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Equilibrium Welfare and Government Policy with Quasi-geometric Discounting[1]

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We consider a representative–agent equilibrium model where the consumer has quasi-geometric discounting and cannot commit to future actions. We restrict attention to a parametric class for preferences and technology and solve for time-consistent competitive equilibria globally and explicitly. We then characterize the welfare properties of competitive equilibria and compare them to that of a planning problem. The planner is a consumer representative who, without commitment but in a time-consistent way, maximizes his or her present-value utility subject to resource constraints. The competitive equilibrium results in strictly higher welfare than the planning problem whenever the discounting is not geometric. *Journal of Economic Literature* Classification Numbers: E21, E61, E91.

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1. INTRODUCTION

Experimental psychology has gathered a significant body of evidence that preference reversals are a common occurrence in decision making over time (see, e.g., Ainslie [1]). One expression of these findings is that discounting of future rewards is not geometric—the standard case considered in economics—but rather hyperbolic, or quasi-hyperbolic.[2] A view of these findings commonly expressed by economists is that experiments in general are fraught with problems and should be disregarded or, as a less extreme position, that these particular experiments suffer from specific problems, thus including the possibility that there are alternative interpretations of the experimental results that are not in contradiction with our standard assumptions about preferences. However, a dismissal of the psychologists' findings seems hazardous, since they quite strongly suggest a "friction" that may be an important part of economic welfare and, possibly, also one where government intervention might be helpful. In this paper we take the latter view: we admit the possibility that consumption–savings decisions are indeed made by agents whose preferences allow reversals. In particular, we assume quasi-geometric preferences, which is the very simplest kind of departure from our standard assumption on discounting.[3] We assume that the time-inconsistency in preferences is accompanied by an inability of consumers to commit to future actions, but view consumers as fully rational: they are aware of their "internal friction" and do their best to minimize its effects. Our main goal here is the most basic question to an economist: we perform welfare analysis of the market mechanism. The question is whether the invisible hand works as well as in the standard case or whether government intervention would be desirable.

We show, first, that the standard recursive tools used to analyze the neoclassical-growth, general-equilibrium framework can be employed also in the case of quasi-geometric preferences. A consumer in our economy plays a game with his or her future selves, with whom he or she disagrees about how to save, and a key part of the consumer's decisions is about how to manipulate his or her future savings decisions by saving differently today. This is also a way of thinking about the geometric case, with the difference that there no disagreements occur, and manipulation is superfluous. We restrict attention to Markov (time-consistent) equilibria

[2] The term quasi-hyperbolic commonly used in the recent literature—see, e.g., Laibson [11]—refers to the use of a quasi-geometric discounting function in order to approximate a (generalized) hyperbolic function.

[3] Indeed, we look precisely at the class of discounting functions referred to as quasi-hyperbolic. As indicated in the previous footnote, the correct mathematical term for these functions is quasi-geometric; they are quasi-hyperbolic only in the sense that, for certain parameter values, they resemble a generalized hyperbolic function.

which are limits of finite-horizon equilibria, thereby obtaining uniqueness in our specific functional-form examples and allowing straightforward welfare analysis and comparative–static exercises.

For the welfare analysis, we assume that the "visible hand" in our economy—a planner with the ability to command consumption decisions, or a government with the possibility of using taxes to distort these decisions—is subject to the same friction as are consumers: it cannot commit its future behavior. So to the extent that it shares the consumers' preferences, it also will have preference reversals and will want to manipulate its future actions. It is well known that if the government can commit, then it can help consumers achieve higher ex-ante welfare, but it is not at all clear to us how the government could provide a general commitment technology for consumers' future consumption decisions. If it could commit to paths of future tax rates, it would be helpful, but if it is benevolent and shares the (combination of the current and future) consumers' preferences, such choices are not time consistent and are thus not likely to be realized. Although we do think that some government commitment may be feasible, we do not think a full ability to commit is likely. Whatever lack of commitment remains is the subject of this paper.

Our results are rather striking. We find that whenever there is a time-inconsistency in preferences, not only does a benevolent-social-planning economy not deliver the same consumption allocation as does a laissez-faire world, but it delivers strictly lower welfare. Thus, we find a new argument for why the market mechanism is a particularly good one. The key insight regards price-taking behavior, as opposed to taking into account the impact of one's decisions on prices (or aggregate allocations). An alternative interpretation of the result is that a decentralization with many identical Robinson Crusoes each operating their own production technology would do as poorly as the visible-hand world: a separation of consumption and production decisions is desirable in this economy.

The mechanism behind our main finding is intuitive and can be understood as follows. Suppose that the social planner's preferences coincide exactly with those of the current consumer. They would then both regard their respective future selves as saving too little (the argument focuses on this particular bias for illustration). As a consequence, they would both want to manipulate their future selves. The manipulation occurs via savings (which is their only decision variable here): when an extra dollar is saved, the extra income next period will influence savings next period. With time-consistent preferences, the current saver agrees with his or her future self about next period's savings decision, and an envelope argument allows him or her to ignore this effect when deciding on current savings. Here, in contrast, an extra dollar saved today has an additional benefit next period, since (so long as the marginal savings propensity is not negative or zero) it induces more future savings. The difference between the decentralized allocation and the central planning allocation originates in how these induced future increases in savings are perceived: they are perceived as being larger by a price-taker than by a planner. The reason for this is that the price-taker's future behavioral response is "more linear:" the added income he or she obtains next period is the higher savings times the return, which he or she perceives as constant. The planner, on the other hand, sees that for every additional dollar saved, the next period's return falls (assuming decreasing marginal productivity to capital). In sum, the incentives to save are higher for the decentralized agents, so they save more, and more saving is better in this economy: it takes us closer to the full commitment outcome.[4]

The most closely related literature to this paper is the set of studies started by Strotz [21] and Phelps and Pollak [18] and then further developed recently in a set of papers Laibson, e.g., Laibson [13]. Laibson [14] discusses some aspects of government taxation, but does not consider the case when the government cannot commit its future tax rates, the main focus of our analysis here. In many of Laibson's setups (e.g., Laibson [15]), there are added market frictions, such as credit constraints, which makes illiquid assets play an important role. As a consequence, Ricardian equivalence does not hold. Another common assumption is the existence of partially uninsurable idiosyncratic shocks (Harris and Laibson [6]). Here, the setup is the most basic one possible: the only friction is the internal friction in consumers' preferences (naturally accompanied with a lack of commitment), and there is no uncertainty. In particular, all long- and short-term asset markets are operative; although it would be beneficial to close asset markets in the future, without the commitment power to do so, ex post it will always be in the interest of the group of consumers as a whole to keep current markets open. As a result, in our environment, Ricardian equivalence holds. Another example of an added friction is the one considered in O'Donoghue and Rabin [17], where in one version of their model consumers are not rational, but are rather constantly surprised at their preference reversals.

The analysis in the paper builds fundamentally on recursive methods and moreover uses specific functional forms so as to allow manageable closed-form solutions. The dynamic game thus played between the current and future selves is perhaps the simplest tractable example of a time-consistent equilibrium available in the literature. This literature began in Kydland and Prescott [12] but has not allowed simple closed-form examples of Markov equilibria. We consider the case with a saver with time-inconsistent preferences as very natural grounds for illustrating the main forces at work.

[4] The argument is parallel when the time inconsistency takes the form of excessive short-run patience.

Central to this illustration are the recursive setup and the derivation and discussion of the generalized Euler equation. Section 2 describes the basic setup and the principles we follow in our analysis. Section 3 introduces recursive competitive equilibrium (without policy), and Section 4 considers the planning problem, including the comparison with the competitive equilibrium outcome. Section 5 looks at time-consistent policy: we show that the planning outcome is also the outcome of a government policy game, provided the government has a sufficiently large set of instruments. Section 6 concludes, discusses some weaknesses in the analysis, and mentions an alternative approach.

2. THE SETUP

2.1. *Primitives*

Time is discrete and infinite and begins at time 0; there is no uncertainty. An infinitely lived consumer derives utility from a stream of consumption at different dates. We assume that the preferences of this individual are time-additive, and that they take the form of a sequence of preference profiles:

$$U_0 = u_0 + \beta(\delta u_1 + \delta^2 u_2 + \delta^3 u_3 + \cdots)$$
$$U_1 = u_1 + \beta(\delta u_2 + \delta^2 u_3 + \cdots)$$
$$U_2 = u_2 + \beta(\delta u_3 + \cdots).$$

When $\beta = 1$, we have standard, time-consistent, geometric preferences. When $\beta \neq 1$, there is a time inconsistency: at date 0, the trade-off between dates 1 and 2 is perceived differently than at date 1, and so on. When $\beta < 1$, we have excessive short-run impatience: the individual thinks "I want to save, just not right now;" when $\beta > 1$, excessive short-run patience is expressed as "I want to consume, just not right now." We refer to this class of preferences as quasi-geometric, as they are a one-period deviation from the standard geometric case.[5] Successively more complicated extensions are straightforward to analyze within our framework.[6] Figure 1 illustrates the different cases.

