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Consistent Intertemporal Decision Making [1, 2]

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I. INTRODUCTION

A plan for an intertemporal consumer (or society) is a (constrained optimum) specification of consumption behaviour from the present to the end of his life (or time horizon). If an individual cannot dictate his future behaviour, he may be inconsistent (Strotz [12]); that is, he may, as time passes, revise his specified future behaviour. Among the many unpleasant features of inconsistent planning is that if an individual behaves myopically (" naively ", cf. Pollak [10]) by continually executing the present portion of his plan, his behaviour, *ex post*, makes no sense from any point of view.

This paper presents alternative ways of looking at intertemporal behaviour, and examines the conditions under which such behaviour is consistent. In Section III, we show that naive intertemporal optimization is consistent only if intertemporal preferences are structured so that the future is functionally separable from the present. In Section IV, we discuss " sophisticated solutions ", which have been suggested, [1], [10] and [12], as a planning strategy when the intertemporal preference ordering does not satisfy the necessary condition for consistent naive planning. We prove an existence and uniqueness theorem for sophisticated solutions and find the necessary and sufficient conditions for the " sophisticated " choice functions to be generated by conventional utility maximization.

II. NOTATION AND DEFINITIONS

We will consider a planning entity (an individual or a society) with an m-period horizon. In each period ($t = 1, \dots, m$), the commodity bundle is $X^t \in \Omega^n$ (the n-dimensional non-negative Euclidean orthant) [3] and consumption of a single commodity is denoted x^t_j. A programme is denoted ${}_1X = (X^1, \dots, X^m) \in \Omega^{nm}$ and the tth continuation of a programme is denoted ${}_tX = (X^t, \dots, X^m)$. The tth continuation of a programme chosen in the sth period is denoted ${}_t\overset{*}{X}_{(s)}$ and $\overset{*}{X}{}^t_{(s)}$ is the component vector of ${}_t\overset{*}{X}_{(s)}$ relating to period t. The notation for prices (which are assumed strictly positive) is analogous. We also denote

1 *First version received June* 1971; *final version received September* 1972 (*Eds.*).

2 We wish to thank Robert Pollak and the editor and referees of this journal for many helpful comments.

3 The assumption that the number of commodities is the same in each period is made only for notational convenience.

the set of subscript indices in the tth period by I^t so that $i \in I^t$ indicates that the subscripted variable refers to the tth period.

Prices throughout are " forward " prices; i.e., cross-temporal exchange ratios in terms of the first-period numeraire (hence all interest rates are own rates). Initial wealth is w and wealth in the tth period is w_t.

Definition. A *utility function* $U(\cdot)$ is a continuous, real valued, non-decreasing, strictly quasi-concave function defined on the non-negative Euclidean orthant.

Definition. A pair of utility functions on the same domain, $U(\cdot)$ and $U'(\cdot)$, are *equivalent*,

$$U(\cdot) \sim U'(\cdot),$$

if they induce the same pre-ordering on that domain.

Definition. Let (χ^s, χ^c) be a partition of the set of variables, χ, and let $\hat{X}^s$ be any set of non-negative values of the variables in χ^s. The group of variables, χ^s, is *separable* in U from a variable χ_k if and only if the correspondence, β, defined by

$$\beta(\hat{X}^s, X^c) = \{X^s \mid U(X^s, X^c) > U(\hat{X}^s, X^c)\}$$

is independent of x_k, the value of the variable χ_k.

Remarks. The condition of this definition originally appeared in an equivalence lemma by Stigum [11, p. 352].

If U is differentiable, the condition of the definition is equivalent to the condition that the marginal rates of substitution,

$$M_{ij} \equiv \frac{\partial U/\partial x_i^s}{\partial U/\partial x_j^s}$$

be independent of x_k for all $i, j \in I^s$. If $U(X)$ is twice differentiable, this is equivalent to the condition that

$$\frac{\partial M_{ij}}{\partial x_k} = 0$$

for all $i, j \in I^s$. This is the condition originally introduced by Leontief [6].

