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DOES MODEL UNCERTAINTY JUSTIFY CAUTION? ROBUST OPTIMAL MONETARY POLICY IN A FORWARD-LOOKING MODEL

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This paper proposes a general method based on a property of zero-sum two-player games to derive *robust optimal monetary policy rules*—the best rules among those that yield an acceptable performance in a specified range of models—when the true model is unknown and model uncertainty is viewed as uncertainty about parameters of the structural model. The method is applied to characterize robust optimal Taylor rules in a simple forward-looking macroeconomic model that can be derived from first principles. Although it is commonly believed that monetary policy should be less responsive when there is parameter uncertainty, we show that *robust optimal Taylor rules* prescribe in general a stronger response of the interest rate to fluctuations in inflation and the output gap than is the case in the absence of uncertainty. Thus model uncertainty does not necessarily justify a relatively small response of actual monetary policy.

Keywords: Monetary Policy Rules, Parameter Uncertainty, Robust Control

1. INTRODUCTION

A considerable recent literature has sought to characterize desirable monetary policies in terms of interest-rate feedback rules, that is, guides for setting at each period the policy instrument, such as the federal funds rate in the United States, in response to economic conditions. Many computations of optimal policy rules in the context of one or another econometric model—such as those collected by Taylor (1999)—imply that an optimal rule would involve stronger responses of the federal funds rate to fluctuations in inflation (and perhaps also in output) than are

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implied by estimated Fed reaction functions, or by Taylor's (1993) much-discussed characterization of recent Fed policy. However, the specific equations favored by various authors are still significantly different. This raises the question of how a policy rule should be selected in the face of uncertainty about the correct model of the economy.

A common intuition first proposed by Brainard (1967) is that parameter uncertainty should lead one to choose a more "cautious" policy: Policymakers should compute the optimal change of their instrument as if they knew the functioning of the economy with certainty, and then move their instrument by less [see Blinder (1998)].[1] Some commentators have therefore proposed that the strong responses of the instrument required by optimal policy in the context of an econometric model depend upon assuming that estimated model coefficients are known to be true, whereas taking proper account of one's actual uncertainty about the true coefficients should justify gentler responses, perhaps closer to current policy.[2]

This paper seeks to formally evaluate this argument. We seek to characterize optimal monetary policy rules that are *robust* to uncertainty about the proper model of the economy when all of the models considered are similar, though not identical. This can be modeled as uncertainty about the *parameters* that numerically specify the economic model. Uncertainty of this kind necessarily exists in practice because researchers do not know with certainty all parameters of their model. In contrast to the standard Bayesian approach followed by, for example, Brainard (1967), Chow (1975), Clarida et al. (1999), and Rudebusch (2001), we assume that the policymaker has multiple priors about the probability distribution of the true model, and that he is uncertainty averse. The result is that the best policy rule is a *robust optimal monetary policy rule* of the kind advocated recently by Hansen and Sargent (1999a,b), Sargent (1999), Stock (1999), and Onatski and Stock (2002). Such a rule is designed to avoid an especially poor performance of monetary policy in the event of an unfortunate parameter configuration, and guarantees to yield an acceptable performance in the specified range of models.[3]

We propose a method to characterize robust optimal policy rules in a broad class of models. Whereas most studies of optimal policy in the face of model uncertainty focus on backward-looking models, we use our method to determine robust optimal policy rules in a simple forward-looking macroeconomic model. As in Woodford (1996, 1999c), the model is composed of a monetary policy rule and two structural equations—an intertemporal IS equation and an aggregate supply equation—that are based on explicit microeconomic foundations. Because it can be derived from first principles, the model is not subject to the famous Lucas (1976) critique for the evaluation of policy. An important property of this model is that the policymaker faces a trade-off between the stabilization of inflation and the output gap on one hand and the nominal interest rate on the other hand.

A comparison of the robust optimal rule to the optimal policy in the absence of uncertainty allows us to determine whether Brainard's (1967) result generalizes to the class of models considered here. In contrast to the "conventional wisdom," we obtain that robust optimal monetary policy generally commands a *stronger* response of the interest rate to fluctuations in goal variables such as inflation and the output gap than is the case in the absence of uncertainty. In fact, model uncertainty affects the trade-off facing the policymaker in a way that places more weight on the stabilization of inflation and the output gap, and relatively less weight on the stabilization of the nominal interest rate. This is because the robust optimal rule, which is designed to perform well in those instances in which exogenous shocks have particularly large effects on the goal variables, requires the interest rate to respond by enough to guarantee that exogenous perturbations have only a limited effect on the economy. It is therefore far from clear that model uncertainty can provide a justification for the kind of policies implied by estimates of current policy.[4]

Similar results have been obtained recently with a different approach and in other frameworks by Hansen and Sargent (1999b), Söderström (1999), Sargent (1999), Stock (1999), Kasa (2002), and Onatski and Stock (2002). Whereas Sargent (1999) applies robust control theory to a backward-looking model, Hansen and Sargent (1999b) apply it to an optimal monetary policy problem that is similar (except for the nature of model uncertainty) to the one treated here. However, they specify a broad, nonparametric set of additive model perturbations that represent deviations of the model actually used from the true model, and bound uncertainty in terms of a bound upon the possible size of this additive term. We instead assume uncertainty about the values of coefficients of the linear equations of the structural model. This type of uncertainty seems to us more intuitive, and it seems more likely that modelers should be able to quantify their degree of confidence in that way. We furthermore obtain an analytical characterization of the robust optimal policy, which helps us to clarify the circumstances under which robust optimal policy is more aggressive than the policy obtained in the absence of model uncertainty. Stock (1999) and Onatski and Stock (2002) study a type of uncertainty that is similar to ours. Those authors, however, determine robust optimal rules in the backward-looking model of Rudebusch and Svensson (1999), whereas we consider a forward-looking model. Kasa (2002) also seeks to characterize robust policies in a forward-looking model, but uses a frequency-domain approach instead of a time-domain approach.[5]

The rest of the paper is organized as follows. Section 2 reviews the baseline model in the absence of model uncertainty. In Section 3, we introduce model uncertainty and explain how this affects the objective of monetary policy. We next propose a solution procedure to derive the robust optimal policy rule in a general class of models. In Section 4, we apply our solution procedure to characterize analytically robust optimal Taylor rules in the model of Section 2. We conclude in Section 5.

2. MONETARY POLICY IN A SIMPLE OPTIMIZING MODEL WITH KNOWN PARAMETERS

This section reviews the monetary policy design problem in a formal model that can be derived from first principles, when the model parameters are known with certainty. Our baseline framework is taken from Woodford (1996, 1999b,c).[6] We first describe the model that characterizes the behavior of the private sector and then turn to monetary policy.

2.1. A Simple Structural Model

Apart from the monetary policy rule to be discussed later, Woodford's model consists of two structural equations that can be derived as log-linear approximations to equilibrium conditions of an underlying dynamic general equilibrium model with sticky prices. The intertemporal IS equation, which relates spending decisions to the interest rate, is given by

$$x_t = E_t x_{t+1} - \sigma^{-1}\left(i_t - E_t \pi_{t+1} - r_t^n\right), \tag{1}$$

and the aggregate supply equation (or expectational Phillips curve) is given by

$$\pi_t = \kappa x_t + \beta E_t \pi_{t+1,} \tag{2}$$

where x_t denotes the output gap (defined as the deviation of output from its natural level, i.e., the equilibrium level of output under flexible prices), π_t is the inflation rate, and i_t is the deviation of the short-term nominal interest rate from its steady-state value.[7] The composite exogenous disturbance r_t^n represents Wicksell's "natural rate of interest," that is, the real interest rate that equates output to its natural level or, alternatively, the interest rate that would prevail in equilibrium under flexible prices [see Blinder (1998, Ch. 2) and Woodford (1999b,c)]. Perturbations to the natural rate of interest represent all nonmonetary disturbances that affect inflation and the output gap. For instance, a temporary increase in r_t^n could reflect a temporary exogenous increase in aggregate demand or, alternatively, a temporary decrease in the natural level of output. Moreover, because both interest rates enter the structural equations only through the "interest-rate gap" $(i_t - E_t\pi_{t+1}) - r_t^n$, nonmonetary perturbations affect inflation and the output gap only if the interest rate controlled by the central bank is such that the real interest rate, $i_t - E_t\pi_{t+1}$, departs from the natural rate of interest.

While (1) can be viewed as a log-linear approximation to the representative household's Euler equation for optimal timing of consumption in the presence of complete financial markets, (2) can be interpreted as a log-linear approximation to the first-order condition for the supplier's optimal price-setting decision. All variables are assumed to be bounded.[8] The structural parameters σ and κ are both positive by assumption. The parameter σ represents the inverse of the intertemporal elasticity of substitution ($-\sigma$ is the slope of the intertemporal IS curve), and κ, which is the slope of the short-run aggregate supply curve, can be interpreted as a measure of the speed of price adjustment. Finally, $\beta \in (0, 1)$ can be interpreted as the time discount factor of the price setters, which is assumed to be the same as the discount factor of the representative household.

Table 1. Calibrated parameter values

Certainty case: baseline model			Parameter uncertainty		
Parameters	Value	Std. error	Parameter	Lower bound	Upper bound
Structural			σ	0.0915	0.2227
β	0.99		κ	0.0168	0.0308
σ	0.1571	(0.0328)			
κ	0.0238	(0.0035)			
Shock process					
ρ	0.35				
sd(r^n)	3.718				
Loss function					
λ_x	0.0483				
λ_i	0.2364				

Rotemberg and Woodford (1997) have shown that an estimated model similar to the one considered here (but slightly more complicated) provides a very good description of the actual behavior of inflation, output, and the quarterly average of the federal funds rate in the United States between 1979 and 1995, in that it is able to replicate accurately the responses of the three endogenous variables to a monetary shock.[9] Their estimated structural parameters are given in Table 1. They are used here, as in Woodford (1999c), to "calibrate" the model in the baseline case.

2.2. Optimal Monetary Policy

We now turn to the objective of monetary policy. Researchers have traditionally assumed that policymakers should seek to minimize a weighted average of some measure of variability of inflation and of the output gap [see, e.g., Walsh (1998, Ch. 8), Woodford (1999b), and Clarida et al. (1999) for a recent discussion]. In the model considered earlier, the policymaker can in fact perfectly stabilize inflation and the output gap by setting $i_t = r_t^n$ in every period, so that the interest rate perfectly tracks the exogenous fluctuations in the natural rate of interest. However, it may be undesirable to vary the nominal interest rate as much as the natural rate of interest.[10] For instance, Friedman (1969) has argued that high nominal interest rates involve welfare costs of transactions. Since it is plausible that the deadweight loss is a convex function of the distortion [see Woodford (1990, 1999b)], it may be desirable not only to reduce the level, but also the variability, of nominal interest rates. Accordingly, we assume the following loss criterion[11]:

$$L_0 = E_0\left\{(1-\beta)\sum_{t=0}^{\infty}\beta^t\left[\pi_t^2 + \lambda_x x_t^2 + \lambda_i i_t^2\right]\right\}, \tag{3}$$

where $\lambda_x, \lambda_i > 0$ are weights that the policymaker places on the stabilization of the output gap and the nominal interest rate, and where $\beta \in (0, 1)$ is the discount factor mentioned earlier.[12]

An implication of this loss criterion is that an equilibrium with complete stabilization of inflation and the output gap is not fully efficient. In fact, exogenous fluctuations in the natural rate of interest require variations in the nominal interest rate to stabilize inflation and the output gap. Hence, welfare costs associated to fluctuations in the nominal interest rate introduce a tension between stabilization of inflation and the output gap on one hand, and stabilization of the nominal interest rate on the other hand.

Following recent studies of monetary policy [see, e.g., Taylor (1999)], we characterize monetary policy in terms of interest-rate feedback rules. Specifically, we assume that the policymaker commits credibly at the beginning of period 0 to a policy rule of the form

$$i_t = \mathcal{P}_t\left(\pi_t, \pi_{t-1}, \ldots, x_t, x_{t-1}, \ldots, i_{t-1}, i_{t-2}, \ldots, r_t^n, r_{t-1}^n, \ldots\right) \tag{4}$$

for each date $t \geq 0$. The policymaker's problem is to determine the functions $\mathcal{P}_t(\cdot)$, $t = 0, 1, 2, \ldots$ to minimize the loss $E[L_0]$ subject to the structural equations (1) and (2). Because the objective is quadratic and the constraints are linear in all variables, we may, without loss of generality, restrict our attention to linear functions $\mathcal{P}_t(\cdot)$.

We denote by $\boldsymbol{\psi}$ the vector of coefficients that completely characterizes $\{\mathcal{P}_t(\cdot)\}_{t=0}^{\infty}$, and we simply call $\boldsymbol{\psi}$ a "policy rule." In practice, however, we always assume that policy rules $\boldsymbol{\psi}$ are drawn from some finite-dimensional linear space $\tilde{\boldsymbol{\Psi}} \subseteq \mathbf{R}^n$. We denote by $\boldsymbol{\theta} = [\theta_1, \theta_2, \ldots, \theta_m]'$ the finite-dimensional vector of structural parameters of the model, and by $\boldsymbol{\Theta} \subseteq \mathbf{R}^m$ the set of possible vectors $\boldsymbol{\theta}$. We also write $\boldsymbol{q_t} = [\pi_t, x_t, i_t]'$ for the vector of endogenous variables at date t, and $\boldsymbol{q}$ for the stochastic process $\{\boldsymbol{q_t}\}_{t=0}^{\infty}$, specifying $\boldsymbol{q_t}$ at each date as a function of the history of exogenous shocks until that date.

To be feasible, the stochastic process $\boldsymbol{q}$ needs to satisfy the structural equations (1) and (2) at all dates t. These can be written compactly as

$$\tilde{\boldsymbol{S}}(\boldsymbol{q}, \boldsymbol{\theta}) = \mathbf{0}. \tag{5}$$

Similarly, the restrictions imposed by the commitment to (4) at all dates $t \geq 0$ can be written as

$$\tilde{\boldsymbol{P}}(\boldsymbol{q}, \boldsymbol{\psi}) = \mathbf{0}. \tag{6}$$

We assume that both $\tilde{\boldsymbol{S}}(\boldsymbol{q}, \boldsymbol{\theta})$ and $\tilde{\boldsymbol{P}}(\boldsymbol{q}, \boldsymbol{\psi})$ are linear in $\boldsymbol{q}$. Similarly, the loss function (3) can be denoted by $L_0(\boldsymbol{q}, \boldsymbol{\theta})$.[13]

A *rational expectations equilibrium* is then defined as a stochastic process $\boldsymbol{q}(\boldsymbol{\psi}, \boldsymbol{\theta})$ satisfying both (5) and (6). In general, many different policy rules may result in the same equilibrium. Some rules may also yield many different equilibria, in which case the set of equilibria *always* includes some with arbitrarily large fluctuations of the endogenous variables.[14] The latter equilibria are therefore arbitrarily bad under the assumed loss criterion, and the policy rules that allow them to occur cannot be optimal in the class of rules $\tilde{\boldsymbol{\Psi}}$.[15] We therefore restrict our attention to a subset $\boldsymbol{\Psi} \subseteq \tilde{\boldsymbol{\Psi}}$ of policy rules that results in a unique bounded rational expectations equilibrium, and let $\boldsymbol{q}(\psi, \boldsymbol{\theta})$ denote this equilibrium. We consider only bounded equilibrium processes because the structural equations (1) and (2) would not provide a reasonable approximation of the true equilibrium conditions in the underlying model if the endogenous variables were not bounded.

