Observational Implications of Non-Exponential Discounting[*]

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1. Introduction

The standard assumption in economics is that individuals' preferences are dynamically consistent. That is, if a plan of action is optimal at time 0, the individual will have no incentive to revise it at future times. If that individual is maximizing a time additively separable utility function over consumption, dynamic consistency implies that the individual must discount the future with an exponential discount function. Yet all attempts to directly identify a discount function suggest that it is not exponential (see Ainslie [1992]). In particular, individuals are prepared to accept a lower interest rate to postpone consumption from t periods hence to $t + 1$ periods hence than to postpone consumption from today to the next period.

This issue was first studied in the economics literature by Strotz [1956]. How should individuals with non-exponential discounting make intertemporal choices, given their dynamic inconsistency? A savings rule was said to be a "consistent" savings rule if it was optimal for an individual to follow the rule if he anticipated that his future selves would follow that rule. Strotz claimed that individuals following such consistent rules would act as if they were maximizing a utility function derived from certain exponential weights. This claim implied that individuals with non-exponential discounting would be observationally equivalent (in a stationary environment) to individuals with exponential discounting. There is a simple intuition for such a result: when an individual makes a saving decision, we learn about the relative weights he puts on the present and the future, but we learn nothing about the relative weights he puts on different periods in the future.

Strotz's particular characterization of consistent rules turned out to be false (see Pollak [1968]). Nonetheless a large literature on dynamically inconsistent individual choice and the formally equivalent problem of intergenerational altruism has built on Strotz's ideas (see Laibson [1996] and O'Donaghue and Rabin [1997] for recent contributions and further references). The intergenerational altruism literature showed

that the observational equivalence claim was false, in at least some settings. In particular, if a consumer has a smooth and strictly concave utility function and discounts the future exponentially, his optimal savings rule must be differentiable. But Kohlberg [1976] described an example (with non-exponential discounting) where no *differentiable* consistent savings rule exists; Leininger [1986] and Bernheim and Ray [1987] showed the *existence* of a consistent savings rule in the same setting. So we have a non-constructive proof of the existence of intertemporal stationary savings problems where non-exponential discounting has observational implications (i.e., non-differentiable savings).

In this note, we provide (for a slightly different setting) an elementary example with a concave utility function where the essentially unique consistent savings rule is discontinuous; such a savings rule could not arise with exponential discounting. We demonstrate this savings rule by construction. The construction has the advantage that it provides simple intuition about what drives the discontinuity.

Examples making similar points have been constructed in related contexts. Laibson [1996] discusses examples with liquidity constraints and log utility where a similar phenomenon occurs: exponential discounters would have continuous increasing consumption rules, but non-exponential discounters have discontinuous and sometimes decreasing consumption rules. O'Donaghue and Rabin [1997] describe a discrete choice example where exponential discounters would have increasing consumption rules, but non-exponential discounters have a decreasing consumption rule.

2. The Intertemporal Consumption Problem

An individual has a continuous, strictly increasing, concave utility function over consumption, $u : \Re_+ \to \Re$, and discounts the future with discount weights $\{\delta_\tau\}_{\tau=0}^{\infty}$. Assume $\sum_{\tau=0}^{\infty} \delta_\tau$ exists and without loss of generality set $\delta_0 = 1$. Thus the utility of the time t self from consumption stream $\mathbf{x} = \{x_\tau\}_{\tau=0}^{\infty}$ is

$$v_t(\mathbf{x}) = \sum_{\tau=0}^{\infty} \delta_\tau u(x_{t+\tau}).$$

A savings rule is a function $s : \Re_+ \to \Re_+$ with $s(y) \leq y$ for all $y \in \Re_+$. Write $s^0(y) \equiv y$ and $s^{k+1}(y) \equiv s(s^k(y))$ for each $k = 0, 1, 2, ...$ If the time t self expects his future selves to follow rule s and currently has wealth y, he will choose savings $x \in [0, y]$ to maximize

$$u(y - x) + \sum_{\tau=1}^{\infty} \delta_\tau u(s^{\tau-1}(x) - s^\tau(x)).$$

Thus savings rule s is a *consistent savings rule* if for all $y \in \Re_+$,

$$s(y) \in \arg\max_{x \in [0,y]} \left\{ u(y - x) + \sum_{\tau=1}^{\infty} \delta_\tau u(s^{\tau-1}(x) - s^\tau(x)) \right\}. \tag{2.1}$$

