12
Permanent Income Models

This chapter describes a class of permanent income models of consumption, which stress a connection between consumption and income implied by present value budget balance, and which generate interesting predictions about the responses of various components of consumption to identifiable shocks to the information sets of economic agents. The models allow us to characterize how consumption of durables act as a form of savings and how habit persistence alters consumption-savings profiles.

Technology

To focus on dynamics induced by the household technology, it serves our purposes to adopt the following technology specification:

$$\begin{aligned} \phi_c \cdot c_t + i_t &= \gamma k_{t-1} + e_t \\ k_t &= k_{t-1} + i_t \end{aligned} \tag{12.1}$$

where ϕ_c is a vector of positive real numbers with n_c elements, e_t is a scalar exogenous endowment of consumption and k_{t-1} is a scalar capital stock. We set $\delta_k = 1$, thereby ignoring depreciation in capital so that i_t is net investment. Introducing depreciation in capital would add nothing to our analysis because we shall eliminate any additional input requirement for making new capital productive. With no intermediate inputs required for investment, even if there were depreciation in the capital stock, a version of the first equation of (12.1) would apply to net investment by suitably altering the marginal product of capital parameter γ.

The empirical counterpart to the scalar endowment process $\{e_t\}$ is typically labor income (e.g., see Flavin 1981, and Deaton 1992). Labor is supplied inelastically and produces e_t units of output independently of the level of capital. The absence of curvature in the technology has some troublesome implications for equilibrium prices that we will discuss later. Nevertheless, this technology provides a good laboratory for studying how the household technology alters consumption-savings profiles. Moreover, this specification has played a prominent role in the empirical literature on the permanent income theory of consumption.

To make the model behave well, we impose the restriction that $(1+\gamma)\beta = 1$. Relaxing this restriction to make capital more or less productive has unpleasant implications for the solution to the model. Thus, *reducing* the marginal product of capital will produce a solution in which the capital stock must eventually become negative.[1] *Increasing* the marginal product of capital typically produces a solution with asymptotic satiation in a deterministic version of the model; stochastic versions yield a marginal utility vector with mean zero in a stochastic steady state. This razor's edge linkage between the marginal product of capital and subjective discount factor is the price we pay for eliminating the role of intermediate goods in making new capital productive.

To put this technology within the general specification of Chapter 3, we include an additional equation

$$\phi_i i_t - g_t = 0, \tag{12.2}$$

where ϕ_i is a small positive number. Strictly speaking, this introduces a form of adjustment cost by requiring a household input be used to make capital productive. This small penalty makes capital satisfy the square summability constraint. When there are multiple consumption goods, to make the resulting matrix $[\Phi_c \; \Phi_g]$ nonsingular, we introduce $n_c - 1$ additional investment goods equal to the last $n_c - 1$ entries of c_t Thus, combining these constraints, when n_c is one, we form

$$\Phi_c = \begin{pmatrix} \phi_c \\ 0 \end{pmatrix} \; \Phi_i = \begin{pmatrix} 1 \\ \phi_i \end{pmatrix}, \Phi_g = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \tag{12.3}$$

$$\Gamma = \begin{pmatrix} \gamma \\ 0 \end{pmatrix}, \Delta_k = 1, \Theta_k = 1;$$

and when n_c is greater than one,

$$\Phi_c = \begin{pmatrix} \phi_c' \\ 0 \\ 0 \;\; -I \end{pmatrix} \; \Phi_i = \begin{pmatrix} 1 & 0 \\ \phi_i & 0 \\ 0 & I \end{pmatrix}, \Phi_g = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \tag{12.4}$$

$$\Gamma = \begin{pmatrix} \gamma \\ 0 \\ 0 \end{pmatrix}, \Delta_k = 1, \Theta_k = (\, 1 \quad 0 \,)\, ;$$

Thus, to embed this model into the general setup of Chapter 3, we have n_c investment goods (the original good and $n_c - 1$ additional ones introduced for technical purposes), and one intermediate good (used to enforce square summability in the capital stock).

[1] A more interesting solution of the model imposes a period-by-period nonnegativity constraint on capital. Instead we limit the terminal behavior of the capital stock by imposing square summability.

Two Implications

We extract two sharp implications of this class of permanent income models of consumption. We obtain the first by substituting $k_t - k_{t-1}$ for i_t in (12.1) and solving the resulting difference equation for k_{t-1}:

$$k_{t-1} = \beta \sum_{j=0}^{\infty} \beta^j (\phi_c \cdot c_{t+j} - e_{t+j}). \tag{12.5}$$

From this formula, it follows that

$$k_{t-1} = \beta \sum_{j=0}^{\infty} \beta^j E(\phi_c \cdot c_{t+j} - e_{t+j}) | J_t. \tag{12.6}$$

Relation (12.6) will help us to find recursive solutions to the model.