[5] The term quasi-hyperbolic is used in the literature as referring to the same preference setup even though, mathematically, hyperbolic functions take an entirely different form. The reason for the use of the term quasi-hyperbolic is that, for certain parameter values—see Fig. 1—the discounting function resembles a (generalized) hyperbola. Here, we are interested in the entire class of quasi-geometric preferences.

[6] One would then, for example, assume that the disagreement between current self and the self k periods later disagree not only on the value of date $t+k$ consumption relative to other goods, but on the relative value of date $t+k+1$ consumption as well. This extension would simply require the introduction of another β. Successive introductions of more βs would then relax the geometric framework more and more.

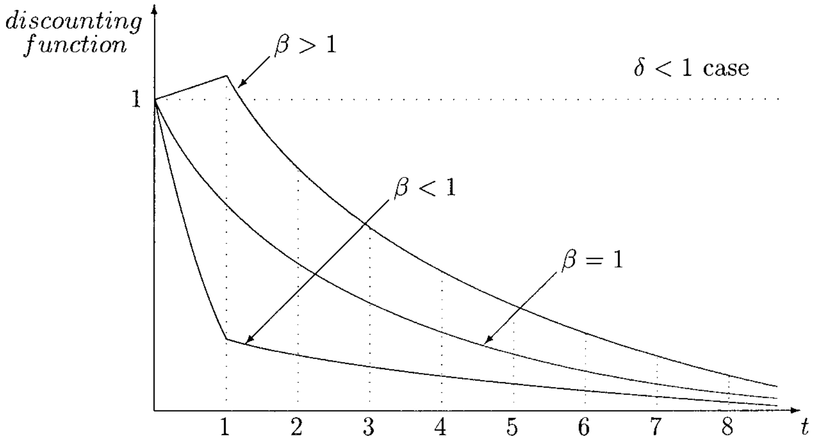

FIGURE 1

The standard infinite-life setup is often interpreted in terms of an infinitely lived dynasty. This interpretation is possible here, too. If c_t refers to the entire consumption of generation t, then the assumption of quasi-geometric discounting implies a version of impure altruism: although the generation t agent cares about the consumption of all his or her descendants, the agent disagrees with his or her descendants on the weights. For example, the agent places a higher weight on the consumption of his or her grandchildren relative to his or her children than do his or her children. Thus, with dynasties in mind, the dynamic game we study here can be thought of as a game between different generations.

2.2. *Modeling Behavior under Time-Inconsistent Preferences*

As time progresses, the individual will change his or her mind about the relative values of consumption at different points in time so long as $\beta \neq 1$. The individual would, therefore, if he or she could, commit his or her future consumption levels. We assume that there is no way for the consumer to do so. This is a rather natural assumption in a framework with time-inconsistent preferences. The reason is that commitment contracts would have to be quite elaborate to avoid renegotiation problems. Suppose two consumers, A and B, agree on a contract whereby consumer B would punish consumer A for any deviation from the planned future consumption. For his or her services, consumer B would be paid some reward. At the future date, however, consumer A would be able to convince consumer B not to carry out the punishment: A would just offer slightly more than the original reward to B for tearing the contract instead of adhering to it.

Hence, unless the two consumers were playing an infinite game, renegotiation would always make both consumers better off ex post, and any forward-looking consumer would not bother to set up a commitment contract. The fact that a truly infinite horizon is necessary for commitment contracts to work is an argument that makes anonymous markets unlikely suppliers of commitment services in practice, and we believe that it is an important reason why we do not observe such a market.[7] In our model, the horizon is infinite, but we make a general restriction to Markov strategies in our work—one that we discuss below—and one consequence of this restriction is that commitment cannot be achieved.

Further, we assume that the consumer realizes that his or her preferences will change and makes the current decision taking this into account—this encapsulates our notion of rationality in this framework.[8] This means that we model the decision-making process as a dynamic game, with the agent's current and future selves as players.

For our game, we focus on (first-order) Markov equilibria: at a moment in time, no histories are assumed to matter for outcomes beyond what is summarized in the current stock of wealth held by the agent. This means that we rule out trigger-strategy equilibria of the kind studied in Laibson [13] and Bernheim *et al.* [3].[9] Further, we restrict attention to those Markov equilibria which are limits of finite-horizon equilibria. This refinement eliminates a large number of equilibria: Krusell and Smith [11] show that there is a large set of equilibria for this game even when attention is restricted to first-order Markov equilibria. How many equilibria remain is in general not known, but for the specific parametric case that allows a closed-form solution, there is only one remaining equilibrium. This can be shown by explicit backward solution of the finite-horizon model. The resulting equilibrium thus generalizes the standard preference setup in a continuous manner. Our ability to establish uniqueness relies on restrictive assumptions on technology and preferences—we use closed-form solutions. However, for a slightly larger parametric class that does not allow closed-form solutions, we solve for equilibria numerically and do not find any evidence that the limits of finite-horizon equilibria are not unique.

2.3. *Assumptions on Primitives*

In order to obtain closed-form solutions, we restrict preferences and technology to specific functional forms. The period utility function is

[7] One could argue that commitment mechanisms are more likely to occur within tight social groups or families. Social networks would then potentially play an important role in accumulation decisions.

[8] Others, such as O'Donoghue and Rabin [17], in addition consider the possibility that the consumer does not realize that his or her preferences will change.

[9] We discuss reasons for our refinement strategy in Section 6.

$u(c) = \log c$. We assume that $0 < \delta < 1$ and that $\beta > 0$. Production is Cobb–Douglas and there is full depreciation so the resource constraint reads

$$c + k' = Ak^{\alpha}.$$

Primes denote consecutive-period values.

Perfect competition implies marginal-product pricing of the capital and labor inputs:

$$r = \alpha Ak^{\alpha-1}$$

$$w = (1-\alpha)\, Ak^{\alpha}.$$

2.4. *Recursive Formulation of the Decision Making*

Assume that the current self perceives future savings decisions to be given by a function $g(k)$:

$$k_{t+1} = g(k_t).$$

Note that, by the Markov assumption, g is time independent and only has current capital as an argument.

The current self solves

$$V_0(k) \equiv \max_{k'}\ u(rk + w - k') + \beta\delta V(k'),$$

where

$$V(k) = u(rk + w - g(k)) + \delta V(g(k)).$$

Notice that successive substitution of V into the objective generates the right objective if the expectations of future behavior are given by the function g.

A solution to the current self's problem is denoted $\tilde{g}(k)$. We have a solution to the agent's game (i.e., to the game between the different selves) if the fixed-point condition $\tilde{g}(k) = g(k)$ is satisfied for all k.

Parenthetically, the commitment solution would be obtained if the expression for V instead satisfied the standard dynamic programming functional equation

$$V(k) = \max_{k'}\ u(rk + w - k') + \delta V(k'),$$

which would yield a g function that differs from $\tilde{g}$ so long as $\beta \neq 1$; $\tilde{g}$ would be used at time 0 and the g forever after.

2.5. *Constant Prices: An Explicit Solution*

If the prices r and w in the previous section are constant and exogenous, then, with our parametric assumptions, we can solve explicitly for the equilibrium decision rule and corresponding value function. In particular, the value function takes the form $V(k)=a+b\log(k+\frac{w}{r-1})$, where $k+\frac{w}{r-1}$ is proportional to the present value of the agent's lifetime wealth $W\equiv rk+w\sum_{i=0}^{\infty}1/r^{i}$. The equilibrium decision rule takes the form $g(k)=s(rk+\frac{r}{r-1}w)-\frac{w}{r-1}$, where $s=\frac{\beta\delta}{1-\delta(1-\beta)}$. It is not too hard to see that this decision rule implies that $W'=srW$; that is, the agent saves a constant fraction of his or her wealth in each period.

On a stationary point, where the individual's capital stock does not change, we have $g(k)=k$ and $W'=W$. This implies $r=\frac{1}{s}$, a requirement for a steady state in the general equilibrium model, which in turn equals $\frac{1-\delta(1-\beta)}{\beta\delta}$. This means that $\frac{w}{r-1}=s\frac{r}{r-1}w$. The left-hand side of this equality is present value of future labor income; the right-hand side is equal to the savings out of the present value of total labor income. These two need to be equal to each other in equilibrium. As we will show, this kind of condition will also hold off the steady state in the economy with our particular functional-form assumptions.

3. RECURSIVE COMPETITIVE EQUILIBRIUM

Before we move on to the formal definition of a recursive competitive equilibrium, we need to describe the market structure. Our assumption here is that the consumer rents his or her capital and labor services to firms, treating prices parametrically. Further, the consumer makes the accumulation decisions for capital. If we assumed that firms made these decisions and that consumers had access to markets for one-period loans, the results would not change. The addition of multiperiod assets would also not change our results: these assets would be priced using arbitrage and their returns would be given by the returns on the relevant one-period assets. If one-period assets did not exist, the results would change. However, with a similar argument as the one used above, it would always be in the interest of consumers ex post to open one-period asset markets, and the spirit we follow here is to treat consumers as fully rational at any moment in time. Thus, the natural benchmark for us is to assume that one-period asset markets are open.

