On the Existence of a Consistent Course of Action when Tastes are Changing [1,2]

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I. INTRODUCTION

Consider an economic agent whose preferences change over time. In this paper we would like to address ourselves to the question of how such an agent might behave and whether or not he can escape the predicament of yesterday's actions being non-optimal, when viewed from the vantage point of today's preferences.

Before we embark on this discussion, we must acknowledge the fact that, from the methodological point of view, the whole question of preferences that change over time is, at the outset, rather troublesome. An agent's preference ordering is nothing more than a summary of his choices, when confronted with dichotomous alternatives. As such, preferences are an *ex-post* concept, and there is a real methodological difficulty in talking today about tomorrow's preferences, since tomorrow's preferences only become meaningful after tomorrow's potential choices are known. A case can therefore be made to the effect that the assumption of constant preferences is the only possible one in economic theory. Nevertheless, we shall speak here of preferences that change in a way that is known in advance, thus knowingly placing ourselves on shaky methodological ground. But changing tastes, such as through the formation of fads, fashions, and habits, is a real phenomenon and we feel that it is worthy of examination, even at the cost of a certain amount of methodological deficiency.

It is possible to give an alternative interpretation to the discussion that follows. Rather than talk about a single decision-maker whose tastes are subject to change, one could talk about a sequence of decision-makers, one for each period, without any of the arguments being affected in any way. Each decision-maker has his own preferences which depend, however, on the consumption rates in all periods. Unfortunately, this alternative interpretation also gives rise to certain methodological difficulties, as discussed, for example, by Phelps and Pollak [4].

Below we shall formulate two illustrative models describing the behaviour of an economic agent through time. In one of these models, the agent is a producer-consumer (to be interpreted, perhaps, as a central planning board in a planned economy) with investment in one period affecting output in the next period. In the other model, the agent is a consumer operating successively in a sequence of competitive markets, with saving in one period affecting income in the next period. In both models, the notion of a *feasible consumption plan* may easily be defined. It has the form of a sequence

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2 This essay is closely related to an unpublished paper [3] by Bezalel Peleg.

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$\boldsymbol{y} = \langle y(0), y(1), \ldots\rangle$ of vectors satisfying certain constraints, where $y(t)$ is to be interpreted as the commodity bundle to be consumed in period t. The set of all feasible consumption plans will, in both models, be denoted by the symbol $\boldsymbol{Y}$. We shall assume that, in period t, the agent has a preference ordering that is representable by a utility function u_t. If preferences were unchanging over time, then u_t would be the same function, say u, for all t, and the agent's behaviour would be describable as a straightforward maximization problem, namely to select in $\boldsymbol{Y}$ a plan $\boldsymbol{y}^*$ such that $u(\boldsymbol{y}^*) \geqq u(\boldsymbol{y})$, for all $\boldsymbol{y}$ in $\boldsymbol{Y}$. However, when preferences do change over time, it becomes apparent that the agent's behaviour can no longer be described by means of a simple maximization problem. The agent's problem is to pick a feasible consumption plan $\boldsymbol{y}^*$ in such a way that, in some sense, $\boldsymbol{y}^*$ would have an optimality property consistent with *all* the utilities, and it goes without saying that a plan $\boldsymbol{y}^*$ which maximizes all the functions $u_0, u_1, u_2, \ldots$ simultaneously does not, in general, exist. How, then, should the agent's optimal behaviour be described? Which plan, in the set $\boldsymbol{Y}$ of feasible plans, should he select? An answer to this question was proposed by Strotz [6] and by Pollak [5]. To use the words of these authors, the agent should pick " the best plan that he would actually follow ". The next section is devoted to a spelling out of this suggestion.

II. THE STROTZ-POLLAK SOLUTION

Finding " the best plan that an agent would actually follow " turns out to be a dynamic programming problem. Specifically, what Strotz and Pollak are saying is that the agent pick a plan $\boldsymbol{y}^*$ in $\boldsymbol{Y}$ having the property that, for each t, $y^*(t)$ is the best action in period t, on the assumption that future actions shall be optimal. Thus, the plan $\boldsymbol{y}^*$ can, in principle, be determined by means of a backwards recursion. It might be worthwhile to illustrate this procedure in the framework of Strotz's and Pollak's own specific example.

Assume, for simplicity, that there is only a finite number, say $N+1$, of planning periods, and consider a consumer operating in a world of a single commodity. Let $y(t)$ be the amount of the commodity to be consumed in period t, so that a consumption plan is now an $(N+1)$-tuple of real numbers of the form $\langle y(0), \ldots, y(N)\rangle$. Define the set of feasible consumption plans by

$$\boldsymbol{Y} = \left\{ \langle y(0), \ldots, y(N)\rangle \;\middle|\; \begin{array}{l} y(t) \geqq 0 \text{ for } t = 0, \ldots, N \\ \text{and } \sum_{t=0}^{N} y(t) \leqq K \end{array} \right\}$$