The optimal monetary policy rule that is optimal relative to the subset of rules $\boldsymbol{\Psi}$ can, in turn, be defined as follows.

DEFINITION 1. *In the case of known structural parameters* $\boldsymbol{\theta}$, *let* $\boldsymbol{\Psi}$ *be a set of policy rules such that there is a unique bounded equilibrium. Then, an* optimal monetary policy rule *is a vector* ψ^0 *that solves*

$$\min_{\psi \in \boldsymbol{\Psi}} E[L_0(\boldsymbol{q}(\psi, \boldsymbol{\theta}), \boldsymbol{\theta})]$$

where the unconditional expectation is taken over all possible histories of the disturbances $\{r_t^n\}$.

Note that the stochastic process $\boldsymbol{q}(\psi, \boldsymbol{\theta})$ is a specification of the endogenous variables for all possible initial conditions r_0^n, as well as for all possible realizations of the exogenous shocks. Since the unconditional expectation in Definition 1 is also taken over the exogenous initial states r_0^n, monetary policy is evaluated here without reference to any particular initial conditions.

We shall determine such optimal policy rules in Section 4. First, in the next section, we describe how the introduction of uncertainty about the structural parameters alters the objective of monetary policy, and we propose a general method for finding the optimal policy rule in the presence of parameter uncertainty.

3. MODEL UNCERTAINTY AND ROBUST OPTIMAL MONETARY POLICY: GENERAL FRAMEWORK

In the preceding section, the parameters that specify the model are supposed to be constant and known with certainty by all economic agents. The only uncertainty is due to exogenous perturbations to the natural rate of interest. In reality, however, central banks and researchers do not know the parameters of their models or the exogenous disturbances with certainty. They can extract estimates of model parameters from their data sets, but as long the sample is finite, there is no way one can be sure about the value of most structural parameters. This parameter uncertainty may well have an effect on the optimal monetary policy rule. It is precisely this effect that we analyze here.

The underlying framework we have in mind is the same as the one in the model mentioned above, except that the private sector (or representative household) can be one of many different types. We assume that the household's type is determined in period 0; the household knows its type, but the central bank does not.[16]

We assume that the policymaker commits at the beginning of period 0 to a policy rule ψ (or, equivalently, to functions $\mathcal{P}_t(\cdot)$ for each date $t \geq 0$). We assume that the commitment to such a rule is credible, and in particular that the policymaker does not revise it at later dates using additional information he might have gathered about unknown model parameters.[17] Given the (publicly known) policy rule and the structural equations describing the behavior of the representative household (which knows the structural parameters), a rational expectations equilibrium can be determined. Since the policymaker does not know the true parameter vector, however, he does not know which equilibrium will be realized (for given exogenous disturbances).

3.1. Objective of Monetary Policy with Model Uncertainty

To characterize parameter uncertainty, we assume that the vector $\boldsymbol{\theta}$ of structural parameters lies in a given (known) compact set $\boldsymbol{\Theta} \subset \mathbf{R}^m$, and that the distribution of $\boldsymbol{\theta}$ is unknown. Instead of *assuming* a particular prior distribution over $\boldsymbol{\Theta}$ and deriving the policy rule that minimizes the expected loss, as is usually done in the standard Bayesian approach, we let the policymaker consider many probability measures over $\boldsymbol{\Theta}$, including the possibility that any given element $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ holds with certainty. Moreover, we assume that the policymaker has aversion toward uncertainty in the sense axiomatized by Gilboa and Schmeidler (1989). Their results show that if the policymaker has multiple priors on $\boldsymbol{\Theta}$, and his preferences satisfy uncertainty aversion in addition to the axioms of standard expected utility theory, the policymaker's problem is to minimize his loss in the worst-case scenario, that is, when the prior distribution is the worst distribution in the set of possible distributions.[18] The optimal policy rule is then the robust rule defined as follows.

DEFINITION 2. *Let $\boldsymbol{\Psi}$ be a set of policy rules such that there is a unique bounded equilibrium process $\boldsymbol{q}(\psi, \boldsymbol{\theta})$ for all $\psi \in \boldsymbol{\Psi}$, $\boldsymbol{\theta} \in \boldsymbol{\Theta}$. In the case of parameter uncertainty, a* robust optimal monetary policy rule *is a vector ψ^* that solves*

$$\min_{\psi \in \boldsymbol{\Psi}} \left\{ \max_{\boldsymbol{\theta} \in \boldsymbol{\Theta}} E[L_0(\boldsymbol{q}(\psi, \boldsymbol{\theta}), \boldsymbol{\theta})] \right\}, \tag{7}$$

where the unconditional expectation is taken over all possible histories of the disturbances $\{r_t^n\}$.[19]

Given that the unknown vector of structural parameters is in $\boldsymbol{\Theta}$, the policymaker can guarantee that the loss is no higher than the one obtained in the following "minmax" equilibrium.

DEFINITION 3. *A* minmax equilibrium *is a bounded rational expectations equilibrium $\boldsymbol{q}^* = \boldsymbol{q}(\psi^*, \boldsymbol{\theta}^*)$, where $\psi^* \in \boldsymbol{\Psi}$ is a robust optimal monetary policy rule and $\boldsymbol{\theta}^*$ maximizes the loss $E[L_0(\boldsymbol{q}(\psi^*, \boldsymbol{\theta}), \boldsymbol{\theta})]$ on the constraint set $\boldsymbol{\Theta}$.*

3.2. Robust Optimal Policy Rule: Solution Method

The method that we propose to characterize the robust optimal policy rule can, in principle, be applied to any model in which the feasibility constraints can be expressed as in (5), possible policy rules may be parametrized as in (6), and the robust optimal monetary policy rule solves (7). Therefore, it is not limited to the model presented in Section 2. This method is based on the relation between the solution to problem (7) and the equilibrium of a zero-sum two-player game.

Consider the game of pure strategies $\Gamma = \langle\{P, N\}, (\boldsymbol{\Psi}, \boldsymbol{\Theta}), (-L(\psi, \boldsymbol{\theta}), L(\psi, \boldsymbol{\theta}))\rangle$, where $L(\psi, \boldsymbol{\theta}) \equiv E[L_0(\boldsymbol{q}(\psi, \boldsymbol{\theta}), \boldsymbol{\theta})]$. In this game, the policymaker (P) chooses the policy rule $\psi^* \in \boldsymbol{\Psi}$ to minimize his loss, $L(\psi, \boldsymbol{\theta})$, knowing that a malevolent Nature tries to hurt him as much as possible. The other player, Nature (N), chooses the vector of structural parameters $\boldsymbol{\theta}^* \in \boldsymbol{\Theta}$ to maximize the policymaker's loss, knowing that the policymaker is going to minimize it. The Nash equilibrium (NE) of this game is a profile of strategies $(\psi^*, \boldsymbol{\theta}^*)$ such that

$$\psi^* \in \arg\max_{\psi\in\boldsymbol{\Psi}}\{-L(\psi, \boldsymbol{\theta}^*)\} = \arg\min_{\psi\in\boldsymbol{\Psi}} L(\psi, \boldsymbol{\theta}^*), \tag{8}$$

$$\boldsymbol{\theta}^* \in \arg\max_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} L(\psi^*, \boldsymbol{\theta}). \tag{9}$$

We shall look for a profile of strategies $(\psi^*, \boldsymbol{\theta}^*)$ that solves both (8) and (9). If such a profile exists, then the following property of zero-sum games guarantees that the policy rule ψ^* obtained in the NE is the robust optimal policy rule that we are seeking to determine.

PROPOSITION 1. *Suppose that* Γ *has an NE. The profile* $(\psi^*, \boldsymbol{\theta}^*)$ *is an NE of* Γ *if and only if the action of each player is a maxminimizer*; *that is*,

$$\psi^* \in \arg\max_{\psi\in\boldsymbol{\Psi}}\left\{\min_{\boldsymbol{\theta}\in\boldsymbol{\Theta}}[-L(\psi, \boldsymbol{\theta})]\right\} = \arg\min_{\psi\in\boldsymbol{\Psi}}\left\{\max_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} L(\psi, \boldsymbol{\theta})\right\},$$

$$\boldsymbol{\theta}^* \in \arg\max_{\boldsymbol{\theta}\in\boldsymbol{\Theta}}\left\{\min_{\psi\in\boldsymbol{\Psi}} L(\psi, \boldsymbol{\theta})\right\}.$$

Proof. See Osborne and Rubinstein (1994), Proposition 22.2(a) and (c). ■

Rather than defining conditions (5) and (6) for general stochastic processes $\boldsymbol{q}$, it is typically convenient to restrict attention to a particular linear subspace of processes that satisfy additional linear constraints besides (5) and (6). These additional constraints do not exclude outcomes that might result from policies in $\boldsymbol{\Psi}$, but they restrict *all* of the equilibria resulting from policies $\psi \in \tilde{\boldsymbol{\Psi}}$ so that no optimal plan is infeasible given the class of policies $\tilde{\boldsymbol{\Psi}}$ considered. (For example, in Section 4, it is assumed that the interest rate is set according to a standard Taylor rule; this implies that an optimal plan cannot be feasible if any of the endogenous variables depends upon lagged variables.)[20] It is then convenient to parameterize this subspace of possible processes by an alternative parameter vector $\boldsymbol{f}$. The stochastic process corresponding to any parameters $\boldsymbol{f}$ is given by $\boldsymbol{q}(\boldsymbol{f})$. The restrictions (5) and (6) can then be rewritten as

$$\boldsymbol{S}(\boldsymbol{f}, \boldsymbol{\theta}) = \mathbf{0}, \tag{10}$$

$$\boldsymbol{P}(\boldsymbol{f}, \psi) = \mathbf{0}, \tag{11}$$

and the vector $\boldsymbol{f}$ that solves (10) and (11) is given by $\boldsymbol{f}(\psi, \boldsymbol{\theta})$.

We consider in turn the policymaker's problem (8), for any given vector of structural parameters $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, and Nature's problem (9). Instead of solving the policymaker's problem directly, it is convenient to proceed in two steps, as in Woodford (1999c): First, we determine the vector $\boldsymbol{f}^*(\boldsymbol{\theta})$ parameterizing the feasible equilibrium $\boldsymbol{q}(\boldsymbol{f}^*(\boldsymbol{\theta}))$ that minimizes the loss criterion for any given $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, and second, we look for a policy rule $\boldsymbol{\psi}^*(\boldsymbol{\theta})$ in the set $\boldsymbol{\Psi}$ that *implements* this optimal equilibrium. Formally, we first determine $\boldsymbol{f}^*(\boldsymbol{\theta})$ to minimize

$$\hat{L}(\boldsymbol{f}, \boldsymbol{\theta}) \equiv E[L_0(\boldsymbol{q}(\boldsymbol{f}), \boldsymbol{\theta})] \tag{12}$$

subject to the restrictions (10) imposed by the structural equations for any $\boldsymbol{\theta} \in \boldsymbol{\Theta}$. The policymaker's Lagrangian can thus be written as

$$\mathcal{L}^P(\boldsymbol{f}, \boldsymbol{\phi}; \boldsymbol{\theta}) = \hat{L}(\boldsymbol{f}, \boldsymbol{\theta}) + \boldsymbol{\phi} \cdot \boldsymbol{S}(\boldsymbol{f}, \boldsymbol{\theta}), \tag{13}$$

where $\boldsymbol{\phi}$ is a row vector of Lagrange multipliers. The solution $\boldsymbol{f}^*(\boldsymbol{\theta})$ and the optimal Lagrange multipliers $\boldsymbol{\phi}^*(\boldsymbol{\theta})$ solve the first-order necessary conditions

$$\frac{\partial \hat{L}(\boldsymbol{f}^*(\boldsymbol{\theta}), \boldsymbol{\theta})}{\partial \boldsymbol{f}'} + \boldsymbol{\phi}^*(\boldsymbol{\theta}) \cdot \frac{\partial \boldsymbol{S}(\boldsymbol{f}^*(\boldsymbol{\theta}), \boldsymbol{\theta})}{\partial \boldsymbol{f}'} = \mathbf{0}, \tag{14}$$

and the constraints

$$\boldsymbol{S}(\boldsymbol{f}^*(\boldsymbol{\theta}), \boldsymbol{\theta}) = \mathbf{0} \tag{15}$$

for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}$. In (14), $\partial \boldsymbol{S}/\partial \boldsymbol{f}'$ refers to the Jacobian matrix with ij-element $\partial \boldsymbol{S}_i/\partial f_j$.[21] Equations (14) and (15) allow us to determine the optimal equilibrium $\boldsymbol{q}(\boldsymbol{f}^*(\boldsymbol{\theta}))$ for any given $\boldsymbol{\theta}$.

In the second step, we look for a policy rule $\boldsymbol{\psi}^*(\boldsymbol{\theta}) \in \boldsymbol{\Psi}$ that satisfies

$$\boldsymbol{P}(\boldsymbol{f}^*(\boldsymbol{\theta}), \boldsymbol{\psi}^*(\boldsymbol{\theta})) = \mathbf{0}. \tag{16}$$

If such a policy exists in $\boldsymbol{\Psi}$ (so that it results in a unique equilibrium), then it *implements* the optimal equilibrium parameterized by $\boldsymbol{f}^*(\boldsymbol{\theta})$. As made clear in the following lemma, such a policy is the policymaker's best response to the vector of structural parameters $\boldsymbol{\theta}$. In particular, given an *equilibrium* vector $\boldsymbol{\theta}^*$, the policy rule $\boldsymbol{\psi}^*(\boldsymbol{\theta}^*)$ solves (8), and hence is part of an NE.

LEMMA 1. *Suppose that* $\boldsymbol{f}^*(\boldsymbol{\theta})$ *minimizes* (12) *subject to* (10) *for any given* $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, *and that there exists* $\boldsymbol{\psi}^*(\boldsymbol{\theta}) \in \boldsymbol{\Psi}$ *that solves* (16) *for all* $\boldsymbol{\theta} \in \boldsymbol{\Theta}$. *Then* $\boldsymbol{\psi}^*(\boldsymbol{\theta}) \in \arg\min_{\boldsymbol{\psi} \in \boldsymbol{\Psi}} L(\boldsymbol{\psi}, \boldsymbol{\theta})$.

Proof. See Appendix A.1.

