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The Existence of Perfect Equilibria in a Model of Growth with Altruism between Generations

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An intertemporal model of consumption and bequest behaviour is specified and analysed as a game between generations. The main feature of this game is that no a priori restrictions (like linearity) are placed on the strategy choice of generations. The paper gives an existence proof for perfect (Nash) equilibria in finite and infinite horizon versions of the model and determines characteristic properties of equilibrium strategies. The main result is to demonstrate existence of stationary perfect equilibrium if the time horizon is infinite.

INTRODUCTION

The purpose of this paper is to give an existence proof for perfect equilibria in the context of a highly aggregative model of growth. The model which is widely used in the literature on the theory of growth (see Arrow (1973), Peleg and Yaari (1973), Dasgupta (1974), Kohlberg (1976) and Lane and Mitra (1981)) has the special feature that preferences of different generations display altruism towards the next generation. The existence results are derived under very general assumptions on production and preference structures; in particular, no *a priori* restrictions are placed on strategies a generation may use. The question whether such *a priori* restrictions are necessary for the existence of equilibria is answered in the negative.

Game-theoretical equilibrium concepts were introduced into growth theory by Phelps and Pollak (1968) with reference to a model in which the intertemporal preference structure incorporates an imperfect kind of altruism between generations. They are used to describe what might happen in such an economy in the presence of intertemporal conflicts about the evaluation of different consumption and production programs by different generations.

This idea was taken up by Dasgupta (1974) to give the Rawlsian "just savings rule" the interpretation of a Nash equilibrium. This proposal yields a time-consistent solution to the Rawlsian problem because in a Nash equilibrium state no generation acting alone can do better and all generations act so as to fulfill their expectations of the future savings pattern. These equilibria also possess the alternative interpretation of time-consistent planning behaviour over several periods by an individual planner who holds different views at different points in time and, therefore, resolve the classical problem posed by Strotz (1955).

However, all these papers (with the notable exceptions of Kohlberg (1976) and Bernheim and Ray (1983)) specify a model in which each generation is faced with the problem of choosing a (constant) savings *ratio* given initial capital and the savings ratios later generations may use. This, of course, is—as Dasgupta (1974) noted—

> "tantamount to allowing each generation to select its own savings *schedule* (i.e. its total savings as a function of the capital stock) on the stipulation that the schedule be linear and that it should pass through the origin, the two together implying that the marginal and the average savings rates be equal".

We will argue that this a priori restriction of strategy selection has no economic justification and may well produce "artificial" equilibria. Peleg and Yaari (1973) employ it (in a more general model) to derive an existance result for Nash equilibria, but—as Goldman (1980) points out—these equilibria may not be perfect in the sense of Selten (1975). He provides an existence proof for perfect Nash equilibria in models with a finite time horizon. The perfectness problem is extensively discussed in Lane and Leininger (1984a). They show in the context of the infinite horizon model used in this paper that the linearity restriction precisely implies non-perfectness of the equilibria obtained if the restriction is a binding one. Moreover, they demonstrate that it is non-binding in only very special cases.

The basic underlying problem pointed to by Peleg and Yaari (1973) which led them to introduce the a priori assumption of linearity into their model may be described as follows:

The sequential character of the decision-making process of successive generations leads to an indeterminacy causing a problem known from the theory of dynamic programming as "violation of the preservation property". The actual structural properties of the objective function for a present stage problem crucially depend on the structural properties of optimal plans for future (or past) stages once they are substituted into the present stage problem. This may result in non-uniqueness of the present stage optimal plan and in a plan having structural properties that are too weak to ensure existence at the next stage.[1]

So one has to search for structural properties of strategies strong enough to yield existence of optimal reactions and which are preserved through the maximization exercise, i.e. which are inherited by the derived optimal plan.

When discussing the problems of enlarging the strategy spaces Dasgupta (1974) noted: "But once we define a generation's strategy as a complete savings schedule (and not a savings ratio) there is nothing a priori to limit the functional form". This motivates our research strategy. We adopt a very general selection theory approach starting with no restrictions placed on the functional or structural form of the elements of the strategy spaces (besides being functions) and ask what minimal structural properties are necessary to ensure existence. The argument is based on a generalization of Berge's "Maximum Theorem" due to Leininger (1984).

The existence results for the infinite horizon version of our model presented in Sections 4 and 5 below were independently and simultaneously obtained by Bernheim and Ray (1983). So it is of interest to examine the connection between their analysis and the one presented here. This is done in Section 6 below.

The paper is organized as follows: Section 1 introduces the model and focuses on the intertemporal optimization problem for each generation. Section 2 contains an existence result for finite horizon truncations of the economy under consideration. In Section 3 we derive and discuss properties of equilibrium schedules which will be used in Section 4 to prove existence of a perfect Nash equilibrium in the infinite horizon case. This result is strengthened to the existence of stationary equilibria in Section 5.

1. THE MODEL

In each time period $t (t = 0, 1, 2, \dots)$ a decision maker (generation), whose preferences are altruistic towards the next generation, decides on the optimal use of the resources available to him. The only good in the economy is capital which can also be consumed.

Generation t's utility function is given by

$$u_t = u(c_t, c_{t+1}) : R_+^2 \to R$$

where u is assumed to be continuous and increasing in both components. c_t denotes its own consumption, while c_{t+1} stands for the consumption of generation $t+1$.

Production possibilities are described by a production function

$$f : R_+ \to R_+ \quad \text{such that } f(0) = 0$$

which is assumed to be continuous and increasing.

Given any initial stock of capital y the consumption decision of any generation is constrained by $0 \leqq c \leqq y$, which defines a *feasibility constraint correspondence*

$$\beta : R_+ \to P(R_+) \quad (P(R) \text{ denoting the power set of } R)$$

$$y \to [0, y]$$

determining the set of feasible actions given the state of the system, y. β obviously is non-empty, compact-valued, and continuous. The resulting "law of motion" of the economy is then given by the following *transition function*

$$g : R_+ \times R_+ \to R_+$$

$$(y, c) \to f(y-c) = g(y, c).$$

g is continuous in (y, c) since f is continuous. The function g determines the new state of the system as a function of the previous state and the action taken.

A feasible *strategy* (for any generation) is a function

$$c : R_+ \to R_+ \quad \text{s.t. } c(y) \leqq y \quad \text{for all } y \text{ (resp. } c(y) \in \beta(y) \text{ for all } y).$$

A generation's maximization problem—expecting or knowing the next generation shall use strategy $c(y)$—consists in solving

$$(P) \qquad \max_{\bar{c} \in \beta(y)} \{u(\bar{c}, c(f(y-\bar{c})))\} \quad \text{for all } y.$$

Actual growth in the economy is determined by the Nash equilibrium concept:

Definition (*). A sequence of strategies $\{c_0^*(y), \ldots, c_t^*(y), \ldots\}$ is called a *perfect Nash equilibrium* if for all $t \geqq 0$ and for all y

$$u(c_t^*(y), c_{t+1}^*(f(y - c_t^*(y)))) \geqq u(c_t(y), c_{t+1}^*(f(y - c_t(y)))),$$

$c_t(y)$ being an arbitrary, feasible strategy. If $c_t^*(y) = c^*(y)$ for all t we call the sequence a *stationary* perfect equilibrium.