where K is a fixed positive real number. Thus, we have here a pure storage model, in which a given stock (of size K) of the commodity is to be divided up for consumption in the various periods. Let u_t be the consumer's utility in period t, and assume that u_t depends only on the consumption in periods from t onwards, that is, assume that the arguments of u_t are $y(t), \ldots, y(N)$. Assume also that, for each t, u_t is continuous, concave, and non-decreasing. The Strotz-Pollak suggestion may now be stated as follows: let $K(t)$ be the stock of the commodity at the beginning of period t (in particular, $K(0) = K$) and look first at the Nth period. Solve the problem of maximizing $u_N(y(N))$, under the restriction $0 \leqq y(N) \leqq K(N)$. Since u_N is monotone, we may assume without loss of generality that $y(N) = K(N)$ is a solution. Now look at the $(N-1)$st period, and solve the problem of maximizing $u_{N-1}(y(N-1), y(N))$, under the restrictions $0 \leqq y(N-1) \leqq K(N-1)$ and $y(N) = K(N-1) - y(N-1)$. In view of this last condition, the maximization takes place only over $y(N-1)$, and its solution must be of the form

$$y(N-1) = h_{N-1}(K(N-1)),\; y(N) = K(N-1) - h_{N-1}(K(N-1)),$$

for some real function h_{N-1}. Now look at period $(N-2)$, and solve the problem of maximizing $u_{N-2}(y(N-2), y(N-1), y(N))$, under the restrictions

$$0 \leqq y(N-2) \leqq K(N-2),\; y(N-1) = h_{N-1}(K(N-2) - y(N-2)),$$

and $y(N) = K(N-2)-y(N-2)-h_{N-1}(K(N-2)-y(N-2))$. The last two equations imply that the maximization takes place only over $y(N-2)$, and a solution must therefore be of the form $y(N-2) = h_{N-2}(K(N-2))$, for some real function h_{N-2}. It is clear now that this procedure can be repeated successively, for the periods $N-3$, $N-4$, and so on, until we reach the period 0, in which we must solve the problem of maximizing $u_0(y(0), \ldots, y(N))$, under a set of restrictions having among them the equations

$$y(1) = h_1(K(1)),\ y(2) = h_2(K(2)),\ \ldots,\ y(N-1)) = h_{N-1}(K(N-1)),$$

and $y(N) = K(N)$, where $h_1, h_2, \ldots, h_{N-1}$ are known functions, determined in previous stages of the process. Since, by definition, $K(t) = K - \sum_{s=0}^{t-1} y(s)$ for $t>0$, we conclude that $y(t)$ is a known function of $y(0)$, so the maximization of u_0 takes place over $y(0)$ alone. This procedure thus provides an appealing way to define what is meant by a consumption plan being optimal with respect to *all* the utilities, $u_0, u_1, \ldots, u_N$.

It is not immediately clear how to generalize the Strotz-Pollak procedure to the case where there are infinitely many planning periods, or to the case where the utility in period t depends also on consumption in periods before t. Both things can, however, be done. In Section IV, we shall give a formal definition of the Strotz-Pollak solution for this general case.

III. EXISTENCE OF A STROTZ-POLLAK SOLUTION

Does an optimal plan, in the Strotz-Pollak sense, always exist? In terms of the specific example that we have just discussed, we might ask whether the functions h_t (for $t = 0, \ldots, N-1$), which associate the optimal consumption in period t with the stock of the commodity at the beginning of period t, are always guaranteed to exist. Unfortunately, the answer to this question, in general, is negative. To see this, we turn to a four-period example, for which a Strotz-Pollak solution fails to exist.

Consider the pure storage model of the previous section, and let $N = 3$. We begin by writing $u_3(y(3)) = y(3)$, which leads to $y(3) = K(3)$ as being optimal in the last period. Now, for period 2, let u_2 be given by

$$u_2(y(2), y(3)) = \min\left(2y(2), \frac{y(2)+3}{2}\right) + y(3).$$

Then, we find that $h_2(K(2)) = \min(K(2), 1)$. Proceeding to period 1, we let u_1 be given by

$$u_1(y(1), y(2), y(3)) = \min\left(2y(1), \frac{y(1)+3}{2}\right) + y(3),$$

which leads to *two* policies, h_1^α and h_1^β, *both* of which are optimal. Specifically, we have

$$h_1^\alpha(K(1)) = \begin{cases} K(1) & \text{for} \quad 0 \leqq K(1) \leqq 3 \\ 1 & \text{for} \quad K(1) > 3 \end{cases}$$

and

$$h_1^\beta(K(1)) = \begin{cases} K(1) & \text{for} \quad 0 \leqq K(1) < 3 \\ 1 & \text{for} \quad K(1) \geqq 3. \end{cases}$$

Moving now to period 0, we write

$$u_0(y(0), y(1), y(2), y(3)) = (y(0)y(1)y(2))^{\frac{1}{3}} + y(1).$$

It is now easily seen that an optimal policy for period 0 does not exist. Using the previously computed optimal policies, h_1 and h_2, to substitute for $y(1)$ and $y(2)$ in u_0, we arrive at the following result: If, in period 1, the policy h_1^α is used, then we get

$$u_0 = \begin{cases} y(0)^{\frac{1}{3}}+1 & \text{for} \quad 0 \leqq y(0) < K-3 \\ K-y(0) & \text{for} \quad K-3 \leqq y(0) \leqq K. \end{cases}$$

And if, in period 1, the policy h_1^β is used, then

$$u_0 = \begin{cases} y(0)^{\frac{1}{3}}+1 & \text{for} \quad 0 \leqq y(0) \leqq K-3 \\ K-y(0) & \text{for} \quad K-3 < y(0) \leqq K. \end{cases}$$

