15
Recursive Risk Sensitive Control

Introduction

This chapter describes a class of preferences with a single additional parameter that permits us to represent altered attitudes toward risk *vis a vis* our earlier quadratic specification. We build on work by Jacobson (1973) and Whittle (1990), who proposed an exponential type of risk adjustment that, in linear–quadratic environments with Gaussian disturbances, preserves most of the computational conveniences of the standard *undiscounted* linear quadratic dynamic programming problem. But Whittle's (1981, 1990) way of introducing discounting into this problem had the unpleasant feature of introducing time dependence into the optimal decision rules, with the time dependence causing the effects of the risk parameter σ to wear off as the planning horizon is extended. By using a recursive specification of utility along the lines of Koopmans, Lucas and Stokey, Epstein and Zin, and Weil, Hansen and Sargent (1992) altered Whittle's specification to model discounting in a way that implies time invariance of optimal decision rules.

In this chapter we describe this preference specification and how it can be implemented with versions of our earlier formulas for computing equilibria and asset prices. The principal features of our specification are that: (a.) the equations for solving the optimal linear regulator problem are modified by replacing the operator associated with the usual matrix Riccati equation with the composition of that operator with another operator that is easily computed and interpreted; (b.) optimal decision rules remain linear in the state; (c.) 'certainty equivalence' no longer holds, i.e., the decision rules, although linear in the state, depend on the conditional covariance matrix of the innovations to the state; (d.) asset pricing formulas have an additional layer of 'risk adjustments,' because the covariance conditional covariance matrix of the innovations to the state now enter the quadratic form in the state in our asset pricing formulas.

A Control Problem

This section describes a 'discounted linear quadratic exponential Gaussian control problem that we shall eventually use for the social planning problem of our standard economy. This is an infinite horizon control problem associated with iterations to convergence on the following equation in the value functions $V_j(x)$:

$$V_{j+1}(x) = \max_{u,x'}\{u'Qu + x'Rx + \frac{2\beta}{\sigma}\log E(\exp(\sigma V_j(x')/2)|J)\}, \tag{15.1}$$

where the maximization is subject to

$$x' = Ax + Bu + Cw,$$

where w is a Gaussian random vector with $Eww' = I$, and where $\beta \in (0,1)$ is a discount factor. Associated with the solution of this control problem are three operators:

$$\begin{aligned} D(V) &= V + \sigma VC(I - \sigma C'VC)^{-1}C'V \\ T(W) &= R + A'(\beta W - \beta^2 WB(Q + \beta B'WB)^{-1}B'W)A \\ d(k, V) &= \beta k - \left(\frac{\beta}{\sigma}\right)\log\det(I - \sigma C'VC). \end{aligned} \tag{15.2}$$

The operator T is the usual one associated with the matrix Riccati difference equation for the discounted optimal linear regulator problem. We shall give an interpretation of the new operator D shortly.

The value function associated with the solution of the infinite horizon control problem is

$$V_\infty(x) = x'U_1x + U_0 \tag{15.3}$$

where

$$U_1 = \lim_{j\to\infty}(T \circ D)^j(0)$$

$$U_0 = \lim_{j\to\infty} d^j(0,0)$$

The optimal decision rule is time invariant $u_t = -Fx_t$ where

$$F = \beta[Q + \beta B'D(U_1)B]^{-1}B'D(U_1)A. \tag{15.4}$$

These formulas can be derived by solving the maximization problem on the right side of (15.1) and using a Lemma stated by Jacobson (1973) on a property of the Gaussian distribution. For details, see Hansen and Sargent (1992).[1]

The optimal decision rule is linear in the state. When $\sigma = 0$, we get the standard linear–quadratic situation, because in that case $D = I$, so that the

[1] This specification of discounting differs from the one used by Whittle (1990) in a way designed to assure time-invariant decision rules.

operator $T \circ D = T$. When $\sigma = 0$, the feedback rule F is independent of the conditional covariance parameters in C, so that 'certainty equivalence' holds when $\sigma = 0$. When $\sigma \neq 0$, the decision rule F depends on the parameters in C.

Pessimistic Interpretation

For some purposes, it is convenient to use an interpretation of the D operator due to Jacobson (1973) and Whittle (1990). Evaluate the following 'aggregator function' associated with the functional equation (15.1):

$$\mathcal{A}(-Fx, x, y'Vy + k|J) = x'R^*x + \frac{2\beta}{\sigma}\log E\{\exp(\sigma(y'Vy+k)/2)|J\} \tag{15.5}$$

where $A^* = A - BF, R^* = F'QF + R,$ and

$$y = A^*x + Cw.$$

We obtain

$$\mathcal{A}(-Fx, x, y'Vy + k|J) = x'[R^* + \beta A^{*\prime}D(V)A^*]x + d(k).$$

The piece $x'[R^* + \beta A^{*\prime}D(V)A^*]x$ has an interpretation in terms of 'pessimism,' to be seen as follows.