Although policies satisfying (16) are necessarily in the class $\tilde{\boldsymbol{\Psi}}$ of policy rules—as $\boldsymbol{f}$ is a parameterization of the subspace of possible processes resulting from policies in $\tilde{\boldsymbol{\Psi}}$—they need not be in the set $\boldsymbol{\Psi}$ of policies that results in a unique bounded equilibrium. Therefore, we shall need to verify that the obtained policy rules are indeed in $\boldsymbol{\Psi}$.[22]

To characterize the equilibrium structural parameters, we consider $\boldsymbol{\theta}^*$ that solves (9) or, equivalently,

$$\max_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} \hat{L}(\boldsymbol{f}(\boldsymbol{\psi}^*, \boldsymbol{\theta}), \boldsymbol{\theta}) \tag{17}$$

for a given policy rule $\boldsymbol{\psi}^* \in \boldsymbol{\Psi}$. Let us form the Lagrangian for Nature,

$$\mathcal{L}^N\big(\boldsymbol{\theta}, \boldsymbol{\mu_1}, \boldsymbol{\mu_2}; \boldsymbol{\psi}^*\big) = \hat{L}(\boldsymbol{f}(\boldsymbol{\psi}^*, \boldsymbol{\theta}), \boldsymbol{\theta}) - \boldsymbol{\mu_1} \cdot (\boldsymbol{\theta} - \bar{\boldsymbol{\theta}}) + \boldsymbol{\mu_2} \cdot (\boldsymbol{\theta} - \underline{\boldsymbol{\theta}}), \tag{18}$$

where $\boldsymbol{\mu_1}, \boldsymbol{\mu_2}$ are row vectors of Lagrange multipliers and $\underline{\boldsymbol{\theta}}, \bar{\boldsymbol{\theta}}$ are some finite vectors satisfying $[\underline{\boldsymbol{\theta}}, \bar{\boldsymbol{\theta}}] = \boldsymbol{\Theta}$. From Kuhn–Tucker's theorem, we know that necessary conditions for $\boldsymbol{\theta}^* \in \boldsymbol{\Theta}$ to solve this problem are given by the first-order conditions

$$\frac{d\hat{L}(\boldsymbol{f}(\boldsymbol{\psi}^*, \boldsymbol{\theta}^*), \boldsymbol{\theta}^*)}{d\boldsymbol{\theta}'} = \boldsymbol{\mu}_1^* - \boldsymbol{\mu}_2^*, \tag{19}$$

the complementary slackness conditions

$$\boldsymbol{\mu}_1^* \cdot (\boldsymbol{\theta}^* - \bar{\boldsymbol{\theta}}) = 0, \qquad \boldsymbol{\mu}_2^* \cdot (\boldsymbol{\theta}^* - \underline{\boldsymbol{\theta}}) = 0, \tag{20}$$

and the requirement that all elements of $\boldsymbol{\mu}_1^*, \boldsymbol{\mu}_2^*$ be nonnegative. In general, it is difficult to compute the left-hand side of (19) directly because it involves differentiation of the vector $\boldsymbol{f}$ with respect to $\boldsymbol{\theta}$, and it requires knowledge of the optimal policy rule $\boldsymbol{\psi}^*$. Thus, it will be convenient to rewrite (19) using the following lemma.

LEMMA 2. *If $\boldsymbol{\psi}^*(\boldsymbol{\theta})$ is a best response to $\boldsymbol{\theta}$, then*

$$\frac{d\hat{L}(\boldsymbol{f}(\boldsymbol{\psi}^*(\boldsymbol{\theta}), \boldsymbol{\theta}), \boldsymbol{\theta})}{d\boldsymbol{\theta}'} = \frac{\partial\mathcal{L}^P(\boldsymbol{f}^*(\boldsymbol{\theta}), \boldsymbol{\phi}^*(\boldsymbol{\theta}); \boldsymbol{\theta})}{\partial\boldsymbol{\theta}'}, \tag{21}$$

where $\mathcal{L}^P(\boldsymbol{f}, \boldsymbol{\phi}; \boldsymbol{\theta})$ is the policymaker's Lagrangian (13).

Proof. See Appendix A.2.

This lemma implies that the derivative of the loss function at the NE $(\boldsymbol{\psi}^*, \boldsymbol{\theta}^*)$ can be computed by *partially* differentiating the *policymaker's* Lagrangian (13) with respect to $\boldsymbol{\theta}$, and setting $\boldsymbol{\theta}, \boldsymbol{f}, \boldsymbol{\phi}$ at their equilibrium values $\boldsymbol{\theta}^*$, $\boldsymbol{f}^* \equiv \boldsymbol{f}^*(\boldsymbol{\theta}^*)$, $\boldsymbol{\phi}^* \equiv \boldsymbol{\phi}^*(\boldsymbol{\theta}^*)$. Note that we do not need to differentiate $\boldsymbol{f}$ with respect to $\boldsymbol{\theta}$ any more. Thus, we can write (19) as

$$\boldsymbol{Z}(\boldsymbol{\theta}^*) = \boldsymbol{\mu}_1^* - \boldsymbol{\mu}_2^*, \tag{22}$$

where we define

$$\boldsymbol{Z}(\boldsymbol{\theta}) \equiv \frac{\partial\mathcal{L}^P(\boldsymbol{f}^*(\boldsymbol{\theta}), \boldsymbol{\phi}^*(\boldsymbol{\theta}); \boldsymbol{\theta})}{\partial\boldsymbol{\theta}'}. \tag{23}$$

Then, the Kuhn–Tucker conditions can be written entirely in terms of $\boldsymbol{\theta}^*$, $\boldsymbol{\mu}_1^*$, and $\boldsymbol{\mu}_2^*$, which is particularly useful because they no longer require knowledge of the optimal policy rule $\boldsymbol{\psi}^*$.

Because (17) is not a concave problem, in general, the first-order conditions (22) and the complementary slackness conditions (20) are *necessary but not sufficient* to guarantee that the resulting parameter vector $\boldsymbol{\theta}^*$ maximizes the loss criterion. These conditions are useful, however, in restricting the set of possible solution *candidates*. They allow us to determine a *local* NE $(\psi^*, \boldsymbol{\theta}^*)$, that is, a situation in which each player's strategy is at least locally a best response to the other player's strategy. The following lemma states formally that in a local NE, Nature chooses the highest possible value of a parameter when the loss is increasing in that parameter, whereas it chooses the lowest possible parameter value when the loss is decreasing.

LEMMA 3. *Let* $\theta_i^* \in [\underline{\theta}_i, \bar{\theta}_i]$ *be the* ith *element of* $\boldsymbol{\theta}^*$, *and let* $Z_i^* \equiv \partial \mathcal{L}^P(\boldsymbol{f}^*, \boldsymbol{\phi}^*; \boldsymbol{\theta}^*)/\partial\theta_i$ *be the corresponding element of* $\boldsymbol{Z}(\boldsymbol{\theta}^*)$, *for* $i = 1, \ldots, m$. *If* $\boldsymbol{\theta}^*$ *is part of a local NE* $(\psi^*, \boldsymbol{\theta}^*)$, *then*

$$\theta_i^* = \begin{cases} \underline{\theta}_i, & \text{if} \quad Z_i^* < 0 \\ \bar{\theta}_i, & \text{if} \quad Z_i^* > 0. \end{cases}$$

If $Z_i^* = 0$, *then* θ_i^* *can be any value in* $[\underline{\theta}_i, \bar{\theta}_i]$ *that is consistent with* $Z_i^* = 0$.

Proof. See Appendix A.3.

To verify that $(\psi^*, \boldsymbol{\theta}^*)$ is not only a *local* but also a *global* NE, we need to check that the solution candidate $\boldsymbol{\theta}^*$ is indeed Nature's best response to the policymaker's optimal policy ψ^* on the whole constraint set $\boldsymbol{\Theta}$. This can be done by verifying numerically that there is no vector $\boldsymbol{\theta}^\dagger \in \boldsymbol{\Theta}$ such that

$$L(\psi^*, \boldsymbol{\theta}^\dagger) > L(\psi^*, \boldsymbol{\theta}^*), \tag{24}$$

given the policy rule ψ^*.

In summary, our solution strategy involves the four following steps.

1. Optimal equilibrium for given $\boldsymbol{\theta}$. We determine the parameterization $\boldsymbol{f}^*(\boldsymbol{\theta})$ of the equilibrium process that solves the policymaker's problem, minimizing the loss (12) subject to (10) for any given $\boldsymbol{\theta} \in \boldsymbol{\Theta}$.
2. Candidate minmax equilibrium. We construct the vector $\boldsymbol{Z}(\boldsymbol{\theta})$ obtained by partially differentiating the policymaker's Lagrangian (13) with respect to $\boldsymbol{\theta}$, and use Lemma 3 to determine a candidate worst-case parameter vector $\boldsymbol{\theta}^*$. Using the results of step 1, we determine the vector $\boldsymbol{f}^*(\boldsymbol{\theta}^*)$ parameterizing the *candidate* minmax equilibrium $\boldsymbol{q}(\boldsymbol{f}^*(\boldsymbol{\theta}^*))$.
3. Robust optimal policy rule. We look for a policy rule ψ^* that implements the candidate minmax equilibrium, i.e., that solves $\boldsymbol{P}(\boldsymbol{f}^*(\boldsymbol{\theta}^*), \psi^*) = \mathbf{0}$. We verify that $\psi^* \in \boldsymbol{\Psi}$, i.e., that the policy rule results in a *unique* bounded equilibrium process $\boldsymbol{q}(\psi^*, \boldsymbol{\theta})$ for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}$.
4. Existence of global NE. We verify that $(\psi^*, \boldsymbol{\theta}^*)$ is a *global* NE by checking that the candidate worst-case parameter vector $\boldsymbol{\theta}^*$ maximizes the loss $L(\psi^*, \boldsymbol{\theta})$ on the whole constraint set $\boldsymbol{\Theta}$, i.e., that there is no vector $\boldsymbol{\theta}^\dagger \in \boldsymbol{\Theta}$ satisfying (24), given the policy rule ψ^*.

Although this final step requires calculation of the loss at all points on a grid intended to cover the entire constraint set $\boldsymbol{\Theta}$, we note that this is simpler, in practice, than a brute-force evaluation of the objective (7) at all points on a grid covering $\boldsymbol{\Psi}$ would have been. First, in our applications, $\boldsymbol{\Theta}$ is a low-dimensional set, whereas we may wish to allow for complex families of possible policy rules. Second, it is not necessary to solve a maximization problem at each grid point in order to evaluate $L(\boldsymbol{\psi}^*, \boldsymbol{\theta})$ for the candidate Nash equilibrium policy $\boldsymbol{\psi}^*$. Finally, it is not necessary to consider an extremely fine grid in order to obtain an accurate approximation to the robust optimal policy rule. This is because the candidate policy $\boldsymbol{\psi}^*$ has already been computed in step 3; the grid search is merely a check that the conjectured NE involves globally, and not just locally, optimal behavior on the part of Nature. For this it suffices that all regions of the constraint set $\boldsymbol{\Theta}$ be given at least minimal attention. If one finds no evidence of other choices $\boldsymbol{\theta}^\dagger$ that are nearly as good as $\boldsymbol{\theta}^*$ (except other choices near $\boldsymbol{\theta}^*$ itself), there is no practical need for a fine grid search.

We have argued above that, for given $\boldsymbol{\theta}^*$, steps 1 and 3 yield a policy rule $\boldsymbol{\psi}^*$ that solves (8). We have also shown that, for given policy rule $\boldsymbol{\psi}^*$, step 2 yields a parameter vector $\boldsymbol{\theta}^*$ that solves (9), provided that step 4 is verified. Hence, a profile $(\boldsymbol{\psi}^*, \boldsymbol{\theta}^*)$ that is consistent with steps 1 to 4 is an NE of the game Γ, and Proposition 1 guarantees that $\boldsymbol{\psi}^*$ is the desired robust optimal policy rule. However, if we find a $\boldsymbol{\theta}^\dagger \in \boldsymbol{\Theta}$ satisfying (24), then $\boldsymbol{\theta}^*$ cannot be an equilibrium vector of structural parameters, and $(\boldsymbol{\psi}^*, \boldsymbol{\theta}^*)$ is not a global NE, so that $\boldsymbol{\psi}^*$ may or may not be the robust optimal policy rule. Note that there need not exist any NE, even though a robust optimal policy rule should still exist. However, in applications, a global NE will often exist when parameters take reasonable values, as we show below.

4. ROBUST OPTIMAL TAYLOR RULES

In this section, we use the method presented earlier to characterize robust optimal Taylor rules in the framework of Section 2. Formally, we restrict $\tilde{\boldsymbol{\Psi}}$ to the class of policy rules $\boldsymbol{\psi} = [\psi_\pi, \psi_x]'$ satisfying

$$i_t = \psi_\pi \pi_t + \psi_x x_t \tag{25}$$

at all dates $t \geq 0$.[23] Policies of this form have received considerable attention in recent research [see, e.g., contributions collected by Taylor (1999)], especially after being proposed by Taylor (1993).[24] They are called noninertial policies because they involve no response to lagged variables. We seek to determine the optimal coefficients ψ_π and ψ_x in the model of Section 2, assuming that the two critical structural parameters σ and κ that specify the slope of the IS and the aggregate supply equations are known only to be in given intervals $[\underline{\sigma}, \bar{\sigma}]$ and $[\underline{\kappa}, \bar{\kappa}]$ respectively, where $0 < \underline{\sigma} < \bar{\sigma} < \infty$, and $0 < \underline{\kappa} < \bar{\kappa} < \infty$. To keep the analysis as simple as possible, we abstract from uncertainty about the intercept of these curves so that the steady-state level of endogenous variables is assumed to be known, and the policy rules specify percent deviations of the interest rate from the known steady state. We choose not to consider uncertainty about the time discount factor β because there is substantial theoretical and empirical evidence that it corresponds to a number slightly below 1 (say, 0.99). Furthermore, for simplicity and clarification of the mechanisms at hand, we suppose that the weights λ_x and λ_i that characterize the policymaker's preferences are known to the policymaker.

Apart from its popularity, this simple class of policy rules is of interest because it allows a simple analytical characterization of robust optimal policy.[25] However, as explained by Woodford (1999c), policymakers who choose optimal actions by disregarding their past actions and past states of the economy, do not achieve the best equilibrium when the private sector is forward-looking. The characterization of robust optimal rules of a more general form—in particular, rules that would implement the best equilibrium if the parameters were known with certainty—is taken up by Giannoni (2001).[26]

Following our solution strategy, we determine first the equilibrium processes for the endogenous variables (inflation, output, and the interest rate) that achieve the lowest value of the loss criterion (3) for a given parameter vector $\boldsymbol{\theta} = [\sigma, \kappa]'$. Second, we characterize the minmax equilibrium process by determining the structural parameters that obtain in the NE. Finally, we look for a policy rule that implements the minmax equilibrium.