Now, in the former case, u_0 fails to attain a maximum for values of K satisfying $K > 11$. In the latter case, u_0 fails to attain a maximum for values of K satisfying $3 \leqq K < 11$. In either case, an optimal policy for period 0 fails to exist. Note that it is not possible to resolve this difficulty by having the consumer decide in advance (that is, in period 0) to use the policy h_1^α in period 1 if K satisfies $3 \leqq K < 11$, and to use the policy h_1^β in period 1 if $K > 11$. Suppose, for example, that $K > 11$. A decision in advance to use the policy h_1^β in period 1 works against the consumer's own interest in period 1, because it results in $K(1)$—and consequently also u_1—being *lower* than it would be if it were known in period 0 that the policy h_1^α would be used in period 1. Specifically, with $K > 11$, a decision in advance to use h_1^β results in $u_1 = 3$, whereas a decision in advance to use h_1^α results in $u_1 = 3+\varepsilon$, for some $\varepsilon > 0$.

It is important to note also that, while the utilities in the above example are not strictly concave, this has nothing to do with the non-existence of a Strotz-Pollak solution. In fact, the very same example can be modified slightly, to make all the utilities strictly concave, and a Strotz-Pollak solution will still fail to exist.[1]

We see, then, that the Strotz-Pollak procedure cannot be relied upon to yield a solution to the problem of characterizing optimal behaviour when tastes are changing. To the best of our knowledge, the only case where the Strotz-Pollak solution can be shown to exist is that of stationary preferences.[2] By stationary preferences we mean that the problem has infinitely many planning periods and, in period t, the utility u_t is defined on the rates of consumption from period t onwards, i.e. on $y(t)$, $y(t+1)$, ..., and is given by the same function for all t. But this assumption, on preferences being stationary, negates the very phenomenon—changing tastes—that gives rise to the present discussion. Thus, it seems that a solution concept other than that suggested by Strotz and Pollak must be sought, in order to describe optimal behaviour under changing tastes.

IV. NASH EQUILIBRIUM AS A WAY TO DESCRIBE OPTIMAL BEHAVIOUR WHEN TASTES ARE CHANGING

Consider, once again, an economic agent whose tastes in period t are described by a utility u_t ,where u_t is a real function defined on the set Y of all feasible consumption plans. A typical consumption plan in Y is a sequence of the form $\langle y(0), y(1), \ldots\rangle$, where $y(t)$ is the commodity bundle to be consumed in period t. Now consider the non-cooperative game in which the set of players is given by the non-negative integers, and in which player t's move is to pick $y(t)$, and his payoff, after all players have made their moves, $y(0)$, $y(1)$, ..., is given by $u_t(y(0), y(1), \ldots)$. Player t's move (that is, his choice of $y(t)$) is, of course, restricted in such a way that the sequence $\langle y(0), y(1), \ldots\rangle$, describing the moves made by all the players, shall belong to the set Y. A *strategy* for player t is a function s_t that associates a feasible move $y(t)$ for player t with every t-tuple $\langle y(0), \ldots, y(t-1)\rangle$ of feasible moves by players 0, ..., $t-1$. Thus, specifying a sequence $\langle s_0, s_1, \ldots\rangle$ of strategies for all players determines a feasible consumption plan uniquely. Let $\langle s_0^*, s_1^*, \ldots\rangle$ be a

[1] A referee, to whom we are very much indebted, has suggested an elaboration of the following sort at this point: suppose that the functions u_0, ..., u_3 are all strictly concave in their arguments. This does not imply that they remain even weakly concave, after future optimal policies are substituted in them, in the variable over which the maximization takes place. In particular, the only way to guarantee that u_1 be concave in $y(1)$, after substitution of h_2 and h_3 as functions of $y(1)$, is to have $h_2(K(2))$ linear, which is obviously not the case in general. Now, if, as a function of $y(1)$, u_1 is not concave, then its maximum may occur at a set of points which is not connected, so that $h_1(K(1))$ fails to be continuous. When, in turn, h_1 is substituted in u_0, one finds that u_0 is not continuous in $y(0)$, so the maximum fails to exist.

[2] If production is allowed, then it, too, must be stationary.

sequence of strategies, and let $\langle y^*(0), y^*(1), \ldots\rangle$ be the consumption plan determined by it. It is common to say that $\langle s_0^*, s_1^*, \ldots\rangle$ is an *equilibrium point* (in the sense of Nash) if, for $t = 0, 1, \ldots$, we have $u_t(y^*(0), y^*(1), \ldots) \geqq u_t(y(0), y(1), \ldots)$, for any consumption plan $\langle y(0), y(1), \ldots\rangle$ determined by a sequence of (feasible) strategies of the form

$$\langle s_0^*, \ldots, s_{t-1}^*, s_t, s_{t+1}^*, \ldots\rangle.$$

That is, $\langle s_0^*, s_1^*, \ldots\rangle$ is an equilibrium point if, for each t, s_t^* maximizes u_t, over all feasible choices of s_t, under the restriction that player τ, for all $\tau \neq t$, sticks to the strategy s_τ^*. If $s^* = \langle s_0^*, s_1^*, \ldots\rangle$ is an equilibrium point, then we shall refer to the associated consumption plan, $y^* = \langle y^*(0), y^*(1), \ldots\rangle$, as an *equilibrium consumption plan*. An equilibrium consumption plan is optimal, in the sense that it has the following consistency property: the agent has no motivation to change his action in period t, nor does he have reason to regret his action in any period, given his actions in other periods. The notion of equilibrium in the sense of Nash thus provides a way to describe consistent behaviour in the face of changing tastes. But before this way of describing behaviour can be of any use, it is necessary to show that an equilibrium consumption plan exists in a sufficiently rich setting. It is to this question that we wish to devote the remainder of the present investigation.