In order to analyze a general equilibrium, on and off its steady state, we need to be explicit about state variables. The agent makes his or her decision taking as given the prices as functions of the aggregate state $\bar{k}$,

$(r(\bar{k}), w(\bar{k}))$, the law of motion for the aggregate state, $\bar{k}' = G(\bar{k})$, and the decision rules of his or her future selves; $g(k, \bar{k})$. The recursive equilibrium requires two state variables for the individual: one for the individual's own capital holdings, k, and one for the average capital holdings in the economy, $\bar{k}$, the latter reflecting prices. The current self's problem can be formulated in a similar way to before:

$$V_0(k, \bar{k}) = \max_{k'} \log(r(\bar{k})\, k + w(\bar{k}) - k') + \beta\delta V(k', \bar{k}').$$

The solution to this problem is given by $\tilde{g}(k, \bar{k})$, where $V(k, \bar{k})$ satisfies

$$V(k, \bar{k}) = \log(r(\bar{k})\, k + w(\bar{k}) - g(k, \bar{k})) + \delta V(g(k, \bar{k}), \bar{k}').$$

Formally, we have

Definition 1. A recursive competitive equilibrium for this economy consists of a decision rule, $g(k, \bar{k})$, a value function, $V(k, \bar{k})$, pricing functions $r(\bar{k})$ and $w(\bar{k})$, and a law of motion for aggregate capital, $\bar{k}' = G(\bar{k})$, such that

1. given $V(k, \bar{k})$, $g(k, \bar{k})$ solves the maximization problem above;
2. given $g(k, \bar{k})$, $V(k, \bar{k})$ satisfies the functional equation above;
3. firms maximize, i.e., $r(\bar{k}) = f'(\bar{k})$ and $w(\bar{k}) = (1-\alpha)\, f(\bar{k})$;
4. and the law of motion for aggregate capital resulting from the current self's decision is consistent with the law of motion for aggregate capital; i.e., $g(\bar{k}, \bar{k}) = G(\bar{k})$.

We have

Proposition 1. *For our parametric economy, the recursive competitive equilibrium is given by:*

1. $V(k, \bar{k}) = a + b \log \bar{k} + c \log (k + \varphi \bar{k})$, *where* $c = \frac{1}{1-\delta}$, $b = \frac{\alpha - 1}{(1-\delta)(1-\alpha\delta)}$, *and* $\varphi = \frac{(1-\alpha)(1-\delta(1-\beta))}{\alpha(1-\delta)}$;
2. $g(k, \bar{k}) = \frac{\delta\beta}{1-\delta(1-\beta)}\, r(\bar{k})\, k$; *and*
3. $G(\bar{k}) = g(\bar{k}, \bar{k}) = \frac{\alpha\delta\beta}{1-\delta(1-\beta)}\, A\bar{k}^{\alpha}$.

Proof. See Appendix A.

The proof uses the intuition developed in the partial equilibrium case. Notice that the equilibrium savings rates in the partial equilibrium and the general equilibrium solutions are the same. We have also suppressed the discussion of other equilibria; all equilibria calculated in this paper are limits of finite-horizon equilibria, and as such are unique.

We now turn to the equivalence of the aggregate statistics across models with time-consistent and time-inconsistent preferences (this is a result parallel with that found in Barro [2] for a continuous-time model).

PROPOSITION 2. *The laws of motion for capital are the same for any two models* $(\hat{\beta}, \hat{\delta})$ *and* (β, δ) *such that* $\frac{\hat{\delta}\hat{\beta}}{1-\hat{\delta}(1-\hat{\beta})} = \frac{\delta\beta}{1-\delta(1-\beta)}$.

The proposition implies that it is not possible to estimate (β, δ) by looking at aggregates (or disaggregated variables, for that matter) for the class of preferences and technology that we concentrate on. As a special case, note that the outcome of the model with (β, δ) is identical to the outcome of the standard growth model with $\hat{\delta} = \frac{\delta\beta}{1-\delta(1-\beta)}$.

The observational equivalence simplifies presentation here, but does not eliminate the issue we are interested in: the welfare properties of equilibria, and policy analysis. As we will see, welfare properties across models with different underlying preferences but the same equilibrium laws of motion can be very different.

4. WELFARE PROPERTIES

Welfare properties are usually discussed in terms of the Pareto criterion. Since we are considering an economy with lack of commitment as a central element, it is difficult to formalize this criterion: moving allocations freely around over time violates the assumption of a lack of commitment. We will phrase our discussion in terms of a central planning problem; we will define such a problem and then compare its solution to the competitive equilibrium. In contrast to a Pareto problem, the central planning problem we formulate does not necessarily have a solution with "good" welfare properties.

There is no obvious best notion of a planner here, since the consumer's different selves disagree. However, although one could think of a meta-planner placing positive weights on the lifetime utilities of more than the current self, we do assume here that the planner simply shares the preferences of the current self.[10] That is, by a central planning solution we simply mean a solution which would be chosen by a benevolent representative of the consumers who had the ability to manipulate the current economic choice variables costlessly. Thus, we want to think of the analysis of the benefits of markets as a simple "invisible hand" versus "visible hand" comparison, in the spirit of Adam Smith.

[10] We briefly discuss implications of having meta-planners below.

An important reason for adopting this specific planner is found in Section 5. There we show that if one formalizes the notion of a government with a sufficiently large set of tax instruments representing the preferences of its current electorate, the resulting allocation coincides with the one we obtain in our fictitious planning economy. Finally, and as motivated above, we further assume that the planner cannot directly affect his or her future choices, thereby having to play a game with his or her future selves, just like the consumer does in a competitive equilibrium.

4.1. *The Planning Problem*

In summary, in this section we assume the following: (i) the planner is a consumer representative; i.e., he or she inherits his or her (time-inconsistent) preferences; (ii) the planner faces the same problem as the consumer: he or she cannot commit to future actions; (iii) we require a time-consistent solution to the planner's problem (future planners' reactions are taken into account in a rational way).

The differences between the consumer's equilibrium problem and the planner's problem are thus as follows: (i) the consumer takes prices as given, whereas the planner has a resource constraint; and (ii) the equilibrium consumer deals with different future players (the consumer's future price-taking selves) than the planner does (the planner's future selves).

The problem of the planner's current self can be formulated in the following way

$$V_0(k) \equiv \max_{k'} u(f(k)-k') + \beta\delta V(k'),$$

where

$$V(k) = u(f(k)-h(k)) + \delta V(h(k)).$$

A solution to the problem of the planner's current self is denoted $\tilde{h}(k)$. We have a solution to the planner's game (i.e., to the game between the planner's different selves) if the fixed-point condition $\tilde{h}(k) = h(k)$ is satisfied for all k.

It is straightforward to show that the following functions are solutions to the planner's game

$$k' = h(k) = \frac{\beta\delta}{1-\alpha\delta(1-\beta)} \alpha A k^{\alpha}$$

and

$$V(k) = a + b \log k,$$

where a and b have simple closed-form solutions in terms of our parameters.

4.2. *The Planning Outcome vs the Competitive Outcome*

Apparently, whenever $\alpha < 1$, the competitive equilibrium and the planning problem produce different outcomes. When $\beta < 1$, the price-taking agent saves more than the planner does; when $\beta > 1$, the planner saves more. Neither the planner's solution nor the competitive equilibrium outcome coincide with the full commitment solution. The commitment solution for this economy has one savings rate at time 0 and another, higher, one at all future times. The time-0 commitment rate is equal to the savings rate of the planning problem without commitment; the subsequent savings rate is higher (lower) than both the competitive and the no-commitment planning outcomes when $\beta < (>) 1$.

Turning to a welfare comparison between the competitive equilibrium and the planning solution, we have the following.

PROPOSITION 3. *The competitive outcome results in strictly higher welfare for the consumer than the planning outcome does, whenever $\beta \neq 1$ and $\alpha < 1$.*

Proof. See Appendix A.

Thus, markets outperform a benevolent social planner in this economy; in particular, competitive behavior results in higher savings (when $\beta < 1$), and since there is undersaving relative to the full commitment case, higher savings moves the economy in the right direction. What explains this finding? As pointed out above, it should not be surprising that the two allocations are different. The planning problem is not a standard planning problem; in particular, the planner faces a different environment than do the competitive consumers: they face different future players, whom they cannot fully control. In order to explain why the planner faces future players that induce worse outcomes, we will make use of the generalized Euler equation in each case.