Formula (12.5) also has important implications for how consumption and (endowment) income respond to the underlying shocks, as displayed by a set of dynamic multipliers - or impulse responses – $\{\chi_j\}$ and $\{\epsilon_j\}$ for $\{c_t\}$ and $\{e_t\}$, respectively, where $\chi_j w_t$ gives the response of c_{t+j} to w_t and $\epsilon_j w_t$ the response of e_{t+j} to w_t. Since the capital stock k_{t-1} cannot depend on w_t it follows from (12.5) that the shock must be present-value neutral. In other words, the impact of w_t on current and future values of e_t must be offset in a present-value sense by its impact on current and future values of c_t:

$$\sum_{j=0}^{\infty} \beta^j (\phi_c)' \chi_j = \sum_{j=0}^{\infty} \beta^j \epsilon_j. \tag{12.7}$$

Equality (12.7) is precisely the present-value relation studied by Flavin (1981), Hamilton and Flavin (1986), Sargent (1987), Hansen, Roberds and Sargent (1991), and Gali (1991).

The second implication pertains to the martingale behavior of the shadow price vector for consumption and capital. To begin, note that the forward evolution equation for the shadow price of capital is

$$\mathcal{M}_t^k = E(\beta \mathcal{M}_{t+1}^k | J_t) + E(\beta \gamma \mathcal{M}_{t+1}^e | J_t). \tag{12.8}$$

The first-order conditions for the first component of investment imply that

$$\mathcal{M}_t^e + \phi_i \mathcal{M}_t^g = \mathcal{M}_t^k, \tag{12.9}$$

where the left side captures the cost of an additional unit of investment and right side the benefit. The second term on the left side reflects the adjustment cost, and is zero in the limiting $\phi_i = 0$ case. By substituting (12.9) into (12.8) and using the fact that $\beta(1+\gamma) = 1$, we obtain the martingale implication for the shadow price of capital:

$$\mathcal{M}_t^k = E(\mathcal{M}_{t+1}^k | J_t), \tag{12.10}$$

and likewise for the multiplier process $\{\mathcal{M}_t^e\}$.

Finally, the shadow price of consumption is given by:

$$\mathcal{M}_t^c = (\Phi_c)' \mathcal{M}_t^d = \phi_c \mathcal{M}_t^e, \tag{12.11}$$

since the last $n_c - 1$ components of $\mathcal{M}_t^d$ are zero because these are the multipliers on a set of bookkeeping identities. Hence the shadow price process for consumption depends on a single scalar multiplier process $\{\mathcal{M}_t^e\}$, a martingale that we call the marginal utility process for income. We shall pursue the present-value budget balance and martingale implications further, and use them to find and represent the solution of the model.

Solution

To solve the model, we begin by deriving allocation rules for consumption and investment which can be represented in terms of the scalar martingale process $\{\mathcal{M}_t^e\}$. Then we use present-value relation (12.6) to compute $\mathcal{M}_t^e$. Our focus on the marginal utility of income imitates an aspect of the analysis in Bewley (1977).

To accomplish the first step, we use the notion of a canonical household technology. Recall that the household technology determines the sequence of consumption services associated with a given sequence of consumption goods and an initial condition for the household capital stock. When the household technology is canonical, we can construct an *inverse system* which maps a given sequence of consumption services and an initial condition on the household capital stock uniquely into a sequence of consumption goods required to support that service sequence. For a household technology to be canonical, there must be the same number of services as goods, the matrix Π must be nonsingular, and the absolute values of the eigenvalues of the matrix $(\Delta_h - \Theta_h \Pi^{-1} \Lambda)$ must be strictly less than $\beta^{-1/2}$. Under these restrictions, the inverted system can be represented recursively as:

$$\begin{aligned} c_t &= \Lambda^* h_{t-1} + \Pi^* s_t \\ h_t &= \Delta_h^* h_{t-1} + \Theta_h^* s_t \end{aligned} \tag{12.12}$$

where

$$\Lambda^* \equiv -\Pi^{-1}\Lambda, \ \Pi^* \equiv \Pi^{-1};$$

and

$$\Delta_h^* \equiv (\Delta_h - \Theta_h \Pi^{-1} \Lambda), \ \Theta_h^* \equiv \Theta_h \Pi^{-1}.$$

In Chapter ?, we showed that there always exists a representation of the induced preferences for consumption goods in terms of a canonical technology.