It should be noted that if a consumption plan y is optimal in the Strotz-Pollak sense, then it is also an equilibrium plan. To show this, we give a definition of optimality in the Strotz-Pollak sense, this time in terms of the game-theoretic concepts introduced above. A sequence $\langle s_0^*, s_1^*, \ldots\rangle$ of strategies is optimal in the Strotz-Pollak sense if, for each t, s_t^* maximizes u_t, over all feasible choices of s_t, under the restriction that the players $t+1$, $t+2, \ldots$ stick to the strategies $s_{t+1}^*, s_{t+2}^*, \ldots$ and the players $0, 1, \ldots, t-1$ use any feasible strategies. In particular, therefore, we have that s_t^* maximizes u_t when the players $0, 1, \ldots$, $t-1$ choose the strategies $s_0^*, s_1^*, \ldots, s_{t-1}^*$. This argument makes it clear that the converse of our assertion does not hold: an equilibrium plan need not be optimal in the Strotz-Pollak sense. For this reason, an existence theory for equilibrium plans may have a better chance of success than an existence theory for plans that are optimal in the Strotz-Pollak sense.

V. SHORTCOMINGS OF THE CONCEPT OF AN EQUILIBRIUM CONSUMPTION PLAN

An equilibrium consumption plan is not necessarily Pareto-efficient. In other words, if y^* is an equilibrium consumption plan, then there may exist a feasible plan y, such that $u_t(y) \geqq u_t(y^*)$ for all t, with a strict inequality for some t. This fact may be viewed by some readers as a serious drawback of the concept of an equilibrium consumption plan. However, we feel that the consistency property that characterizes equilibrium plans is a minimal property. A consumption plan that does not have this property cannot be advocated seriously as a reasonable course of action for an agent with changing tastes. Ideally, one should look for a plan that is simultaneously consistent and Pareto-efficient. The existence of such plans depends on the specific model under consideration, and it will not be examined here.

Perhaps a more serious shortcoming of the concept of an equilibrium consumption plan lies in the enormous amount of information which it requires the economic agent to possess. Specifically, in each period, the agent is assumed to know the utilities of all periods, since the concept of equilibrium in the sense of Nash assumes that each player knows the pay-off functions of *all* the players. In other words, the agent knows in advance precisely how his tastes are going to change, throughout the future. This is surely rather far fetched. Some day, the theory of games with incomplete information might be brought to bear upon this question, and this might lead to results that would be more immune to this criticism. As for us, we can offer only a partial consolation. It has to do with the method that we shall use to prove our main existence theorem (Theorem 6.2). To establish the existence of an equilibrium sequence of strategies, we shall use a convergence argument in which, for some integer T, the players $T+1$, $T+2$, ... use some arbitrary fixed strategies and only the players 0, 1, ..., T optimize. Then we shall take the limit, as $T \to \infty$, to obtain the desired equilibrium sequence of strategies. What this procedure amounts to is that, up to an approximation, the agent is required only to know what tastes will be in a number of periods hence. Beyond that, he can assume any arbitrary behaviour and, no matter what he assumes, he is guaranteed to be close to the equilibrium action in the present period. This means, roughly speaking, that the farther into the future we look, the less the amount of information which the agent is required to possess.

VI. EXISTENCE OF EQUILIBRIUM POINTS FOR GAMES WITH COUNTABLY MANY PLAYERS

Before turning to a description of consistent behaviour in two specific economic models with changing tastes, we would like to state and prove a general existence theorem. For games with finitely many players, the following result is known (see, for example, Friedman [1]):

Lemma 6.1. *Let $S_1, ..., S_n$ be non-empty convex compact subsets of finite-dimensional Euclidean spaces. For each i, $i = 1, ..., n$, let H_i be a continuous real function defined on $S = S_1 \times S_2 \times ... \times S_n$. If, for each i, $i = 1, ..., n$, and for every choice of $s_j^* \in S_j$, for $j \neq i$, the function $H_i(s_1^*, ..., s_{i-1}^*, s_i, s_{i+1}^*, ..., s_n^*)$ is quasi-concave in s_i on S_i, then the game* [1] *$\langle S_1, ..., S_n; H_1, ..., H_n\rangle$ has an equilibrium point.*

The proof of this assertion can be obtained by a straightforward application of Kakutani's fixed point theorem.

The theorem that we need is a generalization of Lemma 6.1 to games with countably many players.