4.3. *The Generalized Euler Equation*

A simple derivation of the generalized Euler equation (GEE), which can be used whenever the value function and the policy function are differentiable, goes as follows. Consider the problem of the planner. His or her first-order condition reads

$$u'(f(k)-h(k)) = \beta\delta V'(h(k)).$$

To eliminate the unknown function V', take derivatives of the functional equation for V:

$$V'(k) = u'(f(k)-h(k))(f'(k)-h'(k)) + \delta V'(h(k))\, h'(k).$$

Substitute $V'(h(k))$ from the first of these equations into the second, then update the second and substitute the new expression for $V'(h(k))$ back into the first equation. This gives

$$\begin{aligned} &u'(f(k)-h(k)) \\ &\quad = \beta\delta u'(f(h(k))-h(h(k)))\left(f'(h(k))+\left(\frac{1}{\beta}-1\right)h'(h(k))\right). \end{aligned}$$

This is our key behavioral equation: it is a functional equation in the unknown savings function $h(k)$. For readability, consider this equation in sequential form:

$$u'(c_t) = \beta\delta u'(c_{t+1})\left(f'(k_{t+1})+\left(\frac{1}{\beta}-1\right)h'(k_{t+1})\right).$$

Notice the $h'(k_{t+1})$ term on the right-hand side: you do not agree with your future self about savings propensities and therefore value giving your future self more wealth (if $\beta < 1$). When β is equal to 1, this term does not appear: the envelope theorem (which allows the second equation above to be written simply as $V'(k) = u'(f(k)-h(k))\, f'(k)$) dictates that, since you agree with your future self, you in essence make the future decision yourself, so the indirect effects on savings next period, as captured by h', are second order. Specifically, for every additional unit of saving, consumption next period decreases—by definition—by exactly the marginal propensity to save, $h'(k_{t+1})$. In the standard, $\beta = 1$, case, this utility loss is exactly outweighed by—equal to—the utility increase from additional utility flows after period $t+1$, so the two effects cancel. Here, however, the current self appreciates any utility flows after time $t+1$ exactly $1/\beta$ times more than the self at $t+1$—hence the term $(1/\beta-1)\, h'(k_{t+1})$ must appear as an additional "return" from savings at t. This additional return to saving is crucial in what follows.

Similarly, for the competitive equilibrium we obtain

$$\begin{aligned} &u'(f(k)-G(k)) \\ &\quad = \beta\delta u'(f(G(k))-G(G(k)))\left(f'(G(k))+\left(\frac{1}{\beta}-1\right)g_1(G(k),\, G(k))\right), \end{aligned}$$

or, in sequential form,

$$u'(c_t) = \beta\delta u'(c_{t+1})\left(f'(k_{t+1}) + \left(\frac{1}{\beta} - 1\right) g_1(k_{t+1}, k_{t+1})\right).$$

The two GEE's look identical except for the derivative term (h' and g_1, respectively). Here is the key difference: h has decreasing returns to its argument,

$$h(k) = \hat{s}\alpha A k^{\alpha},$$

if $\alpha < 1$, whereas g is linear in its first argument,

$$g(k, \bar{k}) = sk\alpha A\bar{k}^{\alpha-1}.$$

So, (if $\beta < 1$) the competitive equilibrium consumer sees a higher benefit from extra saving today than does the planner: everything else equal, the planner sees another unit of savings as yielding a smaller increase in future savings than does the competitive-equilibrium consumer. This makes the competitive agent save more than the planner, because the future selves are undersaving and extra future saving is now a good thing. The argument works for $\beta > 1$ as well. Then, the planner saves more than the competitive equilibrium, which saves too much. An extra saved unit increases future savings more as perceived by the consumer than as perceived by the planner. As a result, the consumer saves less than the planner since extra future savings is now a bad thing.

Behind these arguments is the main difference between a planner and a competitive individual: the planner understands that he or she affects the "prices," that is, the return to savings, whereas a price-taker does not. This implies that the marginal propensity to save for the planner is decreasing, whereas it is constant for the competitive agent. This is the key difference behind the planner's and the competitive agent's decision rules.

4.4. *Utility Comparisons for Other Agents*

One might take the view that the consumer's future selves, and their different preferences, ought to be respected and taken separately into account in the comparison between the two allocations above. Suppose, therefore, that we evaluate the utility of the self next in line as he or she perceives it. Would he or she prefer the competitive or the central planning allocation?

It is clear from the arguments in the preceding section that if the next self were given the same amount of capital to start with in both situations, then he or she would prefer the competitive allocation; it provides higher savings at all times, which is perceived as better (if $\beta < 1$). However, capital is not constant across the two allocations. Depending on the value of β, it is either higher or lower. If $\beta < 1$, it is higher, and the competitive equilibrium dominates, not only in the next period, but in all future periods as well. This is the case most commonly emphasized in the literature on time-inconsistent preferences. In this case, therefore, the competitive equilibrium allocation Pareto dominates the central planning allocation. If, on the other hand, $\beta > 1$, there is an effect in favor of the central planning allocation for all future selves, and the net result is not clear.

4.5. *Robustness: Some Examples*

The savings and utility comparisons above across the centralized and decentralized economies use a particular parameterization. We will now briefly discuss a slightly broader class of economies—one with isoelastic utility and less than full depreciation of capital, in which case no closed-form solution can be obtained. First, we compare steady-state capital stocks; it is possible to show that the planning outcome always gives a lower long-run capital stock.[11] Second, we consider utility: we compare the current-self utility at a decentralized steady state to that given by the planning outcome starting from the same capital stock as in the decentralized steady state.

4.5.1. *Steady states.* Recursive competitive equilibria for the present class of economies can be characterized by two functions, $\lambda(\bar{k})$ and $\mu(\bar{k})$, whenever utility is isoelastic (that is, independent of the production structure). In particular, Hercowitz and Krusell [7] showed that the policy function of the individual satisfies

$$g(k, \bar{k}) = \mu(\bar{k}) + \lambda(\bar{k})\, k.$$

The shape of the two unknown functions depends on the specifics of preferences and technology.[12] That is, savings are affine in the individual's wealth with coefficients that depend only on aggregate variables—an aggregation result that is expected given isoelastic utility. Moreover, at the steady-state equilibrium capital stock, $\bar{k}_E$, the functions satisfy $\mu(\bar{k}_E) = 0$ and $\lambda(\bar{k}_E) = 1$: the permanent income hypothesis holds (any increase in initial capital is saved forever, leaving a constant increase in consumption equal to the return on the added capital). This means that the competitive equilibrium steady state must satisfy

$$1 = \beta\delta\left(f'(\bar{k}_E) + \left(\frac{1}{\beta} - 1\right) g_1(\bar{k}_E, \bar{k}_E)\right) = \beta\delta\left(f'(\bar{k}_E) + \frac{1}{\beta} - 1\right),$$

[11] Throughout this section, we assume that $\beta < 1$.

[12] They satisfy a pair of functional equations.

where the last equality follows from the condition $g_1(\bar{k}_E, \bar{k}_E) = \lambda(\bar{k}_E) = 1$ just stated. This delivers

$$\bar{k}_E = (f')^{-1}\left(\frac{1-\delta(1-\beta)}{\beta\delta}\right).$$

We note that the steady-state capital stock is independent of the curvature of the utility function, as in the model where $\beta = 1$.[13]

The steady state for the planning problem cannot be derived analytically; the value of the constant capital stock, $\bar{k}_P$, satisfies

$$1 = \beta\delta\left(f'(\bar{k}_P) + \left(\frac{1}{\beta} - 1\right) h'(\bar{k}_P)\right),$$

but $h'(\bar{k}_P)$ has no closed-form expression. However, we know that it has to be less than 1 for the steady state to be stable (which we regard as a minimal requirement for being interested in a steady state). Thus, assuming that the steady state is stable, and following the same algebra as above, we see that $\bar{k}_P$ has to satisfy

$$\bar{k}_P = (f')^{-1}\left(\frac{1-\delta(1-\beta)\, h'(\bar{k}_P)}{\beta\delta}\right).$$

Because of stability,

$$\frac{1-\delta(1-\beta)\, h'(\bar{k}_P)}{\beta\delta} > \frac{1-\delta(1-\beta)}{\beta\delta}.$$

Thus, whenever $(f')^{-1}$ is strictly decreasing (which we also assume, given the neoclassical focus here), $\bar{k}_P < \bar{k}_E$ must hold.

4.5.2. *Numerical solution of the planner's problem.* For specific utility comparisons, we need to use numerical methods, which requires us to specify a production technology and give values to parameters. Before proceeding to results, we need to comment on the numerical methods that we use. There are two reasons for this: (i) we have developed new methods, since some standard methods fail; and (ii) these new methods are likely to be useful also in other applications where the decision-maker's commitment solution is not time consistent and a time-consistent solution is sought, such as in many macroeconomic problems of optimal policy determination.

