Risk-Sensitive Control with HARA Utility

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***Abstract*—In this paper, a control methodology based on the hyperbolic absolute risk averse (HARA) utility function is presented as an alternative to the exponential-of-an-integral approach to finding robust controllers. This work is inspired by the intuition that HARA controllers, while being robust, may give better performance than exponential controllers in normal situations. The HARA problem is shown to be equivalent to a certain differential game, and the asymptotic properties of the HARA problem and this differential game are studied. As an example, a linear-quadratic HARA problem is studied, where the problem of finding a robust HARA controller is proved to be equivalent to solving a standard linear-quadratic problem for a system with a higher noise intensity. This reveals an interesting relationship between robustness and uncertainty.**

***Index Terms*—Differential games, HARA utility function, risk-sensitive control, upper/lower Isaacs equations, viscosity solutions.**

I. Introduction

IN THIS PAPER, we propose an approach based on the hyperbolic absolute risk averse (HARA) utility function [$U(x) = (1/\gamma)x^\gamma$, for $x \geq 0$ and $\gamma > 0$] as an alternative to the exponential utility [$U(x) = \gamma \exp(\gamma x)$, for $\gamma \geq 0$] approach to finding robust controllers. This work is inspired by the intuition that HARA controllers, while being robust, may perform better than exponential controllers when applied to a system that is operating under normal conditions.

Suppose that the evolution of a given system is determined by the following stochastic differential equation (SDE):

$$\begin{cases} dx(t) = b\big(t, x(t), u(t)\big)\, dt \\ \qquad\quad + \sigma\big(t, x(t), u(t)\big)\, dB(t), \qquad t \in [s, T] \\ x(s) = x \end{cases} \tag{1}$$

where (s, x) represents the initial time and state. Typically, the objective is to find a control input $u(\cdot)$ such that the system (1) with input $u(\cdot)$ satisfies a given set of performance specifications; for example, certain specifications on the minimum rise time, constraints on the maximum overshoot, etc., may need to be satisfied. In recent years, many control methodologies (e.g., proportional-integrator-differentiator (PID), linear-quadratic-regulator (LQR), and H_∞, just to name a few) have been proposed as alternative techniques for finding controllers, each having its own advantages, disadvantages and characteristics. However, we emphasize that irrespective of the methodology used to come up with a given controller, the controller is designed as the input for the system (1) and for this reason, the *same* set of performance specifications are used to evaluate whether or not it is suitable. For example, suppose we have two controllers $u_1(\cdot)$ and $u_2(\cdot)$ such that $u_1(\cdot)$ minimizes an H_∞ norm while $u_2(\cdot)$ minimizes an H_2 norm. When evaluating $u_1(\cdot)$ and $u_2(\cdot)$ and comparing their performance, the crucial issue is the behavior of the system (1) under each input; that is, both controllers are tested on the same system (1) and evaluated according to how well the *original* performance criteria are met (e.g., "Does it satisfy the rise time specifications?" and "Does it meet the constraints on maximum overshoot?" etc.), and not the cost functions (e.g., H_∞ or H_2) that may have been used to determine it.

If the system being controlled is not too complicated, then finding a controller which satisfies all of the specifications is reasonably straightforward. For example, if (1) is a linear, single-input–single-output (SISO), infinite-horizon, time-invariant system, then, engineering insight together with numerical techniques arising from optimization theory (see [3] and [28]) can be used to tune the parameters. However, things may not be so easy when dealing with more complex systems (e.g., nonlinear, time-varying, stochastic). In this case, the following simplifications are made. Rather than introducing a *set* of performance specifications, one assumes that *all* performance specifications are summarized by a *single* performance measure

$$J\big(s, x;\ u(\cdot)\big) = E\left\{\int_s^T f\big(t, x(t), u(t)\big)\, dt + g\big(x(T)\big)\right\} \tag{2}$$

and the following convention is adopted: a controller which makes (2) small satisfies the performance specifications better than controllers that makes (2) large. That is, the set of performance specifications is replaced by a single performance measure and, as in the classical case, all controllers, irrespective of how they are determined, are evaluated using the same performance measure (2).