Consider the following *deterministic* minimum problem

$$\min_{w,y}\{-\frac{\beta}{\sigma}w'w + x'R^*x + \beta y'Vy\}$$

subject to

$$y = A^*x + Cw.$$

In this problem, w is treated as a control vector. As long as $(I - \sigma C'VC)$ is positive definite, this problem has a unique solution. The minimized value of the criterion is verified to be $x'[R^* + \beta A^{*\prime}D(V)A^*]x$, which is the quadratic in x part of $\mathcal{A}(-Fx, x, y'Vy + k|J)$. The 'solution' for w is

$$w = \sigma(I - \sigma C'VC)^{-1}C'VA^*x.$$

Thus, the optimum problem gives an interpretation of the operator D and a piece $\sigma(I - \sigma C'VC)^{-1}C'VA^*x$ in terms of a 'pessimistic' (when $\sigma < 0$) or 'optimistic' (when $\sigma > 0$) adjustment to the random term w_t in the law of motion for the state.

Recursive Preferences

The preceding generalization of linear–quadratic control theory can readily be applied in the context of representative agent versions of the class of economic models that we have been studying. We use the usual household technology

$$
\begin{aligned}
s_t &= \Lambda h_{t-1} + \Pi c_t \\
h_t &= \Delta_h h_{t-1} + \Theta_h c_t.
\end{aligned}
$$

We define the information process z_t as usual by

$$z_{t+1} = A_{22} z_t + C_2 z_t,$$

and let the law of motion for the complete state x_t be denoted

$$x_{t+1} = A x_t + C w_{t+1},$$

where now w_t is a Gaussian white noise. We define a utility index recursively by

$$U_t = -(s_t - b_t) \cdot (s_t - b_t)/2 + \beta \mathcal{R}_t(U_{t+1}) \tag{15.6}$$

where

$$\mathcal{R}_t(U_{t+1}) = \left(\frac{2}{\sigma}\right) \log E[\exp(\sigma U_{t+1}/2) | J_t].$$

Hansen, Sargent, and Tallarini (1993) have shown how to construct the prices induced by this preference specification. We describe these prices first in the context of an endowment economy.[2]

Endowment economy

Consider a pure endowment economy in which $c_t = U_c z_t$ is exogenous. Infinite recursions on (15.6) lead to the quadratic form

$$U_t^e = x_t' \Omega x_t + \rho,$$

where Ω, ρ are the limits as $j \to \infty$ of recursions on

$$
\begin{aligned}
\Omega_{j+1} &= -(S_s - S_b)'(S_s - S_b)/2 + \beta A' D(\Omega_j) A \\
\rho_{j+1} &= d(\rho_j, \Omega_j).
\end{aligned}
$$

For endowment economies, we use Ω as one of the ingredients in constructing a probability measure appropriate for the inner-product representation for asset pricing.

[2] The Matlab program `solvex.m` computes equilibrium quantities and prices in one of our economies with this preference specification; `solvex.m` calls `doublex.m`, which implements a doubling algorithm to solve the control problem.

Asset Pricing

We want to represent asset prices in the usual way as a conditional mathematical expectation of an infinite discounted sum of an inner product of a 'scaled Arrow-Debreu price' vector and the payout of the asset. Hansen, Sargent, and Tallarini (1993) show that when $\sigma \neq 0$, the required conditional expectations operator for representing asset prices corresponds to a probability distribution that is distorted relative to the 'objective' one. In particular, they show that to attain the inner product representation of asset prices it is appropriate to construct a conditional expectations operator $\mathcal{F}_t$ defined by

$$\mathcal{F}_t U_{t+1} = E(V_{t+1} U_{t+1} | J_t) / E(V_{t+1} | J_t),$$

where

$$V_{t+1} = \exp(\sigma U^e_{t+1}/2).$$

The operator $\mathcal{F}_t$ behaves like a conditional expectation operator in that it is linear, monotone, and maps bounded random variables that are measurable with respect to J_{t+1} into bounded random variables that are measurable with respect to J_t.[3] We can use $\beta \mathcal{F}_t$ to value a contingent claim to time $t+1$ utility, and $\mathcal{M}^s_t$ for the equilibrium valuation of services.