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3. Observational Equivalence with Constant Relative Risk Aversion Utility Functions

Phelps and Pollak [1968] observed that in at least some environments, it is not possible to distinguish exponential from non-exponential discounting. Consider the case where u satisfies constant relative risk aversion (CRRA); i.e., with coefficient $\rho \in (0, \infty)$,

$$u(x) = \begin{cases} \frac{x^{1-\rho}-1}{1-\rho}, \text{ if } \rho \neq 1 \\ \ln(x), \text{ if } \rho = 1 \end{cases}.$$

Let λ^* solve

$$\frac{\lambda^\rho}{1-\lambda} = \sum_{\tau=1}^{\infty} \delta_\tau \lambda^{(1-\rho)(\tau-1)} \tag{3.1}$$

(such a λ^* always exists[1]).

Lemma 1. *There is a consistent saving rule with $s(y) = \lambda^* y$.*

This can be shown by substituting a linear saving rule into equation (2.1) and solving first order conditions.[2] This solution coincides with the solution of Phelps and Pollak [1968] in the case where, for all $\tau \geq 1$, $\delta_\tau = \beta \delta^\tau$ for some $\beta \leq 1$ and $\delta < 1$.[3] There is observational equivalence here since observing λ^* does not tell the observer if the individual has exponential discount weights or not. This equivalence is an artifact of the CRRA assumption, which ensures that an individual's relative marginal weights on current and future consumption are not influenced by any linear rule allocating future consumption among different periods. In the next section, we show that the equivalence does not survive without the CRRA assumption.

4. Discontinuous Savings

Consider the concave, piecewise linear utility function

$$u(x) = \begin{cases} x, \text{ if } x \leq 1 \\ 1 + \beta(x-1), \text{ if } 1 \leq x \leq 2 \\ 1 + \beta + \gamma(x-2), \text{ if } 2 \leq x \end{cases}$$

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[1] Consider what happens as λ increases from 0 to 1. The left hand side increases continuously from 0 to ∞. The right hand side varies continuously from either a positive number (if $\rho \leq 1$) or ∞ (if $\rho > 1$) to $\sum_{\tau=1}^{\infty} \delta_\tau$.

If $\rho > 1$, the right hand side is strictly decreasing, so there is a unique solution. There is also a unique solution in the case of log utility (i.e., $\rho = 1$): $\lambda^* = \left(\sum_{\tau=1}^{\infty} \delta_\tau\right) / \left(1 + \sum_{\tau=1}^{\infty} \delta_\tau\right)$. There may be multiple solutions if $\rho < 1$.

[2] Assuming future selves follow linear savings rule $s(y) = \lambda y$, utility from consuming x out of current wealth y is equal to $\frac{1}{1-\rho}\left[(y-x)^{1-\rho} + \sum_{\tau=1}^{\infty} \delta_\tau \left((1-\lambda)\lambda^{\tau-1}x\right)^{1-\rho}\right]$. This is maximized setting $\left(\frac{x}{y-x}\right)^\rho = (1-\lambda)^{1-\rho} \sum_{\tau=1}^{\infty} \delta_\tau \lambda^{(1-\rho)(\tau-1)}$. Setting $x = \lambda y$ gives equation (3.1).

[3] Setting $\delta_\tau = \beta \delta^\tau$ in equation (3.1), we have $\frac{\lambda^\rho}{1-\lambda} = \frac{\beta\delta}{1-\delta\lambda^{1-\rho}}$; re-arranging gives $\lambda^\rho = \delta(\beta + (1-\beta)\lambda)$; this is equation (46b) of Phelps and Pollak [1968] (with different notation; they also allowed for a linear production technology).