The risk-sensitive methodology using the exponential utility function is one approach to finding robust controllers for the system (1) [with performance measure (2)]. Such controllers are obtained by minimizing criteria of the form

$$J\big(s, x;\ u(\cdot)\big) = E \exp\left[\frac{1}{\epsilon}\left(\int_s^T f\big(t, x(t), u(t)\big)\, dt + g\big(x(T)\big)\right)\right]. \tag{3}$$

Again, we emphasize that the *performance* of the optimal exponential controller $\overline{u}(\cdot)$ [which minimizes (3)] is evaluated using

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the cost functional (2): it is "good" if (2) is small, and "bad" if (2) is large. One limitation of the exponential approach is that optimal risk-sensitive controllers, while being robust, may result in poor performance under normal conditions. The simplest explanation for this is that the exponential utility emphasizes the large values of the (random) exponent

$$\int_s^T f\big(t, x(t), u(t)\big)\, dt + g\big(x(T)\big) \tag{4}$$

thus greatly amplifying the contribution of the large values of (4) in the cost (3). That is, optimal exponential controllers are conservative in that they are designed so that the largest (or "worst case") values of (4), should they occur, are kept small. For this reason, however, the performance of the optimal exponential controller in a "less than worst case" environment (which often corresponds to "normal conditions") may not be satisfactory. Alternatively, one can look at the close relationship between exponential risk-sensitive control and the so-called H_∞ approach to robust control; see [22]. H_∞ controllers are designed to perform well in a "worst case disturbances" environment. Consequently, optimal H_∞ controllers (and therefore, optimal risk-sensitive controllers) may perform poorly, according to (2), in normal conditions.

In this paper, we study certain issues related to the performance and robustness of controllers obtained by minimizing the HARA utility of the cost. In particular, for the system (1) with performance measure (2), the HARA cost functional is given by

$$\begin{aligned} &J\big(s, x;\, u(\cdot)\big) \\ &\quad = E\left\{ \left(\int_s^T f\big(t, x(t), u(t)\big)\, dt + g\big(x(T)\big) \right)^{1/\epsilon} \right\} \end{aligned} \tag{5}$$

where $\epsilon > 0$ is a parameter. Since the HARA utility function is of polynomial rather than exponential order, the resulting controllers should be still robust but less risk-averse than optimal exponential controllers. For this reason, HARA controllers may perform better in normal conditions that optimal exponential controllers which are obtained by minimizing (3). While this remains to be verified conclusively, certain weaker, though related, comparisons between the value functions of the HARA and exponential problems are obtained in this paper. In particular, we show how this may be viewed, in some sense, as a comparison between the *guaranteed performance* of HARA controllers and exponential controllers. In addition, we show that the HARA problem is equivalent to a class of stochastic differential games which are characterized by a cost function that contains a logarithmic term that acts like a weight on different components of the cost. We argue that the logarithmic term in the cost may be viewed as making the disturbance player less aggressive; that is, the (HARA) controller is designed in a "less than worst case" environment. This is consistent with the intuition that HARA controllers may perform better than exponential controllers when operating in normal conditions. In addition, the asymptotic properties (small noise limits) of the HARA problem and the associated differential game are also studied.

Our results on the relationship between the HARA problem and differential games as well as the asymptotic properties of the HARA problem are in line with those presented in [22] for the exponential-of-an-integral problem. However, there are certain differences which we wish to point out. In this paper, we consider a broader class of nonlinear systems. In particular, our results apply to systems in which the diffusion term may be control dependent and/or degenerate, and the drift term may depend nonlinearly on the control. (In fact, our analysis can be modified to obtain parallel results for the exponential case for the same class of nonlinear systems that we study in this paper.) These results are obtained using results from nonsmooth analysis and viscosity solutions. The inclusion of systems with control dependent diffusions in our analysis has particular relevance to finance applications; see [16], [18], [25], [32].

