c_0) - v(c_0 - \eta) < \delta v(c_1 + f(k_0 + \eta) - f(k_0)) - \delta v(c_1).$$

Rearranging,

$$v(c_0) + \delta v(c_1) < v(c_0 - \eta) + \delta v(c_1 + f(k_0 + \eta) - f(k_0)).$$

Thus, if 0 increases his investment by $\eta$ and 1 consumes all the incremental proceeds, 0 is strictly better off (trivially, so is 1), and no one else is affected adversely. ‖

For the remaining results, we will use the following conventions and notation. $\langle y_t, k_t, c_t \rangle_0^\infty$ denotes a feasible program from $y_0$ generated by a bequest equilibrium $\langle s_t^* \rangle_0^\infty$. The sequence $\langle \hat{y}_t, \hat{k}_t, \hat{c}_t \rangle_0^\infty$ denotes a hypothetical feasible program which Pareto dominates it. Define the set $\boldsymbol{T} = (t_1, t_2, \ldots)$ as follows: $t \in \boldsymbol{T}$ iff $\hat{c}_t < c_t$. $\boldsymbol{T}$ must be non-empty under Assumptions 1, 2, 3.2, 5, 5.2, and 6, by Theorem 4.1. Define

$$P(c^1, c^2) = \{(a, b) \mid v(a) + \delta v(b) \geqq v(c^1) + \delta v(c^2)\}$$
$$F(y, k) = \{(c^1, c^2) \mid f(y - c^1) - c^2 \geqq k\}.$$

**Lemma 5.9.** *Under Assumptions* 1, 2, 3.1, *and* 5, *if* $(c_{t-1}', c_t') \in P(c_{t-1}, c_t) \cap F(y_{t-1}, k_t)$, *then* $c_{t-1}' \leqq c_{t-1}$ *and* $c_t' \geqq c_t$.

*Proof.* By definition of the set $P(c_{t-1}, c_t)$,

$$[v(c_{t-1}') - v(c_{t-1})] + \delta[v(c_t') - v(c_t)] \geqq 0.$$

By strict concavity of $v(\cdot)$,

$$(c_{t-1}' - c_{t-1})v'(c_{t-1}) + \delta(c_t' - c_t)v'(c_t) > 0.$$

Rearranging,

$$c_t' > c_t - (c_{t-1}' - c_{t-1}) \frac{v'(c_{t-1})}{\delta v'(c_t)} \tag{5.18}$$

By definition of the set $F(y_{t-1}, k_t)$,

$$f(y_{t-1} - c_{t-1}') - c_t' \geqq k_t = f(y_{t-1} - c_{t-1}) - c_t,$$

or

$$[f(k_{t-1}+(c_{t-1}-c'_{t-1}))-f(k_{t-1})]+[c_t-c'_t]\geqq 0.$$

By the concavity of $f(\cdot)$,

$$(c_{t-1}-c'_{t-1})f'(k_{t-1})+(c_t-c'_t)\geqq 0.$$

Rearranging,

$$c'_t\leqq c_t-(c'_{t-1}-c_{t-1})f'(k_{t-1}). \tag{5.19}$$

Now, suppose, contrary to the lemma, that $c'_{t-1}>c_{t-1}$. Then, using Lemma 5.7 and equation 5.19,

$$c'_t\leqq c_t-(c'_{t-1}-c_{t-1})\frac{v'(c_{t-1})}{\delta v'(c_t)}$$

But this contradicts (5.18), so $c'_{t-1}\leqq c_{t-1}$. Since $(c'_{t-1}, c'_t)\in P(c_{t-1}, c_t)$, we must then have $c'_t\geqq c_t$, as desired. ||

**Lemma 5.10.** *Under Assumptions* 1, 2, 3.1 *and* 5, *if* $0\notin T$, *then for all* $t\in T$,

(i) $\hat{c}_{t-1}>c_{t-1}$
(ii) $\hat{y}_{t-1}\leqq y_{t-1}$
(iii) $\hat{k}_t<k_t$.