where $1 > \beta > 2\gamma > 0$. Let $\delta_\tau = 0$ for all $\tau \geq 2$ and δ_1 satisfy $1 > \delta_1 > \beta > \delta_1\beta > 2\gamma > 2\delta_1\gamma$. With commitment, the optimal consumption rule (see figure 1) would be

$$c^*(y) = \begin{cases} y, \text{ if } 0 \leq y \leq 1 \\ 1, \text{ if } 1 \leq y \leq 2 \\ y - 1, \text{ if } 2 \leq y \leq 3 \\ 2, \text{ if } 3 \leq y \leq 4 \\ y - 2, \text{ if } 4 \leq y \end{cases}.$$

$\boxed{\text{Figure 1 about here}}$

The corresponding savings rule (see figure 2) is

$$s^*(y) = \begin{cases} 0, \text{ if } 0 \leq y \leq 1 \\ y - 1, \text{ if } 1 \leq y \leq 2 \\ 1, \text{ if } 2 \leq y \leq 3 \\ y - 2, \text{ if } 3 \leq y \leq 4 \\ 2, \text{ if } 4 \leq y \end{cases}.$$

$\boxed{\text{Figure 2 about here}}$

Lemma 1. *The essentially unique consistent savings rule s is:*

$$s(y) = \begin{cases} 0, \text{ if } 0 \leq y \leq 1 \\ y - 1, \text{ if } 1 \leq y \leq 2 \\ 1, \text{ if } 2 \leq y < 4 + \frac{\gamma}{\delta_1\beta - \gamma} \\ 1 \text{ or } 2 + \frac{\gamma}{\delta_1\beta - \gamma}, \text{ if } y = 4 + \frac{\gamma}{\delta_1\beta - \gamma} \\ y - 2, \text{ if } 4 + \frac{\gamma}{\delta_1\beta - \gamma} < y \leq 5 \\ 3, \text{ if } 5 \leq y \end{cases}.$$

See figure 3.

$\boxed{\text{Figure 3 about here}}$

Proof. Suppose the individual could commit to any optimal consumption rule; he would choose to consume everything this period or next. In particular, his desired consumption next period would be equal to his commitment savings, $s^*(y)$. Thus *any* consistent savings rule has $s(y) = 0$ for all $0 \leq y \leq 1$. But then setting $s(y) = \begin{cases} y - 1, \text{ if } 1 \leq y \leq 2 \\ 1, \text{ if } 2 \leq y \leq 3 \end{cases}$ must uniquely achieve the first best. Now let $v(x)$ be the value to the individual of leaving savings x to the next period under savings rule s. We must have

$$v(x) = \begin{cases} \delta_1 x, \text{ if } 0 \leq x \leq 1 \\ \delta_1, \text{ if } 1 \leq x \leq 2 \\ \delta_1(1 + \beta(x - 2)), \text{ if } 2 \leq x \leq 3 \end{cases}.$$

While we have not identified $s(y)$ for $y \geq 3$, we know that the marginal utility to the current self is at most $\delta_1\gamma$ for any extra units passed on, so $v(x) \leq \delta_1(1 + \beta + \gamma(x - 3))$ for all $x \geq 3$. The individual's problem becomes one of maximizing $u(y - x) + v(x)$ (where v is *not* concave). Note that at a maximum x may

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never exceed 2 (where the marginal utility of savings is $\delta_1 \gamma$) and may never be in the interval $(1,2)$ (where the marginal utility of savings is 0). Thus for $y \geq 3$, savings must be 1 or in the interval $[2,3]$. If $3 \leq y \leq 4$, this is maximized setting $x = 1$; if $5 \leq y$, this is maximized setting $x = 3$. If $4 \leq y \leq 5$, local maxima have $x = 1$ and $x = y - 2$. The former gives utility $1 + \beta + \gamma (y - 3) + \delta_1$, while the latter gives utility $1 + \beta + \delta_1 + \delta_1 \beta (y - 4)$. The former exceeds the latter if $y \geq 4 + \frac{\gamma}{\delta_1 \beta - \gamma}$. ∎

The corresponding consumption rule (see figure 4) is

$$
c(y) = \begin{cases}
y, & \text{if } 0 \leq y \leq 1 \\
1, & \text{if } 1 \leq y \leq 2 \\
y - 1, & \text{if } 2 \leq y < 4 + \frac{\gamma}{\delta_1 \beta - \gamma} \\
2 \text{ or } 3 + \frac{\gamma}{\delta_1 \beta - \gamma}, & \text{if } y = 4 + \frac{\gamma}{\delta_1 \beta - \gamma} \\
2, & \text{if } 4 + \frac{\gamma}{\delta_1 \beta - \gamma} < y \leq 5 \\
y - 3, & \text{if } 5 \leq y
\end{cases}
$$

$$\boxed{\text{Figure 4 about here}}$$

Let us make a few observations about this example.