Theorem 6.2. *For $i = 1, 2, ...$, let S_i be a non-empty convex compact subset of E^{k_i}, the Euclidean space of dimension k_i. For each i, $i = 1, 2, ...$, let H_i be a continuous real-valued function on the (topological) product space $S = \bigtimes_{i=1}^{\infty} S_i$. If, for $i = 1, 2, ...$, and for each choice of $s_j^* \in S_j$, $i \neq j$, the function $H_i(s_1^*, ..., s_{i-1}^*, s_i, s_{i+1}^*, ...)$ is quasi-concave in s_i on S_i, then the game* [2] *$\langle S_1, S_2, ...; H_1, H_2, ...\rangle$ has an equilibrium point.*

Proof. Let s^0 be an arbitrary point of S. For each natural number n, consider the n-person game $\langle S_1, ..., S_n; H_1^n, ..., H_n^n\rangle$, where the function H_i^n is defined by

$$H_i^n(s_1, ..., s_n) = H_i(s_1, ..., s_n, s_{n+1}^0, s_{n+2}^0, ...)$$

for $s_1, ..., s_n$ satisfying $s_j \in S_j$, $j = 1, ..., n$. By Lemma 6.1, this game has an equilibrium point $\langle \bar{s}_1^n, ..., \bar{s}_n^n\rangle$. For $n = 1, 2, ...$, define

$$\bar{s}^n = \langle \bar{s}_1^n, ..., \bar{s}_n^n, s_{n+1}^0, s_{n+2}^0, ...\rangle.$$

Since the sets S_j are compact, we may use the Cantor diagonal process and obtain a subsequence $\langle \bar{s}^{n_j}\rangle$, for $j = 1, 2, ...$, which converges to a point $\bar{s}$ of S. We claim that $\bar{s}$ is an equilibrium point of the original game. For suppose, contrary to this claim, that there exists a natural number q and that there exists a strategy $s_q \in S_q$, such that

$$H_q(\bar{s}) < H_q(\bar{s}_1, ..., \bar{s}_{q-1}, s_q, \bar{s}_{q+1}, ...).$$

[1] The reference here is to games in normal form, that is, to games that are completely specified by the strategy spaces of the players and by the pay-off functions.

[2] See the previous footnote.

Since H_q is continuous, there exists an integer j such that $n_j \geqq q$ and

$$H_q(\bar{s}^{n_j}) < H_q(\bar{s}_1^{n_j}, \ldots, \bar{s}_{q-1}^{n_j}, s_q, \bar{s}_{q+1}^{n_j}, \ldots),$$

which is impossible, since $\langle \bar{s}_1^{n_j}, \ldots, \bar{s}_{n_j}^{n_j} \rangle$ is, by construction, an equilibrium point of the finite game of size n_j. This completes the proof of the theorem.

VII. A MODEL OF ECONOMIC GROWTH WITH CHANGES IN THE PLANNER'S PREFERENCES

Our first application of Theorem 6.2 will be in the analysis of an optimal growth model, in which utility changes over time. The problem of describing consistent behaviour in such a setting has been discussed by Inagaki [2] and by Phelps and Pollak [4].

The model is made up of three components. First, a non-negative vector x_0, describing the initial resources of the economy; second, a sequence $\langle f_0, f_1, \ldots \rangle$ of production functions, where f_t describes the efficient way of using resources in period t to obtain resources in period $t+1$; third, a sequence $\langle u_0, u_1, \ldots \rangle$ of utility functions, where u_t describes the preferences prevailing in period t.

There is no need to assume that the number of commodities is the same in every period. Indeed, everything that will be said here is valid also if we let n_t be the number of commodities in period t, and the sequence $\langle n_t \rangle$ of positive integers need not even be bounded. However, so as not to create a typographical nuisance, we shall write the model as though the number of commodities is a fixed positive integer, say n. We shall let the non-negative orthant of the Euclidean space of dimension n be denoted E_+^n.

For each t, the production function f_t is defined on E_+^n to E_+^n, and we shall assume that f_t is non-decreasing, concave, and continuous. If we let $x(t)$ stand for the vector of capital goods that the economy has in the beginning of period t, and if we let $y(t)$ be the vector describing total consumption in the economy in period t, then the evolution of the economy is given by

$$\begin{aligned} x(t+1) &= f_t(x(t)) - y(t), \quad t = 0, 1, \ldots \\ x(0) &= x_0. \end{aligned} \qquad \ldots(1)$$

This specification views the consumption $y(t)$ as taking place at the *end* of period t. The difference between this specification and others is merely notational.

Let Y be the set of all sequences $\langle y(0), y(1), \ldots \rangle$ of vectors in E^n which satisfy (1) and, in addition, satisfy the inequalities

$$0 \leqq y(t) \leqq f_t(x(t)), \qquad \ldots(2)$$

for $t = 0, 1, \ldots$. Thus, Y is the set of all feasible consumption plans. It is a convex, compact subset of the space formed by taking the countable Cartesian product of E^n with itself.

The utility functions $u_0, u_1, \ldots$ are assumed defined on the set Y to the real numbers. The function u_t describes the planner's preferences in period t among feasible consumption programmes. We shall make the following assumptions on the utility functions. For each t, $t = 0, 1, \ldots, u_t$ is non-decreasing, quasi-concave, and continuous. It should be noted that, while continuity of u_t in the product topology is usually regarded as a rather strong assumption, we are assuming continuity on the compact set Y only, and this is considerably weaker than continuity on the entire space. Thus, for example, if Y happens to be a compact subset of 1_∞, then our continuity assumption is equivalent to 1_∞-continuity, in the norm topology.