In contrast to the standard model, numerically solving the present model —even in partial equilibrium, and even when β is very close to 1—

[13] This fact was noted in Barro [2].

can be a daunting task, because many standard algorithms fail.[14] Specifically, techniques relying on value-function iteration fail to converge and instead tend to cycle. We strongly suspect that these problems have their roots in the indeterminacy finding of Krusell and Smith [11]: with an infinite set of Markov solutions, any numerical routine not relying on specific properties of the solution will have severe difficulties.[15]

Although Euler equation methods might prove useful, we use instead a new method that can be viewed in some respects as a variation on the regular perturbation method described in Chapter 13 of Judd [8]. Specifically, it is based on repeated differentiation of the GEE and therefore assumes that the policy function is differentiable many times. As a first step toward understanding our algorithm, notice that the GEE, evaluated at the steady state, contains, in effect, two unknowns: the level of the policy function, $h(\bar{k}_P) = \bar{k}_P$, and its derivative, $h'(\bar{k}_P)$. Thus the GEE, evaluated at the steady state, can be viewed as a single equation in these two unknowns. To generate another equation, one can proceed as in Judd [8]: differentiate the GEE, which holds for all values of the state variable, and evaluate it at the steady state. However, this equation now in addition contains another unknown, $h''(\bar{k}_P)$. In a first-order approximation to the policy function, we set $h''(\bar{k}_P) = 0$, thereby giving us two equations in the two unknowns $\bar{k}_P$ and $h'(\bar{k}_P)$.

To compute a second-order approximation to the policy function, we can differentiate the GEE again, yielding an equation that now contains $h'''(\bar{k}_P)$. By setting this third derivative equal to 0, we obtain three equations in the three unknowns $\bar{k}_P$, $h'(\bar{k}_P)$, and $h''(\bar{k}_P)$. Evidently, every differentiation of the GEE produces an additional equation but also an additional unknown, which we set to 0 in order to compute an approximation of a given order.

The fact that an additional unknown appears when we differentiate the GEE means that applying the regular perturbation method to this problem is computationally more demanding than applying it to the standard growth model. For the standard model, the steady-state capital stock can be computed directly from the Euler equation without knowing anything

14 This is not a new observation; see Laibson, *et al.* [16].

15 Two comments are in order. First, the fact that we are searching for a continuous solution here might seem enough for ruling out the large set of solutions uncovered in Krusell and Smith [11], since the latter are discontinuous by construction. However, the discontinuities are countable in number and seem hard to detect with global methods. Second, one might think that backward solution, given a finite-horizon problem as in Laibson, *et al.* [16], would work. However, the horizon does not have to be very long for a multitude of "near-solutions" to cause problems for standard algorithms: the indeterminacy result implies existence of these near-solutions.

about the derivatives of the policy function. Moreover, successive differentiations of the Euler equation simply yield one additional unknown, namely, the next higher-order derivative of the policy function, evaluated at the steady state. Consequently, in the standard model, one can compute the sequence of derivatives of the policy function iteratively. In our model, by contrast, we must solve simultaneously both for the steady state and for the derivatives of the policy function.

To solve for the steady state and the derivatives of the policy function at the steady state, we use an algorithm that is inspired by the perturbation ideas described above, but which differs in the details of implementation. Specifically, we let the policy function be approximated by a polynomial of order $n-1$ (with the logarithm of the capital stock as the state variable). The goal of the algorithm is to choose the n coefficients of this polynomial, together with the steady state, to satisfy $n+1$ restrictions. One of these restrictions is that the steady state be a stationary point of the approximate decision rule. The remaining n restrictions are that the GEE and its first $n-1$ derivatives equal 0 when evaluated at the steady state. Thus, we are in effect choosing the level of the decision rule and its first n derivatives to solve the set of equations generated by the perturbation method described above. Appendix B describes our algorithm in much greater detail.

4.5.3. *Utility comparisons.* We assume a Cobb–Douglas specification, use a capital share of 0.36, and assume a depreciation rate of 0.10. Setting $\delta = 0.95$ and $\beta = 0.90$, we then check robustness with respect to the intertemporal substitution elasticity parameter: with $u(c) = (c^{1-\sigma}-1)/(1-\sigma)$, we vary σ from 0.5 to 7.

Table I reports the key results. The table lists capital stocks and utilities, where the planning problem has been solved using a decision rule that takes the form of a third-order polynomial in the logarithm of the capital stock:

$$h(\bar{k}) \cong \exp(a_0 + a_1 \log(\bar{k}) + a_2 \log(\bar{k})^2 + a_3 \log(\bar{k})^3).$$

The utility for the competitive equilibrium is simply the utility of self 0 experienced in the steady-state equilibrium; the utility for the planning problem is the utility obtained from starting at $\bar{k}_E$ and then converging to the planning steady state $\bar{k}_P$. We see from Table 1 that, as shown in the previous section, the planning steady state is always below the competitive steady state and that the former depends on the curvature of u, whereas the latter does not. We also see that the planning steady state becomes closer to the equilibrium steady state as the curvature increases: it is well known that increased curvature leads to slower convergence, that is, to a higher derivative of the decision rule at steady state, which we know from the above

TABLE I

Steady-State Comparisons

σ	$\bar{k}_P$	$\bar{k}_E$	$V_0^P(\bar{k}_E)$	$V_0^E(\bar{k}_E)$
0.5	2.87	3.60	3.73	3.88
1	3.09	3.60	3.62	3.69
2	3.25	3.60	4.51	4.56
3	3.33	3.60	5.05	5.09
4	3.37	3.60	5.67	5.70
5	3.40	3.60	6.38	6.41
6	3.43	3.60	7.21	7.23
7	3.44	3.60	8.16	8.19

section makes the steady states closer. Utility is always higher in equilibrium than in the planning solution. That is, our theoretical findings above are robust at least to the variations presented here.

5. POLICY ANALYSIS

The planning problem in the previous section can perhaps be viewed as artificial. In this section we consider a government with taxation abilities. There are no government expenditures and we impose budget balance; moreover, we assume that the government is benevolent in that it is the representative of the consumer. As the planner, the government cannot commit its future self: future tax rates are set by future governments, and future governments have different preferences, as they are representatives for the futures selves of the consumer.

Assume that the government can tax income and investment at proportional rates τ_y and τ_i, respectively. The first of these is essentially a lump-sum tax in this model, since income and substitution effects cancel with logarithmic utility. The investment tax, on the other hand, is distortionary.

We first consider the full commitment case for illustration. We then turn to time-consistent taxation. Time-consistent taxation in this model leads to taxes which are constant over time (as are savings rates).

5.1. *Full Commitment to Future Taxes*

Since the consumer's future selves are undersaving from the perspective of the current self, the government will want to subsidize investment in all future periods. Given the stationarity of the problem and the log/Cobb–Douglas assumptions, these rates will be constant and given by

$$\tau_i' = \frac{(1-\delta)(\beta-1)}{1-\delta+\delta\beta-\alpha\delta\beta} \quad \text{and} \quad \tau_y' = \frac{\alpha\delta(1-\delta)(1-\beta)}{1-\delta+\delta\beta-\alpha\delta\beta}.$$

These tax rates for future periods will give a savings rate of δ for these periods. For the current period, the government sets the tax rates

$$\tau_i = \frac{\delta(1-\alpha)(1-\beta)}{1-\alpha\delta-\delta(1-\alpha)(1-\beta)} \qquad \text{and}$$

$$\tau_y = \frac{\alpha\delta^2\beta(1-\alpha)(1-\beta)}{\alpha\delta^2\beta(1-\alpha)(1-\beta)-(1-\alpha\delta)(1-\delta(1-\beta))}.$$

This gives a savings rate of $\frac{\delta\beta}{1-\alpha\delta(1-\beta)}$ for the current period. This tax sequence generated by fully committed governments is not time consistent. This creates incentives for future governments to deviate. We now move on to time consistent taxation.

5.2. *No Commitment: Time-Consistent Policy*

We define a subgame-perfect Markov equilibrium for the government problem. This definition parallels the definition of subgame-perfect Markov equilibria: a tax function describes the tax outcome as a function of the aggregate state, and to support this tax function it is necessary to consider one-period deviations from the equilibrium path.[16] The competitive equilibrium is defined as above for given taxes. Before we go through the definition of the time-consistent policy equilibrium, let us point out that taxes in this equilibrium will be constant, due to our special parametric assumptions. To support them as being constant, however, it is necessary to formally verify all the conditions of a subgame-perfect Markov equilibrium.

DEFINITION 2. A time-consistent policy equilibrium is defined in several parts; the elements are listed below.

First, the behavior on the equilibrium path ("outcomes") are as follows:

- Tax outcomes are given by $\tau(\bar{k}) = (\tau_y(\bar{k}), \tau_i(\bar{k}))$.
- Given this tax function, the law of motion for aggregate capital is given by $G(\bar{k})$.
- Given the tax function and the law of motion for aggregate capital, the individual's decision rule is given by $g(k, \bar{k})$.

Second, the one-period deviations to tax rates $\tilde{\tau} = (\tilde{\tau}_y, \tilde{\tau}_i)$ for the current period, with future taxes given by the tax outcome functions evaluated at the capital stocks implied by the current tax rates and the implied capital accumulation, are given by the following:

[16] For a definition in the context of a typical growth model, see Krusell and Ríos-Rull [9].