Governing the multipliers is an analogous dual system:

$$\mathcal{M}_t^h = E[\beta(\Delta_h)'\mathcal{M}_{t+1}^h|J_t] + E[\beta\Lambda'\mathcal{M}_{t+1}^s|J_t] \tag{12.13}$$

$$\mathcal{M}_t^c = (\Theta_h)'\mathcal{M}_t^h + \Pi'\mathcal{M}_t^s;$$

and an associated inverse system:

$$\mathcal{M}_t^h = E[\beta(\Delta_h^*)'\mathcal{M}_{t+1}^h|J_t] - E[\beta(\Lambda^*)'\mathcal{M}_{t+1}^c|J_t] \tag{12.14}$$

$$\mathcal{M}_t^s = -(\Theta_h^*)'\mathcal{M}_t^h + (\Pi^*)'\mathcal{M}_t^c.$$

Since $\{\mathcal{M}_t^c\}$ is a martingale sequence, it follows from the inverse dual system that $\{\mathcal{M}_t^h\}$ and $\{\mathcal{M}_t^s\}$ are both martingales. In fact they are linear combinations of the scalar martingale sequence $\mathcal{M}_t^e$. For instance,

$$\mathcal{M}_t^s = M_s\mathcal{M}_t^e \tag{12.15}$$

where

$$M_s \equiv \{(\Pi^*)' + (\Theta_h^*)'[I - \beta(\Delta_h^*)']^{-1}\beta\Lambda^{*\prime}\}\phi_c \tag{12.16}$$

Consequently, we can solve for the service sequence in terms of the scalar martingale $\{\mathcal{M}_t^e\}$ from the simple link between the vector of services and the corresponding marginal utility vector:

$$s_t = -M_s\mathcal{M}_t^e + b_t. \tag{12.17}$$

From this relation and from the inverse household technology (12.12) it follows that

$$c_t = \Lambda^* h_{t-1} - \Pi^* M_s\mathcal{M}_t^e + \Pi^* b_t \tag{12.18}$$

$$h_t = \Delta_h^* h_{t-1} - \Theta_h^* M_s\mathcal{M}_t^e + \Theta_h^* b_t.$$

To characterize the decision rule for investment, we solve (12.1) for i_t and substitute the right-hand side of (12.18) for c_t:

$$i_t = \gamma k_{t-1} + e_t - \phi_c \cdot c_t \tag{12.19}$$

$$= \gamma k_{t-1} + e_t - (\phi_c)'\Lambda^* h_{t-1} - (\phi_c)'\Pi^*(b_t - M_s\mathcal{M}_t^e).$$

So far, we have derived a recursive representation for consumption, investment and household capital in terms of the scalar multiplier process $\{\mathcal{M}_t^e\}$. However, we have not shown how to initialize this sequence, and we do not yet have a formula relating the time t increment of this process to the underlying martingale difference sequence $\{w_t\}$. We now derive formulas for both of these objects.

To find an expression for the marginal utility of income process, we exploit the present-value budget balance restriction (12.6). In light of the inverse system (12.12) for the household technology, we are led to compute the expected discounted sum of services and household capital:

$$\sum_{j=0}^{\infty} \beta^j E(s_{t+j}|J_t) = -[1/(1-\beta)] M_s \mathcal{M}_t^e + \sum_{j=0}^{\infty} \beta^j E(b_{t+j}|J_t), \tag{12.20}$$

and

$$\beta \sum_{j=0}^{\infty} \beta^j E(h_{t+j}|J_t) = \beta \sum_{j=0}^{\infty} \beta^j E(\Delta_h^* h_{t+j-1} + \Theta_h^* s_{t+j}|J_t). \tag{12.21}$$

Rewriting (12.21), we see that

$$(I - \beta\Delta_h^*) \sum_{j=0}^{\infty} \beta^j E(h_{t+j-1}|J_t) = h_{t-1} + \beta\Theta_h^* \sum_{j=0}^{\infty} \beta^j E(s_{t+j}|J_t),$$

or

$$\begin{aligned} &\sum_{j=0}^{\infty} \beta^j E(h_{t+j-1}|J_t) = \\ &(I-\beta\Delta_h^*)^{-1} h_{t-1} + \beta (I-\beta\Delta_h^*)^{-1} \Theta_h^* \sum_{j=0}^{\infty} \beta^j E(s_{t+j}|J_t). \end{aligned} \tag{12.22}$$

Since $\phi_c \cdot c_t = (\phi_c)'(\Lambda^* h_{t-1} + \Pi^* s_t)$, it follows from equation (12.12) and (12.20) that

$$\begin{aligned} (1+\gamma)k_{t-1} - (\phi_c)'\Lambda^*(I-\beta\Delta_h^*)^{-1} h_{t-1} = \\ - M_s' M_s \mathcal{M}_t^e/(1-\beta) + \sum_{j=0}^{\infty} \beta^j E(M_s' b_{t+j} - e_{t+j}|J_t). \end{aligned} \tag{12.23}$$

Solving for $\{\mathcal{M}_t^e\}$ results in

$$\begin{aligned} \mathcal{M}_t^e = 1/(M_s' M_s)[(1-\beta) \sum_{j=0}^{\infty} \beta^j E(M_s' b_{t+j} - e_{t+j}|J_t) \ - \\ \gamma k_{t-1} + (1-\beta)(\phi_c)'\Lambda^*(I-\beta\Delta_h^*)^{-1} h_{t-1}]. \end{aligned} \tag{12.24}$$

To interpret this solution, it is useful to decompose the right-hand side of (12.24) into three components. First, we follow the permanent income literature by defining permanent income to be that amount of income to be spent on consumption that can be expected to persist in the future and still satisfy (12.6):

$$y_t^p \equiv \gamma k_{t-1} + (1-\beta) \sum_{j=0}^{\infty} E(\beta^j e_{t+j} | J_t). \tag{12.25}$$

Formally, this is obtained by letting $\{y_t^p\}$ be a hypothetical expenditure process for consumption, assuming it is a martingale, substituting y_{t+j}^p for $\phi_c \cdot c_{t+j}$ in equation (12.6), and solving for y_t^p.