On the other hand, the robust control literature has centered, by and large, around the exponential utility function. This is due to its relationship to differential games and H_∞ control; see [5], [22]. However, the HARA utility function and its associated differential game are alternatives that should be kept in mind, especially when exponential and H_∞ controllers are found to be too conservative.

Interestingly, the HARA utility function has been used in several optimal control formulations of problems in mathematical finance. For example, the problem of long term investment is formulated in [18] as an ergodic control problem where the cost functional involves the HARA utility of wealth. In [13], the integral, over time, of the (discounted) HARA utility of consumption is used as a measure of total consumption (that is to be maximized). For recent work in mathematical finance which uses the exponential utility function, we refer the reader to the papers [6], [7].

We also note that the classical investment problem of maximizing the expected (HARA) utility of terminal wealth is studied in [16]. The lower value of a certain differential game is identified as the large deviation rate of the associated value function. It should be noted however that the analysis in [16] relies heavily on the unique structure associated with the finance application; in particular, the state is scalar, the dynamics are linear in the state, there is no running cost and the terminal cost is linear. The asymptotic results in [16] depend on these assumptions being satisfied. On the other hand, the focus of this paper is the *performance comparison* between HARA controllers and exponential controllers for which we adopt the *standard* assumptions from the risk-sensitive literature. Note however that these standard assumptions require the drift, diffusion, running and terminal costs to be uniformly bounded. That is, the asymptotic results in this paper do not cover those in [16], and vice versa.

The paper is organized as follows. In Section II, we introduce the HARA optimal control problem and present the associated Hamilton–Jacobi–Bellman (HJB) equation. In Section III, we introduce a class of differential games which are related to the problem of finding robust controllers and in Section IV, we show the equivalence between the HARA problem and this family of differential games. In Section V, we study the small noise asymptotic properties of the HARA problem. In Section VI, we obtain certain comparisons between the value function of the

HARA problem and that of the exponential problem. This may be interpreted, in some sense, as a comparison between the guaranteed performance of HARA controllers and exponential controllers. In Section VII, we examine the LQR case of the HARA problem. This example is interesting because it exhibits a relationship between robustness and uncertainty. In particular, we show that the problem of finding a robust controller is equivalent to solving an LQR problem with a larger noise intensity. In Section VIII, we end with some concluding remarks. For the convenience of the reader, we have summarized in the Appendix some key results from the theories of viscosity solutions and nonsmooth analysis which are used in the paper.

II. HARA Stochastic Optimal Control

In this section, we introduce the HARA control problem. Suppose that $s \in [0, T)$ is fixed, $(\Omega, \mathcal{F}, \{\mathcal{F}_t^s\}_{t\geq s}, P)$ is a filtered probability space, and $B(\cdot)$ an $\mathbf{R}^k$-valued standard Brownian motion defined on this space. Suppose that $x \in \mathbf{R}^n$ and $\epsilon > 0$ are given. We consider systems with dynamics given by the following SDE:

$$\begin{cases} dx(t) = b\big(t, x(t), u(t)\big)\,dt + \sqrt{\epsilon}\,\sigma\big(t, x(t), u(t)\big)\,dB(t), \\ \qquad\qquad t \in [s, T] \\ x(s) = x. \end{cases} \tag{6}$$

In this equation, $u(\cdot)$ is a U-valued process, referred to as the control input. (The precise definition of admissible controls will be given later.) The associated cost functional is given by

$$J\big(s, x;\, u(\cdot)\big) = E\left\{\left(\int_s^T f\big(t, x(t), u(t)\big)\,dt + g\big(x(T)\big)\right)^{1/\epsilon}\right\} \tag{7}$$

where $x(\cdot)$ is the solution of the SDE (6) corresponding to (s, x) and $u(\cdot)$. We introduce the following assumptions.