Definition (Lady and Nissen [5]). A function is *strongly recursive* (SR) in the ordered partition $(\chi^1, \ldots, \chi^m)$ if and only if each pair (χ^r, χ^{r+1}) is separable from the variables in $(\chi^1, \ldots, \chi^{r-1})$.

Remark. A theorem of Gorman [3] states that the union of overlapping separable sets is separable. Therefore, a function is SR if and only if each continuation, $\chi^r = \bigcup_{s=r}^{m} \chi^s$, is separable from all preceding variables.

Theorem 1. *A utility function, U, is strongly recursive in the ordered partition $(\chi^1, \ldots, \chi^m)$ if and only if there exist functions $f^r(\cdot)$ such that*

$$f^r \equiv f^r(X^r, f^{r+1}),\ r = 1, \ldots, m-1;\ f^m \equiv f^m(X^m);\ U \equiv f^1.$$

Proof. See Lady and Nissen [5]. ||

III. CONSISTENT INTERTEMPORAL BEHAVIOUR

An intertemporal preference structure may belong to a society or to an individual. In either case, in discussing the consistency of an entity which survives m periods, it is convenient, for present purposes, to speak of an intertemporal society of m generations whose behaviour is related by

(i) a wealth inheritance mechanism, and
(ii) a specification of the relation between generations' preferences.

In what follows, we assume complete perfect markets and perfect foresight, described by forward prices (exchange ratios in terms of the first period numeraire),

$$_1P = (P^1, \ldots, P^m),$$

and initial wealth, w.

Definition. An *intertemporal society* is a collection of m generations with utility functions U^t, $t = 1, \ldots, m$, each defined on the space of consumption continuations, $_tX$.

Remark. As the utility functions are defined on the space of consumption continuations, $_tX$, the consumption of future generations, $_{t+1}X$, affects the welfare of the tth generation. If the intertemporal society is an individual at different points in time, this assumption is natural. If the society is, in fact, a succession of generations, the utility representation reflects altruistic (or malevolent) concern for future generations. Alternatively, we could justify preferences over consumption continuations by assuming that the tth generation survives through the mth period but has power only in the tth period.

Definition. A *naive intertemporal society* is an intertemporal society in which each generation plans by solving

$$\max_{_tX} U^t(_tX)$$

$$\text{s.t. } w_t \geqq {_tP} \cdot {_tX}$$

to obtain

$$_t\overset{*}{X}_{(t)} = {_t\phi_{(t)}}(w_t, {_tP})$$

and behaves myopically (naively) by executing $\overset{*}{X}{}^t_{(t)}$ so that the wealth inheritance mechanism is

$$w_{t+1} = w_t - P^t \cdot \overset{*}{X}{}^t_{(t)}; \; t = 1, \ldots, m-1; \; w_1 = w. \qquad \ldots(1)$$

Definition. A society is *consistent* when

$$_t\overset{*}{X}_{(1)} = {_t\overset{*}{X}_{(t)}}, \quad t = 1, \ldots, m, \qquad \ldots(2)$$

for all (w, P).

We now describe, in three naive intertemporal societies, the relation between individual preferences required by consistency.[1]

Society I (*The Ex Post Society*). In this society preferences are inherited (or imposed) That is, each individual specifies the preferences of the next.

Theorem 2. *The ex post society is consistent if and only if the preference inheritance mechanism is*

$$U^t(_tX) \sim U^{t-1}(\overset{*}{X}{}^{t-1}_{(t-1)}; {_tX}), \quad t = 2, \ldots, m. \qquad \ldots(3)$$

Proof. The condition is equivalent (by recursion) to

$$U^t(_tX) \sim U^1(\overset{*}{X}{}^1_{(1)}, \overset{*}{X}{}^2_{(2)}, \ldots, \overset{*}{X}{}^{t-1}_{(t-1)}; {_tX}), \quad t = 2, \ldots, m.$$

Further, the condition (3) together with the wealth inheritance mechanism (1) makes adjacent choice problems equivalent, so that, e.g.,

$$_t\overset{*}{X}_{(t)} = {_t\overset{*}{X}_{(t-1)}}, \quad t = 2, \ldots, m,$$

implying consistency, (2); and condition (3) is sufficient for consistency. Further, by the properties of the utility function each point will be chosen by some (w, P), so the condition is necessary.