Since each generation is only concerned with its own and its direct descendant's consumption and the consumption schedule of the next generation is treated as a constraint in the intertemporal maximization problem this definition of an equilibrium also constitutes what elsewhere is called a "sophisticated equilibrium" or "Strotz-Pollak" solution. It therefore prescribes a time-consistent course of action. A perfect equilibrium is the result of successively "internalising" all externalities imposed on the present generation by later players' behaviour. It thus induces a Nash equilibrium of all subgames, i.e. *for all states at all points in time.*

Note that the objective function of the present generation in (P) *after* the next generation's schedule has been internalised, $u(\bar{c}, c(g(y, \bar{c})))$, is state-dependent and has the single decision variable $\bar{c}$. Existence of a perfect equilibrium requires that there is a solution to (P) for all possible values of the state variable y. Clearly, there always exists a solution if $c(y)$ is assumed to be continuous, since then $u(\bar{c}, c(g(y, \bar{c})))$ is a continuous function in $\bar{c}$ restricted to the compact interval $[0, y]$. The problem is that this solution may not be unique and therefore there may not exist a continuous optimal reaction $\bar{c}(y)$; i.e., continuity is not preserved. Uniqueness could be restored by assuming $c(y)$ to be concave, because then the objective function u would be concave in $\bar{c}$. The optimal reaction then would be continuous but might not be concave, i.e. concavity would not be preserved. Stronger and stronger assumptions on $c(y)$ do not break this vicious cycle unless we demand linearity of $c(y)$ and severely restrict utility and production functions (see Lane and Leininger (1984a)).

One therefore has to accept the non-uniqueness problem and try to deal with it. In what follows we describe the optimal response of generation t to a proposed schedule of generation $(t+1)$ by a correspondence rather than a function. The question then is what structural properties this correspondence will have, dependent on the structure of the next generations schedule $c(y)$, and whether it will permit a selection having the same structural properties as the schedule it is derived from.

The correspondence in question is defined by

Definition (C)

$$\Phi_c : R_+ \to P(R_+)$$

$$y \to \{\bar{c} \mid \bar{c} \in \beta(y) \text{ and } \bar{c} \text{ maximizes } u(\bar{c}', c(g(y, \bar{c}))) \text{ over } \beta(y)\}.$$

By definition of Φ_c we have $\Phi_c(y) \subset \beta(y)$ for all y. A selection of Φ_c, i.e. a function $\hat{c}$ satisfying $\hat{c}(y) \in \Phi_c(y)$ for all y, is called an *optimal reaction to c.*

A general answer to the question what structural properties Φ_c will have provided $u(\bar{c}, c(g(y, \bar{c})))$ is continuous is given by Berge's (1963) Maximum Theorem. It states that Φ_c will be non-empty, compact-valued, and upper hemicontinuous. This continuity concept, however, is not strong enough to yield existence of a continuous selection.

The relevant continuity concept for continuous selection theory is lower hemi-continuity of a correspondence (see Michael (1956)). But this property is not implied by the Maximum Theorem. The Maximum Theorem only yields the existence of an upper semi-continuous selection from Φ_c. But if $c(y)$ in (P) is only upper semi-continuous then u itself is only upper semi-continuous and this is not strong enough to salvage the conclusions of the Maximum Theorem. However, if u—while only being semi-continuous—interplays with the constraint correspondence β in a certain way, which we call graph-continuous, the conclusion is still valid. This is the content of the Generalized Maximum Theorem proved in Leininger (1984).

2. EXISTENCE OF PERFECT NASH EQUILIBRIA—THE FINITE HORIZON

We cite the definition of graph-continuity and the generalized Maximum Theorem and show its applicability to our problem.

Definition. $u: R_+ \times R_+ \to R$ is called *graph-continuous w.r.t.* $\beta: R_+ \to P(R_+)$ if the following is true:

For all $(y, c) \in \text{graph}(\beta) \subset R_+ \times R_+$ there exists a selection $r(\cdot, c): R_+ \to R_+$ from β such that for all $\varepsilon > 0$ there exists a $\delta > 0$ for which

$$|y - y'| < \delta \quad \text{implies} \quad |u(y', r(y', c)) - u(y, c)| < \varepsilon.$$

The interpretation of this definition is the following: whenever there is a small change in the state variable from y to y' there always exists a feasible decision, $r(y', c)$, to restore the pay-off or utility of state y (if decision c is taken) "up to ε" in state y' by taking action $r(y', c)$.

The generalized Maximum Theorem (Leininger, 1984) asserts that if u is upper semi-continuous and graph-continuous with respect to β, where β is compact-valued and continuous, then the optimal choice correspondence Φ_c as defined in (C) is non-empty, compact-valued, and upper hemi-continuous. All we have to show therefore is that $u(\bar{c}, c(g(y, \bar{c})))$ is graph-continuous w.r.t. $\beta(y)$ and upper semi-continuous in $(y, \bar{c})$.[2]

Proposition 1. *Let $c(\cdot)$ be continuous from the left. Then $u(\bar{c}, c(g(y, \bar{c})))$ is graph-continuous* w.r.t. β.

Proof. (i) Let $(\bar{y}, \bar{c})$ be an element from R_+^2 lying below the diagonal, such that $\bar{c} > 0$; then define $r(y, \bar{c}) := \bar{c} + (y - \bar{y}) =: \hat{c}$.

Because of $\hat{c} = \bar{c} + (y - \bar{y}) = y(\bar{c} - \bar{y})$ we have $\hat{c} \leqq y$ since $\bar{c} \in \beta(\bar{y})$, i.e. $\bar{c} \leqq \bar{y}$. In order to have $\hat{c} \in \beta(y)$ we only need $\hat{c} \geqq 0$; but this is certainly true if $|y - \bar{y}| \leqq \delta = \bar{c} > 0$. Moreover, by construction we have

$$y - \hat{c} = \bar{y} - \bar{c} \tag{1}$$

and from this it follows that

$$u(\bar{c}, c(f(\bar{y} - \bar{c}))) - u(\hat{c}, c(f(y - \hat{c}))) = u(\bar{c}, \bar{\bar{c}}) - u(\hat{c}, \bar{\bar{c}}) \tag{2}$$

where $\bar{\bar{c}} := c(f(\bar{y} - \bar{c})) = c(f(y - \hat{c}))$. By continuity of $u(\cdot, \bar{\bar{c}})$ we know that

$$|u(\bar{c}, c(f(\bar{y} - \bar{c}))) - u(\hat{c}, c(f(y - \hat{c})))| < \varepsilon \text{ whenever } |\bar{c} - \hat{c}| < \delta_\varepsilon$$

but by (1) we have $|\bar{c} - \hat{c}| = |\bar{y} - y|$ and therefore the conclusion follows from $|\bar{y} - y| < \delta_\varepsilon$ as well. If we set $\tilde{\delta}_\varepsilon = \min\{\delta, \delta_\varepsilon\}$ feasibility is also met.

(ii) If $(\bar{y}, \bar{c})$ is such that $\bar{c} = 0$ define

$$\hat{c} = r(y, 0) := \begin{cases} y - \bar{y} & \text{if } y \geqq \bar{y} \\ 0 & \text{if } y < \bar{y} \end{cases}.$$

This definition satisfies the argument given in (i) for $y \geqq \bar{y}$. If $y < \bar{y}$ then $r(y, 0) = 0 \in \beta(y)$ for all y and

$$|u(\bar{c}, c(f(\bar{y} - \bar{c}))) - u(\hat{c}, c(f(y - \hat{c})))| = |u(0, c(f(\bar{y}))) - u(0, c(f(y)))| < \varepsilon$$

whenever $|\bar{y} - y| < \delta_\varepsilon$ and $\bar{y} > y$ since $c(\cdot)$ (and therefore u) is continuous from the left. ||

Comment. Part (i) of the proof does not need any structural property of the schedule $c(y)$, because the device consists of neutralizing any change in the state with respect to the bequest to the next generation, so that the next generation's schedule is evaluated at the same point in both situations. This is best seen from (1). It says that in both situations bequests are the same. But for "boundary" values of $\bar{c}$, i.e. $\bar{c} = 0$, an arbitrarily small decline in the state variable y cannot be offset by a corresponding reduction in consumption, since this would violate feasibility. It is here where we need continuity from the left of the schedule $c(y)$.