Now let us consider the game G such that the set of players is given by the non-negative integers, and such that a *strategy* for player t is a function which associates a feasible capital vector $x(t+1)$, for period $t+1$, with every feasible choice of capital vectors $x(1), \ldots, x(t)$, for the periods $1, \ldots, t$. More precisely, we have the following: a strategy for player 0 is a vector s_0, satisfying the requirement

$$0 \leqq s_0 \leqq f_0(x_0), \qquad \ldots(3)$$

and a strategy for player t, for $t > 0$, is a function s_t defined for every t-tuple $\langle x(1), \ldots, x(t) \rangle$ that satisfies the inequalities

$$0 \leqq x(q) \leqq f_{q-1}(x(q-1)), \quad q = 1, \ldots, t, \qquad \ldots(4)$$

with $x(0) = x_0$, and the function s_t itself satisfies

$$0 \leqq s_t(x(1), \ldots, x(t)) \leqq f_t(x(t)). \qquad \ldots(5)$$

Let $\boldsymbol{s} = \langle s_0, s_1, \ldots \rangle$ be a sequence of strategies for all the players. This sequence determines a unique feasible consumption plan, to be denoted $\boldsymbol{y}^{\boldsymbol{s}}$, in the following manner:

$$y^s(t) = f_t(x(t)) - s_t(x(1), \ldots, x(t)),$$

for $t = 1, 2, \ldots$, and $y^s(0) = f_0(x_0) - s_0$. Thus, it is possible to define the *pay-off* to player t, associated with the strategy sequence $\boldsymbol{s}$, by

$$v_t(\boldsymbol{s}) = u_t(\boldsymbol{y}^{\boldsymbol{s}}).$$

We now have the game G completely described (in normal form) and we may proceed to inquire about the existence of an equilibrium sequence of strategies.

In order to be able to make use of Theorem 6.2, and also in order to eliminate the pathological case where the behaviour consisting of completely ignoring the intertemporal structure turns out to be an equilibrium, we now proceed in the following manner: an *a priori* restriction on the choice of a strategy for each player will be imposed, and an equilibrium sequence of strategies will be sought, in the resulting restricted class of strategies. The question of the existence of an equilibrium (and particularly, a non-pathological one) in a wider class of strategies remains open.

Define a set $S \subset E^n_+$, in the following manner:

$$S = \{\langle \sigma_1, \ldots, \sigma_n \rangle \in E^n \mid 0 \leqq \sigma_i \leqq 1, \ i = 1, \ldots, n\}.$$

Also, for $i = 1, \ldots, n$, let $f_{t,\,i}(x(t))$ be the ith component of the vector $f_t(x(t))$. We shall assume that player t's strategy s_t (for $t > 0$) is restricted to be of the form

$$s_t(x(1), \ldots, x(t)) = \langle \sigma_1 f_{t,\,1}(x(t)), \ldots, \sigma_n f_{t,\,n}(x(t)) \rangle, \qquad \ldots(6)$$

for some $\langle \sigma_1, \ldots, \sigma_n \rangle \in S$. For player 0, we assume, likewise, that

$$s_0 = \langle \sigma_1 f_{0,\,1}(x_0), \ldots, \sigma_n f_{0,\,n}(x_0) \rangle, \qquad \ldots(6')$$

for some $\langle \sigma_1, \ldots, \sigma_n \rangle \in S$. In other words, player t's strategy depends directly only on $x(t)$, and not on $x(1), \ldots, x(t-1)$, and the dependence on $x(t)$ takes the following simple form. The player picks an n-tuple $\langle \sigma_1, \ldots, \sigma_n \rangle$ of constants, all lying in the unit interval, and sets the ith component of s_t equal to a fraction σ_i of the ith component of $f_t(x(t))$.

By the notation $s_t \in S$, we mean that s_t is a strategy for player t, satisfying (6)—or (6′). We shall use the symbol $\mathscr{S}$ to denote the countable Cartesian product, $S \times S \times \ldots$, of S with itself. Thus, $\mathscr{S}$ is taken to be the strategy space of the game G.

It should be noted that restricting the strategy space of the game G to the set $\mathscr{S}$ does not, in any way, restrict the set of attainable consumption plans. That is, for each $\bar{\boldsymbol{y}} \in Y$, there exists a sequence of strategies $\boldsymbol{s}$, with $\boldsymbol{s} \in \mathscr{S}$, such that $\boldsymbol{y}^{\boldsymbol{s}} = \bar{\boldsymbol{y}}$.

Let $\boldsymbol{s}$ be a sequence of strategies in $\mathscr{S}$, and let $\sigma_t = \langle \sigma_{1t}, \ldots, \sigma_{nt} \rangle$ belong to the set S. We shall write $\boldsymbol{s} \mid \sigma_t$ for the strategy sequence $\boldsymbol{s}'$ given by

$$\begin{aligned} s'_q &= s_q \quad \text{for} \quad q \neq t \\ &= \sigma_t \quad \text{for} \quad q = t. \end{aligned}$$

A sequence $s^* \in \mathscr{S}$ is said to be an equilibrium sequence if, for each t, $t = 0, 1, \ldots$, we have

$$v_t(s^* \mid \sigma_t) \leqq v_t(s^*), \quad \text{for all } \sigma_t \in S.$$

Theorem 7.1. *Under our assumptions on the production functions $f_0, f_1, \ldots$, and on the utility functions $u_0, u_1, \ldots$, there exists an equilibrium sequence of strategies in $\mathscr{S}$.*

We shall prove this theorem by proving two lemmas which, together with two corollaries, imply that the game G satisfies the conditions of Theorem 6.2.