Note that this measure of permanent income does not adjust for risk in the endowment sequence and hence even when divided by $(1-\beta)$ is distinct from equilibrium wealth. Nevertheless, it is an important component of the solution to the model. In fact, $c_t = y_t^p$ is the solution for consumption in the in the special case of single good, no preference shocks (b_t constant), and time separable preferences (Λ zero), which is Hall's (1978) permanent income model of consumption. In this simple case, the marginal utility process for endowment income is formed by translating the negative of permanent income: $\mathcal{M}_t^e = b - y_t^p$.

More generally, when the preference shock process is not expected to be constant, the term of interest is a 'permanent' measure of the preference shock sequence:

$$b_t^p \equiv (1-\beta) \sum_{j=0}^{\infty} E[\beta^j (M_s)' b_{t+j} | J_t]. \tag{12.26}$$

Finally, when preferences are not separable over time, the household capital stock also enters the solution for the marginal utility of endowment income. To interpret its coefficient, consider the sequence of consumption goods required to support zero consumption services from now into the future. To compute this sequence, simply feed a sequence of zeros into the inverse of the household technology. Discounting the resulting consumption sequence and premultiplying by $(\phi_c)'$ results in:

$$y_t^h \equiv (1-\beta)(\phi_c)' \Lambda^* (I - \beta \Delta_h^*)^{-1} h_{t-1}. \tag{12.27}$$

Hence y_t^h adjusts the permanent income measure to account for implicit consumption associated with a "baseline" zero service sequence.

To summarize, the marginal utility of endowment income can be represented as:

$$\mathcal{M}_t^e = (1-\beta)(1/M_s' M_s)(b_t^p - y_t^p + y_t^h). \tag{12.28}$$

Relation (12.28) gives a decomposition of marginal utility of income into three components: b_t^p, y_t^p and y_t^h. Increases in b_t^p result in an outward shift in preference for consumption, which increases the marginal utility of income; increases in e_t^p correspond to an outward shift of permanent income, which reduces the marginal utility of income; and alterations in y_t^h reflect movements in the initial household capital stock. The measurement of this third component is the discounted endowment-equivalent consumption sequence associated with a baseline (zero) sequence of services. Increasing it has an opposite impact on the marginal of income from increasing permanent income.

The final task of this section is to deduce a formula for the increment of $\mathcal{M}_t^e$ of the form μw_t for some μ. Note that y_t^h depends on time $t-1$ information. Hence only the b_t^p and y_t^p terms enter into consideration. Let $\{\psi_j\}$ denote the sequence of matrices of dynamic multipliers for the preference shock process $\{b_t\}$. It follows from (12.24) that

$$\mu = [(1-\beta)/(M_s'M_s)] \sum_{j=0}^{\infty} \beta^j [(M_s)'\psi_j - \epsilon_j]. \tag{12.29}$$

The dynamic multipliers $\{\chi_j\}$ for consumption can then be computed recursively from (12.18), and by construction satisfy the present-value budget balance restriction (12.7).

Deterministic Steady States

It is useful to study consumption in a deterministic steady state, partly to verify that there exist configurations of the model for which consumption of all goods is positive in this steady-state. Otherwise, the stochastic versions of the model would likely have some perverse implications. We consider cases in which $\{b_t\}$ and $\{e_t\}$ are constants set at the values b and e, respectively.

For a deterministic version of the model, the marginal utility of income is constant over time. Of course, we are only interested in initial conditions such that the initial marginal utility is positive and hence the entire sequence is positive. Thus from equation (12.28), we are lead to require that:

$$\mathcal{M}_0^e = (1/M_s'M_s)(M_s'b - e - \gamma k_{-1} + y_0^h) > 0. \tag{12.30}$$

Since $1/(M_s'M_s)$ is positive by construction, we only need to be concerned with requiring that $(M_s'b - e - \gamma k_{-1} + y_0^h)$ be positive. Any changes in (b, e, h_{-1}, k_{-1}) that alter the right side of (12.30) will clearly change the marginal utility of income (in all time periods).

Since the preference shifter sequence is fixed at a constant level, the constant marginal utility of income sequence implies a constant service sequence given by:

$$s = b - M_s\mathcal{M}_0^e. \tag{12.31}$$

The corresponding sequences of consumption goods and household capital need not be constant, and we now investigate the limiting behavior of these objects.