[1] This relationship is also discussed by Hammond [4]. An alternative approach is to specify preferences *a priori*, but relax the assumption of naive behaviour and let individuals recognize the consistency problem. Phelps and Pollak [9] follow this approach to an elegant second-best solution for a particular class of infinite dimensional utility functions. We analyze this approach in Section V below.

Remarks. First, by the construction in the proof, condition (3) is equivalent to

$$U^t({}_tX) \sim U^1(X^1_{(1)}, \ldots, X^{t-1}_{(1)}, {}_tX), \quad t = 2, \ldots, m; \qquad \ldots(4)$$

that is, *for consistency, the induced pre-ordering on any continuation must be the same for any point of view.* This is the heart of Strotz's contribution [12].[1]

Second, the consistent plan achieved by this society is what Pollak [10] would call the commitment path. Here commitment is achieved by the preference inheritance mechanism.

Society II (The Ex Ante Society). If imposed preferences are distasteful, we may demand the ability to describe preferences before inspecting planned behaviour. In the ex ante society, no generation can impose its preferences on succeeding generations.

Theorem 3. *The ex ante society is consistent if and only if each utility function is strongly recursive with the representation*

$$U^t({}_tX) \equiv F^t[X^t, U^{t+1}({}_{t+1}X)], \quad t = 1, \ldots, m-1. \qquad \ldots(5)$$

Proof. Assume that U is differentiable (but see the remark below). Consider the dual to the tth period maximization problem:

$$\min_{{}_tX} {}_tP \cdot {}_tX$$

$$\text{s.t. } U^t({}_tX) \geqq U^t, \qquad \ldots(6)$$

yielding the compensated demand functions,

$${}_t\overset{*}{X}_{(t)} = {}_t\hat{\phi}_{(t)}({}_tP, U^t).$$

The expenditure function is

$$h^t({}_tP, U^t) = {}_tP \cdot {}_t\hat{\phi}_{(t)}(P, U^t)$$

and the compensated demands are given by

$${}_{(t)}x^s_i = \frac{\partial h^t}{\partial p^s_i} \; \forall i \in I^s, \quad s = t, t+1, \ldots, m. \qquad \ldots(7)$$

Intertemporal consistency (2) is therefore equivalent to

$$\frac{\partial h^t}{\partial p^{t+1}_i} \equiv \frac{\partial h^{t+1}}{\partial p^{t+1}_i} \; \forall i \in I^{t+1}, \quad t = 1, \ldots, m-1, \qquad \ldots(8)$$

where $\partial h^{t+1}/\partial p^{t+1}_i$ is evaluated at

$$U^1 = U^1[{}_1\hat{\phi}_{(2)}({}_1P, w)]; \; U^{t+1} = U^{t+1}[{}_{t+1}\hat{\phi}_{(t)}({}_tP, U^t)], \quad t = 1, \ldots, m-1.$$

Taking the total differential of

$$h^t({}_tP, U^t) = \min\,[P^t \cdot X^t + h^{t+1}({}_{t+1}P, U^{t+1})]$$

yields

$$\sum_{i \in I^t} \frac{\partial h^t}{\partial p^t_i} dp^t_i + \sum_{s=t+1}^{m} \sum_{i \in I^s} \frac{\partial h^t}{\partial p^s_i} dp^s_i + \frac{\partial h^t}{\partial U^t} \sum_{i \in I^t} \frac{\partial U^t}{\partial x^t_i} dx^t_i + \frac{\partial h^t}{\partial U^t} \sum_{s=t+1}^{m} \sum_{i \in I^s} \frac{\partial U^t}{\partial x^s_i} dx^s_i$$

$$= \sum_{i \in I^t} x^t_i dp^t_i + \sum_{i \in I^t} p^t_i dx^t_i + \sum_{s=t+1}^{m} \sum_{i \in I^s} \frac{\partial h^{t+1}}{\partial p^s_i} dp^s_i + \frac{\partial h^{t+1}}{\partial U^{t+1}} \sum_{s=t+1}^{m} \sum_{i \in I^s} \frac{\partial U^{t+1}}{\partial x^s_i} dx^s_i. \qquad \ldots(9)$$