4.1. Optimal Equilibrium Process for Given Parameters

To characterize the class of possible optimal plans corresponding to policy rules $\psi \in \tilde{\boldsymbol{\Psi}}$, we use (25) to substitute for the interest rate in the structural equations (1) and (2), and rewrite the resulting difference equations in matrix form as follows:

$$E_t z_{t+1} = \boldsymbol{A} z_t + \boldsymbol{a} r_t^n, \tag{26}$$

where $z_t \equiv [\pi_t, x_t]'$. Since both π_t and x_t are nonpredetermined endogenous variables at date t, and the process $\{r_t^n\}$ is assumed to be bounded, the dynamic system (26) admits a unique bounded solution if and only if both eigenvalues of $\boldsymbol{A}$ lie outside the unit circle, as explained by Blanchard and Kahn (1980). If we restrict our attention to the usual case in which ψ_π and ψ_x are nonnegative, then it is shown in Appendix B that the policy rule results in a determinate equilibrium if and only if

$$\psi_\pi + \frac{1-\beta}{\kappa}\psi_x > 1. \tag{27}$$

When the structural parameters are unknown and $\kappa \in [\underline{\kappa}, \bar{\kappa}]$, the parameter κ is replaced by $\bar{\kappa}$ in (27).[27]

For simplicity, we consider the case in which the exogenous shocks r_t^n follow an autoregressive process

$$r_t^n = \rho r_{t-1}^n + \varepsilon_t, \tag{28}$$

where $0 \leq \rho < 1$ and $\{\varepsilon_t\}$ is a martingale difference sequence of perturbations. Because optimal policy rules necessarily result in a unique bounded equilibrium (see definitions in Section 3), (26) can be solved forward. Using (28), one realizes that possible optimal plans corresponding to $\tilde{\boldsymbol{\Psi}}$ are of the form

$$\pi_t = f_\pi r_t^n, \qquad x_t = f_x r_t^n, \qquad i_t = f_i r_t^n, \tag{29}$$

where $\boldsymbol{f} = [f_\pi, f_x, f_i]'$ is the vector of response coefficients that parameterizes the equilibrium process. The feasibility restrictions on the response coefficients corresponding to (10), obtained by substituting (29) into the structural equations (1) and (2), are

$$(1-\rho) f_x + \sigma^{-1}(f_i - \rho f_\pi - 1) = 0, \tag{30}$$

$$(1-\beta\rho) f_\pi - \kappa f_x = 0. \tag{31}$$

To solve the policymaker's problem, we choose the plan of the form (29) and consistent with (30)–(31) to minimize the loss criterion $E[L_0]$. Because we consider noninertial plans, we may as well minimize

$$\hat{L}(\boldsymbol{f}, \boldsymbol{\theta}) = f_\pi^2 + \lambda_x f_x^2 + \lambda_i f_i^2$$

subject to the constraints (30)–(31). The policymaker's Lagrangian is

$$\mathcal{L}^P(\boldsymbol{f}, \phi; \boldsymbol{\theta}) = \left(f_\pi^2 + \lambda_x f_x^2 + \lambda_i f_i^2\right) + \phi_1\left[(1-\rho) f_x + \sigma^{-1}(f_i - \rho f_\pi - 1)\right] + \phi_2[(1-\beta\rho) f_\pi - \kappa f_x].$$

The response coefficients parameterizing the optimal feasible equilibrium, for given parameter vector $\boldsymbol{\theta}$, are given by

$$f_\pi^*(\boldsymbol{\theta}) = \lambda_i[\sigma(1-\rho)(1-\beta\rho) - \rho\kappa]\frac{\kappa}{h}, \tag{32}$$

$$f_x^*(\boldsymbol{\theta}) = \lambda_i[\sigma(1-\rho)(1-\beta\rho) - \rho\kappa]\frac{(1-\beta\rho)}{h}, \tag{33}$$

$$f_i^*(\boldsymbol{\theta}) = \frac{\lambda_x(1-\beta\rho)^2 + \kappa^2}{h}, \tag{34}$$

where

$$h \equiv \lambda_i[\sigma(1-\rho)(1-\beta\rho) - \rho\kappa]^2 + \lambda_x(1-\beta\rho)^2 + \kappa^2.$$

It is clear from (34) that $0 < f_i^*(\boldsymbol{\theta}) \leq 1$ for any vector $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, and any positive weights λ_i, λ_x. Thus, the optimal noninertial plan involves an adjustment of the nominal interest rate in the same direction as the perturbation to the natural interest rate, but in general by less than the natural rate. Equations (32) and (33) reveal that the response coefficients $f_\pi^*(\boldsymbol{\theta})$, $f_x^*(\boldsymbol{\theta})$ are positive if and only if

$$\frac{\sigma}{\kappa} > \frac{\rho}{(1-\rho\beta)(1-\rho)}, \tag{35}$$

that is, whenever the fluctuations in the natural rate are not too persistent (relative to the ratio σ/κ). Thus, when (35) holds, a positive shock to the natural rate stimulates both the output gap and inflation. In the special case in which the interest rate does not enter the loss function ($\lambda_i = 0$), or when the persistence of the perturbations is such that $\sigma(1-\rho)(1-\beta\rho) = \rho\kappa$, we obtain $f_\pi^*(\boldsymbol{\theta}) = f_x^*(\boldsymbol{\theta}) = 0$ and $f_i^*(\boldsymbol{\theta}) = 1$; the central bank optimally moves the interest rate by the same amount as the natural rate in order to stabilize the output gap and inflation completely. In contrast, when the disturbances to the natural rate are sufficiently persistent (ρ large enough but still smaller than 1) for the inequality (35) to be reversed, inflation and the output gap decrease in the face of an unexpected positive shock to the natural rate in the optimal noninertial plan $[f_\pi^*(\boldsymbol{\theta}),\, f_x^*(\boldsymbol{\theta}) < 0]$. Even if the nominal interest rate increases less than the natural rate, optimal monetary policy is restrictive in this case, because the real interest rate $(i_t - E_t\pi_{t+1})$ is higher than the natural rate of interest r_t^n.

Following Woodford (1999c), we calibrate the baseline model using the parameter values estimated by Rotemberg and Woodford (1997). The baseline calibration is reported in Table 1.[28] For these parameter values, (35) holds, and it continues to hold for any parameter values that are close to these, so that an increase in r_t^n raises both the output gap and inflation in the optimal noninertial plan.

4.2. Equilibrium Structural Parameters and Minmax Equilibrium

To characterize the minmax equilibrium associated with the class of noninertial Taylor rules, we need to determine the parameter vector $\boldsymbol{\theta}^* = [\sigma^*, \kappa^*]'$ that maximizes the policymaker's loss on the given constraint set $\boldsymbol{\Theta}$, that is, when $\sigma^* \in [\underline{\sigma}, \bar{\sigma}]$ and $\kappa^* \in [\underline{\kappa}, \bar{\kappa}]$. As in step 2 of our solution procedure, we compute

$$Z_1^* \equiv \frac{\partial \mathcal{L}^P(\boldsymbol{f}^*(\boldsymbol{\theta}^*), \boldsymbol{\phi}^*(\boldsymbol{\theta}^*); \boldsymbol{\theta}^*)}{\partial \sigma} = -\phi_1^*(\boldsymbol{\theta}^*)\left[f_i^*(\boldsymbol{\theta}^*) - \rho f_\pi^*(\boldsymbol{\theta}^*) - 1\right](\sigma^*)^{-2} \quad \textbf{(36)}$$

$$Z_2^* \equiv \frac{\partial \mathcal{L}^P(\boldsymbol{f}^*(\boldsymbol{\theta}^*), \boldsymbol{\phi}^*(\boldsymbol{\theta}^*); \boldsymbol{\theta}^*)}{\partial \kappa} = -\phi_2^*(\boldsymbol{\theta}^*) f_x^*(\boldsymbol{\theta}^*). \quad \textbf{(37)}$$

From Lemma 2, we know that Z_1^* and Z_2^* correspond to the slopes of the loss function with respect to σ and κ respectively, evaluated at the candidate NE($\boldsymbol{\psi}^*, \boldsymbol{\theta}^*$). It follows from Lemma 3 that, at a local NE,

$$\sigma^* = \begin{cases} \underline{\sigma}, & \text{if} \quad Z_1^* < 0 \\ \bar{\sigma}, & \text{if} \quad Z_1^* > 0 \end{cases}, \qquad \kappa^* = \begin{cases} \underline{\kappa}, & \text{if} \quad Z_2^* < 0 \\ \bar{\kappa}, & \text{if} \quad Z_2^* > 0. \end{cases}$$

When $Z_1^* = 0$, σ^* can be any value in $[\underline{\sigma}, \bar{\sigma}]$ that is consistent with $Z_1^* = 0$. Similarly, when $Z_2^* = 0$, κ^* can be any value in $[\underline{\kappa}, \bar{\kappa}]$ that is consistent with $Z_2^* = 0$. Intuitively, this means that, at a local NE, Nature chooses a high value—in fact, the highest possible value—for σ^* or κ^* when the loss is increasing in the respective structural parameter (i.e., Z is positive), whereas it chooses a low value for σ^* or

κ^* when it is decreasing. Recall that because Nature's problem is nonconcave, this characterization is *not sufficient* to determine the parameter vector $\boldsymbol{\theta}^*$ that maximizes $\hat{L}(\boldsymbol{f}(\psi^*, \boldsymbol{\theta}), \boldsymbol{\theta})$ *globally*. However, it allows us to determine all possible *solution candidates*.

The candidate noninertial minmax equilibrium $\boldsymbol{q}^* \equiv \boldsymbol{q}^*(\boldsymbol{\theta}^*)$ is characterized in the following proposition.

PROPOSITION 2. *When* $\tilde{\boldsymbol{\Psi}}$ *is restricted to the class of Taylor rules* [*satisfying* (25)], *and the structural parameters* $\sigma \in [\underline{\sigma}, \bar{\sigma}]$ *and* $\kappa \in [\underline{\kappa}, \bar{\kappa}]$ *are uncertain, where* $\underline{\sigma}, \underline{\kappa} > 0$ *and* $\bar{\sigma}, \bar{\kappa} < \infty$, *then the structural parameters* σ^*, κ^* *that are part of a local NE and the candidate noninertial minmax equilibrium* $\boldsymbol{q}^* = \{\pi_t, x_t, i_t\}$ *are characterized by*

$$\sigma^* = \begin{cases} \underline{\sigma}, & \text{if} \quad \eta < \underline{\sigma}/\bar{\kappa} \\ \bar{\sigma}, & \text{if} \quad \bar{\sigma}/\underline{\kappa} < \eta \end{cases}, \qquad \kappa^* = \begin{cases} \bar{\kappa}, & \text{if} \quad \eta < \underline{\sigma}/\bar{\kappa} \\ \underline{\kappa}, & \text{if} \quad \bar{\sigma}/\underline{\kappa} < \eta \end{cases}$$

where

$$\eta \equiv \frac{\rho}{(1-\rho)(1-\beta\rho)}, \tag{38}$$

and

$$\pi_t = f_\pi^*(\boldsymbol{\theta}^*) r_t^n, \qquad x_t = f_x^*(\boldsymbol{\theta}^*) r_t^n, \qquad i_t = f_i^*(\boldsymbol{\theta}^*) r_t^n, \tag{39}$$

with equilibrium response coefficients

$$f_\pi^*(\boldsymbol{\theta}^*) = \lambda_i[\sigma^*(1-\rho)(1-\beta\rho) - \rho\kappa^*]\frac{\kappa^*}{h^*}, \tag{40}$$

$$f_x^*(\boldsymbol{\theta}^*) = \lambda_i[\sigma^*(1-\rho)(1-\beta\rho) - \rho\kappa^*]\frac{(1-\beta\rho)}{h^*}, \tag{41}$$

$$f_i^*(\boldsymbol{\theta}^*) = \frac{\lambda_x(1-\beta\rho)^2 + \kappa^{*2}}{h^*}, \tag{42}$$

where $h^* \equiv \lambda_i[\sigma^*(1-\rho)(1-\beta\rho) - \rho\kappa^*]^2 + \lambda_x(1-\beta\rho)^2 + \kappa^{*2}$.

When $\underline{\sigma}/\bar{\kappa} \leq \eta \leq \bar{\sigma}/\underline{\kappa}$, *an NE is obtained for any combination of structural parameters* σ^*, κ^* *satisfying* $\sigma^*/\kappa^* = \eta$. *In this case, the equilibrium response coefficients are*

$$f_\pi^*(\boldsymbol{\theta}^*) = f_x^*(\boldsymbol{\theta}^*) = 0 \quad \text{and} \quad f_i^*(\boldsymbol{\theta}^*) = 1.$$

Proof. See Appendix A.4.

When ρ is small enough (such that $\eta < \underline{\sigma}/\bar{\kappa}$), the worst situation for the policymaker is achieved when κ is made as large as possible and σ is made as small as possible. To understand this, recall that the output gap and inflation depend upon the interest rate and the natural rate only through the real-interest-rate differential $(i_t - E_t\pi_{t+1}) - r_t^n$. On one hand, a lower σ and a higher κ imply stronger effects of the perturbations to the natural rate on the output gap and inflation. On the other hand, they render monetary policy more effective because changes in i_t have a stronger effect on π_t and x_t. When $\eta < \underline{\sigma}/\bar{\kappa}$, the real interest rate moves less than the natural rate in the optimal noninertial plan, so that the first effect dominates. Thus, for a given real-interest-rate differential, a lower σ and a higher κ are responsible for larger fluctuations of inflation and the output gap, and make the policymaker worse off. These larger changes in inflation and output gap induce the central bank to move its interest rate closer to the natural rate. Since the policymaker also dislikes variability in the interest rate, though, he does not change the interest rate by enough to cancel the effect of a perturbation to the natural rate. In contrast, when the perturbations are so persistent that $\eta > \bar{\sigma}/\underline{\kappa}$, the worst situation is obtained when $\kappa^* = \underline{\kappa}$, $\sigma^* = \bar{\sigma}$. Finally, when the persistence of the perturbations is such that $\underline{\sigma}/\bar{\kappa} \le \eta \le \bar{\sigma}/\underline{\kappa}$, the response of the interest rate in the minmax equilibrium is given by $f_i^*(\boldsymbol{\theta}^*) = 1$, which completely neutralizes the shocks to the natural rate of interest. As a result, inflation and the output gap remain at their steady-state level whether the economy is affected by shocks or not.

In the baseline parameterization, $\eta = 0.824$. As long as the upper bound for κ is less than 0.191 (i.e., eight times the baseline value), and the lower bound for σ is above $0.824\bar{\kappa}$, the condition $\eta < \underline{\sigma}/\bar{\kappa}$ is satisfied. Thus, if the baseline parameterization is an appropriate approximation of the true model of the economy, and the uncertainty about the structural parameters is small enough, the worst-case situation is obtained when $\kappa^* = \bar{\kappa}$, $\sigma^* = \underline{\sigma}$.