Recursively define the sequence of expectations operators

$$\mathcal{S}_{t,\tau} = \mathcal{F}_t \mathcal{F}_{t+1} \mathcal{F}_{t+2} \cdots \mathcal{F}_{t+\tau-1},$$

where $\mathcal{F}_{t,0} = I$. Then the valuation of a stream of consumption services $\{s_t\}$ is just

$$\sum_{t=0}^{\infty} \mathcal{S}_{t,\tau} \mathcal{M}^s_{t+\tau} \cdot s_{t+\tau}.$$

To compute equilibrium asset prices, we have to evaluate consumption rates c_t. We compute a multiplier $\mathcal{M}^c_t$ from $\mathcal{M}^s_t$ in the usual way, except that we substitute the conditional expectation operators $\mathcal{S}_{t,\tau}$ for the usual ones in equation (6.*) (refer to asset pricing chapter):

$$\mathcal{M}^c_t = \Pi' \mathcal{M}^s_t + \Theta'_h \sum_{\tau=1}^{\infty} \beta^\tau (\Delta_h)^{\tau\prime} \Lambda' \mathcal{S}_{t,\tau} (\mathcal{M}^s_{t+\tau}).$$

[3] $V_{t+1}/EV_{t+1}|J_t$ is used as a Radon-Nikodym derivative in this construction.

Characterizing the Pricing Expectations Operator

Hansen, Sargent, and Tallarini show how to evaluate the conditional expectations operators $\mathcal{S}_{t,\tau}$ by constructing a distorted probability measure for the state x_t.[4] In particular, consider the phony law of motion for x_t

$$x_{t+1} = \hat{A}x_t + \hat{C}w_{t+1}$$

where

$$\begin{aligned}\hat{A} &= [I + \sigma C(I - \sigma C'\Omega C)^{-1}C'\Omega]A \\ \hat{C}\hat{C}' &= C(I - \sigma C'\Omega C)^{-1}C'.\end{aligned}$$

Then we can compute $\mathcal{M}_t^c = M_c x_t$ where

$$M_c = [\Pi' + \Theta_h' \sum_{\tau=1}^{\infty} \beta^\tau \Delta_h^{\tau\prime} \Lambda' \hat{A}^\tau](S_b - S_s).$$

We can evaluate a claim on a stream $d_t = S_d x_t$ by

$$\begin{aligned}\sum_{\tau=0}^{\infty} \beta^\tau \mathcal{S}_{t,\tau} \mathcal{M}_{t+\tau}^c \cdot d_{t+\tau} &= \hat{E}_t \sum_{\tau=0}^{\infty} \beta^\tau \mathcal{M}_{t+\tau}^c \cdot d_{t+\tau} \\ &= x_t' \sum_{\tau=0}^{\infty} \beta^\tau (\hat{A}')^\tau S_d' M_c \hat{A}^\tau x_t \\ &+ \left(\frac{\beta}{1-\beta}\right) \text{trace}[S_d' M_c \sum_{\tau=0}^{\infty} \beta^\tau \hat{A}^\tau \hat{C}\hat{C}'(A')^\tau].\end{aligned}$$

Production Economies

The preceding formulas also apply to asset pricing in production economies, with the understanding that the matrix Ω now corresponds to the 'quadratic-form' matrix in the value function for the social planning problem.

Risk-Sensitive Investment under Uncertainty

For sake of illustration, we now consider a one consumption good, one capital good production economy that is a version of a Brock-Mirman stochastic growth model with adjustment costs. This delivers a linear-quadratic version of Lucas

[4] These calculations are implemented in the program `assetxq.m`.

and Prescott's (1971) theory of investment under uncertainty. Consumption and investment satisfy:

$$c_t + i_t = \gamma k_{t-1} + d_t \tag{15.7}$$

where the capital stock k_t evolves according to:

$$k_t = \delta_k k_{t-1} + i_t \tag{15.8}$$

and $\{d_t\}$ is an exogenous endowment process. Labor input is required to adjust the capital stock, reflected in a quadratic adjustment cost in the preferences of the fictitious social planner. Also, there is an exogenous ideal consumption level process $\{b_{1t}\}$. The time t contribution to preferences is:

$$-(c_t - b_{1t})^2 - \phi^2 {i_t}^2 = -(\gamma k_{t-1} + d_t - i_t - b_{1t})^2 - \phi^2 {i_t}^2$$

which is quadratic in the control i_t, the endogenous state variable k_{t-1} and the exogenous states b_{1t} and d_t.

We begin with a parameter specification that implies no investment in a deterministic steady state, an unrealistic but useful starting point that we adopt to compare two alternative 'explanations' for investment. One is that altering the risk parameter σ induces a precautionary savings motive into the preference ordering. The other is that capital is more productive than in the benchmark economy, increasing the physical return to investment.