- The utility function is rather special. But it could easily be perturbed (for example, in such a way that it was continuously differentiable) without altering the qualitative features of this example. Similarly, we could allow strictly positive but small discount weights δ_τ for $\tau \geq 2$ without qualitatively changing the results.

- There is a simple intuition for the discontinuity. The current self wishes to save only if the savings will be consumed by his next period self. At some point, he will refrain from saving until he has enough resources to ensure increased consumption in the next period. But then he will save a lot (and his savings will jump discontinuously).

- Following Phelps and Pollak [1968], discount weights of the form $\delta_\tau = \beta \delta^\tau$ (for some $\beta \leq 1$ and $\delta < 1$) have been used in the literature to capture the idea of "hyperbolic" discounting (e.g., Laibson [1996]). In this case, non-exponential discounting corresponds to the assumption $\beta < 1$. We can similarly use piecewise linear utility functions to construct examples with discontinuous savings schedules within this class of discount weights (with non-exponential discounting). In this case, the intuition is reversed. The current self wishes to save only if the savings will be *passed on* beyond his next period self. At some point he will refrain from saving until he has enough resources to ensure increased *saving* in the next period, at which point he will save a lot.

- By focussing on consistent savings rules, we are restricting attention to stationary Markov perfect equilibria of the underlying infinite savings game (i.e., subgame perfect equilibria where strategies depend only on current wealth and not on either past payoff-irrelevant history or the calendar date). But the assumption of bounded marginal utility of consumption at zero in our example

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ensures that *all* subgame perfect equilibria will be essentially identical to the one we study.