Armed with the consumption service sequence, the consumption and household capital sequences can be computed from the inverse household technology.

The absolute values of the eigenvalues of Δ_h^* are less than $\beta^{-1/2}$ but can be greater than or equal to one. Without further restricting the eigenvalues of Δ_h^* to have absolute values that are strictly less than one, the consumption sequence may not converge to a steady state.[2] With the additional restriction that the absolute values of eigenvalues are strictly less than one, the consumption and household capital sequences will converge with limits:

$$\begin{aligned} h_\infty &= (I - \Delta_h^*)^{-1}\Theta_h^* s \\ c_\infty &= \Lambda^* h_\infty + \Pi^* s \end{aligned} \tag{12.32}$$

where variables with subscript ∞ denote limit points. By combining (12.31) and (12.32), it can be checked whether the limiting consumption vector is strictly positive.

As an illustration, consider a setting with a single consumption good, a single physical capital stock, and the following household technology:

$$\begin{aligned} h_t &= \delta_h h_{t-1} + (1 - \delta_h) c_t \\ s_t &= \lambda h_{t-1} + c_t, \end{aligned} \tag{12.33}$$

where δ is a depreciation factor between zero and one. Notice that the household capital stock is constructed to be a weighted average of current and past consumption. The inverse system is given by:

$$\begin{aligned} h_t &= [\delta_h - \lambda(1 - \delta_h)]h_{t-1} + (1 - \delta_h) s_t \\ c_t &= -\lambda h_{t-1} + s_t. \end{aligned} \tag{12.34}$$

In this simple case, the eigenvalue of Δ_h^* is simply the coefficient on the lagged capital stock in the evolution equation for household capital. This coefficient has an absolute value less than one if:

$$-1 < \lambda < (1 + \delta)/(1 - \delta). \tag{12.35}$$

When these inequalities are satisfied consumption and the household capital stock both converge to $s/(1 + \lambda)$.

Negative values of λ that violate the inequalities in (12.35) display a form of 'rational addiction' as analyzed by Becker and Murphy (1988). For instance, when λ is -1, the coefficient on lagged capital is unity, and the consumption sequence required to support a constant service sequence must grow linearly over

[2] As emphasized by Becker and Murphy (1988), configurations with explosive eigenvalues can arise in models of 'rational addiction'.

time. Lower values of λ (i.e., negative ones with larger absolute values) imply geometric growth in consumption.

Simply requiring the limiting value of consumption to be positive guarantees that consumption will be positive for initial levels of household capital close to the steady state, but it would be nice to obtain a stronger result. One strategy would require entries of the matrices of the inverse household technology all to be positive. Unfortunately, this restriction that would eliminate some important examples in which there is substitutability across goods or over time. For instance, in (12.33), when λ is positive as in the case of a durable good, the inverse household technology has a negative coefficient. Nevertheless, starting from an initial level of household capital below the steady state will result in a positive consumption sequence.

Cointegration

A key feature of our solution is that the marginal utility of income process is a martingale, which implies via (12.17) that the consumption service process is nonstationary. If in addition the preference shock process $\{b_t\}$ is asymptotically stationary, then the service process and consumption are each cointegrated.

Suppose that the preference shock process $\{b_t\}$ is asymptotically stationary, but unobservable to the econometrician. This implies that there are $n_s - 1$ linear combinations of consumption services that are asymptotically stationary. To show this, take any vector ψ that is orthogonal to M_s. It follows from (12.17) that

$$\psi' s_t = \psi' b_t. \tag{12.36}$$

Evidently there are $n_s - 1$ linearly independent ψ's. Each such ψ is referred to as a *cointegrating vector*, in the terminology of Granger and Engle (19??).

An extensive literature treats the efficient estimation of cointegrating vectors. However, what interest us are not the cointegrating vectors but rather the vector M_s that is orthogonal to all of the cointegrating vectors for consumption services.

The cointegrating vectors for consumption services differ from the cointegrating vectors for consumption goods. To deduce the cointegrating vectors for the consumption flows, we shall build upon the deterministic steady-state calculations reported in (12.32). From (12.32), we know that for a deterministic steady state

$$c_\infty = [\Lambda^*(I - \Delta^*)^{-1}\Theta^* + \Pi^*]s_\infty. \tag{12.37}$$

The matrix on the right-hand side of (12.37) also gives the transformation mapping a date t shock to consumption services to the limiting response of consumption. Any vector ψ satisfying

$$\psi'[\Lambda^*(I - \Delta^*)^{-1}\Theta^* + \Pi^*]M_s = 0 \tag{12.38}$$

is a cointegrating vector for consumption. Let Ψ denote an $n_c - 1$ by n_c matrix with rows that are linearly independent cointegrating vectors. Notice that the implied model for Ψc_t and $c_{1,t} - c_{1,t-1}$ contains only stationary endogenous variables, so that it can be estimated using methods that require asymptotic stationarity, like the frequency-domain methods of BLANK. An estimation strategy based on recursive formulations of the Gaussian conditional likelihood function would not require such a transformation to a stationary set of endogenous variables, but would require confronting the nonexistence of an asymptotically stationary distribution of the state vector from which to draw an initial estimator of the state. In chapter BLANK, we described a method based on ideas of Kohn and Ansley (1985) designed to construct an initial estimator of the state in such a circumstance.