Proposition 2. *Let $c(\cdot)$ be upper semi-continuous. Then $u(\tilde{c}, c(g(y, \tilde{c})))$ is upper semi-continuous (in $(y, \tilde{c})$).*

Proof. $c(g)$ is upper semi-continuous in $(y, \tilde{c})$ and u is continuous and increasing in its second component. ||

Propositions 1 and 2 ensure that the optimal choice correspondence Φ_c is non-empty, compact-valued and upper hemi-continuous as a consequence of the Generalized Maximum Theorem (if the schedule $c(\cdot)$, defining the objective function u, is upper semi-continuous and continuous from the left).

An upper hemi-continuous correspondence Φ allows an upper semi-continuous selection. The Generalized Maximum Theorem therefore yields the desired "preservation property" if we can also guarantee that the selection can be chosen to be continuous from the left. But a schedule can only be simultaneously upper semi-continuous and continuous from the left if it is either continuous or all discontinuities consist of "downward" jumps. That is, the schedule c^* selected from Φ should (in addition to upper semi-continuity) also satisfy the following condition:

Condition (L)

$$c^*(y+\Delta)-c^*(y) \leqq \Delta \quad \text{for all } y \geqq 0, \qquad \Delta > 0.$$

Equivalently, $s(y) := y - c^*(y)$ should be non-decreasing in y.

This then is a precise economic requirement for our existence result: the preference structure of the present generation, after the externality imposed by the next generation's schedule has been internalised, should yield bequests as a "normal" good (i.e. (optimal) bequests do not decrease with increasing initial capital holdings).[3]

Most importantly, requirement (L) can be guaranteed by assumptions about the basic preferences *before* internalisation of the externality. That is to say, there is a class of utility functions $u(c_t, c_{t+1})$ such that—whenever $u(c_t, c(g(y, c_t))$ defines a non-empty valued Φ_c—*any* selection of Φ_c satisfies (L). This class of (continuous and in both variables increasing) utility functions is called U. The next proposition shows that U contains all additively separable functions satisfying the condition:

Condition (S): $u(c_t, c_{t+1}) = v(c_t) + bv(c_{t+1})$, $b > 0$ and $v: R \to R$ is strictly concave and increasing.

Proposition 3. *Assume that $u(c_t, c_{t+1})$ satisfies (S) and let $c(\cdot)$ denote any feasible strategy. Then any selection c^* of Φ_c satisfies (L); i.e.*

$$c^*(y+\Delta)-c^*(y) \leqq \Delta \quad \textit{for all } y \textit{ and all } \Delta > 0.$$

Proof. Assume there is a selection c^* that does not satisfy (L). Then there exist y_1 and $\Delta > 0$ such that $c^*(y_1+\Delta) - c^*(y_1) > \Delta$. This is equivalent to

$$c_2 := c^*(y_1+\Delta) > c^*(y_1) + \Delta =: c_1 + \Delta; \tag{3}$$

then the following inequalities hold:

$$\begin{aligned} v(c_2) + bv(\tilde{c}(f(y_1+\Delta-c_2))) &\geqq v(c_1+\Delta) + bv(\tilde{c}(f((y_1+\Delta)-(c_1+\Delta)))) \\ &= v(c_1+\Delta) + bv(\tilde{c}(f(y_1-c_1))) \\ &= v(c_1+\Delta) - v(c_1) + v(c_1) - bv(\tilde{c}(f(y_1-c_1))). \end{aligned} \tag{4}$$

This inequality holds because c_2 maximizes for initial capital $(y_1+\Delta)$ by assumption and $(c_1+\Delta)$ is feasible under $(y_1+\Delta)$ because of (3). Write this as

$$v(c_2)-v(c_1)\geqq v(c_1+\Delta)-v(c_1)+b[v(\tilde{c}(f(y_1-c_1)))-v(\tilde{c}(f(y_1+\Delta-c_2)))]$$
$$\geqq v(c_1+\Delta)-v(c_1)+[v(c_2-\Delta)-v(c_1)]. \tag{5}$$

The last inequality follows from feasibility of $c_2-\Delta$ under y_1 (c_2 is feasible under $y_1+\Delta$) and the fact that c_1 maximizes for initial capital y_1; i.e.

$$v(c_1)+bv(\tilde{c}(f(y_1-c_1)))\geqq v(c_2-\Delta)+bv(\tilde{c}(f(y_1+\Delta-c_2))).$$

(5) reduces to

$$v(c_2)-v(c_2-\Delta)\geqq v(c_1+\Delta)-v(c_1). \tag{6}$$

But (6) contradicts strict concavity of v since $c_2\geqq c_1+\Delta$ by hypothesis! ||

Condition (L), which we repeatedly exploit in the following sections, is also required to hold for the method of proof proposed by Bernheim and Ray (1983). They identify a different subclass of the set U.[4] The complete characterization of the set U is not known.

We are then in a position to piece together the results of this section: Let $u(\cdot,\cdot)\in U$; i.e. for any feasible $c(\cdot)$ any selection of Φ_c (if it exists) satisfies (L). The Generalized Maximum Theorem then guarantees existence of an upper semi-continuous selection of Φ_c if c is upper semi-continuous and continuous from the left. Since this particular selection also satisfies (L) it is automatically continuous from the left.[5] The argument thus can be applied repeatedly.

Theorem 1. *Let f be continuous and increasing and $u_t=u(c_t,c_{t+1})\in U$ for $t=0,1,\ldots,T$. Then there exists a perfect Nash equilibrium. Equilibrium strategies are upper semi-continuous and continuous from the left (and hence satisfy (L)).*

Proof. Let without loss of generality the consumption schedule of generation T be given by $c_T^*(y)=y$, which is continuous. Generation $(T-1)$ reacts to it with an upper semi-continuous (and continuous from the left) schedule $c_{T-1}^*(y)$ by the above derivation. This leaves a well-defined problem to generation $(T-2)$ which reacts optimally to $c_{T-1}^*(y)$ and so on back to generation 0. ||

By Proposition 3, Theorem 1 implies

Corollary. *If f is continuous and increasing and u additively separable according to (S) then there exists a perfect Nash equilibrium.*

Remarks

(i) The fact that all generations evaluate consumption decisions with the same utility function $u(\cdot;\cdot)$ has not been used. Theorem 1, therefore, generalizes to the case of changing preferences; each generation may have its own distinct utility function. The model may then be viewed as a subcase of Goldman's (1980) more general model. However, Theorem 1 is not contained in Goldman's existence result on "consistent plans" which is stated for history-dependent strategies and does not yield any regularity properties of equilibrium strategies.

(ii) No stationarity of the production function f is required. This allows the introduction of technical progress.