[1] Though it is somewhat obscured by the mathematical details, Strotz gets a constant " rate of time preference " by requiring his discount function to be congruent with any continuation of itself, a property unique to exponential functions (cf. Nissen [7]).

The first terms on each side of the equality in (9) are equal by (7) and the third term on the left cancels with the second term on the right by the first order conditions of the dual,

$$p_i^t - \frac{\partial h^t}{\partial U^t} \cdot \frac{\partial U^t}{\partial x_i^t} = 0, \quad \forall i \in I^t,$$

where $\partial h^t/\partial U^t = \lambda^t$ is the equilibrium value of the multiplier in the Lagrangean problem.

By (8), if the solution is intertemporally consistent, the second term on the left equals the third term on the right for all sets of prices. Thus we are left with the following necessary condition for consistency:

$$\sum_{s=t+1}^{m} \sum_{i \in I^s} \frac{\partial U^t({}_tX)}{\partial x_i^s} dx_i^s \equiv \frac{\lambda^{t+1}}{\lambda^t} \sum_{s=t+1}^{m} \sum_{i \in I^s} \frac{\partial U^{t+1}({}_{t+1}X)}{\partial x_i^s} dx_i^s, \quad t = 1, \ldots, m-1. \quad \ldots(10)$$

This holds for arbitrary dx^s only if

$$\frac{\partial U^t}{\partial x_i^s} \equiv \frac{\lambda^{t+1}}{\lambda^t} \frac{\partial U^{t+1}}{\partial x_i^s}, \forall i \in I^s, s = t+1, \ldots, m; \ t = 1, \ldots, m-1,$$

which implies the Leontief separability condition for a system of strongly recursive functions.

Similarly, strong recursivity with a consistent representation implies (10) so that (9) reduces to

$$\sum_{s=t+1}^{m} \sum_{i \in I^s} \frac{\partial h^t({}_tP, U^t)}{\partial p_i^s} dp_i^s \equiv \sum_{s=t+1}^{m} \sum_{i \in I^s} \frac{\partial h^{t+1}({}_{t+1}P, U^{t+1})}{\partial p_i^s} dp_i^s.$$

This holds identically in ${}_tP$, U^t and U^{t+1} for any dp^s only if

$$\frac{\partial h^t}{\partial p_i^s} \equiv \frac{\partial h^{t+1}}{\partial p_i^s} \forall i \in I^s, \quad s = t+1, \ldots, m-1,$$

which is equivalent to intertemporal consistency. ||

Remarks. In the interest of clarity, the above proof uses differentiability of the utility functions but the theorem can be proved without this restriction. In fact, the consistency condition (4) together with the assumption that preferences are not imposed comprise the separability conditions characterizing a strongly recursive structure.

Second, note that Theorem 3 implies that the consistent ex ante society acts as if it were maximizing a single utility function, or " social welfare function ", if and only if the intertemporal preference structure is strongly recursive. That is, the observed demand functions, $\phi_{(t)}^t({}_tP, w_t)$ are integrable into a utility function, $U({}_1X)$, if and only if the preferences are so structured.

The representation (5) suggests that it may be fruitful to consider directly an intertemporal society whose members are altruistic.

Society III (The Altruistic Society). Suppose utility functions have the form,

$$U^t = U^t({}_tX, U^{t+1}, \ldots, U^m).$$[1]

Theorem 4. *The altruistic society is consistent if and only if each utility function is strongly recursive with the representation of Theorem* 3.