- $\tilde{G}(\bar{k}, \tilde{\tau})$ describes the law of motion for aggregate capital for the one-period deviation.
- $\tilde{g}(k, \bar{k}, \tilde{\tau})$ describes the individual's decision rule for the one-period deviation.

Third, we have competitive pricing functions $r(\bar{k})$ and $w(\bar{k})$ (equal to the marginal products off the aggregate production function).

These equilibrium elements have to satisfy:

- Individual optimization: $\tilde{g}(k, \bar{k}, \tilde{\tau})$ solves

$$V_0(k, \bar{k}, \tilde{\tau}) \equiv \max_{k'} \left[\log((r(\bar{k})\,k + w(\bar{k}))(1-\tilde{\tau}_y) - k'(1+\tilde{\tau}_i)) + \beta\delta V(k', \tilde{G}(\bar{k}, \tilde{\tau}))\right]$$

and $V(k, \bar{k})$ satisfies

$$V(k, \bar{k}) = \log((r(\bar{k})\,k + w(\bar{k}))(1-\tau_y(\bar{k})) - g(k, \bar{k})(1+\tau_i(\bar{k}))) + \delta V(g(k, \bar{k}), G(\bar{k})).$$

Note that these requirements imply, as a special case, that $\tilde{g}(k, \bar{k}, \tau(\bar{k})) = g(k, \bar{k})$.

- Consistency between individual and aggregate actions: $\tilde{g}(\bar{k}, \bar{k}, \tilde{\tau}) = \tilde{G}(\bar{k}, \tilde{\tau})$, which implies as a special case that $g(\bar{k}, \bar{k}) = G(\bar{k})$.
- The government maximizes: $\tau(\bar{k}) = (\tau_y(\bar{k}), \tau_i(\bar{k}))$ solves

$$\max_{(\tilde{\tau}_y, \tilde{\tau}_i)} V_0(\bar{k}, \bar{k}, \tilde{\tau})$$

subject to

$$-\tilde{G}(\bar{k}, \tilde{\tau})\,\tilde{\tau}_i = A\bar{k}^{\alpha}\tilde{\tau}_y.$$

Solutions to the problems above can be obtained in the same manner as we derived competitive equilibria above. We conjecture that $\tau_y(\bar{k}) = \tau_y$, $\tau_i(\bar{k}) = \tau_i$, i.e., that the tax functions will be constant. This conjecture is straightforward to verify. The one-period deviation decision rule does not depend on future tax rates. If future tax rates were a nontrivial function of aggregate capital then today's tax policy would affect the future tax rates. With the conjecture that the tax outcome function is constant, a number of derivatives become zero. The constancy of the tax function in particular implies that the current capital stock has no importance for how taxes are set; this would not be true in a calibrated growth model. There, the tax function would depend on capital, taxes would change along the transition path, and a one-period deviation in tax rates would alter the tax rates forever after. Needless to say, the parametric assumptions here simplify the analysis tremendously.

PROPOSITION 4. *The time-consistent tax rates are given by*

$$\tilde{\tau}_i = \frac{\delta(1-\alpha)(1-\beta)}{1-\alpha\delta-\delta(1-\alpha)(1-\beta)} > (<)\, 0 \qquad \textit{if} \quad \beta < (>)\, 1$$

and

$$\tilde{\tau}_y = \frac{\alpha\delta^2\beta(1-\alpha)(1-\beta)}{\alpha\delta^2\beta(1-\alpha)(1-\beta)-(1-\alpha\delta)(1-\delta(1-\beta))} < (>)\, 0 \qquad \textit{if} \quad \beta < (>)\, 1.$$

Proof. See Appendix A.

It is straightforward to verify that the time-consistent tax rates, perhaps not surprisingly, reproduce the allocations that solve the planner's game: the government has enough instruments to manipulate current decisions freely and so chooses the same outcome as if it were a central planner.

As a positive theory of taxation, the model implies

$$\tau_i > (<)\ 0 \qquad \text{if} \quad \beta < (>)\ 1$$

and

$$\tau_y < (>)\ 0 \qquad \text{if} \quad \beta < (>)\ 1.$$

That is, we have positive tax rates on investment when $\beta < 1$. The social planner saves much too little (less than the laissez-faire equilibrium) and so wants to move the equilibrium in the "wrong" direction.

6. SUMMARY AND CONCLUDING REMARKS

We have studied the performance of the market mechanism relative to a mechanism with a benevolent and potentially active government in what we believe is an interesting new case for economists to study: an economy where consumers have time-inconsistent preferences. Our analysis comes out surprisingly strongly in favor of laissez-faire; an initial reader of the literature on time-inconsistent preferences may get the impression that the consumer needs help and that the government can provide the help. We argue here that this is really only true if the government, or social planner, can help alleviate the commitment problem in a direct way or indirectly by being able to commit to future tax rates. Further, under the assumption that the government cannot do anything about the commitment problem, we should strictly prefer laissez-faire. It would be important not to have a government that can tax. Although our goal here is not to send a libertarian message, the results may perhaps be interpreted as a caution against tax policy activism: taxes should not be used to try to correct problems whose underlying cause is a lack of commitment to which the government is also subject.

Could the government provide commitment mechanisms to aid consumers with time-inconsistent preferences? To the extent that it could close short-term credit markets in the future, in effect "creating illiquidity" in the future, then it should, *ceteris paribus*. However, we argued that ex post—when the future arrives—it will be in everybody's interest not to close the markets, and it is unclear how a commitment mechanism for the closing of markets in the future could be created. Moreover, providing illiquidity by offering assets such as the 401(k)—with penalty for early withdrawals—might be helpful, but only if other restrictions on trading and borrowing are simultaneously and credibly put in place. In general, commitment is needed for consumption, and it is not enough to provide assets which are illiquid. Therefore, we take the present analysis as an entirely relevant case.

We found that our tax policy analysis bears resemblance to the rules vs discretion literature. Our government here is benevolent—a samaritan—but it cannot help but adopt policies that are not good. What is the solution to our samaritan's dilemma? Short of some form of commitment, there is no solution. Along the lines of Rogoff's [20] suggestion in the monetary literature of electing a conservative to head the central bank in the future, one could imagine in our context electing, for an extended period, a planner who places independent weight on utility as perceived by the future selves of the consumer. With appropriate such weights—they have to be large enough—outcomes could be improved for all selves. However, the mere idea of being able to elect a planner for the future, or tomorrow's central banker, is one of commitment. Absent institutional rules (that are set in stone) it would then be difficult to explain why it is possible to commit to a future decision maker but not to future policies.

One weakness of the present approach to representing the preference reversals documented in the experimental psychology literature is that it makes a rather drastic conceptual departure from standard Arrow–Debreu analysis. In particular, it does not build up from axioms of choice for the individual. What would seem to be appropriately captured by an optimization problem—the consumer's observed behavior—is instead modeled here as the solution to a dynamic game. As such, all the usual problems of (dynamic) game theory appear; for example, indeterminacy of equilibrium is the rule rather than the exception. We resolve this problem by focusing on the limit of the equilibria of finite-horizon economies. Several comments are in order.

First, many practical applications (such as those considered in Laibson's papers) do have a finite horizon, and our results are directly relevant here: although we do not explicitly consider finite-horizon problems, our results apply if the time horizon is long enough, and they are likely to carry over also to short horizons (e.g., it is straightforward to derive results for a three-period model).

Second, although we do not find the other equilibria uninteresting, we find it particularly instructive to study equilibria without the reputation effects that typically underly the indeterminacy of equilibria in dynamic games. In particular, the economic incentives upon which we focus in this paper will be present whether or not there are additional incentives arising from punishment schemes. In other words, the reputation-free equilibrium is very interesting in itself here, unlike in, say, the repeated prisoner's dilemma model.

Third, we show that there are basic economic forces leading the decentralized (laissez-faire) equilibrium to perform well in our model. Although there surely exist decentralized equilibria with reputation effects that perform worse than specific planning equilibria (and vice versa), we do not have an insight as to whether the consideration of reputation-based equilibria would systematically alter these comparisons.

Fourth and finally, it is useful for our welfare comparisons that the equilibria upon which we focus here (i.e., those that are limits of finite-horizon equilibria) are unique. Under particular assumptions about preferences and technology, such equilibria are unique. Moreover, our computational experiments that generalize the utility function and consider less than full depreciation show no indication of multiplicity. Finally, we suspect that it is not an easy task to generate examples with multiple solutions to finite-horizon games in the context of the neoclassical growth model with quasi-geometric discounting.

An alternative approach to the one we follow here, and one which may ultimately turn out to be more fruitful, is undertaken in a set of papers by Gul and Pesendorfer [4,5]. These authors use decision theory, based on axioms over sets of consumption bundles, and arrive at recursive (time-consistent) preference representations of consumer behavior when temptation and self-control are represented axiomatically. In a related paper (Krusell *et al.* [10]), we are considering the effects of policy on equilibrium allocations and welfare when there is temptation and self-control, as in Gul and Pesendorfer's framework.