Benchmark Economy with No Steady-State Investment

We set $\beta = 1/1.05$, $\gamma = .1, \delta_k = .95, \phi = .5$. With this parameter configuration and constant values of d_t and b_{1t}, the model implies steady state values of zero for capital and investment. This follows from the fact that $\gamma + \delta_k = \beta^{-1}$, which equates the physical rate of return to investment net of adjustment costs to the subjective rate of time discount. Since adjustment costs are also present, reflected by a positive value of ϕ, physical investment in new capital becomes 'unattractive.' As a consequence, asymptotically there are no savings in the deterministic version of this model.

To see this, the Euler equation for capital is given by

$$\begin{aligned} E[(1 - L^{-1})(1 - \beta^{-1}L)k_t + \phi^2(1 - \beta\delta_k L^{-1})(1 - \delta_k L)k_t \mid J_t] = \\ E[(1 - L)(b_{1t+1} - d_{t+1}) \mid J_t] \end{aligned} \tag{15.9}$$

where L is the lag operator. For future reference we have permitted there to be uncertainty in the forcing processes for $\{b_{1t}\}$ and $\{d_t\}$. The solution to this stochastic difference equation has representation:

$$k_t = \lambda k_{t-1} + \psi[(d_t - b_{1t}) - (1 - \beta\lambda) \sum_{j=0}^{\infty} (\beta\lambda)^j E(d_{t+j} - b_{1t+j} \mid J_t)]. \tag{15.10}$$

where $0 < \lambda < 1$, and $0 < \psi$. In contrast to the familiar permanent income model in which λ and ψ are one, the presence of adjustment costs lowers λ. Notice that the term multiplying ψ is the difference between $d_t - b_{1t}$ and a geometric average of current and future values of $d_t - b_{1t}$. This simple link to the forcing processes is a result of the $\gamma + \delta_k = \beta^{-1}$ restriction. For a model with b_{1t} and d_t constant, the term multiplying ψ will be zero and the capital stock sequence will converge to zero.

An Economy with Risk Sensitivity

As a precursor to illustrating investment induced by risk sensitivity, we now introduce a specific model of uncertainty in the endowment and preference shock processes. Let an exogenous state vector process $\{z_t\}$ follow a first-order vector autoregression:

$$z_{t+1} = A_{22} z_t + C_2 w_{t+1},$$

and suppose that $b_{1t} = S_b z_t$ and $d_t = S_d z_t$, so that both the endowment and the preference shock are linear functions of exogenous state vector. Construct

$$A_{22} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & .8 & 0 \\ 0 & 0 & .5 \end{bmatrix} \quad C_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & .1 \end{bmatrix}, \tag{15.11}$$

and set $S_b = [\,30 \quad 0 \quad 1\,], S_d = [5\ 1\ 0]$. We initialize the first component of z_t to be one, which replicates itself over time. Consequently, the preference shock process has mean 30 and the endowment shock process has mean 5.

For the $\sigma = 0$ version of this economy, ordinary certainty equivalence applies. Since $\{d_t\}$ and $\{b_{1t}\}$ are asymptotically stationary, so are the endogenous processes for consumption, capital, and investment. It again follows from (15.10) that investment and capital both have mean zero in the stochastic steady state, and consumption has a mean equal to the mean of the endowment shock d_t. Because of certainty equivalence, no precautionary savings occur in this model.

When σ is less than zero, ordinary certainty equivalence no longer holds, and mean investment is positive in a stochastic steady state. Table 2.1 reports the means and standard deviations computed with respect to both the objective probability distribution, and the appropriate distorted distribution for $\sigma = -.005$.

TABLE 2.1: Means and Standard Deviations for the Adjustment Cost Economy with $\sigma = -.005$

	True Process		*Pessimistic Process*	
	mean	standard deviation	mean	standard deviation
c_{1t}	9.83	1.37	2.07	1.57
i_t	4.83	1.22	0.00	1.32
b_{1t}	30.00	.12	30.01	.12
d_t	5.00	1.67	2.07	1.71

The mean of consumption is now notably higher than the mean endowment and the mean of investment is positive. In effect, there is a precautionary savings motive at work here.

A version of the pessimistic certainty equivalence principle derived by Jacobson (1973) and Whittle (1981) applies here. Thus, an alternative way to obtain the same solution is to endow the fictitious social planner with a pessimistic view of the world. The stochastic difference equation (15.9) and its solution (15.10) still apply except that the distorted expectation operator, call it $\hat{E}$, is used in place of E. As originally suggested by Jacobson (1973), we can imagine there being a second agent, say distinct from the social planner, that picks future values of the shocks in a pessimistic fashion. The degree of pessimism is governed by the value of σ, and the difference equations associated with these pessimistic forecasts for b_{1t+j} and d_{t+j} used for the time t decisions are given by:

$$
\begin{aligned}
(1-\beta^{-1}L)k_{t+j} - d_{t+j} + b_{1t+j} + (100/\sigma)(1-.5\beta L^{-1})(1-.5L)(b_{1t+j}-30)] &= 0 \\
-(1-\beta^{-1}L)k_{t+j} + d_{t+j} - b_{1t+j} + (1/\sigma)(1-.8\beta L^{-1})(1-.8L)(d_{t+j}-5)] &= 0.
\end{aligned}
\tag{15.12}
$$

for $j = 1, 2, \ldots$, where k_{t-1}, b_{1t} and d_t are given initial conditions. In effect, we can think of these as the Euler equations for the second agent. Notice that the capital stock enters these difference equations, and hence the pessimistic forecasts of future values of d_t and b_{1t} depend on k_{t-1}.

Certainty equivalence works here because the optimal choice for k_t can be obtained by (i) shifting (15.9) forward $j-1$ periods for $j = 1, 2, \ldots$ and eliminating the conditional expectation operator; (ii) combining the resulting difference equation with (15.12); and (iii) solving the composite system of difference equations for k_t, b_{1t+1}, d_{t+1} as a function of the state vector k_{t-1}, b_{1t} and d_t, imposing the appropriate terminal conditions. The composite solution gives the evolution equation for the pessimistic forecasts of b_{1t} and d_t, and the solution for k_t gives the optimal decision rule for k_t. This latter equation, when combined with the actual law of motion for b_{1t} and d_t, governs the evolution of the optimal capital stock process.

While λ is .9852 for this economy, once this forecasting dependence is incorporated, the feedback of k_t onto k_{t-1} drops slightly to .9817. A more dramatic implication is that there is now a positive constant term (1.766) in the decision rule for capital, whereas in the $\sigma = 0$ economy the constant term is zero. Table 2.1 also reports the pessimistic means and standard deviations for forcing processes and for consumption and investment. Notice that the perceived means for d_t and c_{1t} are considerably less than their true means. The mean of investment for the perceived process is again zero because of the applicability of (15.10) with distorted expectations. Hence the perceived long run average of the capital stock is not altered by changing expectations operators. Not surprisingly, the perceived standard deviations are larger than the true ones.

An Approximating Economy with $\sigma = 0$

Next we show that we can mimic the quantity implications of the preceding model by setting $\sigma = 0$ and modifying some of its parameters. The parameters that we alter are the mean of b_{1t}, the mean of d_t and the productivity parameter for capital γ. In light of our previous discussion, to make average investment positive it is necessary to relax the restriction that $\delta_k + \gamma = \beta^{-1}$ by making capital more productive.

We used the Kullback-Leibler (1951) information criterion as a device for picking the three parameters because of its well known link to maximum likelihood estimation (*e.g.*, see Akaike (1973), Ljung (1978), and White (1982)).[5] The information criterion was constructed in the same manner as in Hansen and Sargent (1993) using the consumption and investment processes implied by the original model and the approximating $\sigma = 0$ economy. The parameters that make the approximate ($\sigma = 0$) model as close as possible to the original model are .1032 for the productivity parameter γ and 47.0522 and 4.6812 for the means of b_{1t} and d_t. In addition to increasing γ to compensate for setting $\sigma = 0$, we were led to increase substantially the mean of b_{1t} and decrease slightly the mean of d_t.

[5] Let each model be a member of our class of models, with parameters of the first model being denoted by a vector δ and those of the second model being denoted by α. For the first model, let the mean vector for the observables be $\nu(\delta)$, and the spectral density matrix be $\bar{S}_y(\omega, \delta)$. For the second model let the spectral density matrix be $S_y(\omega, \alpha)$, and the mean vector be $\mu(\alpha)$. The parameters α that make the second model as close as possible to the first are those that minimize the criterion

$$\begin{aligned}\Big\{ &- \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \det S_y(\omega, \alpha)\, d\omega \\ &- \frac{1}{2\pi} \int_{-\pi}^{\pi} \operatorname{trace}[S_y(\omega, \alpha)^{-1} \bar{S}_y(\omega, \delta)]\, d\omega \\ &- [\nu - \mu(\alpha)]\, S_y(0, \alpha)^{-1} [\nu - \mu(\alpha)]' \Big\} .\end{aligned}$$

It turns out that these three parameter adjustments are sufficient to make the quantity implications for the two models to be extremely close. This is shown in Figure 2.1 which displays the ratios of the spectral densities of consumption and investment, respectively, for the approximating economy to the corresponding spectral densities for the original economy. Departures from unity are small, and confined to very low frequencies. Departures of this small magnitude and frequency-location would be virtually impossible to detect, via say a likelihood ratio test, without an extremely large time series sample. The adjustments in the means of b_{1t} and d_t are sufficient to make the means for consumption and investment the same for both economies. Hence, from the standpoint of data on consumption and investment, the precautionary savings version of the model is (almost) observationally equivalent to a specification in which savings are induced by making capital more productive.