Proof. The proof of Theorem 3 goes through for Theorem 4.

Remarks. First, note that altruism as represented here means that any two persons are cardinally comparable from the point of view of a third person.

[1] Note that this representation allows for malevolent concern for future generations as well as for altruistic concern.

Second, the representation (5) means that planning consistency requires a consistent altruistic structure in the sense that the pre-ordering on the set of utility continuations is independent of viewpoint. It further requires a kind of noninterference property in that another's experience affects one only through the other's utility function (only X^t, not ${}_tX$, appears in $F^t(X^t, U^{t+1})$; cf. Winter [13].)

Third, there is implicit in the representation,

$$U^t({}_tX) \equiv F^t(X^t, U^{t+1}),$$

a kind of nice " responsibility inheritance mechanism ". The representation $F^t(\cdot)$ must be carefully constructed by " marking up " the effect of U^{t+1} to account for subsequent generations, but once that is done, the detailed planning of ${}_{t+1}X$ may be passed on to future generations. That is, the tth plan is embodied in the choice functions, ζ^t and θ_t, defined by

$$X^t = \zeta^t(w_t, {}_tP)$$

and

$$w_{t+1} = \theta_t(w_t, {}_tP)$$

where ζ^t is vector valued (see Lady and Nissen [5]).

Finally, if in the above representation, U^t is homothetic in ${}_tX$ for $t = 2, \ldots, m$, the tth plan is embodied in the choice functions, α^t and β_t, defined by

$$X^t = \alpha^t(w_t, P^t, \Pi^{t+1})$$

and

$$U^{t+1} = \beta_t(w_t, P^t, \Pi^{t+1})$$

where α^t is vector valued,

$$\Pi^{t+1} = \Pi^{t+1}(P^{t+1}, \Pi^{t+2})$$

is the " price " of the composite future " commodity ", f^{t+1}, a monotonic transform of U^{t+1} which is positively linearly homogeneous in ${}_{t+1}X$, and w_{t+1} is determined by

$$w_{t+1} = \Pi^{t+1} . f^{t+1}$$

(see Blackorby, Nissen, Primont and Russell [2]).

V. SOPHISTICATED SOLUTIONS

If a planning entity does not have strongly recursive preferences and is unable to precommit its behaviour, myopic behaviour may then occur. The present portion of the plan is executed and then revised in each period. Planning appears to make little sense in this context. The consumption programme generated by the behaviour is not optimal from any point of view. Hence, it has been suggested that such a society follow a " sophisticated " optimum path [1], [9], [11].

Society IV (*The Sophisticated Society*). Suppose utility functions for the m generations have the form

$$U^t = U^t({}_tX), \quad t = 1, 2, \ldots, m,$$

and commitment is impossible. In the " sophisticated society ", each generation is cognizant of the preferences of future generations and takes these preferences into account in constructing a consistent intertemporal plan. Thus, each generation chooses a plan which, according to its own preferences, is the optimum among the set of plans which will, in fact, be carried out by successive generations.

In order to describe the sophisticated solution, define, for all t, the budget sets,

$$B_t(w_t) \equiv \{{}_tX \in \Omega^{n(m-t+1)} \mid {}_tP \cdot {}_tX \leqq w_t,\ w_t \in \Omega\},$$

and

$$A_t = \{X^t \in \Omega^n \mid X^t = \xi^t(w_t, {}_tP) \forall w_t \in \Omega\}, \quad t = 2, \ldots, m,$$

where ξ^t is the vector valued choice function of the tth generation. The set, A_t, thus owns all bundles, X^t, that would be chosen by the tth generation for some (non-negative) value of wealth w_t, given the set of prices, ${}_tP$. Thus, this set is nothing more than the projection of the "income consumption curve" of the tth generation on to its own consumption space, Ω^n.