If we were to assume that the preference shock process is itself nonstationary and that there does not exist any nontrivial cointegrating vector for this process, then it would follow that there exist no cointegrating vectors for either the service process or the consumption process. In this case, to utilize estimation methods requiring stationary, we would base parameter estimation on the implications for the differenced processes for consumption and household capital.

Constant Marginal Utility of Income

In the absence of uncertainty, the marginal utility of income process will be constant. In this section we introduce uncertainty in the endowment and preference shock processes, and ask: when will the marginal utility of income process remain constant through time? Constancy of the marginal utility of income is an extreme version of the prediction of permanent income theory that the ability to transfer consumption over time results in "smoothness of consumption" over time. In the absence of preference shocks, a constant marginal utility of income process implies that consumption services will also be constant through time.

We address this question from two angles. Initially, we investigate the limiting behavior as the subjective discount factor β approaches unity, and provide conditions on the stochastic structure sufficient to imply constant marginal utilities of income in the limit. In taking this limit, we will not concern ourselves with interpreting directly the limit economy. Instead we will study the limiting behavior of the solutions to the optimal resource allocation problems along with the associated marginal utility of income processes. The second attack on the question is to characterize the specifications of uncertainty that imply a constant marginal utility of income process for a given β that is strictly less than one.

The initial portion of our investigation will imitate and replicate features of Bewley's (1977) study of the permanent income model of consumption. Our analysis is mechanically simpler than Bewley's, but different. When we drive β to one, we maintain the link between the subjective discount factor and the marginal product of capital. Hence as β tends to one in our analysis, the marginal product of capital as measured by γ is simultaneously being driven to zero. In contrast, Bewley considered setups in which the counterpart to the marginal product of capital is always zero. Since capital is less productive, nonnegativity constraints are a central feature of his analysis of the economies with β strictly less than one.

Suppose that $\{z_t\}$ is a stationary stochastic process and that z_t has a finite second moment. Then it is known that the time series average $(1/N)\sum_{j=0}^{N-1} z_{t+j}$ converges.[3] Moreover, the limit vector is invariant to the starting date t of the average. Consistent with our setup in previous chapters, we assume that both b_t and e_t are linear functions of z_t. Recall that the portion of the solution for the marginal utility of income that is not predetermined (the portion that can respond to a current-period shock) is a linear combination, say ν, of a conditional expectation of the geometric average $(1-\beta)\sum_{j=0}^{\infty} \beta^j z_{t+j}$ where

$$\nu \equiv [1/(M_s' M_s)](M_s U_b - U_e) \tag{12.39}$$

$b_t = U_b z_t$ and $e_t = U_e z_t$ [see (12.24)]. While the simple time-series average and the geometric average will not typically agree, they can be made arbitrarily close by driving N to infinity and β to one. Under both limits, tail terms in the average become relatively more important as the limit point is approached. Formally, it follows from the theory of Cesaro and Abel summability that

$$\lim_{N\to\infty}(1/N)\sum_{j=0}^{N-1} z_{t+j} = \lim_{\beta\to 1}(1-\beta)\sum_{j=0}^{\infty}\beta^j z_{t+j} \tag{12.40}$$

(*e.g.* see Zygmond 1979, Theorem 1.33, page 80). Therefore, as the discount factor tends to one, the right side of (12.40) converges to a vector independent of t. Moreover, for the information structures we impose, the limit vector must be in the initial period information set.[4] The constancy of the marginal utility of income as β goes to unity follows immediately.

The argument just provided relies on stationarity but does not require linearity in the evolution equation for $\{z_t\}$. In fact, stationarity often can be replaced by a weaker notion of "asymptotic stationarity" as we now illustrate using the linear specification

$$z_{t+1} = A_{22} z_t + C_2 w_{t+1} \tag{12.41}$$

[3] The convergence is both with probability one and in mean square, where mean-square convergence is defined using the square root of the second moment as a norm. We use mean-square convergence in our subsequent analysis.

[4] This follows because the limiting random variable has a finite second moment. As long as the forecast error variance is independent of calendar time and information accumulates over time, the forecast error variance must be zero.

imposed elsewhere in this book. This specification can be exploited to obtain an alternative demonstration of the constancy of the marginal utility of income.