Assumptions: (**A1**) (U, d) is a Polish space[1] and $T > 0$.

(**A2**) The maps b: $[0, T] \times \mathbf{R}^n \times U \to \mathbf{R}^n$, σ: $[0, T] \times \mathbf{R}^n \times U \to \mathbf{R}^{n\times k}$, f: $[0, T] \times \mathbf{R}^n \times U \to \mathbf{R}$ and g: $\mathbf{R}^n \to \mathbf{R}$ are uniformly continuous and bounded. f is nonnegative on $[0, T] \times \mathbf{R}^n \times U$ and there exists a constant $C > 0$ such that $g(x) \geq C$ for every $x \in \mathbf{R}^n$. Also, there exists a constant $L > 0$ such that for $\varphi(t, x, u) = b(t, x, u)$, $\sigma(t, x, u)$, $f(t, x, u)$, $g(x)$

$$|\varphi(t, x, u) - \varphi(t, y, u)| \leq L\,|x - y| \qquad \forall\, t \in [0, T], \quad x, y \in \mathbf{R}^n, \quad u \in U. \tag{8}$$

Remark II.1: The assumption that f is nonnegative and g is uniformly positive (i.e., $g \geq C > 0$) is required for the HARA utility function to be well defined. The remaining assumptions in (**A1**) and (**A2**) (and in particular, that of uniform boundedness) are standard; see [5], [22] for risk-sensitive control and [20] for differential games. On the other hand, under the assumption of uniform boundedness, the additional requirement that f and g are nonnegative/uniformly positive is not restrictive. More precisely, recall that our basic aim is to find a controller $u(\cdot)$ for the system (1). If f is *not* nonnegative or g is *not* uniformly positive, we can replace f and/or g by $\overline{f} := f + k$ and $\overline{g} := g + k$ for some sufficiently large constant k and then use the performance measure

$$\overline{J}\big(s, x;\, u(\cdot)\big) = E\left\{\int_s^T \overline{f}\big(t, x(t), u(t)\big)\,dt + \overline{g}\big(x(T)\big)\right\}$$

instead of (2), to evaluate the performance of any given controller. Clearly (2) and $\overline{J}$ are equivalent and (due the assumption of uniform boundedness) k can be chosen so that $\overline{f}$ and $\overline{g}$ satisfy the required conditions. In particular, a risk-averse controller with performance measured by $\overline{J}$ [or equivalently by (2)] can now be obtained by minimizing the HARA cost (7) with f and g replaced by $\overline{f}$ and $\overline{g}$.

In order to study (6) and (7) using dynamic programming, we introduce the following controlled SDE and cost functional, of which (6) and (7) are a special case. Let $\overline{y} \in (0, C)$, where C is the constant in (**A2**). For every $(s, x, y) \in [0, T) \times \mathbf{R}^n \times (-\overline{y}, \infty)$, consider the following SDE:

$$\begin{cases} dx(t) = b\big(t, x(t), u(t)\big)\,dt \\ \qquad\qquad + \sqrt{\epsilon}\,\sigma\big(t, x(t), u(t)\big)\,dB(t) \\ dy(t) = f\big(t, x(t), u(t)\big)\,dt, \quad t \in [s, T] \\ x(s) = x, \quad y(s) = y \end{cases} \tag{9}$$

and cost functional

$$J\big(s, x, y;\, u(\cdot)\big) = E\left\{\big(y(T) + g\big(x(T)\big)\big)^{1/\epsilon}\right\}. \tag{10}$$

Note in particular that since $y \in (-\overline{y}, \infty)$, f is nonnegative, $g \geq C$, and f and g are uniformly bounded [see (**A2**)], there is a constant $C_1 < \infty$ such that

$$\begin{aligned} y + C_1 &> y(T) + g\big(x(T)\big) \\ &= y + \int_s^T f\big(t, x(t), u(t)\big)\,dt + g\big(x(T)\big) \\ &> C - \overline{y} > 0. \end{aligned} \tag{11}$$

Therefore, $y(T) + g\big(x(T)\big) > 0$ (which is required if we are to use the HARA utility function) and the cost (10) is well defined.