The sophisticated path, ${}_1X$, is generated by the following sequence of problems:

$$\max_{{}_tX} U^t({}_tX) \text{ s.t. } {}_tX \in B_t(w_t) \text{ and } X^{t+r} \in A_{t+r}, \quad r = 1, \ldots, m-t$$

to obtain

$$\hat{X}^t = \xi^t(w_t, {}_tP), \quad t = m, \ldots, 1.$$

The w_t's are then found by

$$w_1 = w$$

and

$$w_{t+1} = w_t - P^t \cdot \xi^t(w_t, {}_tP), \quad t = 1, \ldots, m-1.$$

Thus the sophisticated path is generated by a recursive programming procedure, in which the mth choice function is derived first and the first choice function is derived last. This solution is consistent because the choice functions of future generations are treated as constraints in the optimization problem. Although this solution is not Pareto optimal, it has the attractive property that ${}_t\hat{X}$ is preferred by the tth generation to all other paths which will be followed by subsequent generations.

A serious problem with the sophisticated society is that even though U^t is assumed to be strictly quasi-concave, the tth optimum may not be unique, in which case planning is indeterminate at the $(t-1)$th stage.[1] To see this, note that the sets, A_{t+r}, $r = 1, \ldots, m-t$, are not necessarily convex, so that the feasible set for the tth maximization problem is not necessarily convex. Consequently, the optimum might not be unique. Although the tth generation is indifferent between the alternative plans, the $(t-1)$th generation might not be indifferent. In this case, the $(t-1)$th generation would need to know how the "tie" in the tth period is going to be resolved in order to formulate its own plan.

Three possible escapes from this dilemma are as follows:[2] First, we might permit a type of weak pre-commitment, in which the first generation affected by the prospect of future multiple optima commits the relevant generation to a specific plan among the set of optima. Second, we could adopt a probabalistic approach to deal explicitly with the uncertainty regarding the choices of future generations. Finally, we could posit a strong ordering of consumption programmes, ${}_tX$, for $t = 2, \ldots, m-1$, in which case the optimum is unique despite the possibility of a non-convex feasible set.

The problem with the last approach is that a strong ordering cannot be represented by a continuous, order-preserving, real-valued function. The other two approaches do not resolve a more fundamental problem generated by non-convex feasible sets. If, in period t, the feasible set is not convex, the choice function might not be upper semi-continuous, in which case the feasible set in period $t-1$ might not be closed. In this case, an optimum might not exist even if the $(t-1)$th generation can pre-commit future generations in their choice among multiple optima.[3]

1 This fact was first pointed out to us by Robert Pollak. The problem is discussed in Phelps and Pollak [9] and in Peleg and Yaari [8].

2 All three of these solutions have been suggested to us by Robert Pollak.

3 For an example, see Peleg and Yaari [8, pp. 6-7].

In addition to the assumption of strong orderings of consumption programmes sufficient conditions for a unique sophisticated path to exist are embodied in the following theorem:[1]

Theorem 5. *If the utility functions,* U^t, $t = 2, \dots, m$, *are homothetic, a unique sophisticated path,* ${}_1\hat{X}$, *exists for all* ${}_1P \in \Omega^{nm}$ *and for all* $w \in \Omega$.

Proof. As the mth maximization is constrained only by the budget constraint, the assumptions regarding U^m imply that the vector-valued function, ξ^m, exists and is linearly homogeneous in w_m. The proof therefore proceeds by induction.

Suppose that the functions, ξ^r, $r = t+1, \dots, m$, exist and are linearly homogeneous in w_r. Then the sets, A_{t+r}, $r = 1, \dots, m-t$, are rays in Ω^n. The set of constraints,

$$X^{t+r} \in A_{t+r}, \quad r = 1, \dots, m-t,$$

can be expressed equivalently as ${}_tX \in A_t^{\Pi}$ where

$$A_t^{\Pi} \equiv \left(\prod_{r=1}^{m-t} A_{t+r} \right) \times \Omega^n.$$

Clearly, A_t^{Π} is an $(n+1)$-dimensional, closed, convex cone. The feasible set for the tth problem, is therefore