References

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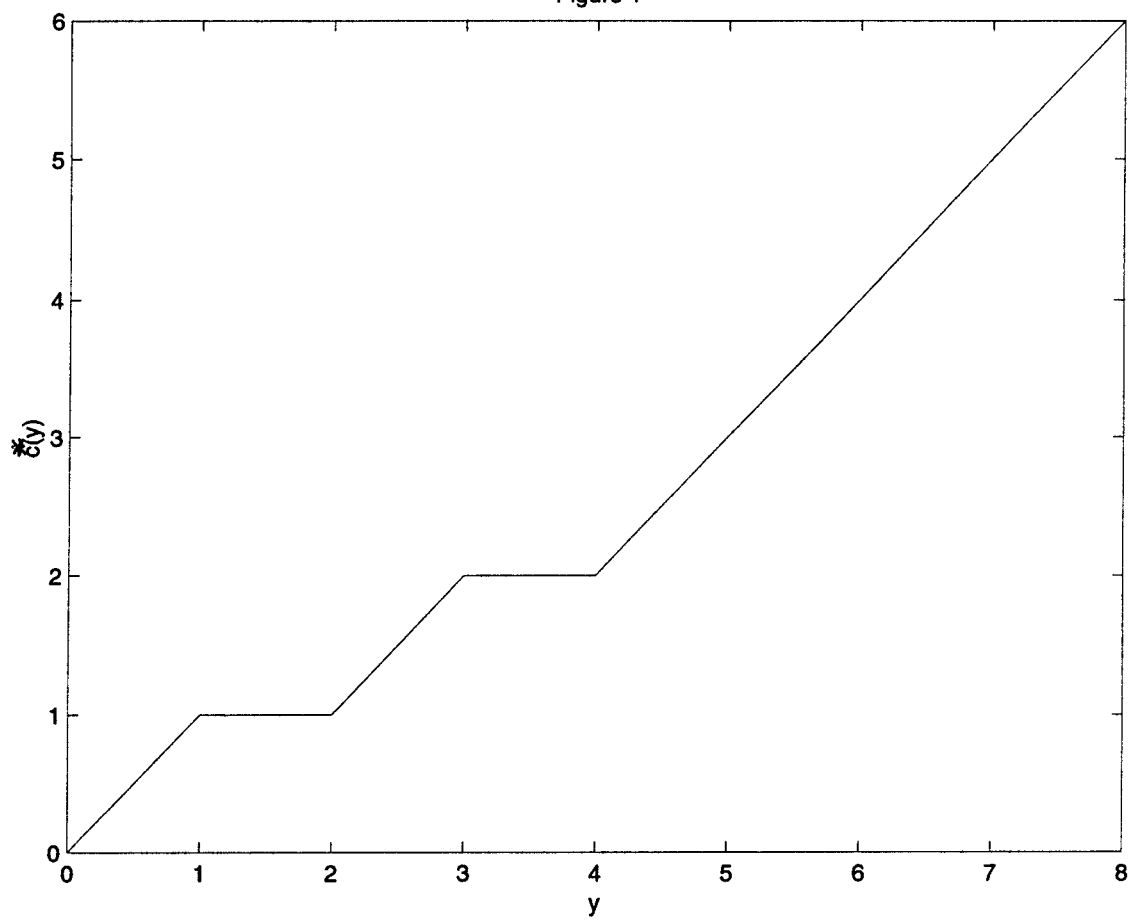[9] Strotz, R. (1956). "Myopia and Inconsistency in Dynamic Utility