4.3. Determining Robust Optimal Taylor Rules

As in step 3 of our solution strategy, we now determine a candidate robust optimal Taylor rule $\boldsymbol{\psi}^*$ that *implements* the noninertial minmax equilibrium characterized in Proposition 2. It is convenient, for technical reasons that will become clear later, to rewrite (25) as

$$i_t = \frac{1}{\hat{\psi}_\pi}\pi_t + \frac{\hat{\psi}_x}{\hat{\psi}_\pi}x_t, \tag{43}$$

where $\hat{\psi}_\pi = 1/\psi_\pi$ and $\hat{\psi}_x = \psi_x/\psi_\pi$, and to determine a robust optimal policy rule $\hat{\boldsymbol{\psi}}^* = [\hat{\psi}_\pi^*, \hat{\psi}_x^*]'$ instead of $\boldsymbol{\psi}^*$. In (43), we assume that $\hat{\psi}_\pi$ and $\hat{\psi}_x$ are finite real numbers so that $\hat{\boldsymbol{\psi}}$ can be used to characterize any rule $\boldsymbol{\psi} \in \tilde{\boldsymbol{\Psi}}$ except those in which $\psi_\pi = 0$.[29] Using the solution (39) to eliminate the endogenous variables in (43), we obtain

$$\hat{\psi}_\pi f_i^*(\boldsymbol{\theta}^*) r_t^n = \left[f_\pi^*(\boldsymbol{\theta}^*) + \hat{\psi}_x f_x^*(\boldsymbol{\theta}^*)\right] r_t^n.$$

Any policy rule $\hat{\boldsymbol{\psi}}^*$ resulting in a unique bounded equilibrium and satisfying

$$\hat{\psi}_\pi f_i^*(\boldsymbol{\theta}^*) = f_\pi^*(\boldsymbol{\theta}^*) + \hat{\psi}_x^* f_x^*(\boldsymbol{\theta}^*) \tag{44}$$

implements the candidate noninertial minmax equilibrium for all exogenous paths of the natural rate of interest.[30] Substituting $f_\pi^*(\boldsymbol{\theta}^*)$, $f_x^*(\boldsymbol{\theta}^*)$, $f_i^*(\boldsymbol{\theta}^*)$ using (40)–(42) and solving for $\hat{\psi}_\pi^*$ yields

$$\hat{\psi}_\pi^* = \frac{\kappa^* + (1-\beta\rho)\hat{\psi}_x^*}{(\kappa^*)^2 + \lambda_x(1-\beta\rho)^2}\lambda_i[\sigma^*(1-\rho)(1-\beta\rho) - \rho\kappa^*], \tag{45}$$

where σ^* and κ^* are determined in Proposition 2.

Whenever η is sufficiently small (so that $\eta < \underline{\sigma}/\bar{\kappa}$) or large (so that $\eta > \bar{\sigma}/\underline{\kappa}$), the result from (45) is that the coefficients of the robust optimal Taylor rule $\psi_\pi^* = 1/\hat{\psi}_\pi^*$ and $\psi_x^* = \hat{\psi}_x^*/\hat{\psi}_\pi^*$ satisfy

$$\psi_\pi^* = \frac{(\kappa^*)^2 + \lambda_x(1-\beta\rho)^2}{\lambda_i\kappa^*[\sigma^*(1-\rho)(1-\beta\rho) - \rho\kappa^*]} - \psi_x^*\frac{(1-\beta\rho)}{\kappa^*}. \tag{46}$$

In fact, any vector $\boldsymbol{\psi}^* = [\psi_\pi^*, \psi_x^*]'$ satisfying (46) implements the candidate noninertial minmax equilibrium, provided that it results in a unique bounded equilibrium. This is the case if $\psi_\pi^* + \psi_x^*(1-\beta)/\bar{\kappa} > 1$, when we restrict ourself to policies with nonnegative coefficients. Note that, by setting $\psi_x^* = 0$, (46) determines the policy rule that implements the candidate noninertial minmax equilibrium without any knowledge of the output gap.

When the uncertainty and the persistence in the perturbations to the natural rate of interest are such that $\underline{\sigma}/\bar{\kappa} \le \eta \le \bar{\sigma}/\underline{\kappa}$, we have $\hat{\psi}_\pi^* = 0$ [recall that $\sigma^*/\kappa^* = \eta$, and that the equilibrium response coefficients are $f_\pi^*(\boldsymbol{\theta}^*) = f_x^*(\boldsymbol{\theta}^*) = 0$, and $f_i^*(\boldsymbol{\theta}^*) = 1$ in this case]. The optimal interest rate responds as much as possible to inflation (and output gap deviations if $\hat{\psi}_x^* \neq 0$), so that equilibrium inflation and output gap remain at their steady state. We shall let $\psi_\pi^* \to +\infty$ in this case.[31]

To verify that the rule $\hat{\boldsymbol{\psi}}^*$ and the equilibrium parameter vector $\boldsymbol{\theta}^*$ determine a *global* NE, and hence that the corresponding rule $\boldsymbol{\psi}^*$ is a robust optimal Taylor rule, we need to verify, as in step 4 of our solution method, that the structural parameters σ^*, κ^* are Nature's best responses to $\hat{\boldsymbol{\psi}}^*$ on the whole constraint set $\boldsymbol{\Theta}$. (Recall that Lemma 3 gives necessary but not sufficient conditions for $\boldsymbol{\theta}^*$ to maximize Nature's objective.) In the numerical example considered here, we assume parameter uncertainty corresponding to the 95% confidence intervals for σ and κ. Because $\eta = 0.824 < \underline{\sigma}/\bar{\kappa}$, Proposition 2 guarantees that, in the local NE, Nature chooses $\sigma^* = \underline{\sigma}$ and $\kappa^* = \bar{\kappa}$. Figure 1 is a contour plot of the loss measure $E[L_0]$ as a function of the structural parameters σ and κ in the specified set $\boldsymbol{\Theta}$, when the policy rule is $\hat{\boldsymbol{\psi}}^*$, the policymaker's best response to $\boldsymbol{\theta}^* = [\underline{\sigma}, \bar{\kappa}]'$, setting $\psi_x = 0.5$.[32] The figure reveals that $\boldsymbol{\theta}^*$ (i.e., the lower right corner) is part of not only a local but also a global NE because it maximizes the loss on the whole set $\boldsymbol{\Theta}$.

We now compare the noninertial minmax equilibrium and the robust optimal Taylor rule with their counterpart in the absence of parameter uncertainty.

4.4. Comparing Equilibria and Taylor Rules in Certainty and Uncertainty Cases

When the structural parameters are known by the policymaker, the optimal equilibrium response coefficient of the interest rate to the natural rate of interest, $f_i^0 \equiv f_i^*(\boldsymbol{\theta}^0)$, satisfies (34) with the vector of structural parameters equal to the true

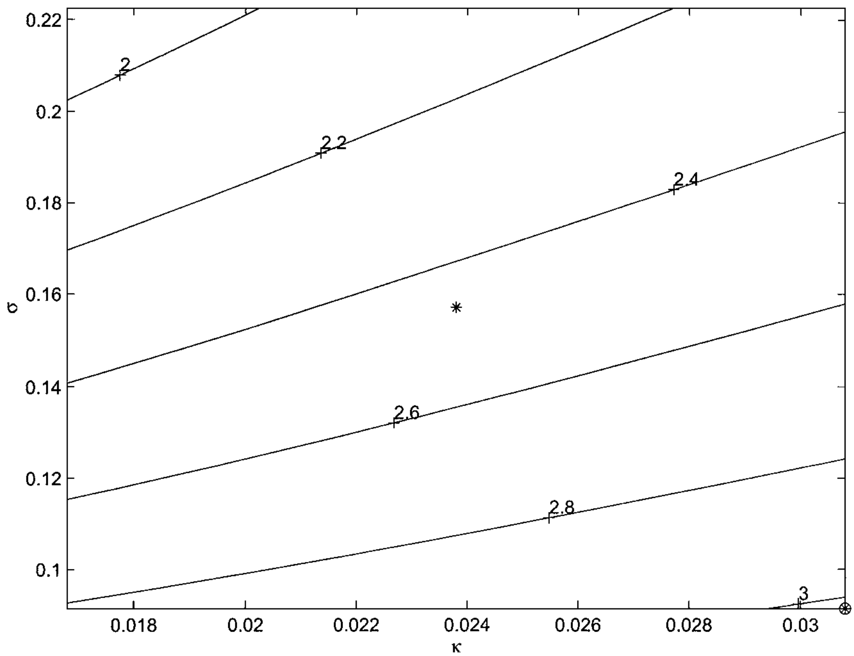

FIGURE 1. $E[L_0]$ with robust optimal Taylor rule.

(known) value $\boldsymbol{\theta}^0$. A comparison of the optimal equilibrium response coefficients reveals that the policymaker lets the interest rate respond *more* strongly to exogenous perturbations in the minmax equilibrium than in the certainty case, regardless of the degree of persistence in the perturbations. The following proposition states this result formally.

PROPOSITION 3. *Let* $f_i^0 \equiv f_i^*(\boldsymbol{\theta}^0)$ [*defined in* (34)] *be the optimal response coefficient of the interest rate in the optimal noninertial plan, when the parameters* $\sigma_0 \in (\underline{\sigma}, \bar{\sigma})$ *and* $\kappa_0 \in (\underline{\kappa}, \bar{\kappa})$ *are known with certainty. Let* $f_i^* \equiv f_i^*(\boldsymbol{\theta}^*)$ (*defined in Proposition* 2) *be the corresponding response coefficient in the minmax equilibrium when the parameters* $\sigma \in [\underline{\sigma}, \bar{\sigma}]$ *and* $\kappa \in [\underline{\kappa}, \bar{\kappa}]$ *are uncertain. Let* $\underline{\sigma}, \underline{\kappa}, \lambda_x, \lambda_i > 0$, *and* $\bar{\sigma}, \bar{\kappa} < \infty$. *If* $\sigma_0/\kappa_0 \neq \eta$, *then*

$$0 < f_i^0 < f_i^* \leq 1.$$

In the special case in which $\sigma_0/\kappa_0 = \eta$, *then* $f_i^0 = f_i^* = 1$.

Proof. See Appendix A.5.

The result of Proposition 3 is illustrated in Figure 2 for the baseline model (see Table 1). Figure 2 displays impulse responses of all three endogenous variables (interest rate, inflation, and output gap) to an unexpected temporary increase in the

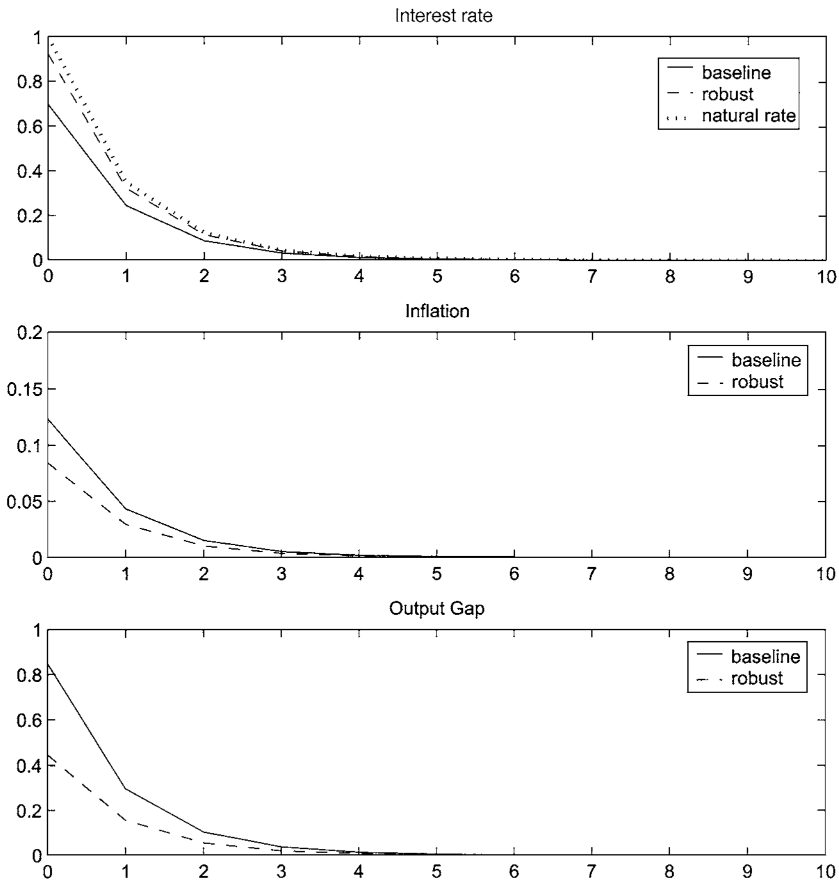

FIGURE 2. Optimal Taylor rules.

natural rate of interest. In the upper panel, the dotted line represents the exogenous path of the natural rate. The solid line represents the impulse response of the interest rate in the optimal noninertial plan, in the absence of uncertainty. The dashed line plots the corresponding impulse response in the minmax equilibrium. It appears clearly that the interest rate reacts more strongly in the presence of uncertainty than when parameters are known.

When the perturbations to the natural rate of interest are sufficiently transitory (so that $\eta < \underline{\sigma}/\bar{\kappa}$) as is the case in Figure 2, the worst case arises when σ is as low as possible and κ is as high as possible, which implies that positive shocks to the natural rate have a larger stimulating effect on inflation and the output gap than is the case in the absence of parameter uncertainty.[33] Thus, the policymaker who seeks to dampen fluctuations in inflation and output gap increases the interest rate by more in the minmax equilibrium than in the certainty case, so that the interest rate moves closer to the natural rate in the minmax equilibrium. The remaining panels confirm that the stronger reaction of the interest rate dampens the effect of the shock upon inflation and the output gap in the presence of parameter uncertainty.

A comparison of optimal Taylor rules in the presence and absence of parameter uncertainty yields a similar result summarized in the following proposition. Note that, in the certainty case, any optimal Taylor rule $\boldsymbol{\psi}^0 = [\psi_\pi^0, \psi_x^0]'$ satisfies an equation of the form (46), but in which the vector of structural parameters $\boldsymbol{\theta}^*$ is replaced with the known vector $\boldsymbol{\theta}^0 = [\sigma_0, \kappa_0]'$.