In concluding this section we should emphasize that the way we have set up our model requires strategies—by definition—to be state-dependent (or of Markov type), i.e. no dependence of actions in period t on values of decision or state variables *before* t is possible. In a more general setting that would allow strategies at period t of the game to depend on the history (or parts thereof) of the game up to period t equilibrium would in general represent a weaker requirement. But note that the Markov perfect equilibria obtained in this model would also represent equilibria in a model which employs the broader definition of (possibly) history-dependent strategies. As Theorem 1 indicates such an enlargement of strategy spaces is not needed to obtain existence of equilibria. The techniques used to prove Theorem 1 might therefore serve as a general device how to find Markov perfect equilibria for other models. Recall that in proving Theorem 1 the application of the Generalized Maximum Theorem (GMT) has been crucial. This Theorem depends on graph-continuity of the (state-dependent) objective function w.r.t. the feasibility constraint correspondence, a property that, roughly speaking, requires the possibly disconnected graph of the objective function to change "continuously" with changes in the state variables. Graph-continuity holds whenever players choose certain well-specified strategies.

The validity of the GMT, however, is not restricted to one-dimensional vectors of state and decision variables. So, in principle, the GMT gives rise to a general existence result for Markov perfect equilibria in games with finitely many players provided one can identify a class of Markov strategies, A, such that

(i) the derived state-dependent pay-off function of the present player is upper semi-continuous and graph-continuous w.r.t. the feasibility constraint correspondence (that is defined on the set of state variables) whenever later players choose strategies from A, and
(ii) the well-defined optimal reaction correspondence of the present player allows a selection which is an element of A.

The principal advantage of the GMT is that it considerably weakens the requirements on A in order to have a well-defined decision problem for the present player. The weaker the requirements on A the easier it is to recover them by way of a selection of optimal reaction correspondences. Recoverability of those properties is essential for the application of backwards inductive reasoning.

3. FURTHER PROPERTIES OF EQUILIBRIA

As indicated above the following Proposition is an immediate consequence of Theorem 1.

Proposition 4. *Let $\langle c_t^*(y)\rangle$ be a sequence of equilibrium schedules ($t=0,\dots,T$). Then the equilibrium savings functions*

$$s_t^*(y_t) := y_t - c_t^*(y_t)$$

are increasing for all t.

This suggests that an equilibrium schedule, $c^*(y)$, is of bounded variation on every interval $[0, \bar{y}]$, $\bar{y} \in R$, and thus is almost everywhere differentiable. Furthermore, if $c^*(y)$

is differentiable at a point y_1 then property (L) is equivalent to

$$\frac{dc^*}{dy}(y_1) \leqq 1;$$

i.e. *the marginal propensity to consume cannot exceed unity.* In the case where consumption schedules have to be linear this is equivalent to the feasibility constraint $c^*(y) \leqq y$ for all y. But in the general case where schedules may be non-linear this becomes an *additional* constraint. It can be binding even when the feasibility constraint is not binding. Note that the linearity restriction also provides for a lower bound on the marginal propensity to consume via the feasibility constraint. This is not true here, the possiblity of a reduction of consumption with increasing y (which increases savings at a high rate) does not even allow us to write down (L) in the "proper" form as

Condition (L')

$$|c^*(y+\Delta) - c^*(y)| \leqq \Delta, \qquad \Delta > 0.$$

However, one can show (L') to hold for a schedule closely connected with $c^*(y)$; this will give us the mathematical structure to carry out a limit argument yielding existence of equilibrium if the horizon is infinite.

Suppose the schedule of generation $(t+1)$ is given by $c(y)$ and compare $c(y)$ with the schedule defined by

Condition (T)

$$\bar{c}(y) := \max_{x \leqq y} c(x).$$

$\bar{c}(y)$ is well-defined ($c(\cdot)$ is upper semi-continuous), non-decreasing and continuous. If $c(y)$ is non-decreasing (and therefore continuous) then $c(y) = \bar{c}(y)$. By definition, $\bar{c}(y) \geqq c(y)$ everywhere.

Illustration

Two possibilities of a non-increasing $c(y)$ are given by Figures 1(a) and 1(b). The dotted lines represent $\bar{c}(y)$ at points where it is different from $c(y)$. We claim that this difference between $c(y)$ and $\bar{c}(y)$ has no effect on the behaviour of a generation optimizing in response to $c(y)$ resp. $\bar{c}(y)$.

Proposition 5. *$c^*(y)$ is an optimal reaction to $c(y)$ if and only if it is an optimal reaction to $\bar{c}(y)$ (as defined in (T)).*

Proof. (i) Assume $c^*(y)$ is an optimal reaction to $c(y)$, that is,

$$u(c^*(y), c(f(y-c^*(y)))) \geqq u(c^*, c(f(y-c^*))) \quad \text{for all } c^* \in [0, y] \text{ and for all } y. \qquad (7)$$

We show that this implies that $c(\cdot)$, the *next* generation's schedule, is never evaluated at a point $y^0 = f(y - c^*(y))$ at which $c(y^0) < \bar{c}(y^0)$.

Assume it is, then there must exist y^1 with $y^1 < y^0$ and

$$c(y^1) \geqq c(y^0). \qquad (8)$$

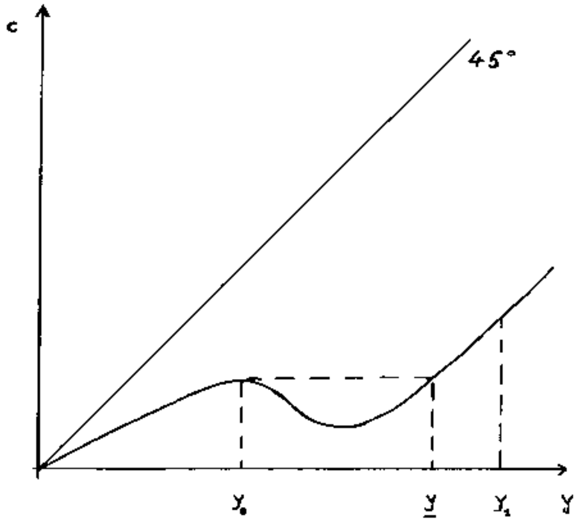

FIGURE 1(a)

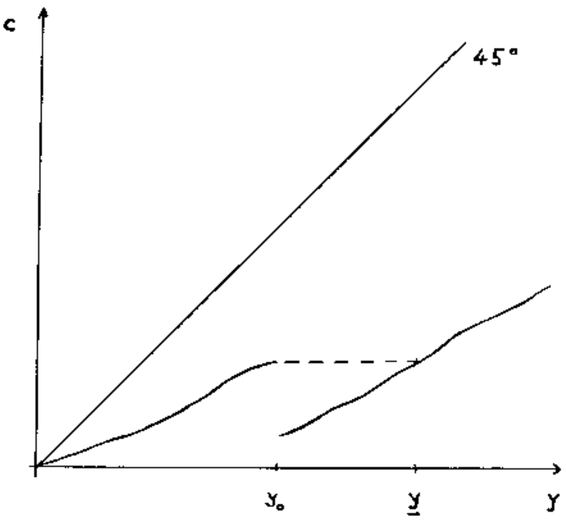

FIGURE 1(b)

This is true by the definition of $\bar{c}(y)$ and the hypothesis. Since the production function f is strictly increasing, y^1 can be realized by the optimizing player by means of a *lower* bequest than that necessary to yield y^0 (in the next period). This, of course, allows for additional direct consumption, say $\Delta c > 0$; that is, consumption $c^*(y)+\Delta c$ leads to $y^1 = f(y-(c^*(y)+\Delta c))$.