$$F_t(w_t) = A_t^{\Pi} \cap B_t(w_t),$$

a compact, convex subset of $\Omega^{n(m-t+1)}$. (In fact, $F_t(w_t)$ is a closed, connected subset of an affine set of dimension $n+1$.) Thus, the non-decreasing, continuous, strictly quasi-concave function, U^t, reaches a unique maximum on the upper boundary of $F_t(w_t)$. Moreover, at the optimum, ${}_t\hat{X}$, the (unique) gradient of the upper boundary of the set $F_t(w_t)$ belongs to the set of subgradients of the convex function whose epigraph is the set,

$$S_t({}_t\hat{X}) \equiv \{{}_tX \mid U^t({}_tX) \geqq U^t({}_t\hat{X})\}.$$

It remains to show that the function, ξ^t, is linearly homogeneous in w_t for $t = m-1, \dots, 2$. As the set

$$\lambda B_t(w_t) = \{{}_tX \in B_t(\lambda w_t) \, \forall \lambda_t \in \Omega\}$$

is a closed, convex cone of dimension $n(m-t+1)$,

$$\lambda F_t(w_t) = A_t^{\Pi} \cap \lambda B_t(w_t)$$

is a closed convex cone of dimension $m-t+n$. Hence, $\lambda F_t(w_t)$ contains the ray

$$\{{}_tX \mid {}_tX = \lambda \cdot {}_t\hat{X} \, \forall \lambda \in \Omega\}.$$

For given ${}_tP$, the gradient of the upper boundary of $F_t(\lambda w_t)$, for $\lambda \in \Omega$, is independent of λ. But, by homotheticity, the sets of subgradients of the functions defined by the epigraphs, $S_t(\lambda \cdot {}_t\hat{X})$, are independent of λ so that $\xi^t(\lambda w_t, {}_tP) = \lambda \hat{X}^t$ for $t = m-1, \dots, 2$. ||

The sophisticated solution (if it exists) would appear to be an improvement over naive planning. By explicitly considering future preferences each generation realizes a level of utility greater than that which is possible under inconsistent naive planning. But despite this improvement, it is still not clear in what sense, if any, the sophisticated solution is optimal, since each generation's utility is maximized subject to some rather unconventional constraints.

To answer this question, it seems reasonable to seek the conditions under which the system of sophisticated demand functions can be integrated into a system of intertemporal utility functions of the form $U^t({}_tX)$, $t = 1, \dots, m$. That is, we seek the conditions under

[1] This theorem is analogous to Theorem 8.1 of Peleg and Yaari [8, p. 25]. The latter authors deal, however, with the existence of a Nash game-equilibrium rather than the sophisticated solution. A set of choice functions is a Nash equilibrium if the choice function of each generation yields maximal utility for that generation, given the choice functions of all other generations. A set of choice functions generates a sophisticated path, on the other hand, if the choice function of each generation yields maximal utility for that generation given the choice functions of all *succeeding* generations for *any* feasible choice functions of *preceding* generations. A sophisticated solution is a game-equilibrium solution, but the converse does not hold (see Peleg and Yaari [8, p. 10]).

which a sophisticated society would act as if it were maximizing a sequence of intertemporal utility functions subject to conventional constraints. This, together with the fact that the demand functions are intertemporally consistent, would then imply integrability into a single utility function, U (see the remarks under Theorem 3). These conditions are given by the following proposition.

Theorem 6. *An intertemporal utility function which generates the demand functions of a sophisticated society exists if and only if the society's preferences are strongly recursive with a consistent representation.*

Proof. Consider the dual to the tth sophisticated maximization problem

$$\min_{{}_tX} {}_tP \cdot {}_tX$$
$$\text{s.t. } {}_tX \in C(U_t); \; x^{t+r} \in D_{t+r}, \quad r = 1, \ldots, m-t,$$

where

$$C(U_t) = \{{}_tX \in \Omega^{n(m-t+1)} \mid U^t({}_tX) \geqq U^t, \; U^t \in \Omega\}$$
$$D_{t+r} = \{X^{t+r} \in \Omega^n \mid X^{t+r} = \xi^{t+r}_{t+r}({}_{t+r}P, U^{t+r}), \; U^{t+r} \in \Omega\}.$$