APPENDIX A

Proof of Proposition 1. The proof follows by using the guess for the value function, $V(k, \bar{k}) = a + b \log \bar{k} + c \log(k + \varphi \bar{k})$, and the guess for the law of motion for aggregate capital, $G(\bar{k}) = s\alpha A\bar{k}^{\alpha}$.[17] Given these guesses we can solve the current self's problem to obtain $g(k, \bar{k}) = \frac{\beta\delta c}{1+\beta\delta c}(rk+w) - \frac{\varphi \bar{k}'}{1+\beta\delta c}$. Using this decision rule we can verify the guess for the value function and obtain $\varphi = \frac{1-\alpha}{\alpha(1-s)}$, $b = \frac{\alpha-1}{(1-\delta)(1-\alpha\delta)}$, and $c = \frac{1}{1-\delta}$. Inserting $\varphi = \frac{1-\alpha}{\alpha(1-s)}$ into the individual decision rules and setting $g(\bar{k}, \bar{k}) = G(\bar{k})$ (which has to hold in competitive equilibrium), we obtain $s = \frac{\delta\beta}{1-\delta(1-\beta)}$. This gives $\varphi = \frac{(1-\alpha)(1-\delta(1-\beta))}{\alpha(1-\delta)}$. Substituting these constants into the agent's decision rule, we obtain $g(k, \bar{k})$ and $G(\bar{k})$.

Proof of Proposition 3. The proof proceeds as follows: first we derive the value function of the current self, $V_0(k)$, given a general law of motion of type $k' = sAk^{\alpha}$. We thus obtain a function of s. We then evaluate this value function at $s_1 = \frac{\alpha\delta\beta}{1-\delta(1-\beta)}$ and at $s_2 = \frac{\alpha\delta\beta}{1-\alpha\delta(1-\beta)}$.

Let us first derive $V(k)$. For the given law of motion for capital, consumption will be given by $c = (1-s)\,Ak^{\alpha}$. Then we can write $V(k)$ as

$$V(k) = \log((1-s)\,Ak^{\alpha}) + \delta \log((1-s)\,Ak'^{\alpha}) + \delta^2 \log((1-s)\,Ak''^{\alpha}) + \cdots.$$

Inserting the law of motion for the capital we obtain

$$V(k) = \frac{1-\alpha\delta}{(1-\alpha\delta)(1-\delta)} \log(1-s) + \frac{\alpha\delta}{(1-\alpha\delta)(1-\delta)} \log s + \frac{\alpha}{(1-\alpha\delta)} \log k + \frac{1}{(1-\alpha\delta)(1-\delta)} \log A.$$

$V_0(k)$ is now given by

$$\begin{aligned} V_0(k) &= \log((1-s)\,Ak^{\alpha}) + \beta\delta V(sAk^{\alpha}) \\ &= \log((1-s)\,Ak^{\alpha}) + \\ &\quad \beta\delta \left[\frac{1-\alpha\delta}{(1-\alpha\delta)(1-\delta)} \log(1-s) + \frac{\alpha\delta}{(1-\alpha\delta)(1-\delta)} \log s + \frac{\alpha}{(1-\alpha\delta)} \log(sAk^{\alpha}) + \frac{1}{(1-\alpha\delta)(1-\delta)} \log A \right] \\ &= \frac{1-\delta(1-\beta)}{1-\delta} \log(1-s) + \frac{\alpha\delta\beta}{(1-\alpha\delta)(1-\delta)} \log s + \cdots. \end{aligned}$$

[17] Where does this guess come from? $\varphi\bar{k}$ is actually equal to discounted value of lifetime wages (discounted to time -1 in this formulation): $\frac{w}{r} + \frac{w'}{rr'} + \frac{w''}{rr'r''} + \cdots$.. If we use $G(\bar{k}) = s\alpha A\bar{k}^{\alpha}$ and plug it into the discounted sum, we obtain $\varphi = \frac{1-\alpha}{\alpha(1-s)}$.

To evaluate $V_0(k)$ at s_1 and s_2, we proceed as follows. We first show that there is a unique s^* that maximizes $V_0(k)$, by showing that $V_0(k)$ is monotone increasing in s for $s < s^*$ and monotone decreasing for $s > s^*$. We then complete the proof by showing that $s_2 < s_1 < s^*$ for $\beta < 1$ and $s_2 > s_1 > s^*$ for $\beta > 1$. To do this, first let us look at the function $F(s) = (1-\delta(1-\beta))\log(1-s) + \frac{\alpha\delta\beta}{1-\alpha\delta}\log s$.

$$
\begin{aligned}
F'(s) &= -\frac{(1-\delta(1-\beta))}{1-s} + \frac{\alpha\delta\beta}{(1-\alpha\delta)\, s} \\
&= -\frac{[(1-\alpha\delta)(1-\delta(1-\beta))+\alpha\delta\beta]\, s - \alpha\delta\beta}{s(1-s)(1-\alpha\delta)}
\end{aligned}
$$

Note that $s^* = \frac{\alpha\delta\beta}{(1-\alpha\delta)(1-\delta(1-\beta))+\alpha\delta\beta}$, $F'(s) < 0$ ($V_0(k)$ is monotone decreasing) for $s > s^*$, and $F'(s) > 0$ ($V_0(k)$ is monotone increasing) for $s < s^*$. We can also easily see that $s_2 < s_1 < s^*$ for $\beta < 1$ and that $s_2 > s_1 > s^*$ for $\beta > 1$.

Proof of Proposition 4. We will demonstrate how to derive the unique solution to the government's problem. The government's problem is

$$
\begin{aligned}
\max_{(\tilde{\tau}_y, \tilde{\tau}_i)} V_0(\bar{k}, \bar{k}, \tilde{\tau}) = \log\left(\left[A\bar{k}^\alpha - \frac{\alpha\delta\beta}{1-\alpha\delta(1-\beta)} A\bar{k}^\alpha\right](1-\tilde{\tau}_y)\right) + \beta\delta a \\
+ \beta\delta \frac{\alpha-1}{(1-\delta)(1-\alpha\delta)} \log\left(\frac{\alpha\beta\delta}{1-\delta+\beta\delta} \frac{1-\tilde{\tau}_y}{1+\tilde{\tau}_i} A\bar{k}^\alpha\right) \\
+ \frac{\beta\delta}{1-\delta} \log\left(\frac{\alpha\beta\delta}{1-\delta+\beta\delta} \frac{1-\tilde{\tau}_y}{1+\tilde{\tau}_i} A\bar{k}^\alpha(1+\varphi)\right).
\end{aligned}
$$

This problem is equivalent to

$$
\max_{(\tilde{\tau}_y, \tilde{\tau}_i)} \frac{1-\alpha\delta+\alpha\delta\beta}{1-\alpha\delta} \log(1-\tilde{\tau}_y) - \frac{\alpha\delta\beta}{1-\alpha\delta} \log(1+\tilde{\tau}_i) + \cdots .
$$

The government budget reduces to

$$
(\tilde{\tau}_y - 1)\, \tilde{\tau}_i \frac{\alpha\delta\beta}{1-\delta(1-\beta)} = \tilde{\tau}_y (1+\tilde{\tau}_i).
$$

Solving for $\tilde{\tau}_y$ in terms of $\tilde{\tau}_i$ from the government budget, the government's problem can be written as

$$\max_{\tilde{\tau}_i} Q(\tilde{\tau}_i) = \log(1+\tilde{\tau}_i) - \frac{1-\alpha\delta+\alpha\delta\beta}{1-\alpha\delta} \log((1-\delta(1-\beta))(1+\tilde{\tau}_i) - \alpha\delta\beta\tilde{\tau}_i).$$

$Q(\tilde{\tau}_i)$ is the government's objective, expressed as a function of $\tilde{\tau}_i$. Taking derivatives with respect to $\tilde{\tau}_i$, we can show that $Q'(\tilde{\tau}_i) > 0$ for $\tilde{\tau}_i < \frac{\delta(1-\alpha)(1-\beta)}{1-\alpha\delta-\delta(1-\alpha)(1-\beta)}$ and that $Q'(\tilde{\tau}_i) < 0$ for $\tilde{\tau}_i > \frac{\delta(1-\alpha)(1-\beta)}{1-\alpha\delta-\delta(1-\alpha)(1-\beta)}$. So there is a unique maximum to the government's problem. It is given by the constructed solution to the first-order condition.