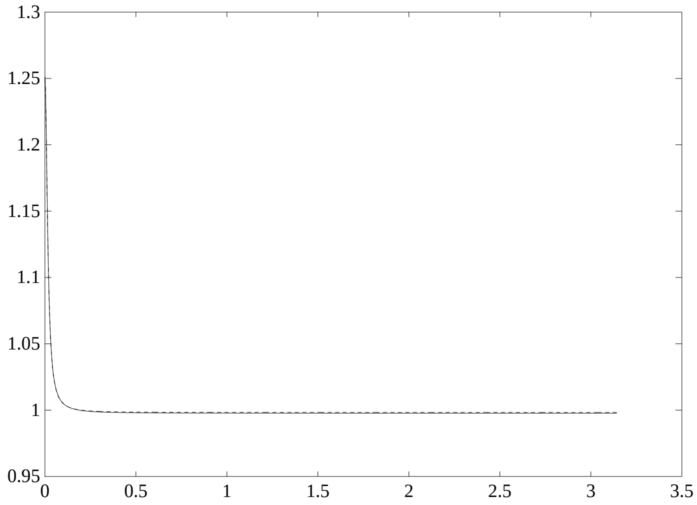

Figure 2.1. Ratio of spectral density of consumption and investment (dotted line) in the approximating economy to the spectral density of consumption and investment in the true economy.

Equilibrium Prices in the Adjustment Cost Economies

We now return to the two adjustment cost economies with their competing explanations for investment. While the quantities for the $\sigma = 0$ 'approximating' economy are, by design, close to those for the 'true' risk-sensitive economy, behaviors of asset prices and rates of return differ markedly. To illustrate this phenomenon, we first consider the implications for the two components of equilibrium wealth: the value of the endowment process from the current period forward (a Lucas tree), and the value of the existing capital stock. Figures 5.1 and 5.2 depict realizations of these two components to wealth for a common realization of the vector of Gaussian noises driving the composite preference and endowment shock process.

The value of the endowment process is always lower and the value of the capital stock higher for the original economy than for the approximating $\sigma = 0$ economy. Hence a bigger fraction of wealth is held in the form of capital in the original (risk-adjusted) economy. This is true even though the mean of the endowment process is higher for the original economy. The mean gross returns are (1.0444, 1.0786) for capital and the endowment process, respectively, in the *true* economy; and (1.0500, 1.0504) in the *approximating* economy.[6] These phenomena trace to the fact that the capital stock is a less risky investment than the endowment and that less risky investments are more highly valued in the original (risk-adjusted) economy than in the approximating economy.

Figure 5.3 shows probability densities for risk-free gross returns for both economies. Notice that the density for the approximating economy is centered around β^{-1}, and the modal value is higher than for the original economy with a risk adjustment. The density for the original (risk-adjusted) economy displays more dispersion than its counterpart for the approximating economy, reflecting in part the smaller mean of b_{1t}.

Consider next the market prices of risk for the two economies. The probability densities for these prices are reported in Figures 5.4 and 5.5. The market price of risk is considerably higher in the risk-sensitive ($\sigma = -.005$) economy.

Finally, Figures 5.6 and 5.7 report spectral densities for the logarithmic one-period returns to holding a claim to the endowment process and to holding capital, respectively. Both returns are more variable in the original (risk-adjusted) economy than in the approximating one. Moreover, the low frequency dip in the spectral density for the return to holding the endowment is substantially more pronounced.

Returning to the decomposition of equilibrium wealth described earlier, it is clear from Figures 5.6 and 5.7 that holding capital is much less risky than holding the endowment for both economies. It is also evident from Figure 5.3 that a riskless security is more valued (commands a lower equilibrium rate of return) in the original economy. This apparently underlies the fact that the capital stock is more highly valued in the original ($\sigma = -.005$) economy as depicted in Figure 5.2. Moreover, as illustrated in Figures 5.4 and 5.5, the market prices of risk tend to be higher in the original economy, so that the equilibrium risk adjustments are more pronounced. This helps to explain why equilibrium values for claims to the endowment process are lower for the original economy as depicted in Figure

[6] These means were computed from simulations of length 100,000.

5.1 even though the mean of the endowment process is higher.