Clearly, (6) and (7) corresponds to the special case of (9) and (10) when $y = 0$. The class of admissible controls (in the weak formulation; see [30, Ch. 2]) is the set of 5-tuples $\big(\Omega, \mathcal{F}, P, B(\cdot), u(\cdot)\big)$ which satisfy the following properties:

1) $(\Omega, \mathcal{F}, P)$ is a complete probability space;
2) $\{B(t)\}_{t\geq s}$ is a $\mathbf{R}^k$-dimensional standard Brownian motion defined on $(\Omega, \mathcal{F}, P)$ over $[s, T]$, and $\mathcal{F}_t^s$ is $\sigma\{B(r)\,|\, s \leq r \leq t\}$ augmented by all the P-null sets in $\mathcal{F}$;
3) u: $[s, T] \times \Omega \to U$ is an $\{F_t^s\}_{t\geq s}$-progressively measurable process on $(\Omega, \mathcal{F}, P)$;
4) Under $u(\cdot)$, for any initial condition $(x, y) \in \mathbf{R}^n \times \mathbf{R}$, the SDE (9) admits a unique weak solution $\big(x(\cdot), y(\cdot)\big)$ on $(\Omega, \mathcal{F}, \{\mathcal{F}_t^s\}_{t\geq s}, P)$.

[1]A Polish space is a separable complete metric space.

If it is clear from the context what $(\Omega, \mathcal{F}, P)$ and $B(\cdot)$ are, we will write $u(\cdot) \in \mathcal{U}[s, T]$ as shorthand for 5-tuple $\big(\Omega, \mathcal{F}, P, B(\cdot), u(\cdot)\big) \in \mathcal{U}[s, T]$.

Assumptions (**A1**) and (**A2**) guarantee the existence of a unique weak solution $\big(x(\cdot), y(\cdot)\big)$ of the SDE (9), for every admissible control $u(\cdot) \in \mathcal{U}[s, T]$. Furthermore, the cost functional (10) is well defined for every $(s, x, y) \in [0, T] \times \mathbf{R}^n \times (-\overline{y}, \infty)$ and $u(\cdot) \in \mathcal{U}[s, T]$. The value function associated with (9) and (10)

$$v(s, x, y) = \inf_{u(\cdot)\in\mathcal{U}[s, T]} J\big(s, x, y; u(\cdot)\big), \quad (s, t, x, y) \in [0, T] \times \mathbf{R}^n \times (-\overline{y}, \infty) \tag{12}$$

is well defined [30], and has a positive lower bound.

The HJB equation associated with (9) and (10) is

$$\begin{cases} v_t + \inf_{u\in U} \left\{ \dfrac{\epsilon}{2} \operatorname{tr}\left[v_{xx}\, \sigma\, \sigma'\right] + v_x' b + v_y\, f \right\} = 0 \\ \quad (t, x, y) \in [0, T) \times \mathbf{R}^n \times (-\overline{y}, \infty) \\ v(T, x, y) = \big(y + g(x)\big)^{1/\epsilon}, \quad (x, y) \in \mathbf{R}^n \times (-\overline{y}, \infty). \end{cases} \tag{13}$$

We have the following result.

Theorem II.1: Assume that (**A1**) and (**A2**) hold. Then, the value function v, as defined by (12), is the unique viscosity solution of (13).