APPENDIX B

Appendix B describes the numerical algorithm that we use to find approximate solutions to the planning problem (see Section 4.1) when we abandon the twin assumptions of logarithmic utility and full depreciation of the capital stock in one period. In particular, we look for an approximation to the decision rule that satisfies the GEE described in Section 4.3. In our numerical algorithm, we let $\tilde{k} \equiv \log(k)$ be the planner's state variable. Define $\tilde{f}(\tilde{k}) \equiv f(\exp(\tilde{k}))$. The GEE then takes the form

$$\begin{aligned} &u'(\tilde{f}(\tilde{k}) - \exp(h(\tilde{k})))\exp(h(\tilde{k})) \\ &\quad - \beta\delta u'(\tilde{f}(h(\tilde{k})) - \exp(h(h(\tilde{k}))))[\tilde{f}'(h(\tilde{k})) \\ &\quad + (\beta^{-1}-1)\, h'(h(\tilde{k}))\exp(h(h(\tilde{k}))] = 0, \end{aligned}$$

where h is the planner's decision rule (in logs). Given h, the GEE can be written more compactly as $H(\tilde{k}) = 0$. Let $H^{(m)}$ denote the mth derivative of H. Since the GEE holds for all $\tilde{k}$, $H^{(m)}(\tilde{k}) = 0$ for all m and all $\tilde{k}$. We use this fact in our numerical algorithm to define restrictions that an approximate solution to the GEE must satisfy.

Let $\hat{h}_\psi$ be an approximation to the unknown decision rule h characterized by an n-dimensional vector of parameters ψ. For example, $\hat{h}$ could be a polynomial of order $n-1$, with ψ being a set of n coefficients. The basic idea of our numerical algorithm is to choose the coefficients ψ so that H and its first $n-1$ derivatives are equal to zero when evaluated at a particular value of $\tilde{k}$.

Specifically, let $\hat{H}_\psi$ be the value of the left-hand side of the GEE when h is replaced by the approximate decision rule $\hat{h}_\psi$. We choose ψ to solve the n equations $\hat{H}_\psi^{(m)}(\tilde{k}^*) = 0$, $m = 0, 1, 2, \ldots, n-1$, where $\hat{H}_\psi^{(m)}$ is the mth derivative of $\hat{H}_\psi$ and $\tilde{k}^*$ is an arbitrarily chosen point in the state space.

Since we are typically interested in dynamic behavior near the steady-state capital stock $\tilde{k}_{ss}$, we would like to evaluate $\hat{H}_\psi$ and its derivatives at $\tilde{k}_{ss}$ (but note that the algorithm is well defined and works well in practice even if we evaluate $\hat{H}_\psi$ and its derivatives at an arbitrary point in the state space). Unlike in the standard model (where $\beta = 1$), we cannot solve for $\tilde{k}_{ss}$ without solving simultaneously for the equilibrium decision rule (since the derivative of the decision rule appears in the GEE). We therefore solve for the decision rule coefficients ψ and the steady-state capital stock $\tilde{k}_{ss}$ at the same time, with $\tilde{k}_{ss}$ being pinned down by the requirement that $\tilde{k}_{ss} - \hat{h}_\psi(\tilde{k}_{ss}) = 0$. In sum, our algorithm boils down to the solution of $n+1$ equations in the $n+1$ unknowns ψ and $\tilde{k}_{ss}$.

We have implemented this algorithm using polynomial decision rules up to order 3. We define $\hat{h}_\psi$ using ordinary polynomials: $\hat{h}_\psi(\tilde{k}) = \sum_{i=0}^{n-1} a_i \tilde{k}^i$. We find that the numerical results change only to a small degree when increasing the order of the polynomial from 2 to 3. In order to avoid collinearity problems, a higher-order implementation of the algorithm would probably require the use of polynomials defined in terms of a set of orthogonal polynomials (such as the Chebyshev polynomials).

To solve the $n+1$ equations, we use the simplex algorithm as described in Chapter 10 of Press *et al.* [19]. Specifically, we choose ψ and $\tilde{k}_{ss}$ to minimize the objective function

$$\sum_{i=0}^{n-1} (\hat{H}_\psi^{(i)}(\tilde{k}_{ss}))^2 + (\tilde{k}_{ss} - \hat{h}_\psi(\tilde{k}_{ss}))^2.$$

In all of the cases that we tried, we were able to find values for ψ and $\tilde{k}_{ss}$ that set this objective function equal to 0. Although we could use a variety of other methods to solve for ψ and $\tilde{k}_{ss}$, this method is reasonably robust and is sufficiently fast.

The simplex algorithm requires an initial simplex consisting of $n+2$ points $(\psi, \tilde{k}_{ss}) \in \mathbf{R}^{n+1}$. We let one of these points correspond to a parameterization of the model economy for which we have a closed-form solution (for example, logarithmic utility and full depreciation of the capital stock in one period). We choose the remaining points in the simplex to be random perturbations of this point. We then solve the $n+1$ equations for a parameterization of the model economy that is close to the parameterization with the closed-form solution. This solution serves in turn as one of the points in the initial simplex when we perturb the model economy's parameters again.

Although the required derivatives could be calculated analytically, we calculate them numerically using the following two-sided finite-difference formulas

$$\hat{H}_{\psi}^{(1)}(\tilde{k}) \approx \frac{\hat{H}_{\psi}((1+\epsilon_1)\,\tilde{k}) - \hat{H}_{\psi}((1-\epsilon_1)\,\tilde{k})}{2\epsilon_1 \tilde{k}}$$

$$\hat{H}_{\psi}^{(2)}(\tilde{k}) \approx \frac{\hat{H}_{\psi}((1+2\epsilon_2)\,\tilde{k}) - 2\hat{H}_{\psi}(\tilde{k}) + \hat{H}_{\psi}((1-2\epsilon_2)\,\tilde{k})}{(2\epsilon_2 \tilde{k})^2}$$

$$\hat{H}_{\psi}^{(3)}(\tilde{k}) \approx \frac{\begin{array}{c}\hat{H}_{\psi}((1+3\epsilon_3)\,\tilde{k}) - 3\hat{H}_{\psi}((1+\epsilon_3)\,\tilde{k}) \\ + 3\hat{H}_{\psi}((1-\epsilon_3)\,\tilde{k}) - \hat{H}_{\psi}((1-3\epsilon_3)\,\tilde{k})\end{array}}{(2\epsilon_3 \tilde{k})^3},$$

with $\epsilon_i = 10^{-6/i}$ for $i = 1, 2, 3$.

REFERENCES

1. G. Ainslie, "Picoeconomics," Cambridge University Press, Cambridge, UK, 1992.
2. R. Barro, Ramsey meets Laibson in the neoclassical growth model, *Quart. J. Econ.* **114** (1999), 1125–1152.
3. D. B. Bernheim, D. Ray, and S. Yeltekin, Self-control, saving, and the low asset trap, manuscript, 1999.
4. F. Gul and W. Pesendorfer, Temptation and self-control, *Econometrica*, in press.
5. F. Gul and W. Pesendorfer, Self-control and the theory of consumption, manuscript, Princeton University, 2000.
6. C. Harris and D. Laibson, Dynamic choices of hyperbolic consumers, *Econometrica*, in press.
7. Z. Hercowitz and P. Krusell, Overheating, manuscript, 2001.
8. K. L. Judd, "Numerical Methods in Economics," MIT Press, Cambridge, MA, 1998.
9. P. Krusell and J.-V. Ríos-Rull, On the size of U.S. Government: Political economy in the neoclassical growth model, *Amer. Econ. Rev.* **89** (1999), 1156–1181.
10. P. Krusell, B. Kuruşçu, and A. A. Smith, Jr., Temptation and Taxation, manuscript, 2001.
11. P. Krusell and A. A. Smith, Jr., Consumption-savings decisions with quasi-geometric discounting, manuscript, 2000.
12. F. Kydland and E. C. Prescott, Rules rather than discretion: The inconsistency of optimal plans, *J. Polit. Econ.* **85** (1977), 473–491.
13. D. Laibson, "Self-Control and Saving," Ph.D. thesis, MIT, 1994.
14. D. Laibson, "Hyperbolic Discount Functions, Undersaving, and Savings Policy," NBER Working Paper 5635, 1996.
15. D. Laibson, Hyperbolic discount functions and time preference heterogeneity, manuscript, 1997.
16. D. Laibson, A. Repetto, and J. Tobacman, Self-control and saving for retirement, *Brookings Pap. Econ. Act.* **1** (1998), 91–196.
17. E. D. O'Donoghue and M. Rabin, Doing it now or doing it later, *Amer. Econ. Rev.* **89** (1999), 103–124.
18. E. Phelps and R. A. Pollak, On second-best national saving and game-equilibrium growth, *Rev. Econ. Stud.* **35** (1968), 185–199.

19. W. H. Press, B. P. Flannery, S. A. Teukolski, and W. T. Vetterling, "Numerical Recipes: The Art of Scientific Computing," Cambridge University Press, Cambridge, UK, 1989.
20. K. Rogoff, The optimal degree of commitment to an intermediate monetary target, *Quart. J. Econ.* **100** (1985), 1169–1190.
21. R. H. Strotz, Myopia and inconsistency in dynamic utility maximization, *Rev. Econ. Stud.* **23** (1956), 165–180.