This yields a solution,[1]

$$\left.\begin{aligned} \hat{X}^t &= \xi^t_t({}_tP, U^t), \\ \hat{X}^{t+r} &= \xi^{t+r}_t({}_tP, U^t), \quad r = 1, \ldots, m-t \end{aligned}\right\} t = m, \ldots, 1.$$

In general, $\xi^s_t(\,.\,)$ is the (vector valued) compensated demand function derived in period t for the bundle of goods to be consumed in period s. It gives the sophisticated path because in each period t, the bundles chosen for future generations lie on their "utility expansion paths", denoted by D_{t+r}.

The compensated demand function, $\hat{X}^m = \xi^m_m({}_mP, U^m)$, is homogeneous of degree zero in ${}_mP \equiv P^m$. Assuming that $\hat{X}^{t+r} = \xi^{t+r}_{t+r}({}_{t+r}P, U^{t+r})$ is homogeneous of degree zero in ${}_{t+r}P$, for $r = 1, \ldots, m-t$, then clearly $\hat{X}^t = \xi^t_t({}_tP, U^t)$ is homogeneous of degree zero in ${}_tP$ since changing all prices by the same proportion has no effect on the feasible set. Hence, by induction, all the compensated demand functions are homogeneous of degree zero in prices.

Intertemporal consistency implies that

$$\xi^s_s({}_sP, U^s) = \xi^s_t({}_tP, U^t), \quad s = t, \ldots, m, \; t = 1, \ldots, m.$$

Hence, the expenditure function for the tth continuation can be written as

$$H^t({}_tP, U^t) = \sum_{s=t}^{m} P^s \xi^s_s({}_sP, U^s) \equiv \sum_{s=t}^{m} P^s \xi^s_t({}_tP, U^t)$$
$$= \sum_{s=t}^{m} \sum_{j \in I^s} p^s_j \xi^s_{tj}({}_tP, U^t), \quad t = 1, \ldots, m,$$

where $\xi^s_{tj}(\,.\,)$ is the compensated demand by the tth generation for the jth good to be consumed in the sth period.

Differentiating (partially) with respect to the price of the ith good to be consumed in period r,

$$\frac{\partial H^t}{\partial p^r_i} = \xi^r_{ti} + \sum_{s=t}^{m} \sum_{j \in I^s} p^s_j \frac{\partial \xi^s_{tj}}{\partial p^r_i}.$$

Integrability is equivalent to the symmetry[2] condition,

$$\frac{\partial \xi^s_{tj}}{\partial p^r_i} = \frac{\partial \xi^r_{ti}}{\partial p^s_j}.$$

[1] As the theorem presumes the existence of a sophisticated solution, the existence problems discussed above are moot in this proof.

[2] That is, the symmetry of the cross-substitution term.

Substituting this condition into the equation above and employing Euler's Theorem yields

$$\frac{\partial H^t}{\partial p_i^r} = \xi_{ti}^r({}_tP, U^t).$$

Hence, consistency implies that

$$\frac{\partial H^t}{\partial p_i^{t+1}} = \frac{\partial H^{t+1}}{\partial p_i^{t+1}}, \forall i \in I^{t+1}, \quad t = 1, ..., m-1.$$

As this is equivalent to (8), the remainder of the proof proceeds exactly as in the proof of Theorem 3. ||

Remarks. Again, while the above proof uses differentiability, the theorem can be proved without this restriction.

The above result indicates that the sophisticated society acts " rationally " under alternative price and wealth configurations if and only if its preferences are strongly recursive. That is, unless the society's intertemporal preferences are of this form, it may violate the transitivity axiom. This, of course, is not surprising since the sophisticated optimum path is a " second-best " solution unless the society's preferences are strongly recursive with a consistent representation.

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