But then we have

$$u(c^*(y)+\Delta c, c(y^1)) > u(c^*(y), c(y^0)) \tag{9}$$

because of (2) and $\Delta c > 0$. This contradicts (1) since $c^*(y)+\Delta c$ is feasible.

Thus, $c(f(y-c^*(y))) = \bar{c}(f(y-c^*(y)))$ for all y if $c^*(\cdot)$ is the optimal reaction to $c(\cdot)$. But this means that $c^*(\cdot)$ is also an optimal reaction to $\bar{c}(\cdot)$, since $c(\cdot)$ in (7) can be replaced by $\bar{c}(\cdot)$.

(ii) By the same argument it is true that $c^*(y)$ must be an optimal reaction to $c(\cdot)$ if it is an optimal reaction to $\bar{c}(y)$. The crucial feature of the above argument, namely, that $\bar{c}(y)$ can be realized at a lower capital level than y if $\bar{c}(y) > c(y)$, is still present. ||

Players always react as if they were playing against *continuous* strategies of other players (yet their optimal reactions may be discontinuous). Actual play always follows a path along which the schedules $c_t^*(y)$ and $\bar{c}_t^*(y)$ coincide (except possibly for $t=0$); i.e. differences between $c_t^*(y)$ and its "levelled" version $\bar{c}_t^*(y)$ can only occur *off* an equilibrium path (at least for $t \geqq 1$).

Let us illustrate this proof with respect to Figure 1(a): $c(y)$ is now interpreted as player $(t+1)$'s strategy to which player t tries to react optimally.

Since generation t will not choose a level of consumption that determines consumption of generation $(t+1)$ as lying in the 'sink' of its consumption schedule there has to be a range of capital levels y_t over which the marginal propensity to consume of generation t exactly equals 1. Over this range bequests are constant and equal $f^{-1}(y_0)$. That is, $c^*(\cdot)$ looks like the solid curve in Figure 2.

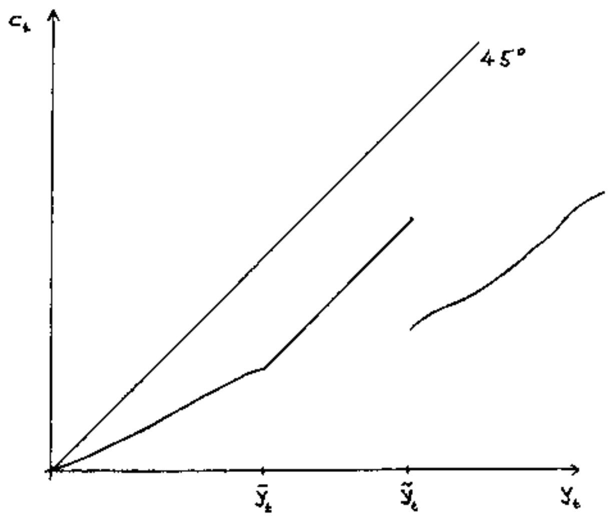

FIGURE 2

In fact, there is a whole "equivalence class" of schedules for $(t+1)$ leading to the same optimal reaction of t. For example, in Figures 1(a) and 1(b) it does not matter for the slope of $c^*(y)$ what the graph of $c(y)$ looks like over $[y_0, y]$ as long as it is defined *below* $\bar{c}(y)$. The transition from equilibrium schedules $c_t^*(y)$ to $\bar{c}_t^*(y)$, $t \geqq 1$, yields the following proposition.

Proposition 6. *$\bar{c}_t^*(y)$ satisfies (L') for all $t \geqq 0$.*

These properties of the schedules $c_t^*(y)$ (resp. $\bar{c}_t^*(y)$) which are derived from relatively simple insights into a player's decision problem will now be used to prove existence of a perfect equilibrium in the infinite horizon case.

4. EXISTENCE OF NASH EQUILIBRIA—INFINITE HORIZON

In order to prove the existence of a sequence of equilibrium schedules if the horizon is infinite we shall use a technique proposed by Peleg and Yaari (1973). We construct a sequence of equilibria for finite horizon games which converges to a limit sequence as the horizon approaches infinity. This limit sequence then is shown to be an equilibrium in the infinite horizon game. The convergence argument is based on the Theorem of Helly. The two versions of this theorem we are going to use are taken from Natanson (1969); they will be applied to schedules $c(y)$ and $\bar{c}(y)$, respectively.

The procedure is as follows:

Start with an arbitrary, *stationary* sequence of feasible and continuous consumption schedules $(c^0(y), c^0(y), \dots, c^0(y), \dots)$; that is, every generation (player) uses the same schedule (strategy), and consider the following finite horizon games:

If the time horizon is given by T then players from $(T+1)$ onwards keep their schedules while generation T reacts optimally to the schedule $c^0(y)$ used by generation $(T+1)$, generation $(T-1)$ reacts optimally to the optimal response of generation T and so on back to generation O; that is, *the first $T+1$ components of this sequence contain a perfect Nash equilibrium for the T-horizon economy.* This produces the pattern in Table I.

TABLE I

Horizon \ Generation	0	1	2	...	T	...
0	$c^0(y)$	$c^0(y)$	$c^0(y)$	...	$c^0(y)$	...
1	$c^1(y)$	$c^0(y)$	$c^0(y)$	...	$c^0(y)$	...
2	$c^2(y)$	$c^1(y)$	$c^0(y)$	...	$c^0(y)$	...
3	$c^3(y)$	$c^2(y)$	$c^1(y)$	...	$c^0(y)$	...
.	.	.	.	.	.	.
.	.	.	.	.	.	.
T	$c^T(y)$	$c^{T-1}(y)$	.	$c^1(y)$	$c^0(y)$	...
.	.	.	.	.	.	.
.	.	.	.	.	.	.
	$\downarrow$	$\downarrow$			$\downarrow$	
	c_0^*	$c_1^*(y)$	.	...	$c_T^*(y)$	...

For $i=0, 1, 2, \dots, T, \dots, c^{i+1}(y)$ is the optimal reaction to $c^i(y)$. Note, that the sequence generated in the first column of Table I reappears in all later columns. All functions in this sequence—besides being upper semi-continuous and continuous from the left—are uniformly bounded on every interval $[0, \bar{y}]$ and they are all of bounded variation. The conditions of Helly's Theorem (Natanson (1969), p. 250) are therefore shown to hold on every interval if we can show that the variation over all functions is uniformly bounded on every interval $[0, \bar{y}]$. This is the content of the following Lemma.

Lemma. *There exists $M \in R$, $M < \infty$, such that*

$$\overset{\bar{y}}{\underset{0}{V}}(c^i) \leqq M$$

for all $c^i|_{[0,\bar{y}]}$ of Table I. ($\overset{\bar{y}}{\underset{0}{V}}(c^i)$ denotes the variation of c^i on $[0, \bar{y}]$).

Proof. Use the fact that $c^i(y) = y - s^i(y)$ to get

$$\overset{\bar{y}}{\underset{0}{V}}(c^i) \leqq \overset{\bar{y}}{\underset{0}{V}}(y) + \overset{\bar{y}}{\underset{0}{V}}(s^i) \leqq 2\bar{y} = M \quad \text{for all } c^i$$

since y and s^i are increasing. ||

Helly's Theorem thus yields the existence of a convergent subsequence of $\{c^i(y)\}_{i=0}^{\infty}$, which converges pointwise *everywhere* to a limit function $c^*(y)$ of bounded variation. This holds for every interval $[0, \bar{y}]$. Denote this convergent subsequence by $\{c^{i_{n(0)}}(y)\}$ and think of it as a subsequence of the sequence in column 0 of Table I. Denote its limit by $c_0^*(y)$. $\{c^{i_{n(0)}}(y)\}$ also determines a subsequence of *rows* of Table I. From this sequence of rows choose a subsequence $c^{i_{n(1)}}$ in column 1 of Table I which converges—again by Helly's Theorem—to a limit function $c_1^*(y)$ of bounded variation. Repetition of this procedure yields a sequence of limit functions $\{c_0^*(y), c_1^*(y), \ldots, c_t^*(y), \ldots\}$.