PROPOSITION 4. *Let $\boldsymbol{\psi}^0 = [\psi_\pi^0, \psi_x]' \in \boldsymbol{\Psi}$ be a Taylor rule that implements the optimal noninertial plan, given some coefficient ψ_x, when the parameters $\sigma_0 \in (\underline{\sigma}, \bar{\sigma})$ and $\kappa_0 \in (\underline{\kappa}, \bar{\kappa})$ are known with certainty. Let $\boldsymbol{\psi}^* = [\psi_\pi^*, \psi_x]' \in \boldsymbol{\Psi}$ be the robust optimal Taylor rule, given the same ψ_x, when the parameters $\sigma \in [\underline{\sigma}, \bar{\sigma}]$ and $\kappa \in [\underline{\kappa}, \bar{\kappa}]$ are uncertain. Suppose $\eta \neq \sigma_0/\kappa_0$. If*

$$\psi_\pi^0, \psi_\pi^*, \psi_x \geq 0$$

and

$$\psi_x > -\frac{\kappa_0\bar{\kappa}(\sigma_0\bar{\kappa} - \kappa_0\underline{\sigma}) + \lambda_x(1-\beta\rho)^2\left[\eta\left(\bar{\kappa}^2 - \kappa_0^2\right) + \kappa_0\sigma_0 - \bar{\kappa}\underline{\sigma}\right]}{\lambda_i(1-\rho)(1-\beta\rho)^2(\sigma_0 - \eta\kappa_0)(\underline{\sigma} - \eta\bar{\kappa})(\bar{\kappa} - \kappa_0)}, \tag{47}$$

then

$$\psi_\pi^* > \psi_\pi^0.$$

Proof. See Appendix A.6.

Proposition 4 states that, for given (and sufficiently large) response to the output gap ψ_x, the policymaker should respond more strongly to inflation deviations in the presence of parameter uncertainty than when parameters are known. Such a stronger reaction to inflation deviations is exactly what is required to make the interest rate move more closely to the natural rate of interest in the presence of uncertainty, and to prevent shocks from having too large an effect on inflation and the output gap in the worst case.[34]

To illustrate this result, we represent in Figure 3 policies that implement the optimal noninertial plan for the baseline parameterization of the model. The solid line represents the *optimal* Taylor rules in the baseline case, that is, the combinations (ψ_π^0, ψ_x^0) satisfying an equation similar to (46), in which the parameter vector $\boldsymbol{\theta}^*$ is replaced with the baseline vector $\boldsymbol{\theta}^0$. The dashed-dotted line plots the corresponding *robust optimal* policies—the combinations (ψ_π^*, ψ_x^*) satisfying (46)—in the presence of the parameter uncertainty given in Table 1. The white region indicates the set of policy rules that result in a determinate equilibrium for *any* value of the parameters in the assumed region $\boldsymbol{\Theta}$ (see Appendix B). In contrast, the gray region indicates combinations (ψ_π, ψ_x) that result in indeterminacy of the equilibrium for at least one value of the parameters σ, κ in $\boldsymbol{\Theta}$. Thus, only optimal policies in the white region may satisfy step 3 of the solution method. The circled star indicates the coefficients of the rule proposed by Taylor (1993) as a good approximation of recent U.S. monetary policy. Figure 3 clearly shows that whenever monetary policy involves a response to the output gap that is strong

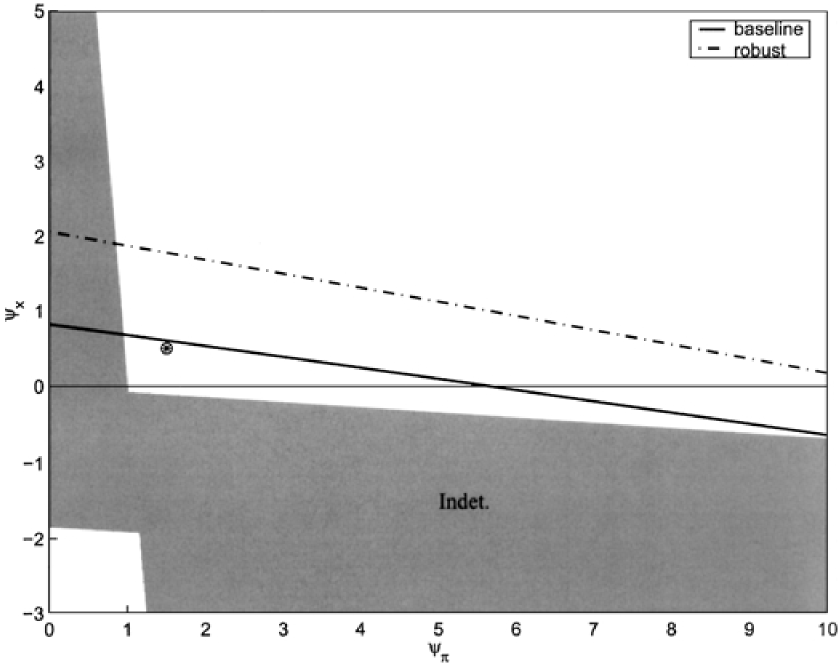

Figure 3. Optimal Taylor rules.

enough, the optimal response to inflation is larger in the presence of uncertainty than when the parameters are known, as predicted by Proposition 4. In fact, the line representing robust optimal policies is steeper and has a higher intercept than the corresponding line representing optimal policy rules in the absence of uncertainty. Condition (47) guarantees that ψ_x lies above the intersection point of the two lines.[35] We have focused on the effect of uncertainty on the response of the interest rate to inflation for a given response to the output gap. As should be clear from Figure 3, the presence of uncertainty calls also for a larger response to the output gap, for any given ψ_π.

Another aspect of the stronger reaction of the interest rate in the presence of model uncertainty is presented in Table 2. This table reports optimal Taylor rules (when ψ_x is set equal to 0.5), the policymaker's loss criterion, and the following measure of variability,

$$V[z] \equiv E\left\{E_0\left[(1-\beta)\sum_{t=0}^{\infty}\beta^t z_t^2\right]\right\},$$

for all three endogenous variables. The latter statistic determines the contribution of each endogenous variable to the loss measure $E[L_0]$. Indeed, $E[L_0]$ is just a weighted sum of $V[\pi]$, $V[x]$ and $V[i]$ with weights being those in the loss

TABLE 2. Optimal Taylor rules and statistics (ψ_x is arbitrarily set at 0.5)

	Policy		Statistics			
Case	ψ_π	ψ_x	$V[\pi]$	$V[x]$	$V[i]$	$E[L_0]$
$(\psi^0, \boldsymbol{\theta}^0)$	2.217	0.5	0.211	9.923	6.720	2.279
$(\psi^0, \boldsymbol{\theta}^*)$	2.217	0.5	0.414	11.640	9.809	3.295
$(\psi^*, \boldsymbol{\theta}^0)$	8.294	0.5	0.069	3.240	9.455	2.461
$(\psi^*, \boldsymbol{\theta}^*)$	8.294	0.5	0.098	2.767	11.782	3.017

function (3).[36] The lines of Table 2 corresponding to the baseline case are indicated by $(\psi^0, \boldsymbol{\theta}^0)$. In contrast, $(\psi^*, \boldsymbol{\theta}^0)$ denotes the case in which the central bank faces uncertainty and follows the robust optimal policy ψ^*, but the actual structural parameters are equal to their values in the baseline case, $\boldsymbol{\theta}^0$. Comparing these two lines again confirms that the central bank lets the interest rate move by more in the presence of uncertainty. The more aggressive monetary policy is then responsible for a decrease in the variability of inflation and the output gap, but an increase in the volatility of i. Overall, switching from the baseline policy rule to the robust optimal rule raises the loss $E[L_0]$ from 2.28 to 2.46 when the true parameters are the ones of the baseline model. However, if the unknown parameters are not at the baseline value, but reach a less favorable combination, such as the worst-case $\boldsymbol{\theta}^*$, the advantage of following the robust policy rule is clear: The maximum loss is reduced from 3.30 to 3.02.

5. CONCLUSION

This paper proposes a general method based on a property of zero-sum two-player games to derive *robust optimal monetary policy rules*—the best rules among those that yield an acceptable performance in a specified range of models—when the true model is unknown. Model uncertainty is viewed as uncertainty about the true structural parameters that numerically specify the model. The method is applied to characterize robust optimal rules in a standard forward-looking macroeconomic model that can be derived from first principles.

Whereas it is commonly believed among economists and central bankers that monetary policy should be less responsive when there is uncertainty about model parameters, we have shown that the opposite is likely to be true in the model considered when the two key structural parameters—the slopes of the intertemporal IS curve and the aggregate supply curve—are subject to uncertainty: The *robust optimal Taylor rule* requires the interest rate to respond *more strongly* in general to fluctuations in inflation or the output gap than is the case in the absence of uncertainty.[37] Yet, the policymaker is *cautious* in our framework. In fact, he is even *more cautious* than in Brainard's model because he cares very much about situations in which monetary policy would perform poorly. In contrast to Brainard's analysis, however, caution induces the policymaker to be more responsive.

The model has the property that the policymaker faces a trade-off between the stabilization of inflation and the output gap on one hand and the nominal interest rate on the other. In the presence of model uncertainty, the robust policymaker seeks to limit the welfare losses, especially in those bad outcomes in which exogenous perturbations (to the natural rate of interest) have a large effect on inflation and the output gap, i.e., when the aggregate supply curve is particularly steep and the intertemporal IS curve is particularly flat. Model uncertainty therefore affects the trade-off facing the policymaker by increasing the weight given to inflation and output gap stabilization relative to the weight given to interest rate stabilization. A more aggressive policy allows the central bank to stabilize inflation and the output gap around their target values more effectively and guarantees that welfare losses will be contained.

NOTES

1. This result holds in Brainard's model in particular when the exogenous disturbances and the parameters that relate the instrument of policy to the target variable are not too strongly correlated.

2. See, for example, Sack (1998), Wieland (1998), Clarida et al. (1999), Estrella and Mishkin (1999), Hall et al. (1999), Martin and Salmon (1999), and Rudebusch (2001).

3. A number of other recent papers have also looked for policy rules that work well across a range of models, though they do not try to actually find the optimal robust rule [see, for example, McCallum (1988, 1999), Taylor (1998), Christiano and Gust (1999), and Levin et al. (1999)].

4. Aoki (1998) derives the optimal time-consistent policy in a model very similar to ours when the structural parameters are known with certainty but when inflation and output are subject to measurement errors. He shows that measurement errors lead to less active monetary policy. Orphanides (1998) obtains a similar conclusion in a different model.

5. Following a Bayesian approach, Söderström (1999) shows that uncertainty about the persistence of inflation induces the policymaker to respond more aggressively to shocks, in the model due to Svensson (1997).

6. This model is similar to other small dynamic macroeconomic models that have been used in recent studies of monetary policy, such as those of Kerr and King (1996), Bernanke and Woodford (1997), Goodfriend and King (1997), Kiley (1998), McCallum and Nelson (1999a,b), and Clarida et al. (1999). It is also a simplified version of the econometric model of Rotemberg and Woodford (1997, 1999).

7. All three variables represent percent deviations from their values in a steady state with zero inflation and constant output growth.

8. The structural equations (1) and (2) provide an accurate approximation to the exact equilibrium conditions in the underlying model only when we restrict our attention to small perturbations around the steady state.

9. Moreover, an aggregate supply curve of the form (2) has found some empirical support from Roberts (1995), Sbordone (1998), and Galí and Gertler (1999).

10. Rotemberg and Woodford (1997) estimate that the standard deviation of the natural rate has been almost 10 times as large as the standard deviation of the federal funds rate, from 1979 to 1995.

11. Note that the welfare costs due to monetary frictions mentioned earlier justify the presence of the interest rate in the loss function even if they have no effect on the structural equations, that is, even if, for example, utility is additively separable in real balances, consumption, and goods supply [see Woodford (1999b)].

12. A similar loss function can also be obtained by performing a second-order Taylor approximation to the expected utility of the representative household in the model that has been used to derive (1) and (2) [see Woodford (1999b,c]. The interest rate's presence in the loss function results, for example,

from the approximation of transaction frictions modeled by the presence of real balances in the utility function. A similar term appears when one takes into account the fact that the nominal interest rate faces a lower bound at zero. There are additional reasons, from which we abstract, that make volatile interest rates undesirable. Williams (1999), for example, argues that policymakers may dislike reversals in the direction of policy because they fear that such actions would be misinterpreted by the public as mistakes on the part of the monetary authority. Finally, variable interest rates may decrease potential output through higher costs of capital because a large variance in expected short-term rates has been observed to raise the term premium [Tinsley (1999)].

13. The second argument in $L_0(\boldsymbol{q}, \boldsymbol{\theta})$ allows for the possibility that coefficients of the loss function such as λ_x, λ_i be functions of elements of the parameter vector $\boldsymbol{\theta}$.

14. Recent contributions that emphasize determinacy of the rational expectations equilibrium in monetary models include Bernanke and Woodford (1997), Christiano and Gust (1999), Rotemberg and Woodford (1999), Woodford (1999a,c), and Clarida et al. (2000). Papers that focus more on the effects of interactions between monetary and fiscal policy include Leeper (1991), Sims (1994), Woodford (1994, 1995, 1996), and Schmitt-Grohé and Uribe (2000).

15. Our concern for choosing a policy rule that does not allow for the worst possible equilibrium to occur is consistent with the approach to robust policy analysis proposed below.

16. In contrast, in Hansen and Sargent (1999a,b) and Sargent (1999), both the policymaker and the private sector face similar uncertainty with respect to the correct model.

17. Note that this formulation also allows for policy rules that involve learning for at least some time on the part of the policymaker because ψ is only restricted to be finite-dimensional. In the application of Section 4, however, we restrict our attention to a family of simple rules that involves no learning.

18. Since we allow for priors such as any given element $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ holding with certainty, the worst-case scenario, for a given policy rule ψ^*, is the parameter vector $\boldsymbol{\theta}^*$ that maximizes the loss $E[L_0(\boldsymbol{q}(\psi^*, \boldsymbol{\theta}), \boldsymbol{\theta})]$ on $\boldsymbol{\Theta}$. Note that the worst case described here does not need to be at all close to the *absolute* worst-case situation, which involves an arbitrarily large loss for the policymaker. Indeed, by choosing a set $\boldsymbol{\Theta}$ that is sufficiently small, the worst-case scenario can be made arbitrarily close to the best-case scenario.

19. Note that there is no loss of generality in restricting $\boldsymbol{\Psi}$ to be a set of rules such that $\boldsymbol{q}(\psi, \boldsymbol{\theta})$ is uniquely defined for all $\psi \in \boldsymbol{\Psi}$, and $\boldsymbol{\theta} \in \boldsymbol{\Theta}$. For if the set of policy rules was a larger set $\hat{\Psi}$, and the policymaker chose a rule $\hat{\psi}$ in $\hat{\boldsymbol{\Psi}}$ but not in $\boldsymbol{\Psi}$, the maximum loss would always be arbitrarily large, so that $\hat{\psi}$ could not possibly be a robust optimal rule.

20. Note, however, that these constraints do not necessarily eliminate all of the equilibria that result from policies in $\tilde{\boldsymbol{\Psi}} \backslash \boldsymbol{\Psi}$, that is, equilibria that are unbounded or that result from policy rules that allow for multiple rational expectations equilibria (see discussion later).