Lemma 7.2. *Let $s \in \mathscr{S}$, and denote the capital sequence associated with s by $\mathbf{x}^s$. Then, for each non-negative integer t, $\mathbf{x}^s$ is a concave function of s_t on S, for fixed $s_q \in S$, $q \neq t$.*

Proof. Let r be a non-negative integer. We must show that $x^s(r)$ is concave, as a function of s_t, with $s_q \in S$ fixed for $q \neq t$. Now, for values of r satisfying $r \leqq t$, we have that $x^s(r)$ is independent of s_t, and therefore, trivially, it is concave in s_t, as asserted. For values of r satisfying $r > t$, we prove our assertion by induction on r. First let $r = t+1$, and assume that $s_t = \langle \sigma_{1t}, \ldots, \sigma_{nt} \rangle$, $0 \leqq \sigma_{it} \leqq 1$. Then, the ith component of $x^s(r)$ is given by $\sigma_{it} f_{t,i}(x^s(t))$, so that $x^s(r)$ is linear in s_t. Now assume that $r \geqq t+1$ and that $x^s(r)$ has been shown to be concave in s_t. We have to show that $x^s(r+1)$ is also concave in s_t. But the ith component of $x^s(r+1)$ is given by $\sigma'_i f_{r,i}(x^s(r))$, for some real number σ'_i satisfying $0 \leqq \sigma'_i \leqq 1$. Since f_r is non-decreasing and concave, and $x^s(r)$ is assumed, by the induction hypothesis, to be concave in s_t, we have the desired result.

Corollary 7.3. *y^s is concave in s_t on S, for fixed $s_q \in S$, $q \neq t$.*

Proof. Let $y_i^s(r)$ be the ith component of $y^s(r)$. Then, $y_i^s(r) = (1-\sigma_{ir}) f_{r,i}(x^s(r))$, for $i = 1, \ldots, n$, $\langle \sigma_{1r}, \ldots, \sigma_{nr} \rangle \in S$ and for $r = 0, 1, \ldots$. If $r \leqq t$, then $f_r(x^s(r))$ is constant for fixed s_q, $q < t$. If $r > t$, then, by Lemma 7.2 and by our assumptions on f_r, we have that $f_r(x^s(r))$ is concave in s_t, as asserted.

Corollary 7.4. *$v_t(s)$ is quasi-concave with respect to s_t on S, for fixed $s_q \in S$, $q \neq t$.*

Proof. Let $s \in \mathscr{S}$, and select σ_t and σ'_t in S, and α in $(0, 1)$. Then,

$$\begin{aligned} v_t(\alpha(s \mid \sigma_t) + (1-\alpha)(s \mid \sigma'_t)) &= u_t(y^{\alpha(s \mid \sigma_t) + (1-\alpha)(s \mid \sigma'_t)}) \\ &\geqq u_t(\alpha y^{s \mid \sigma_t} + (1-\alpha) y^{s \mid \sigma'_t}) \\ &\geqq \min\,[u_t(y^{s \mid \sigma_t}),\ \ u_t(y^{s \mid \sigma'_t})] \\ &= \min\,[v_t(s \mid \sigma_t),\ \ v_t(s \mid \sigma'_t)]. \end{aligned}$$

Lemma 7.5. *y^s is a continuous function of s on $\mathscr{S}$. Therefore, $v_t(s)$ is continuous in s on $\mathscr{S}$.*

We omit the proof, which is straightforward.

Theorem 7.1 now follows directly from Corollary 7.4 and Lemma 7.5, when used in conjunction with Theorem 6.2.

VIII. A CONSUMER OPERATING IN COMPETITIVE MARKETS, WITH CHANGING TASTES

For another application of Theorem 6.2, let us turn now to a description of the decisions made by a consumer unit over time. The unit will be characterized by a sequence $\langle u_0, u_1, \ldots \rangle$ of utility functions, and by a sequence $\langle z(0), z(1), \ldots \rangle$ of positive real numbers, describing the unit's non-interest income in the various periods. The real function u_t, which describes the unit's preferences in period t, is assumed to be defined for every feasible consumption plan $\langle y(0), y(1), \ldots \rangle$ and to be continuous, non-decreasing and quasi-concave.

The environment in which our unit is assumed to operate is given by two sequences: a sequence of price vectors $\langle p(0), p(1), \ldots\rangle$, and a sequence of interest factors,

$$\langle r(0), r(1), \ldots\rangle.$$

The vector $p(t)$, describing market prices in period t, belongs to E^n [1] and we assume that it is positive (in every component) for each t. The real number $r(t)$, describing the rate of exchange of wealth in period $t+1$ for wealth in period t, is also assumed positive for all t.

For $t = 1, 2, \ldots$, the unit's action in period t is to pick a vector $y(t)$ in E^n_+ and a non-negative real number $x(t)$, subject to the obvious restriction—

$$p(t) \cdot y(t) + x(t) = z(t) + r(t-1)x(t-1), \qquad \ldots(1)$$

and for $t = 0$, the unit has to pick $y(0)$ and $x(0)$ under the restriction

$$p(0) \cdot y(0) + x(0) = z(0). \qquad \ldots(2)$$

The unit's action in period t will be called *feasible* if it satisfies (1) [or (2)].