Maximization," *Review of Economic Studies* 23, 165-180.
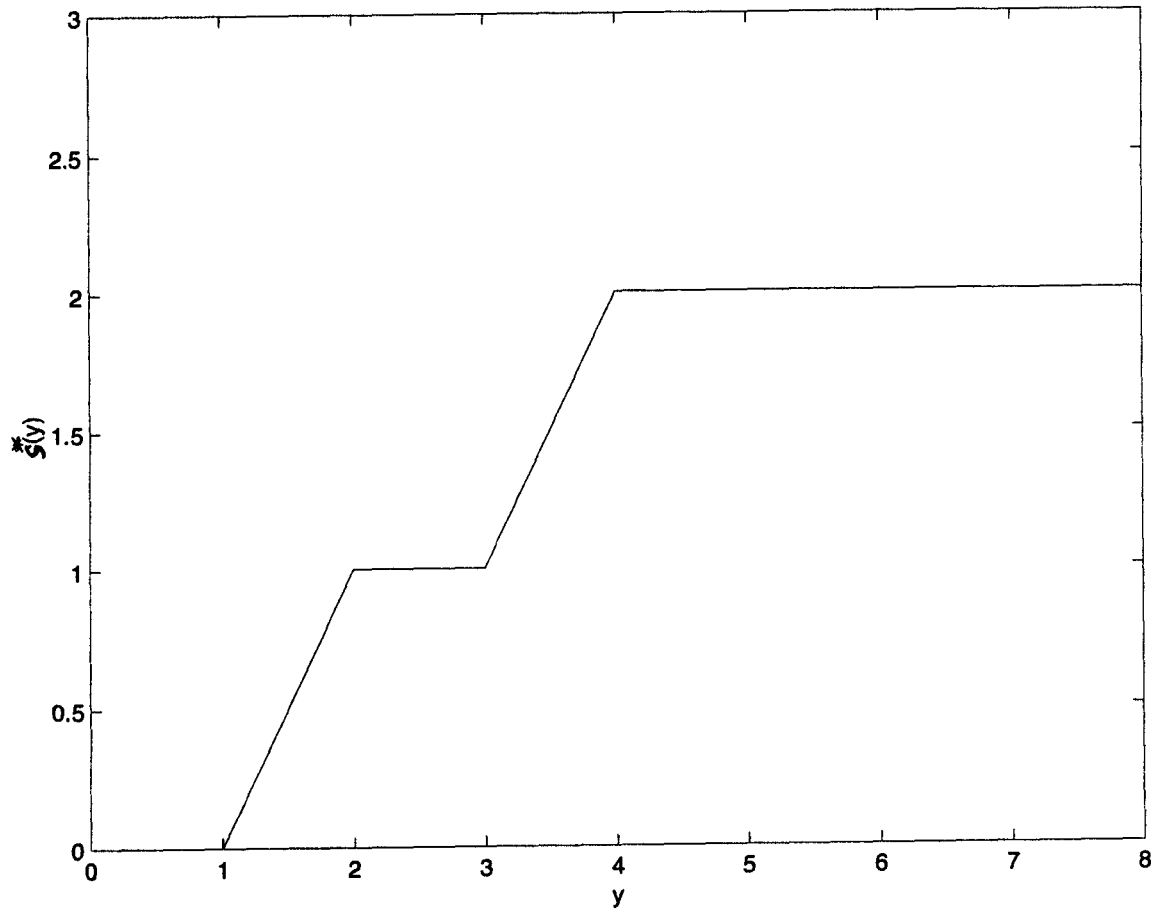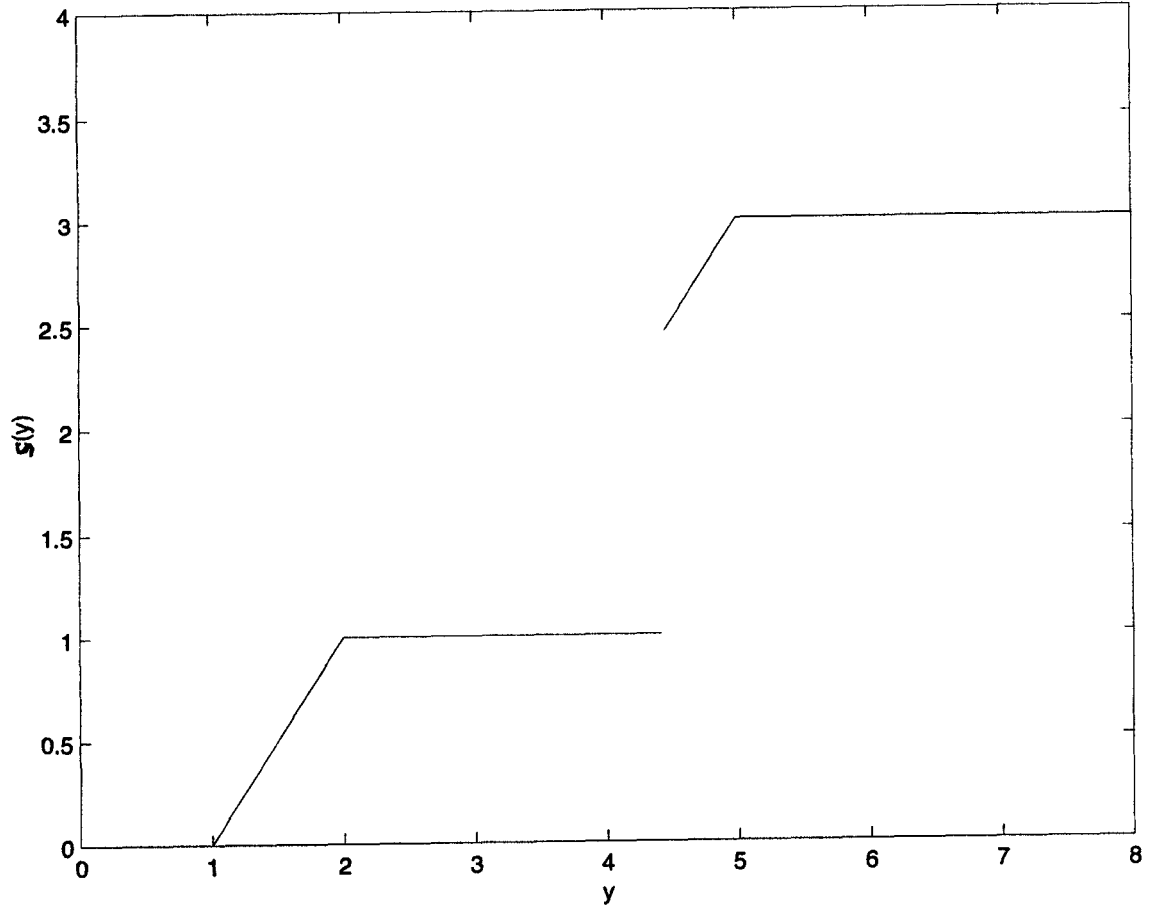
Figure 1

Figure 2

Figure 3

Figure 4