21. If $\boldsymbol{q}(\boldsymbol{f})$ is linear in $\boldsymbol{f}$, then the objective function $\hat{L}(\boldsymbol{f}, \boldsymbol{\theta})$ is convex in $\boldsymbol{f}$ as $E[L_0]$ is convex in $\boldsymbol{q}$, and the constraints are all linear in $\boldsymbol{f}$. Thus, the first-order conditions are also sufficient to guarantee that $\boldsymbol{f}^*(\boldsymbol{\theta})$ achieves the desired minimum. However, if $\boldsymbol{q}(\boldsymbol{f})$ is nonlinear in $\boldsymbol{f}$, second-order conditions are necessary to identify local minima, and numerical methods can be used to determine which of these is a global minimum of $\hat{L}(\boldsymbol{f}, \boldsymbol{\theta})$.

22. It may happen that all policies that allow for the optimal equilibrium to occur are in $\tilde{\boldsymbol{\Psi}}$ but not in $\boldsymbol{\Psi}$, so that they all result in an indeterminate equilibrium. In such situations one could determine policy rules that implement a constrained optimal equilibrium that satisfies additional restrictions upon $\boldsymbol{f}$ such that all possible policies are in $\boldsymbol{\Psi}$.

23. It is probably not very realistic to assume that the central bank can observe the output gap in the current period. However, as shown below, an interest-rate rule that responds only to deviations in observed inflation would be sufficient to implement the optimal noninertial plan in this model. As a result, we could set $\psi_x = 0$ without any loss of generality.

24. Taylor (1993) has argued that such a rule with $\psi_\pi = 1.5$ and $\psi_x = 0.5$ constitutes an appropriate description of U.S. monetary policy under chairman Greenspan. In Taylor (1993), however, the output gap is constructed as the percent deviation of real output from a trend, rather than our variable x.

25. Onatski and Stock (2002) also determine robust optimal Taylor rules. However, they perform their analysis in the backward-looking model of Rudebusch and Svensson (1999), and use numerical methods instead of the solution strategy proposed earlier.

26. Although we could, in principle, allow for families of rules that involve learning on the part of the policymaker, we abstract from this issue here. Note, however, that as long as the set of parameters $\boldsymbol{\Theta}$ is not affected by the learning process, and $\boldsymbol{\theta}$ cannot be inferred with certainty, the rule without learning is optimal at least in the *weak* sense in which other rules (that involve learning) are not better in the worst case. The rule without learning may, however, be suboptimal according to a *stronger* notion of robustness because one could possibly find a rule with learning that is equally good at $\boldsymbol{\theta}^*$, but performs better for other values of $\boldsymbol{\theta}$. We leave this issue for further research.

27. The equilibrium may also be determinate when ψ_π and ψ_x are negative. This is, however, critically due to the discrete-time version of the model. In the continuous-time limit, negative values for either coefficient of the policy rule result in indeterminacy of the equilibrium.

28. I am grateful to Thomas Laubach for providing me with the estimated standard errors for σ and κ. These were computed for the Rotemberg and Woodford (1997) model, using the estimation method explained by Amato and Laubach (1999).

29. The rules in which $\psi_\pi = 0$ are not interesting here because the results shown later indicate that optimal rules of the form (43) do not involve values for $\hat{\psi}_\pi$ that are extremely large.

30. Note that (44) corresponds to $\boldsymbol{P}(\boldsymbol{f}^*(\boldsymbol{\theta}^*), \hat{\boldsymbol{\psi}}^*) = \boldsymbol{0}$ in the general terminology of Section 3.

31. We would obtain the same minmax equilibrium by letting $\psi_\pi^* \to -\infty$. However, as mentioned in note 27, even if the equilibrium is determinate in this case, it would be indeterminate in the continuous-time version of this model.

32. The statistic $E[L_0]$ as well as all statistics in Table 2 are reported in annual terms. The statistics $V[\pi]$, $V[i]$, and $E[L_0]$ are therefore multiplied by 16. Furthermore, the weight λ_x reported in Table 1 is also multiplied by 16 in order to represent the weight attributed to the output gap variability (in annual terms) relative to the variability of annualized inflation and of the annualized interest rate. The coefficients ψ_x reported here are multiplied by 4 so that the response coefficients to the output gap and to annualized inflation are expressed in the same units.

33. Unless $\sigma_0 = \underline{\sigma}$ and $\kappa_0 = \bar{\kappa}$, which we have ruled out in Proposition 3.

34. By assuming ψ_π^0, ψ_π^*, $\psi_x \geq 0$ and $\eta \neq \sigma_0/\kappa_0$ in Proposition 4, we implicitly restrict our attention to situations in which the persistence of the perturbations is small enough for (35) to hold. If instead the shocks to the natural rate are very persistent (so that $\eta > \bar{\sigma}/\underline{\kappa}$, corresponding to $\rho > 0.76$), then a result similar to Proposition 4 holds when ψ_x is large enough, but in this case the optimal response to inflation is more negative in the presence of uncertainty; that is, $\psi_\pi^* < \psi_\pi^0 < 0$. (Recall that the optimal response coefficient $f_\pi^*(\boldsymbol{\theta}) < 0$ when $\sigma/\kappa < \eta$). In this case, however, the optimal policy may yield an indeterminate equilibrium (see note 27). This is in fact an example of a situation in which there may be no Taylor rule that implements the optimal equilibrium: There may be no $\boldsymbol{\psi}^*(\boldsymbol{\theta})$ in $\boldsymbol{\Psi}$ that solves (16).

35. Note that (47) is satisfied for all $\psi_x \geq 0$, whenever $\eta(\bar{\kappa}^2 - \kappa_0^2) + \kappa_0\sigma_0 - \bar{\kappa}\underline{\sigma} \geq 0$ because all other terms in the fraction in the right-hand side of (47) are positive.

36. See note 32.

37. Giannoni (2001) shows that this result generalizes to more flexible policy rules that allow for responses of the interest rate to lagged variables, in the model of Section 2.

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APPENDIX A

A.1. PROOF OF LEMMA 1

First note that since $\psi^*(\boldsymbol{\theta}) \in \boldsymbol{\Psi}$, the latter policy rule results in a unique bounded equilibrium. Suppose as a way of contradiction that there exists a policy rule $\psi^\dagger(\boldsymbol{\theta}) \in \boldsymbol{\Psi}$, $\psi^\dagger(\boldsymbol{\theta}) \neq \psi^*(\boldsymbol{\theta})$, satisfying $L(\psi^\dagger(\boldsymbol{\theta}), \boldsymbol{\theta}) < L(\psi^*(\boldsymbol{\theta}), \boldsymbol{\theta})$. By definition of $L(\cdot)$ and $\hat{L}(\cdot)$ we have $L(\psi, \boldsymbol{\theta}) = \hat{L}(\boldsymbol{f}(\psi, \boldsymbol{\theta}), \boldsymbol{\theta})$ for all $\psi \in \boldsymbol{\Psi}, \boldsymbol{\theta} \in \boldsymbol{\Theta}$, so that $\hat{L}(\boldsymbol{f}(\psi^\dagger(\boldsymbol{\theta}), \boldsymbol{\theta}), \boldsymbol{\theta}) < \hat{L}(\boldsymbol{f}(\psi^*(\boldsymbol{\theta}), \boldsymbol{\theta}), \boldsymbol{\theta}) = \hat{L}(\boldsymbol{f}^*(\boldsymbol{\theta}), \boldsymbol{\theta})$. But then $\boldsymbol{f}^*(\boldsymbol{\theta})$ cannot minimize (12) subject to (10). ■

A.2. PROOF OF LEMMA 2

Because $\boldsymbol{f}^*(\boldsymbol{\theta})$ satisfies (15), and $\psi^*(\boldsymbol{\theta})$ solves (16), we have $\boldsymbol{f}^*(\boldsymbol{\theta}) = \boldsymbol{f}(\psi^*(\boldsymbol{\theta}), \boldsymbol{\theta})$, for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}$. Using this and totally differentiating $\hat{L}(\boldsymbol{f}, \boldsymbol{\theta})$, we obtain

$$\frac{d\hat{L}(\boldsymbol{f}(\psi^*(\boldsymbol{\theta}), \boldsymbol{\theta}), \boldsymbol{\theta})}{d\boldsymbol{\theta}'} = \frac{d\hat{L}(\boldsymbol{f}^*(\boldsymbol{\theta}), \boldsymbol{\theta})}{d\boldsymbol{\theta}'} = \frac{\partial\hat{L}(\boldsymbol{f}^*(\boldsymbol{\theta}), \boldsymbol{\theta})}{\partial \boldsymbol{f}'} \cdot \frac{d\boldsymbol{f}^*(\boldsymbol{\theta})}{d\boldsymbol{\theta}'} + \frac{\partial\hat{L}(\boldsymbol{f}^*(\boldsymbol{\theta}), \boldsymbol{\theta})}{\partial\boldsymbol{\theta}'}. \quad \textbf{(A.1)}$$

Note that any solution $\boldsymbol{f}(\psi, \boldsymbol{\theta})$ to (10) and (11) must always satisfy $\boldsymbol{S}(\boldsymbol{f}(\psi, \boldsymbol{\theta}), \boldsymbol{\theta}) = \mathbf{0}$ for *all* $\psi \in \tilde{\boldsymbol{\Psi}}$, $\boldsymbol{\theta} \in \boldsymbol{\Theta}$. Differentiating these constraints with respect to $\boldsymbol{\theta}$, and evaluating the resulting expression at $\psi^*(\boldsymbol{\theta})$, yields

$$\mathbf{0} = \frac{d\boldsymbol{S}(\boldsymbol{f}(\psi^*(\boldsymbol{\theta}), \boldsymbol{\theta}), \boldsymbol{\theta})}{d\boldsymbol{\theta}'} = \frac{d\boldsymbol{S}(\boldsymbol{f}^*(\boldsymbol{\theta}), \boldsymbol{\theta})}{d\boldsymbol{\theta}'} = \frac{\partial\boldsymbol{S}(\boldsymbol{f}^*(\boldsymbol{\theta}), \boldsymbol{\theta})}{\partial \boldsymbol{f}'} \cdot \frac{d\boldsymbol{f}^*(\boldsymbol{\theta})}{d\boldsymbol{\theta}'} + \frac{\partial\boldsymbol{S}(\boldsymbol{f}^*(\boldsymbol{\theta}), \boldsymbol{\theta})}{\partial\boldsymbol{\theta}'}.$$

Premultiplying this by the vector of Lagrange multipliers $\phi^*(\boldsymbol{\theta})$ associated with the constraints (10) in the policymaker's problem, we obtain

$$\mathbf{0} = \phi^*(\boldsymbol{\theta}) \cdot \left(\frac{\partial\boldsymbol{S}(\boldsymbol{f}^*(\boldsymbol{\theta}), \boldsymbol{\theta})}{\partial \boldsymbol{f}'} \cdot \frac{d\boldsymbol{f}^*(\boldsymbol{\theta})}{d\boldsymbol{\theta}'} + \frac{\partial\boldsymbol{S}(\boldsymbol{f}^*(\boldsymbol{\theta}), \boldsymbol{\theta})}{\partial\boldsymbol{\theta}'}\right).$$

Adding this to (A.1) on both sides and rearranging yields

$$\begin{aligned}\frac{d\hat{L}(\boldsymbol{f}(\psi^*(\boldsymbol{\theta}), \boldsymbol{\theta}), \boldsymbol{\theta})}{d\boldsymbol{\theta}'} = &\left(\frac{\partial\hat{L}(\boldsymbol{f}^*(\boldsymbol{\theta}), \boldsymbol{\theta})}{\partial \boldsymbol{f}'} + \phi^*(\boldsymbol{\theta}) \cdot \frac{\partial\boldsymbol{S}(\boldsymbol{f}^*(\boldsymbol{\theta}), \boldsymbol{\theta})}{\partial \boldsymbol{f}'}\right)\frac{d\boldsymbol{f}^*(\boldsymbol{\theta})}{d\boldsymbol{\theta}'} \\ &+ \frac{\partial\hat{L}(\boldsymbol{f}^*(\boldsymbol{\theta}), \boldsymbol{\theta})}{\partial\boldsymbol{\theta}'} + \phi^*(\boldsymbol{\theta}) \cdot \frac{\partial\boldsymbol{S}(\boldsymbol{f}^*(\boldsymbol{\theta}), \boldsymbol{\theta})}{\partial\boldsymbol{\theta}'}.\end{aligned}$$

Using (14) to eliminate the first term on the right-hand side yields (21), where we note that, in the partial derivative $\partial\mathcal{L}^P(\boldsymbol{f}^*(\boldsymbol{\theta}), \phi^*(\boldsymbol{\theta}); \boldsymbol{\theta})/\partial\boldsymbol{\theta}'$, we maintain $\boldsymbol{f}^*(\boldsymbol{\theta})$ and $\phi^*(\boldsymbol{\theta})$ constant. ■

A.3. PROOF OF LEMMA 3

If the local NE is such that $Z_i^* > 0$, then (22) implies $\mu_{i1}^* - \mu_{i2}^* > 0$, where μ_{i1}^* and μ_{i2}^* are the ith elements of $\boldsymbol{\mu}_1^*$ and $\boldsymbol{\mu}_2^*$, respectively. Since the multipliers satisfy $\mu_{i1}, \mu_{i2} \geq 0$ for all $i = 1, \ldots, m$, we must have $\mu_{i1}^* > 0$. In this case, (20) implies $\theta_i^* = \bar{\theta}_i$. Alternatively, if the local NE is such that $Z_i^* < 0$, then $\theta_i^* = \underline{\theta}_i$. If $Z_i^* = 0$, then (22) implies that $\mu_{i1}^* = \mu_{i2}^*$. Suppose as a way of contradiction that $\mu_{i1}^* \neq 0$. Then we know from (20) that $\theta_i^* = \bar{\theta}_i$. However, because $\mu_{i1}^* = \mu_{i2}^*$, we also have $\mu_{i2}^* \neq 0$, and thus $\theta_i^* = \underline{\theta}_i$. Since θ_i^* cannot equal $\bar{\theta}_i$ and $\underline{\theta}_i$ in the same time, we must have $\mu_{i1}^* = \mu_{i2}^* = 0$. So, when $Z_i^* = 0$ in an NE, then θ_i^* can be any value in $[\underline{\theta}_i, \bar{\theta}_i]$ that is consistent with $Z_i^* = 0$. ■

A.4. PROOF OF PROPOSITION 2

Given the equilibrium parameter vector $\boldsymbol{\theta}^* = [\sigma^*, \kappa^*]'$, the minmax equilibrium is by definition the process $\boldsymbol{q}^*(\boldsymbol{\theta}^*) \equiv \boldsymbol{q}(\psi^*(\boldsymbol{\theta}^*), \boldsymbol{\theta}^*)$. It is thus of the form (29), where the response coefficients $f_\pi^*(\boldsymbol{\theta}^*)$, $f_x^*(\boldsymbol{\theta}^*)$, $f_i^*(\boldsymbol{\theta}^*)$ are given by (32)–(34), evaluating the parameter vector at $\boldsymbol{\theta}^*$.