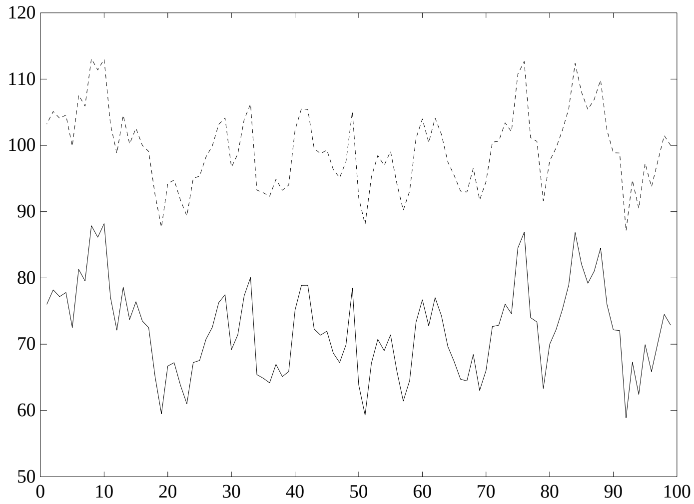

Figure 5.1. Realizations of price of Lucas tree in true economy ($\sigma = -.005$) and approximating ($\sigma = 0$) economy (dashed line).

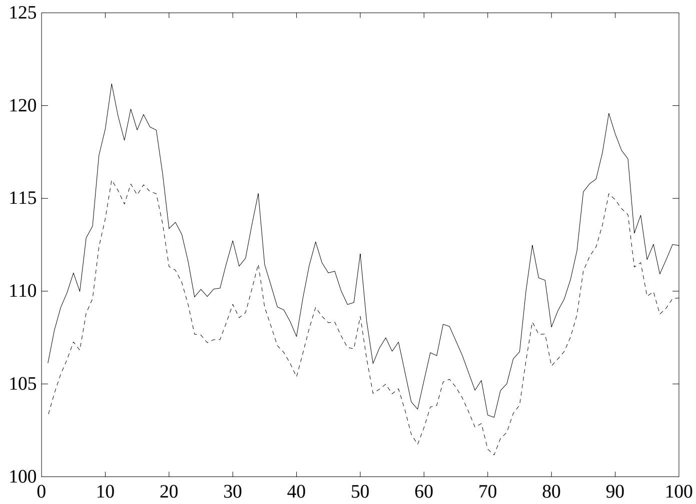

Figure 5.2. Realizations of value of capital stock in true economy($\sigma = -.005$) and approximating ($\sigma = 0$) economy (dashed line).

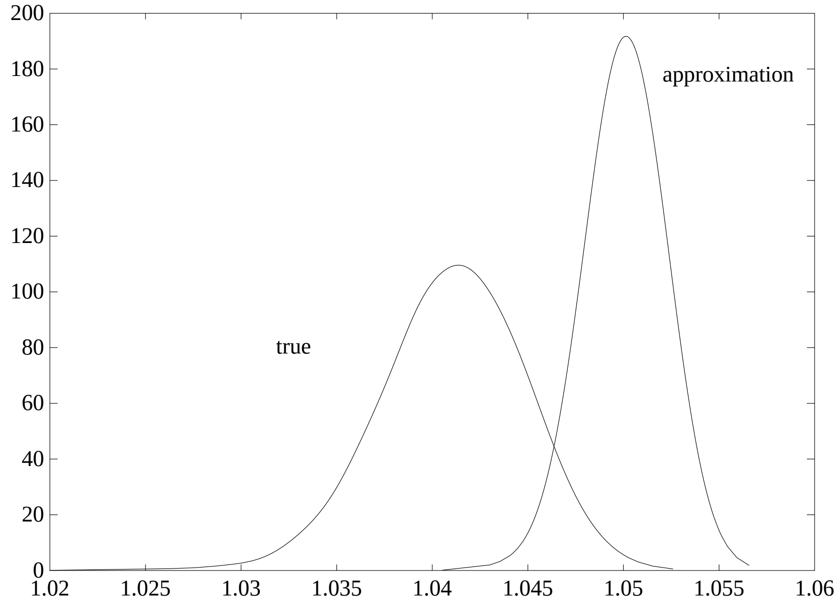

Figure 5.3 Densities of risk-free interest rate for true and approximating economies.