Recall that

$$(1-\beta)\sum_{j=0}^{\infty}\beta^j E(z_{t+j}|J_t) = (1-\beta)(I-\beta A_{22})^{-1}z_t. \tag{12.42}$$

To investigate the limit as β tends to one, it is convenient to uncouple the dynamics according to the eigenvalues. Let

$$A_{22} = TDT^{-1} \tag{12.43}$$

be the Jordan decomposition, and suppose that D can be partitioned as:

$$D = \begin{pmatrix} I & 0 \\ 0 & D_2 \end{pmatrix} \tag{12.44}$$

where the absolute values of the diagonal entries of D_2 are all strictly less than one. Using the Jordan decomposition, it follows that

$$(1-\beta)(I-\beta A_{22})^{-1} = T\begin{pmatrix} I & 0 \\ 0 & (1-\beta)(I-\beta D_2)^{-1} \end{pmatrix}T^{-1}. \tag{12.45}$$

Taking limits, we see that

$$\lim_{\beta\to 1}(1-\beta)(I-\beta A_{22})^{-1} = T\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}T^{-1}. \tag{12.46}$$

Therefore, $(1-\beta)\sum_{j=0}^{\infty}\beta^j E(z_{t+j}|J_t)$ depends only on

$$z_t^* \equiv (\,I \quad 0\,)\,T^{-1}z_t, \tag{12.47}$$

where z_t^* has law of motion

$$z_{t+1}^* = z_t^* + C^* w_{t+1}, \tag{12.48}$$

where

$$C^* \equiv (\,I \quad 0\,)\,T^{-1}C_2. \tag{12.49}$$

Sufficient conditions for the marginal utility of income to be constant are that the Jordan decomposition of A_{22} satisfies (12.44) and C^* be zero. When these restrictions are satisfied, the process $\{z_t\}$ will be asymptotically stationary because the process $\{z_t^*\}$ will be constant over time and because $\{(\,0 \quad I\,)\,T^{-1}z_t\}$ will converge to a stationary process. This latter result follows since the diagonal entries of D_2 have absolute values that are strictly less than unity. Stationarity is only implied when the $\{z_t\}$ is initialized appropriately.

The arguments just given cannot be extended to $\{z_t\}$ processes with more fundamental forms of nonstationarity. For instance, time trends or unit roots in the endowment process would suffice to overturn constancy of the marginal utility of income in the limit. In the case of time trends, the averages may diverge as the limits are taken. For unit root processes (without drifts) the limits are well defined, but the uncertainty in the marginal utility of income process does not vanish.

We now change experiments and hold fixed the subjective discount factor and ask if it is still possible for the marginal utility of income to be constant. The answer to this question turns out to be yes. Assume the Jordan decomposition (12.43) and (12.44) along with restriction (12.49), except that D_2 can now have eigenvalues with absolute values that are equal or even greater than one (but less than $\beta^{-1/2}$). If

$$\nu(I - \beta A_{22})^{-1} = (\,\nu^* \quad 0\,) \tag{12.50}$$

for some vector ν^*, then the marginal utility of income will be constant over time. While this clearly imposes a restriction on the matrix A_{22}, it is satisfied by some stationary and nonstationary specifications of the endowment and preference shock processes.

Consumption Externalities

One of the prime motivations for intertemporal complementarities put forth by Ryder and Heal (1973) is that individual consumers are concerned in part about their consumption relative to the past community average. In other words, there is an externality in consumption. This motivation is in contrast to that given by Becker and Murphy (1988) in which the complementarities are purely private. The solution described previously is applicable even if this consumption externality is present as a solution to an optimal resource allocation problem. However, the link between optimal resource allocation and competitive equilibrium may vanish when there is a consumption externality. We now investigate this issue.

To capture the externality, we endow the consumer with the household technology:

$$\begin{aligned} H_t &= \Delta_h H_{t-1} + \Theta_h C_t \\ s_t &= \Lambda H_{t-1} + \Pi c_t \end{aligned} \tag{12.51}$$

where H_t, H_{t-1} and C_t are interpreted as community-wide vectors that are beyond the control of the private consumers but are equal to their lower case counterparts in equilibrium.

The previous argument justifying the martingale properties of the marginal utilities of income and consumption relied only on the technology specification and still applies when the externality is present. In light of the externality interpretation, the marginal utility of services now satisfies:

$$\begin{aligned} \mathcal{M}_t^s &= (\Pi^*)'\mathcal{M}_t^c \\ &= (\Pi^*)'\phi_c\mathcal{M}_t^e. \end{aligned} \tag{12.52}$$

Recall that Π^* is equal to Π^{-1}. Although the link between the marginal utility of services and marginal utility of consumption goods is altered, the marginal utility of service process remains a martingale. The previous solution method can now be imitated by substituting the matrix $(\Pi^*)'\phi_c$ for M_s given by (12.15).

When there is a single consumption good and the household technology is canonical, the two solutions coincide. This alteration can be verified by taking the previous solution for the marginal utility of income and showing that all of the equilibrium conditions and first-order conditions remain satisfied. While the marginal utility of services is altered by a constant scale factor over time, this clearly has no impact on the implied marginal rates of substitution for consumption services and therefore the original quantity allocation remains intact with the externality interpretation. When there are multiple consumption goods, the quantity allocations can be altered. Also, when the original household technology is not 'canonical,' the quantity allocations can be altered even when there is a single consumption good. While there generally exists a canonical household technology that implies the same induced preferences for consumption goods, the externality version of the specification can give rise to a fundamentally different canonical technology, breaking the simple link between solution to the resource allocation problem and the decentralized economy.