We now proceed to consider the game G', whose set of players is given by the non-negative integers, and such that a strategy for player t is a function associating a feasible choice of $x(t)$ and $y(t)$ with every configuration $\langle x(q), y(q)\rangle$, $q = 0, \ldots, t-1$, of feasible choices for players $0, \ldots, t-1$. To make this notion precise, let us introduce the following notation: if v is a vector in E^{n+1}, then we shall write v^0 for the first component of v, and we shall write $\hat{v}$ for the vector in E^n obtained from v by deleting the first component. Given this, we may define a strategy for player 0 as a vector s_0 in E^{n+1} satisfying

$$p(0) \cdot \hat{s}_0 + s_0^0 = z(0). \qquad \ldots(3)$$

Similarly, a strategy for player t, for $t>0$, is a function s_t, with values in E^{n+1}, defined for every $2t$-tuple of the form $\langle x(0), y(0), \ldots, x(t-1), y(t-1)\rangle$ and satisfying the requirement

$$p(t) \cdot \hat{s}_t(\cdot) + s_t^0(\cdot) = z(t) + r(t-1)x(t-1). \qquad \ldots(4)$$

If $s = \langle s_0, s_1, \ldots\rangle$ is a sequence of strategies for all the players, then we can define the *pay-off* to player t, associated with s, in the following manner:

$$v_t(s) = u_t(y^s), \qquad \ldots(5)$$

where, once again, we let y^s be the sequence of consumption vectors determined by s. This definition completes the description of the game G', in normal form.

In order to be able to apply Theorem 6.2, we must, once again, restrict the strategy space of the game G' quite considerably. The restriction that we need is that $x(t)$ and $y(t)$ should both be concave in $x(t-1)$. But, in view of the budget constraint (1), if *both* $x(t)$ and $y(t)$ are concave in $x(t-1)$, then they must, in fact, be linear as well as homogeneous, not directly in $x(t-1)$, but in $z(t)+r(t-1)x(t-1)$. Thus we are led to restricting the strategy space in such a way that, in each period, the consumer's Engel curves are straight lines, passing through the origin. In other words, the budget constraint, together with the concavity of $x(t)$ and $y(t)$ in $x(t-1)$, lead us to the same kind of restriction on the strategy space that was encountered in the previous section.

Let A be the unit simplex in E^{n+1}. We shall say that a strategy s_t, for player t, belongs to the set A if, for each $j = 0, \ldots, n$, the jth component of the value taken on by s_t is a fraction $\alpha_j/p_j(t)$ of the quantity $z(t)+r(t-1)x(t-1)$, where $p_j(t)$ is the jth component of $p(t)$ and $\langle \alpha_0, \ldots, \alpha_n\rangle$ belongs to the simplex A.[2] Thus, α_0 is the fraction of the consumer's total resources devoted to saving, α_1 is the fraction of total resources devoted to the purchase of the first commodity, and so on. For $t>0$, our assumption is that player t's strategy,

[1] As in Section VII, the assumption that the number of commodities is the same in all periods is merely a typographical convenience.

[2] Here, we define $p_0(t) = 1$, for all t.

s_t, is restricted to belong to the set A. That is, s_t is restricted by

$$s_t(x(0), y(0), \ldots, x(t-1), y(t-1)) = \langle \alpha_0 w(t), \alpha_1 w(t)/p_1(t), \ldots, \alpha_n w(t)/p_n(t) \rangle$$

for some $\langle \alpha_0, \ldots, \alpha_n \rangle \in A$, where $w(t) = z(t) + r(t-1)x(t-1)$ and $p_j(t)$ is the jth component of $p(t)$. As for $t = 0$, we assume that s_0 is restricted to lie in A, i.e. to be of the form

$$s_0 = \langle \alpha_0 z(0), \alpha_1 z(0)/p_1(0), \ldots, \alpha_n z(0)/p_n(0) \rangle,$$

for some $\langle \alpha_0, \ldots, \alpha_n \rangle \in A$. We shall use the symbol $\mathscr{A}$ to denote the countable Cartesian product of A with itself. The reader is referred to Section VII for a definition of what is meant by a strategy sequence $\boldsymbol{s}$ being an equilibrium for the game G' in $\mathscr{A}$.

Theorem 8.1. *Under our assumptions on utilities, on prices, and on interest factors, there exists an equilibrium sequence of strategies for the game G' in $\mathscr{A}$.*

The proof of this assertion is similar to that of Theorem 7.1, and will be omitted.

IX. TWO REMARKS IN CONCLUSION

Our first remark is a technical one. If, in Theorem 7.1, we restrict the choice of a strategy in period t to some non-empty, closed, and convex subset S_t of S, then the Theorem still holds true, with $\mathscr{S}$ defined to be the Cartesian product $S_0 \times S_1 \times \ldots$. Such a restriction may be used to incorporate various kinds of *a priori* information about consumption (such as the existence of a minimum subsistence level) into the model. The proof of the modified theorem remains unchanged. Similarly, in Theorem 8.1, one may redefine the set $\mathscr{A}$ to be a Cartesian product of the form $A_0 \times A_1 \times \ldots$, where A_t is a non-empty, closed, convex subset of A, without affecting the validity of the theorem.

Our second remark is about a recent contribution, [7], by C. C. von Weizsäcker, in which he discusses consumer behaviour under endogenously changing tastes. In Weizsäcker's formulation, the assumption is that today's consumption determines tomorrow's tastes. It should be noted that the framework presented here does not exclude the case discussed by Weizsäcker, since we have certainly allowed the utility in period t to depend upon consumption rates in periods prior to t. However, Weizsäcker is able, in a very elegant fashion, to avoid the whole issue of the consistency of actions and preferences at different points of time. This is done through the adoption of a revealed preference approach that starts from the demand functions. Starting out from the demand functions obviates the need to inquire about their existence.

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