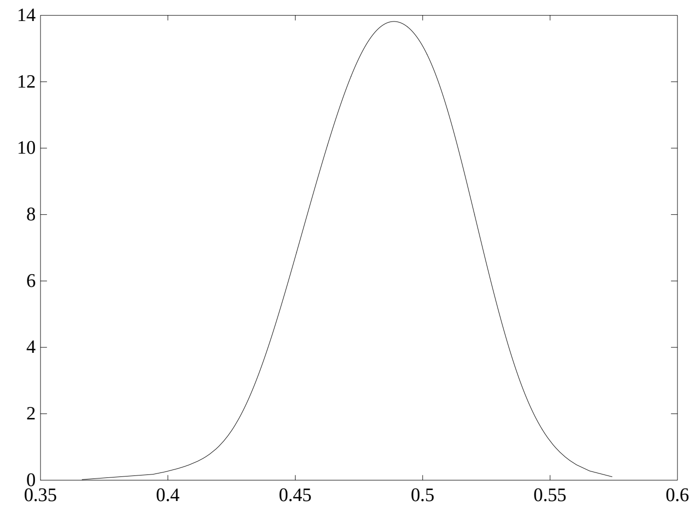

Figure 5.4. Density for market price of risk for true adjustment cost true economy.

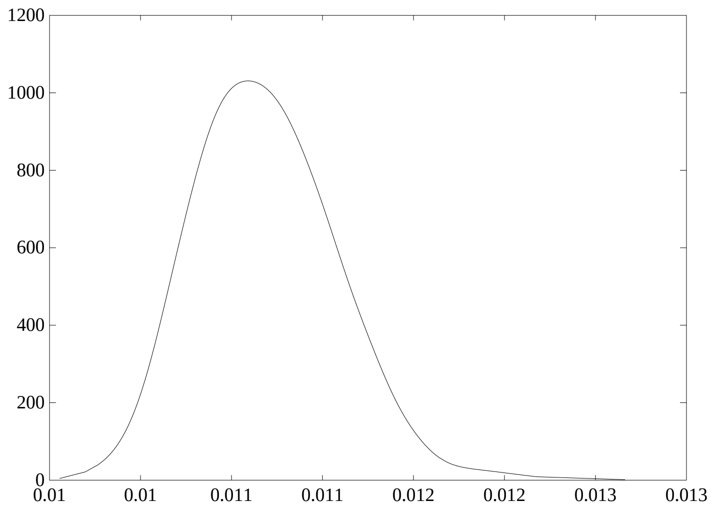

Figure 5.5. Density for market price of risk for approximating adjustment cost economy.

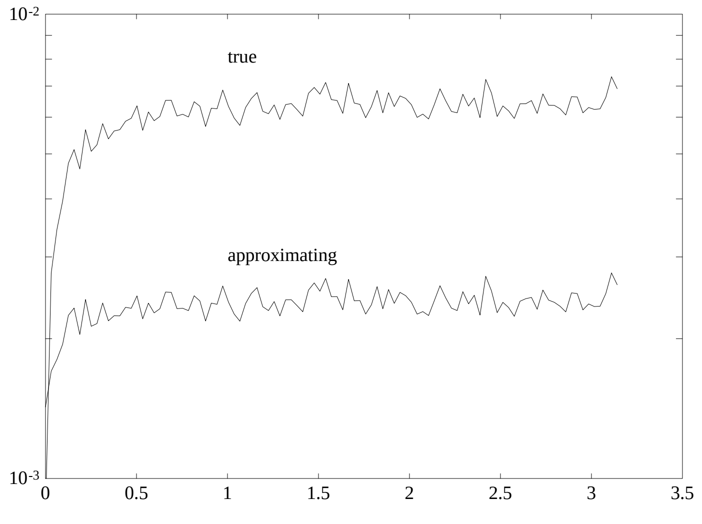

Figure 5.6 Spectral densities of windsorized logarithmic returns to endowment streams for true and approximating adjustment costs economies.

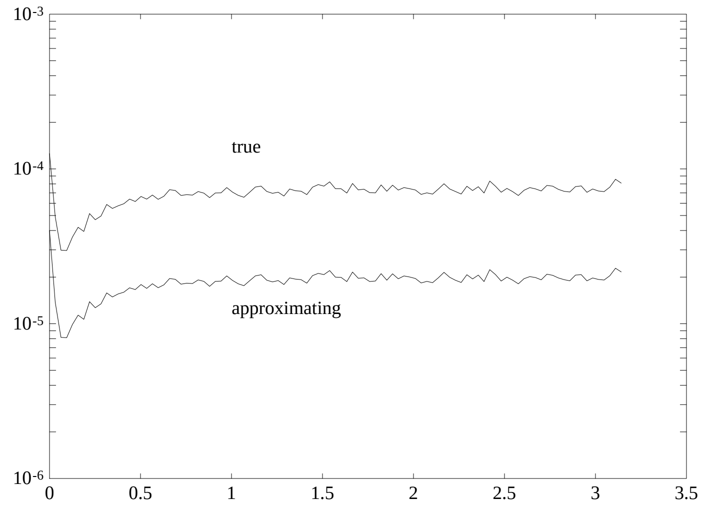

Figure 5.7 Spectral densities of logarithmic returns on capital in true and approximating adjustment costs economies.