To elaborate on this last point, suppose the original household technology is not canonical. In Chapter ? we showed how to find the corresponding canonical technology to be used in solving the optimal resource allocation problem. In the presence of consumption externalities, we can find the analog to a canonical household technology by first noting that

$$b_t - s_t = B_t - \Pi c_t, \tag{12.53}$$

where

$$B_t \equiv b_t - \Lambda H_{t-1}.$$

Consequently,

$$(b_t - s_t)\cdot(b_t - s_t) = B_t\cdot B_t - 2(B_t)'\Pi c_t + (c_t)'\Pi'\Pi c_t. \tag{12.54}$$

Suppose that $\Pi'\Pi$ is nonsingular, and obtain a factorization:

$$\hat{\Pi}'\hat{\Pi} = \Pi'\Pi \tag{12.55}$$

where $\hat{\Pi}$ is nonsingular. Also, define

$$\begin{aligned}\hat{\Lambda} &\equiv \hat{\Pi}'^{-1}\Pi'\Lambda \\ \hat{b}_t &\equiv \hat{\Pi}'^{-1}\Pi' b_t \\ \hat{s}_t &\equiv \hat{\Lambda} H_{t-1} + \hat{\Pi} c_t.\end{aligned} \tag{12.56}$$

Then $(b_t - s_t) \cdot (b_t - s_t)$ and $(\hat{b}_t - \hat{s}_t) \cdot (\hat{b}_t - \hat{s}_t)$ agree except for a term that is not controllable by the individual consumer. Consequently, technology (12.56) and the implied preferences for the original household technology are the same.

For this solution method to apply, we need this transformed version of the household technology to be canonical. Since the matrix $\hat{\Pi}$ is nonsingular by construction, it suffices for the matrix $\Delta_h - \Theta_h \hat{\Pi}^{-1}\hat{\Lambda}$ to have eigenvalues with absolute values that are strictly less than $\beta^{-1/2}$. If this restriction is not satisfied then there fails to exist a competitive equilibrium.

Tax Smoothing Models

By reinterpreting variables, our model can represent a class of linear models of optimal taxation, versions of which were studied by Barro (1979) and Judd (1990). Let τ_t be a vector of taxes collected from various sources (e.g., capital, labor, imports, etc.); G_t a scalar stochastic process of government expenditures; B_{t-1} the stock of risk-free one-period government debt bearing net one-period interest rate of $\gamma = \frac{1}{\beta} - 1$; and def_t the gross-of-interest 0 deficit. Match up variables as follows: $c_t \sim \tau_t, e_t \sim G_t, k_{t-1} \sim B_{t-1}, i_t \sim \text{def}_t$. Set ϕ_c to a vector of ones, so that $\phi_c'\tau_t$ measures total time t government tax revenues. With these associations, (12.1) become

$$\begin{aligned}\text{def}_t &= \gamma B_{t-1} + G_t - \phi_c'\tau_t \\ B_t &= B_{t-1} + \text{def}_t.\end{aligned}$$

The preference ordering is interpreted as minus the loss function associated with taxation, and measures the dynamics of tax distortions.

Consider three special cases of this model, each of which sets b_t to a vector of zeroes.

1. *Random walk taxes.* To capture Robert Barro's specification, set $\phi_c = 1$ (so there is only one kind of tax revenues), $\Lambda = 0, \Pi = 1, \Theta_h = \Delta_h = 0$. This version makes taxes follow a random walk. It is a relabelling of Hall's model of consumption.

2. *White noise taxes.* To capture a one-tax version of Judd's specification, again set $\phi_c = 1$, but now set $\Pi = \Delta_h = 1, \Theta_h = \Lambda = -1$. With these settings, the government's objective function is $-.5E_0 \sum_{t=0}^{\infty} \beta^t (\sum_{j=0}^{t} \tau_{t-j})^2$. This specification is intended to capture the long-lived adverse effects of taxation on capital. The optimal policy makes taxes a white noise process, a feature that characterizes the asymptotic behavior of capital taxation in the model of Chari, Christiano, and Kehoe (1994). To deduce the white noise property for this model, use (12.18) and the relations defining $\Lambda^*, \Pi^*, \Delta_h^*, \Theta_h^*$ under (12.12). In particular, we obtain $\tau_t = -M_s(\mathcal{M}_t^e - \mathcal{M}_{t-1}^e)$.

3. *Two taxes.* Set $\phi_c = [1 \quad 1]$, and specify two taxes whose 'distortion technology' is obtained by stacking the two technologies described in examples 1 and 2. This is the kind of setup advocated by Judd (1990), and makes one tax a random walk, the other a white noise.