*Proof.* If (i) fails, $t-1$ is worse off under the dominating program—a contradiction. For (ii) and (iii), we argue as follows. First, it is obvious that $\hat{y}_{t_1-1}\leqq y_{t_1-1}$. Now suppose that $\hat{k}_{t_1}\geqq k_{t_1}$. Then $F(\hat{y}_{t_1-1}, \hat{k}_{t_1})\subseteq F(y_{t_1-1}, k_{t_1})$. We know that

$$\begin{aligned}(\hat{c}_{t_1-1}, \hat{c}_{t_1})&\in P(c_{t_1-1}, c_{t_1})\cap F(\hat{y}_{t_1-1}, \hat{k}_{t_1})\\ &\subseteq P(c_{t_1-1}, c_{t_1})\cap F(y_{t_1-1}, k_{t_1})\end{aligned}$$

Apply Lemma 5.9 to conclude that $\hat{c}_{t_1-1}\leqq c_{t_1-1}$, which contradicts (i).

Now suppose that (ii) and (iii) are valid for $t_n$. $\hat{k}_{t_n}<k_{t_n}$, and for $t\in(t_n+1,\ldots,t_{n+1}-2)$, $\hat{c}_t\geqq c_t$. Thus, $\hat{y}_{t_{n+1}-1}<y_{t_{n+1}-1}$. Applying the same argument as above, we see that $\hat{k}_{t_{n+1}}<k_{t_{n+1}}$. ||

**Lemma 5.11.** *Under Assumption* 3.2, *if* $y^1\geqq y^2$ *and* $k^1>k^2$, *and if there exists* $(\tilde{c}^1, \tilde{c}^2)\in F(y^2, k^2)-F(y^1, k^1)$, *then for all* $(c^1, c^2)\in F(y^1, k^1)$ *with* $c^1<\tilde{c}^1$, *there exists* $(c^1, c^3)\in F(y^2, k^2)$ *with* $c^3>c^2$.

*Proof.* We know that $f(y^2-\tilde{c}^1)-k^2>f(y^1-\tilde{c}^1)-k^1$, and by concavity of $f$ (along with $c^1<\tilde{c}^1$),

$$f(y^2-\tilde{c}^1)-f(y^2-c^1)<f(y^1-\tilde{c}^1)-f(y^1-c^1)$$

Combining these,

$$c^3\equiv f(y^2-c^1)-k^2>f(y^1-c^1)-k^1\geqq c^2$$

But $(c^1, c^3)\in F(y^2, k^2)$ by construction. ||

*Proof of Theorem* 4.3. It is completely straightforward to show that efficiency is a necessary condition for Pareto optimality. Suppose now that the equilibrium program is not modified Pareto optimal—i.e. there is a dominating program $\langle \hat{y}_t, \hat{k}_t, \hat{c}_t \rangle_0^\infty$ with $0 \notin T$. Choose any $t \in T$. Since $(\hat{c}_{t-1}, \hat{c}_t) \in P(c_{t-1}, c_t)$, and since $\hat{c}_{t-1} > c_{t-1}$ (Lemma 5.10(i)), then by Lemma 5.9, $(\hat{c}_{t-1}, \hat{c}_t) \notin F(y_{t-1}, k_t)$. Further, by Lemma 5.10(ii) and (iii), $\hat{y}_{t-1} \leqq y_{t-1}$, and $\hat{k}_t < k_t$. But then, by Lemma 5.11, there exists some $(c'_{t-1}, c'_t) \in F(\hat{y}_{t-1}, \hat{k}_t)$ such that $(c'_{t-1}, c'_t) > (c_{t-1}, c_t)$. Consider a new consumption program, $\langle \tilde{c}_t \rangle_0^\infty$, where $\tilde{c}_t = c'_t$ if $t$ or $t+1 \in T$, and $\tilde{c}_t = \hat{c}_t$ otherwise. Since $\langle \hat{c}_t \rangle_0^\infty$ was feasible, by construction $\langle \tilde{c}_t \rangle_0^\infty$ is feasible. But since $\langle \tilde{c}_t \rangle_0^\infty$ strictly dominates $\langle c_t \rangle_0^\infty$, $\langle c_t \rangle_0^\infty$ is not efficient. ‖