We claim that this sequence of schedules represents a perfect Nash equilibrium for the infinite horizon game. This will be proved via the levelled schedules $\bar{c}^i(y)$ as defined by (T).

Analogously we can apply a stronger version of Helly's Theorem (Natanson (1969), p. 248) to $\{\bar{c}^i(y)\}_{i=0}^{\infty}$. The following Lemma shows that convergence of these sequences is uniform.

Let $\{i_n\}$ be a subsequence of the natural numbers, then

Lemma 2. *If $\{\bar{c}^{i_n}(y)\}_{n=1}^{\infty}$ converges to $\bar{c}^*(y)$ then it converges uniformly to $\bar{c}^*(y)$.*

Proof. By Proposition 6 all $\bar{c}^i$-functions satisfy the Lipschitz condition (L') with respect to the same Lipschitz-constant $k = 1$. This implies that the family of functions $\{\bar{c}^{i_n}(y)\}_{n=1}^{\infty}$ is equicontinuous. Convergence to the limit $\bar{c}^*(y)$ is therefore uniform. ‖

Furthermore, we claim that one can interchange limit and (T)-operation.

Proposition 7. $\lim_{n\to\infty} \bar{c}^{i_n}(y) = \bar{c}^*(y) = \overline{c^*(y)} = \overline{\lim_{n\to\infty} c^{i_n}(y)}$.

Proof. It is immediate from (T) that for all y we have $\bar{c}^*(y) \geqq \overline{c^*(y)}$. Suppose that for a $y_0 \in R_+$ $\bar{c}^*(y_0) > \overline{c^*(y_0)}$ holds. We then can without loss of generality assume that this happened at a y_0 for which $c^*(y_0) = \overline{c^*(y_0)}$. This implies that there is a sequence $y_n \to y_0$ such that $c^{i_n}(y_n) \to c^*(y_0)$ and $c^{i_n}(y_n) = c^{i_n}(y_n) = \bar{c}^{i_n}(y_n)$ for n big enough. But $\bar{c}^{i_n}(y_n)$ must converge to $\bar{c}^*(y_0)$ and thus strict inequality cannot hold. ‖

We can now prove one of the main results of this paper.

Theorem 2. *$\langle c_t^*(y)\rangle_{t=0}^{\infty}$, the sequence of limit functions from Table I, is a perfect Nash equilibrium of the infinite horizon economy.*

Proof. Assume this is not true. Then there exists at least one realization of y, say y_0, such that some player t could do better by not consuming $c_t^*(y_0)$ i.e. since the objective function of player t is given by $u(c_t, c_{t+1}^*(f(y-c_t)))$ if everyone else uses $\langle c_t^*(y)\rangle$ we have: There must exist $c_0 \in [0, y_0]$ such that

$$u(c_t^*(y_0), c_{t+1}^*(f(y_0 - c_t^*(y_0)))) < u(c_0, c_{t+1}^*(f(y_0 - c_0))). \tag{10}$$

Note that equation (10) is continuous in the values taken by c_t^* and c_{t+1}^*; i.e. there exist positive numbers ε_1, ε_2 and ε_3, and neighbourhoods U_{ε_1}, U_{ε_2}, and U_{ε_3} such that

$$u(c_1, c_2) < u(c_0, c_3) \tag{11}$$

whenever

$$(c_1, c_2, c_3) \in U_{\varepsilon_1}(c_t^*(y_0)) \times U_{\varepsilon_2}(c_{t+1}^*(f(y_0 - c_t^*(y_0)))) \times U_{\varepsilon_3}(c_{t+1}^*(f(y_0 - c_0))).$$

Now divide the interval $[0, 2y_0]$ into two regions, A and B, where

$$A := \{y \mid c_t^*(y) = \bar{c}_t^*(y)\}$$
$$B := \{y \mid c_t^*(y) < \bar{c}_t^*(y)\}$$

and assume

$$\text{(i)} \quad y_0 \in A.$$

Let $\{c^{i_{n(t+1)}}\} =: \{c_{t+1}^{i_n}\}$ be the subsequence chosen in column $t+1$ of Table I, which converges to $c_{t+1}^*(y)$. Pick the *same* subsequence (of rows) in column t. This is possible since by construction $\{i_{n(t+1)}\}$ is a subsequence of $\{i_{n(t)}\}$; call the resulting sequence of schedules $\{c_t^{i_n}\}$. $\{c_t^{i_n}\}$ then is chosen from $\{c^{i_{n(t)}}(y)\}$ and must therefore converge to $c_t^*(y)$.

It is then a consequence of the uniform convergence (Lemma 2) and the uniform continuity of the "levelled" schedules that

$$\begin{aligned}(\bar{c}_t^{i_n}(y_0), \bar{c}_{t+1}^{i_n}(f(y_0 - \bar{c}_t^{i_n}(y_0))), \bar{c}_{t+1}^{i_n}(f(y_0 - c_0)))\\ \in U_{\varepsilon_1}(\bar{c}_t^*(y_0)) \times U_{\varepsilon_2}(\bar{c}_{t+1}^*(f(y_0 - \bar{c}_t^*(y_0))) \times U_{\varepsilon_3}(\bar{c}_{t+1}^*(f(y_0 - c_0)))\end{aligned} \tag{12}$$

for large enough n holds. But $\bar{c}_t^*(y_0) = c_t^*(y_0)$ by hypothesis and $\bar{c}_{t+1}^*(f(y_0 - c_0)) = c_{t+1}^*(f(y_0 - c_0))$, if we assume (without loss of generality) that c_0 is optimal at y_0 against $c_{t+1}^*(\cdot)$, because of Proposition 5; thus (12) becomes

$$\begin{aligned}(\bar{c}_t^{i_n}(y_0), \bar{c}_{t+1}^{i_n}(f(y_0 - \bar{c}_{t+1}^{i_n}(y_0))), \bar{c}_{t+1}^{i_n}(f(y_0 - \bar{c}_{t+1}^{i_n}(f(y_0 - c_0)))\\ \in U_{\varepsilon_1}(c_t^*(y_0)) \times U_{\varepsilon_2}(\bar{c}_{t+1}^*(f(y_0 - c_t^*(y_0))) \times U_{\varepsilon_3}(c_{t+1}^*(f(y_0 - c_0))))\end{aligned} \tag{12'}$$