To determine $\boldsymbol{\theta}^*$, we need to determine the sign of Z_1^* and Z_2^*. Using the first-order conditions to the policymaker's problem, we can express the equilibrium Lagrange multiplier associated with (30) as $\phi_1^*(\theta^*) = -\lambda_i \sigma^* f_i^*(\boldsymbol{\theta}^*)$. Combining this with (36), and using (40), (42) to solve for $f_\pi^*(\boldsymbol{\theta}^*)$, $f_i^*(\boldsymbol{\theta}^*)$, we obtain

$$Z_1^* = \left(\eta - \frac{\sigma^*}{\kappa^*}\right)\chi_1, \tag{A.2}$$

where $\chi_1 > 0$ and $\eta \equiv \rho(1-\rho)^{-1}(1-\beta\rho)^{-1}$. Similarly, using the first-order conditions to the policymaker's problem, we can express the equilibrium Lagrange multiplier associated with (31) as

$$\phi_2^*(\boldsymbol{\theta}^*) = \frac{\lambda_x}{\kappa^*} f_x^*(\boldsymbol{\theta}^*) - \frac{\sigma^*\lambda_i(1-\rho)}{\kappa^*} f_i^*(\boldsymbol{\theta}^*).$$

Using this to substitute for $\phi_2^*(\boldsymbol{\theta}^*)$ in (37), and using (41), (42) to solve for $f_x^*(\boldsymbol{\theta}^*)$, $f_i^*(\boldsymbol{\theta}^*)$, we get

$$Z_2^* = -\left(\eta - \frac{\sigma^*}{\kappa^*}\right)\chi_2 \tag{A.3}$$

where $\chi_2 > 0$.

We need to consider three cases.

Case 1. $\eta < \underline{\sigma}/\bar{\kappa}$. In this case, $\eta < \sigma^*/\kappa^*$ for all σ^*, κ^* in the allowed set $\boldsymbol{\Theta}$. Equations (A.2), (A.3) imply that, for any given structural parameters σ^*, κ^*, the policymaker's best response, ψ^*, is such that $Z_1^* < 0$ and $Z_2^* > 0$. By Lemma 3, Nature's best response is then $\boldsymbol{\theta}^* = [\underline{\sigma}, \bar{\kappa}]'$ in a local NE.

Case 2. $\bar{\sigma}/\underline{\kappa} < \eta$. Symmetrically, $\eta > \sigma^*/\kappa^*$ for all σ^*, κ^* in $\boldsymbol{\Theta}$. Equations (A.2), (A.3) imply that, for any σ^*, κ^*, the policymaker's best response, ψ^*, is such that $Z_1^* > 0$ and $Z_2^* < 0$. By Lemma 3, Nature's best response is then $\boldsymbol{\theta}^* = [\bar{\sigma}, \underline{\kappa}]'$ in a local NE.

Case 3. $\underline{\sigma}/\bar{\kappa} \leq \eta \leq \bar{\sigma}/\underline{\kappa}$. We need to consider three situations.

(i) Suppose first that we have a local NE in which Nature chooses some σ^*, κ^* such that $\sigma^*/\kappa^* < \eta$. We know from (A.2), (A.3) that the policymaker's best response is such that $Z_1^* > 0$ and $Z_2^* < 0$. However, Lemma 3 guarantees that Nature chooses $\sigma^* = \bar{\sigma}$ and $\kappa^* = \underline{\kappa}$ in this case. Because $\eta \leq \bar{\sigma}/\underline{\kappa}$, we obtain a contradiction, and such σ^*, κ^* cannot be part of an NE.

(ii) Suppose, alternatively, that we have a local NE in which Nature chooses some σ^*, κ^* such that $\sigma^*/\kappa^* > \eta$. We know from (A.2), (A.3) that the policymaker's best response is such that $Z_1^* < 0$ and $Z_2^* > 0$. However, Lemma 3 guarantees that Nature chooses $\sigma^* = \underline{\sigma}$ and $\kappa^* = \bar{\kappa}$ in this case. Because $\underline{\sigma}/\bar{\kappa} \leq \eta$, such σ^*, κ^* cannot be part of an NE.

(iii) Suppose, finally, that we have an NE in which Nature chooses some σ^*, κ^* such that $\sigma^*/\kappa^* = \eta$. We know from (A.2), (A.3) that the policymaker's best response is such that $Z_1^* = Z_2^* = 0$. Lemma 3 in turn says that Nature may choose any σ^*, κ^* in $\mathbf{\Theta}$ that is consistent with $Z_1^* = Z_2^* = 0$, that is, that satisfies $\sigma^*/\kappa^* = \eta$.

Thus, when $\underline{\sigma}/\bar{\kappa} \leq \eta \leq \bar{\sigma}/\underline{\kappa}$, Nature's best response is given by any vector $[\sigma^*, \kappa^*]' \in \mathbf{\Theta}$ satisfying $\sigma^*/\kappa^* = \eta$, in a local NE. In this case, (40)–(42) imply that the minmax equilibrium is characterized by $f_\pi^*(\boldsymbol{\theta}^*) = f_x^*(\boldsymbol{\theta}^*) = 0$, and $f_i^*(\boldsymbol{\theta}^*) = 1$. ■

A.5. PROOF OF PROPOSITION 3

First observe that since both h and h^* in (34) and (42) are nonnegative, we have $0 < f_i^0 \equiv f_i^*(\boldsymbol{\theta}^0) \leq 1$, and $0 < f_i^* \equiv f_i^*(\boldsymbol{\theta}^*) \leq 1$. We now show that $f_i^0 \leq f_i^*$ and that $f_i^0 < f_i^*$ when $\sigma_0/\kappa_0 \neq \eta$. We need to consider three cases.

Case 1. $\eta < \underline{\sigma}/\bar{\kappa}$. In this case, Proposition 2 implies $\kappa^* = \bar{\kappa} > \kappa_0$, $\sigma^* = \underline{\sigma} < \sigma_0$. Note that $\eta < \underline{\sigma}/\bar{\kappa}$ can be rewritten as

$$\underline{\sigma}\chi - \rho\bar{\kappa} > 0,$$

where $\chi \equiv (1-\rho\beta)(1-\rho) > 0$. Using (34) and (42), we obtain, after some algebraic manipulations,

$$f_i^* - f_i^0 = \lambda_i \frac{\left[(\sigma_0\chi - \rho\kappa_0)^2 - (\underline{\sigma}\chi - \rho\bar{\kappa})^2\right]\xi + \left[\bar{\kappa}^2(\sigma_0\chi - \rho\kappa_0)^2 - \kappa_0^2(\underline{\sigma}\chi - \rho\bar{\kappa})^2\right]}{h \cdot h^*},$$

where $\xi = \lambda_x(1-\beta\rho)^2 > 0$. Since $(\sigma_0\chi - \rho\kappa_0) > (\underline{\sigma}\chi - \rho\bar{\kappa}) > 0$, we have $(\sigma_0\chi - \rho\kappa_0)^2 - (\underline{\sigma}\chi - \rho\bar{\kappa})^2 > 0$, so that the numerator is positive. Since the denominator is also positive, we have $f_i^* > f_i^0$.

Case 2. $\bar{\sigma}/\underline{\kappa} < \eta$ In this case, Proposition 2 implies $\kappa^* = \underline{\kappa} < \kappa_0$, $\sigma^* = \bar{\sigma} > \sigma_0$. Note that $\bar{\sigma}/\underline{\kappa} < \eta$ can be rewritten as

$$\bar{\sigma}\chi - \rho\underline{\kappa} < 0.$$

This implies $\sigma_0\chi - \rho\kappa_0 < 0$. We now have

$$f_i^* - f_i^0 = \lambda_i \frac{\left[(\sigma_0\chi - \rho\kappa_0)^2 - (\bar{\sigma}\chi - \rho\underline{\kappa})^2\right]\xi + \left[\underline{\kappa}^2(\sigma_0\chi - \rho\kappa_0)^2 - \kappa_0^2(\bar{\sigma}\chi - \rho\underline{\kappa})^2\right]}{h \cdot h^*}.$$

Since $\sigma_0\underline{\kappa} < \bar{\sigma}\kappa_0$, we have $(\sigma_0\underline{\kappa} - \bar{\sigma}\kappa_0)\chi = \underline{\kappa}(\sigma_0\chi - \rho\kappa_0) - \kappa_0(\bar{\sigma}\chi - \rho\underline{\kappa}) < 0$, so that $\underline{\kappa}(\sigma_0\chi - \rho\kappa_0) < \kappa_0(\bar{\sigma}\chi - \rho\underline{\kappa})$, and $\underline{\kappa}^2(\sigma_0\chi - \rho\kappa_0)^2 > \kappa_0^2(\bar{\sigma}\chi - \rho\underline{\kappa})^2$. Because this implies also that $(\sigma_0\chi - \rho\kappa_0)^2 > (\bar{\sigma}\chi - \rho\underline{\kappa})^2$, the numerator is positive. Because the denominator is also positive, we have $f_i^* > f_i^0$.

Case 3. $\underline{\sigma}/\bar{\kappa} \leq \eta \leq \bar{\sigma}/\underline{\kappa}$. In this case, Proposition 2 implies $\sigma^*/\kappa^* = \eta$ so that $f_i^* = 1$. In general, when $\sigma_0/\kappa_0 \neq \eta$, we have $f_i^0 < 1 = f_i^*$. In the special case in which $\sigma_0/\kappa_0 = \eta$, we obtain $f_i^0 = f_i^* = 1$. ■

A.6. PROOF OF PROPOSITION 4

In the certainty case, any optimal Taylor rule $\psi^0 = [\psi_\pi^0, \psi_x^0]'$ satisfies an equation of the form (46), but where the vector of structural parameters $\boldsymbol{\theta}^*$ is replaced with the known vector $\boldsymbol{\theta}^0 = [\sigma_0, \kappa_0]'$; that is,

$$\psi_\pi^0 = \frac{\kappa_0^2 + \lambda_x(1-\beta\rho)^2}{\lambda_i \kappa_0[\sigma_0(1-\rho)(1-\beta\rho) - \rho\kappa_0]} - \psi_x^0 \frac{(1-\beta\rho)}{\kappa_0}. \tag{A.4}$$

Assuming $\psi_\pi^0, \psi_x \geq 0$, it results from (A.4) that $\sigma_0(1-\rho)(1-\beta\rho) - \rho\kappa_0 \geq 0$, or, equivalently, $\sigma_0/\kappa_0 \geq \eta$. We need to consider two cases.

Case 1. $\eta < \underline{\sigma}/\bar{\kappa}$. Using (46) and (A.4), and setting $\psi_x^0 = \psi_x^* = \psi_x$, we obtain, after some algebraic manipulations,

$$\psi_\pi^* - \psi_\pi^0 = \frac{(1-\beta\rho)(\bar{\kappa} - \kappa_0)}{\kappa_0\bar{\kappa}}$$
$$\times \left\{ \psi_x + \frac{\kappa_0\bar{\kappa}(\sigma_0\bar{\kappa} - \kappa_0\underline{\sigma}) + \lambda_x(1-\beta\rho)^2\left[\eta\left(\bar{\kappa}^2 - \kappa_0^2\right) + \kappa_0\sigma_0 - \bar{\kappa}\underline{\sigma}\right]}{\lambda_i(1-\rho)(1-\beta\rho)^2(\sigma_0 - \eta\kappa_0)(\underline{\sigma} - \eta\bar{\kappa})(\bar{\kappa} - \kappa_0)} \right\}.$$

Since the first fraction in the right-hand side is positive, $\psi_\pi^* > \psi_\pi^0$ if and only if (47) holds.

Case 2. $\underline{\sigma}/\bar{\kappa} \leq \eta \leq \sigma/\kappa$. In this case, $\hat{\psi}_\pi^* = 0$, so that $\psi_\pi^* = 1/\hat{\psi}_x^* \to +\infty$. Since ψ_π^0 is finite when $\sigma_0/\kappa_0 \neq \eta$, we have $\psi_\pi^* > \psi_\pi^0$. ■

APPENDIX B. REGION OF DETERMINACY FOR TAYLOR RULES

As mentioned in the text, the dynamic system (26) admits a unique bounded solution if and only if both eigenvalues of $\boldsymbol{A}$ lie outside the unit circle. The characteristic polynomial of $\boldsymbol{A}$ is

$$P(\gamma) = \left[\gamma^2\beta\sigma - (\sigma + \beta\sigma + \kappa + \beta\psi_x)\gamma + \sigma + \psi_x + \kappa\psi_\pi\right](\beta\sigma)^{-1}.$$

Because the roots of $P(\gamma)$ can be represented by complex numbers of the form $\gamma = e^{iv}R$, with modulus R, the characteristic polynomial has one or more roots on the unit circle when $\gamma = e^{iv}$. Since the coefficients of $P(\gamma)$ are all real, we know that if $\gamma = e^{iv}R$ is a root, then its complex conjugate $\bar{\gamma} = e^{-iv}R$ is also a root. Using this result, we can find conditions for at least one eigenvalue of $\boldsymbol{A}$ to be on the unit circle by solving

$$P(e^{iv}) = 0,$$
$$P(e^{-iv}) = 0$$

for v and ψ_π. The solutions are

$$\psi_\pi = 1 - \psi_x(1-\beta)\kappa^{-1}, \qquad v = 0 \tag{B.1}$$

$$\psi_\pi = -2\sigma(1+\beta)\kappa^{-1} - 1 - \psi_x(1+\beta)\kappa^{-1}, \qquad v = \pi \tag{B.2}$$

$$\psi_\pi = -\sigma(1-\beta)\kappa^{-1} - \psi_x\kappa^{-1}, \qquad v = -i\ln(z) \tag{B.3}$$

where z is a root of $z^2\beta\sigma - (\sigma + \beta\sigma + \kappa + \beta\psi_x)\,z + \beta\sigma$. The conditions involving ψ_π, ψ_x determine the boundaries of the region of determinacy. They can be represented by lines in the (ψ_π, ψ_x) plane (see boundaries of the gray region in Figure 3). If we restrict our attention to the case with $\psi_\pi, \psi_x \geq 0$, then only the first boundary is relevant because it is the only one that crosses the positive orthant. Since there is only one eigenvalue of $\boldsymbol{A}$ outside the unit circle when $\psi_\pi = \psi_x = 0$, then the same must be true for all couples (ψ_π, ψ_x) in the positive orthant and below the boundary (B.1), so that the equilibrium is indeterminate in this region. In contrast, all couples above the boundary (B.1) result in a determinate equilibrium because both eigenvalues are outside the unit circle.

In the presence of parameter uncertainty, the set $\boldsymbol{\Psi}$ of policies (in the positive orthant) that result in a determinate equilibrium for all parameter vectors $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ is the intersection of all sets above the boundary (B.1) when σ and κ vary in the respective intervals $[\underline{\sigma}, \bar{\sigma}]$ and $[\underline{\kappa}, \bar{\kappa}]$. Hence, when $\psi_\pi, \psi_x \geq 0$, the region of determinacy is the set of all couples above $\psi_\pi = 1 - \psi_x(1-\beta)/\bar{\kappa}$.