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NOTES

1. This model is adapted from Kohlberg (1976), and is closely related to those used by Arrow (1973), Dasgupta (1974*a*, *b*), Lane and Mitra (1981), Leininger (1983), and Bernheim and Ray (1983).
2. A Markov perfect equilibrium is a subgame perfect equilibrium in which strategies are defined only as functions of current state variables.
3. The second part of Assumption 4 is simply an assumption of *normality* of the second argument of $w(c_t, c_{t+1})$. If a consumer with this utility function chose $c_t$ and $c_{t+1}$ at fixed prices, $c_{t+1}$ would be a normal good.
4. The borderline case $\lim_{t\to\infty} f'(k) = \delta^{-1}$ presents more subtle technical problems which we have not explored.
5. The term originally appears in Arrow (1973).
6. Corollary 3.1 does not preclude the possibility that $\lim_{t\to\infty} k_t^* = \infty$, which is not, strictly speaking, a steady-state property. However, $k^* < \infty$ in a large class of situations (at least, in all situations where the corresponding optimal growth "turnpike" is finite—see below).
7. One can, of course, use a similar argument when feasible utility streams diverge.
8. The proof is omitted. For a more general version of this result in an aggregative context, see Mitra and Ray (1984).
9. Since this result implies Theorem 3.3, and is obtained under only slightly more restrictive conditions, Theorem 3.3 may appear redundant. However, Theorem 3.3 is used in the proof of Theorem 4.1, which in turn yields Theorem 3.4. Consequently, it is necessary to state these results separately.

REFERENCES

ARROW, K. J. (1973), "Rawls' Principle of Just Saving", *Swedish Journal of Economics*, **75**, 323–335.

BERNHEIM, B. D. and RAY, D. (1983), "Altruistic Growth Economies: I. Existence of Bequest Equilibria" (Technical Report No. 419, Institute for Mathematical Studies in the Social Sciences).

BILINGSLEY, P. (1968) *Convergence of Probability Measures* (New York: John Wiley).

BROCK, W. (1971), "Sensitivity of Optimal Growth Paths with Respect to a Change in Target Stocks", *Zeitschrift für Nationalokonomie*, **1**, 73–89.

DASGUPTA, P. (1974*a*), "On Some Problems Arising from Professor Rawls' Conception of Distributive Justice", *Theory and Decision*, **4**, 325–344.

DASGUPTA, P. (1974*b*), "On Some Alternative Criteria for Justice Between Generations", *Journal of Public Economics*, **3**, 405–423.

KOHLBERG, E. (1976), "A Model of Economic Growth with Altruism Between Generations", *Journal of Economic Theory*, **13**, 1–13.

LANE, J. and MITRA, T. (1981), "On Nash Equilibrium Programs of Capital Accumulation Under Altruistic Preferences", *International Economic Review*, **22** (2), 309–331.

LEININGER, W. (1983), "The Existence of Nash Equilibria in a Model of Growth with Altruism Between Generations" (Discusssion Paper No. 126, University of Bonn).

MITRA, T. (1979), "Identifying Inefficiency in Smooth Aggregative Models of Economic Growth: a Unifying Criterion" *Journal of Mathematical Economics*, **6**, 85–111.

MITRA, T. and RAY, D. (1983), "Dynamic Optimization on a Nonconvex Feasible Set: Some General Results for Nonsmooth Technologies", *Zeitschrift für Nationalokonomie*, **44**, 151–175.

PELEG, B. and YAARI, M. (1973), "On the Existence of a Consistent Course of Action When Tastes Are Changing", *Review of Economic Studies*, **40** (3), 391–401.

PHELPS, E. and POLLAK, R. (1968), "On Second-best National Saving and Game-equilibrium Growth", *Review of Economic Studies*, **35** (2), 185–199.

SELTEN, R. (1975), "Reexamination of the Perfectness Concept for Equilibrium Points in Extensive Games", *International Journal of Game Theory*, **4**, 25–55.