This means that we can replace $\bar{c}_t^{i_n}(y_0)$ and $\bar{c}_{t+1}^{i_n}(f(y_0 - c_0))$ by $c_t^{i_n}(y_0)$ and $c_{t+1}^{i_n}(f(y_0 - c_0))$, respectively, by Proposition 7 without violation of the above inclusion relation. Note then, in particular, that continuity of $\bar{c}_{t+1}^{i_n}(\cdot)$ allows to substitute $c_t^{i_n}(y_0)$ for $\bar{c}_t^{i_n}(y_0)$ in the argument of $\bar{c}_{t+1}^{i_n}(\cdot)$; that is, we have

$$\begin{aligned}(c_t^{i_n}(y_0), \bar{c}_{t+1}^{i_n}(f(y_0 - c_t^{i_n}(y_0))), c_{t+1}^{i_n}(f(y_0 - c_0)))\\ \in U_{\varepsilon_1}(c_t^*(y_0)) \times U_{\varepsilon_2}(\bar{c}_{t+1}^*(f(y_0 - c_t^*(y_0)))) \times U_{\varepsilon_3}(c_{t+1}^*(f(y_0 - c_0))).\end{aligned} \tag{12''}$$

Since $c_t^{i_n}(y_0)$ is the optimal reaction at y_0 to $c_{t+1}^{i_n}(\cdot)$ by construction of Table I and the definition of $c_t^{i_n}$ resp. $c_{t+1}^{i_n}$ (they are taken from the same *row*) we finally conclude that $\bar{c}_{t+1}^{i_n}(f(y_0 - c_t^{i_n}(y_0))) = c_{t+1}^{i_n}(f(y_0 - c_t^{i_n}(y_0)))$ by Proposition 5, which yields

$$\begin{aligned}(c_t^{i_n}(y_0), c_{t+1}^{i_n}(y_0 - c_t^{i_n}(y_0))), c_{t+1}^{i_n}(y_0 - c_0)))\\ \in U_{\varepsilon_1}(c_t^*(y_0)) \times U_{\varepsilon_2}(c_{t+1}^*(f(y_0 - c_t^*(y_0)))) \times U_{\varepsilon_3}(c_{t+1}^*(f(y_0 - c_0))).\end{aligned} \tag{12'''}$$

But by definition of the neighbourhoods this means that

$$u(c_t^{i_n}(y_0), c_{t+1}^{i_n}(f(y_0 + c_t^{i_n}(y_0)))) < u(c_0, c_{t+1}^{i_n}(f(y_0 - c_0)))$$

This is contradictory to the fact that $c_t^{i_n}(y_0)$ is an optimal reply to $c_{t+1}^{i_n}(\cdot)$.

Thus y_0 cannot be an element of A.

$$\text{(ii)} \quad \text{so assume } y_0 \in B.$$

Neither can this proposition hold. By Proposition 7, $c_t^*(y_0) < \bar{c}_t^*(y_0)$ implies that $c_t^{i_n}(y_0) < \bar{c}_t^{i_n}(y_0)$ for large enough n. But the proof of Proposition 5 established that the preceding player would never make a bequest leading to the realization of y_0 if he plays against $c_t^{i_n}(\cdot)$. This means that, if y_0 *is* realized, $c_t^{i_n}(y_0) = \bar{c}_t^{i_n}(y_0)$ for all n which implies, by taking limits, that $c_t^*(y_0) = \bar{c}_t^*(y_0)$. This contradicts the initial hypothesis.

From (i) and (ii) it follows that $y_0 \notin A \cup B = [0, 2y_0]$, a contradiction. Thus, y_0 cannot exist and $\langle c_t^*(y)\rangle$ is a perfect Nash equilibrium. ‖

Next we show that the limit schedules c_t^* have the same structural properties as their finite-horizon counterparts. Firstly, since all the c^{i}'s satisfy (L) it is trivial that their (pointwise) limits do. The following argument then shows that the c_t^*-schedules can be assumed to be upper semi-continuous.

Define the schedules $\tilde{c}_t^*(y)$ by

$$\tilde{c}_t^*(y) := c_t^*(y-0) = \lim_{y_n \uparrow y} c^*(y), \qquad t = 0, 1, \ldots$$

$\tilde{c}_t^*(y)$ is well-defined and continuous from the left. Hence $\tilde{c}_t^*(y)$ is upper semi-continuous (since it satisfies (L)). Moreover, $\langle \tilde{c}(y)\rangle$ is a perfect Nash equilibrium:

By Proposition 5 we have for all t that $\Phi_{c_t^*} = \Phi_{\bar{c}_t^*}$ (as defined by (C)); the latter has a closed graph. Thus, if $c_t^*(y)$ is a selection from $\Phi_{c_{t+1}^*}$ so is $\tilde{c}_t^*(y)$. But $\bar{\tilde{c}}_{t+1}^* = \bar{c}_{t+1}^*$, which implies that $\Phi_{c_{t+1}^*} = \Phi_{\bar{c}_{t+1}^*} = \Phi_{\bar{\tilde{c}}_{t+1}^*} = \Phi_{\tilde{c}_{t+1}^*}$ and therefore $\tilde{c}_t^*(y) \in \Phi_{\tilde{c}_{t+1}^*}(y)$ for all y and all t. This proves that (without loss of generality)

Lemma 3. *The equilibrium schedules $\langle c_t^*(y)\rangle$ satisfy (L) and are continuous from the left (hence they are upper semi-continuous and differentiable almost everywhere).*

Again, no use has been made of the stationarity of u and f. Thus the results contained in Theorem 2 and Lemma 3 also hold for non-stationary models.

Theorem 2 does not yield existence of a *stationary* equilibrium since one cannot guarantee that the limits taken in different columns of Table I (w.r.t different subsequences) converge to the *same* limit function.[6] However, one of the main advantages of Theorem 2 is the *constructiveness* of its proof. It shows that the infinite horizon case corresponds to the finite horizon case through a continuous limit process. For a discussion of the advantages of this result see Peleg and Yaari (1973, pp. 395–396). Of course, it would be interesting to know under what conditions the convergence process yields a stationary (resp. non-stationary) equilibrium. That there always exists a stationary equilibrium is proved in the next section in a non-constructive way.

5. STATIONARY EQUILIBRIA

In this section we prove existence of a stationary equilibrium via Schauder's Fixed Point Theorem. Again we make extensive use of the "levelling"-operation (T) and Proposition 5. This enables us to apply the fixed point argument over a space of *continuous* schedules.

Case a. Let us first assume that the production function f crosses the 45°-line at a level $\bar{y} > 0$ (i.e. $f(\bar{y}) = \bar{y}$) and stays below thereafter. We can then restrict attention to $[0, \bar{y}]$; since for any $y_0 \in [0, \bar{y}]$ the evolving capital sequence must stay inside $[0, \bar{y}]$. (The same applies to $y_0 > \bar{y}$ with respect to $[0, y_0]$).

Let $C([0, \bar{y}])$ be the space of continuous, bounded functions from $[0, \bar{y}]$ into R endowed with the sup-norm. Define the following subset:

$$\zeta := \{c \mid c \in C([0, \bar{y}]),\ c \text{ increasing, feasible and satifies } (L)\}.$$

ζ is a convex and compact subset of $C([0, \bar{y}])$. Now define

$$K : \zeta \to \zeta \quad \text{by}$$
$$c \to \overline{h(\Phi_c)}$$

where Φ_c is the optimal reaction correspondence defined by (C), the schedule

$$h(\Phi_c) : [0, y] \to [0, \bar{y}]$$
$$y \to \max\{\Phi_c(y)\}$$

is the "maximal" selection of the correspondence Φ_c (which is upper semi-continuous), and $\bar{\cdot}$ denotes the "levelling"-operation (T). Clearly, $K(c) \in \zeta$ for all $c \in \zeta$ and K is well-defined. The key observation is expressed by the next proposition.

Proposition 8. *A fixed point of K, c^*, generates a stationary perfect equilibrium*

$$\langle c_t^*(y)\rangle = \langle c^*(y)\rangle.$$

Proof. $c^* = K(c^*) = \overline{h(\Phi_{c^*})}$ implies that $h(\Phi_{c^*})$ is an optimal reply to c^* and thus (by Proposition 5) optimal reply to itself. ||

Since ζ is compact and convex all we have to show is that K is continuous. Existence of a fixed point then follows from Schauder's Theorem (Smart (1974)).

Proposition 9. $K : \zeta \to \zeta$ *is continuous.*

Proof. Let $c_n \to c$ in ζ and define $k_n = K(c_n)$. $\{k_n\}$ must contain a convergent subsequence, which we assume to converge to $k \in \zeta$. *Show*: $k = K(c)$, i.e. $k = \overline{h(\Phi_c)}$. Because of (T) it suffices to show that k is the "maximal" reply to c for all y's at which the $\bar{\cdot}$-operation has no effect on k. Let Y denote this set; i.e. $y \in Y$ if there exists a sequence $y_n \to y$ such that $k(y) = \lim_n k_n(y_n)$ *and* $k_n(y_n) = h(\Phi_{c_n})(y_n)$. k is continuous from the left on Y since all the $h(\Phi_{c_n})$'s are continuous from the left. Moreover, k is optimal response to c on Y. If not, there is $y \in Y$ such that $c^* = k(y)$ does *not* maximize $u(c', c(f(y - c')))$; i.e. there is $c^0 \in [0, y]$ such that

$$u(c^0; c(f(y - c^0))) = u(c^*, c(f(y - c^*))) + \varepsilon, \qquad \varepsilon > 0.$$

But then we have by the same argument as in Theorem 2 (uniform convergence of c_n to c, convergence of $k_n(y_n)$ to c^* and uniform continuity of u) that

$$u(c^0, c_n(f(y - c^0))) \geqq u(k_n(y_n), c_n(f(y_n - k_n(y_n)))) + \frac{\varepsilon}{2} \quad \text{for } n \geqq n_0$$

in contradiction to the fact that $k_n(y_n)$ is optimal reply to c_n by construction. Hence, k is optimal reply to c on Y. That K is the "maximal" optimal reply to c is seen from the fact that it is continuous from the left. Because it follows from (L) that the only selection of Φ_c which is continuous from the left is the "maximal" one. ||

Case b. If f does not cross the 45°-line then $f(y) \to \infty$ as $y \to \infty$. Consider then the game with a truncated version of the production f. For any $\bar{y} > 0$ define

$$\bar{f}(y) = \begin{cases} f(y) & \text{if } y \leqq \bar{y}, \\ \bar{y} & \text{if } y \geqq \bar{y}. \end{cases}$$

Note that this truncation does not affect the game over the interval $[0, f^{-1}(\bar{y})]$ and that $f^{-1}(\bar{y}) \to \infty$ as $\bar{y} \to \infty$. This yields the desired stationary equilibrium. Because of Proposition 8 every game with a truncated version of f, $\bar{f}$, has a stationary equilibrium, which over $[0, f^{-1}(\bar{y})]$ reproduces the game with the untruncated original f. By the same continuity arguments as those used in the last section (namely, the uniform convergence of the "levelled" schedules) it follows that the limit (for $\bar{y} \to \infty$) of these stationary equilibria is a stationary equilibrium of the original game.

We have proven

Theorem 3. *In the infinite horizon game there exists a stationary perfect equilibrium. The equilibrium schedule is almost everywhere differentiable, upper semi-continuous and satisfies (L).*

6. COMMENTS ON THE LITERATURE

We first relate Theorem 3 to Kohlberg's paper (1976). Kohlberg defines equilibrium to be stationary and assumes from the beginning that strategies be *differentiable* schedules. He studies the problem in terms of the resulting first-order condition which is a functional differential equation and shows by means of a counter-example, in which no equilibrium exists, that this functional differential equation does not admit a general solution. Specifically, it does not admit a solution *in the space of differentiable schedules.* The reason for this is perfectly clear as has been shown above. Since our model contains Kohlberg's as a special case we get the existence of a not everywhere differentiable schedule (to be precise: an almost everywhere differentiable schedule) whereas he concludes "non-existence".

Bernheim and Ray (1983) have examined a model essentially the same as the one posed in this paper. Their method to prove existence of a (stationary) perfect equilibrium is different from the one proposed here. To overcome the analytical difficulties caused by discontinuities they transform the original problem in terms of consumption schedules into a problem defined in a space of "filled" graphs of correspondences which are derived from the original schedules. This space endowed with the Hausdorff topology is then shown to have the fixed point property. Furthermore, the correspondences in question satisfy the "Keynesian property" (Bernheim and Ray (1983), p. 9) which is *equivalent* to our property (*L*) of *schedules.* In view of the discussion preceding Proposition 3 this implies that their method, too, requires bequests to be a normal good. However, the observation in the present paper that discontinuities of the schedules of other players can be "levelled out" *without provoking any change in the optimal behaviour of a given player* greatly simplifies the mathematics and has a straightforward economic interpretation. It allows (by taking limits) a unified treatment of the existence problem in finite and infinite horizon models. Bernheim and Ray (1983) do not refer to finite horizon models.

7. CONCLUDING COMMENTS

In the light of the constructiveness of the proof of Theorem 2 it would be interesting to know, whether it is always possible to approximate a stationary equilibrium by choosing an appropriate "starting point". One certainly cannot hope that any starting point would do, since there may exist cyclical equilibria. It would also be interesting to know under what conditions equilibria must be stationary and whether they are unique. From examples solved explicitly in Leininger (1983) a likely conjecture is that a uniqueness result for stationary interior equilibria holds (this result is known to hold in case the schedules are continuous for the model used by Kohlberg (1976)), but there are examples in which both stationary and non-stationary equilibria exist. A more promising route probably is to use the method of proof in section 4 to show that under certain conditions there must exist non-stationary equilibria. Finally, a natural question to ask is whether these results generalize to the case of an n-goods economy. Here the main obstacle is to get an equivalent of property (L) that made the application of the Generalized Maximum Theorem in the finite horizon case possible. If that could be achieved, it should also be possible to apply a multi-dimensional version of Helly's theorem to carry over the result to the infinite horizon model. This is a point for future research.

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NOTES

1. A detailed discussion of this point is contained in Blackorby, Primont and Russell (1978), Chapter 10.
2. Note that because of the feasibility constraint u is only defined on the "lower half" of $R_+ \times R_+$ (we could easily extend it in a continuous way to all of the positive orthant of R^2 by defining $u(\bar{c}, g(y, \bar{c}) = u(y, 0)$ if $\bar{c} > k$).
3. For a reformulation and welfare analysis of this problem in terms of "shadow" prices see Lane and Leininger (1984b).
4. Bernheim and Ray (1983, Theorem 3.1.) show that U also contains all continuous and increasing utility functions which instead of (S) satisfy the following condition:

$$\text{for all } c_t, c_t', c_{t+1}, \text{ with } c_t \geqq c_t' \geqq 0, c_{t+1} \geqq c_{t+1}' \geqq 0$$

$$u(c_t, c_{t+1}) - u(c_t', c_{t+1}), \geqq u(c_t, c_{t+1}') - u(c_t', c_{t+1}')$$

5. The requirements (L) and left-continuity are *equivalent* for the set of upper semi-continuous functions.
6. A particularly interesting class of non-stationary equilibria are cyclical equilibria which we found to exist in specific examples. See Leininger (1983), Chapter 1.

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