Proof: Following the proof in [19, Sec.V3 and V9], it can be shown that v is a viscosity solution of (13). (Note that in this proof, continuity of v is all that is assumed.) However, for the uniqueness we cannot immediately apply the results in [19] since the terminal condition of (13) is unbounded due to the presence of y. To get around this, we follow a technique in [27] and consider the following transformation:

$$\overline{v}(t, x, y) = \langle y\rangle^{-(1/\epsilon)}\, v(t, x, y)$$

where $\langle y\rangle := \sqrt{1 + |y|^2}$. Then, (13) becomes

$$\begin{cases} \overline{v}_t + \inf\limits_{u\in U} \left\{ \dfrac{\epsilon}{2} \operatorname{tr}[\overline{v}_{xx}\, \sigma\, \sigma'] \right. \\ \qquad\qquad \left. + \overline{v}_x'\, b + \left(\overline{v}_y + \dfrac{1}{\epsilon}\, \dfrac{y}{\langle y\rangle^2}\, \overline{v} \right) f \right\} = 0 \\ \quad (t, x, y) \in [0, T) \times \mathbf{R}^n \times (-\overline{y}, \infty) \\ \overline{v}(T, x, y) = \left(\dfrac{y + g(x)}{\langle y\rangle} \right)^{1/\epsilon} \\ \quad (x, y) \in \mathbf{R}^n \times (-\overline{y}, \infty). \end{cases}$$

This equation has a uniformly bounded terminal condition and, hence, admits at most one viscosity solution. It follows then that (12) is the unique viscosity solution of (13). ∎

III. Differential Games

In this section, we introduce a class of differential games which can be used to find robust controllers.

A. General Formulation

We shall follow the Elliott–Kalton formulation of two-player, zero-sum differential games [15], [20], [22], a summary of which is as follows. Suppose that $s \in [0, T)$ is given and fixed. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t^s\}_{t\geq s}, P)$ be a fixed filtered probability space, and $B(\cdot)$ a fixed $\mathbf{R}^k$-valued standard Brownian motion on this space. Assume that $\mathcal{F}_t^s$ is $\sigma\{B(r) \,|\, s \leq r \leq t\}$ augmented with all the P-null sets of $\mathcal{F}$. Suppose that the system dynamics are governed by the following SDE:

$$\begin{cases} dx(t) = b\big(t, x(t), u(t), w(t)\big)\, dt \\ \quad +\sigma\big(t, x(t), u(t), w(t)\big)\, dB(t), \qquad t \in [s, T] \\ x(s) = x \end{cases} \tag{14}$$

where $u(\cdot)$ is the U-valued input of player 1 (the control player), and $w(\cdot)$ is the W-valued input of player 2 (the disturbance player, or opponent). The cost functional is given by

$$J\big(s, x;\, u(\cdot), w(\cdot)\big) = E\left\{ \int_s^T f\big(t, x(t), u(t), w(t)\big)\, dt + g\big(x(T)\big) - \frac{1}{2} \int_s^T |w(t)|^2\, dt \right\}. \tag{15}$$

1) Admissible Inputs: The set of admissible controls for player 1 is

$$\mathcal{U}[s, T] = \{u(\cdot)\colon [s, T] \times \Omega \to U \,|\, u(\cdot) \text{ is } \mathcal{F}_t^s\text{-progressively measurable}\}. \tag{16}$$

For any $t \in (s, T]$, two admissible inputs $u_1(\cdot), u_2(\cdot) \in \mathcal{U}[s, T]$ are said to be *equivalent on* $[s, t]$ if $u_1(\cdot) = u_2(\cdot)$ a.e. on $[s, t]$, P-a.s. We shall denote this by $u_1(\cdot) \approx u_2(\cdot)$ on $[s, t]$.

The set of admissible disturbances for player 2 is

$$\mathcal{W}[s, T] = \{w(\cdot)\colon [s, T] \times \Omega \to W \,|\, w(\cdot) \text{ is } \mathcal{F}_t^s\text{-progressively measurable and (14) has a unique solution } \forall\, u(\cdot) \in \mathcal{U}[s, T]\}. \tag{17}$$

As in the case of admissible controls, we consider two admissible disturbances $w_1(\cdot), w_2(\cdot) \in \mathcal{W}[s, T]$ as being equivalent on $[s, t]$ (for a given $t \in (s, T]$) if $w_1(\cdot) = w_2(\cdot)$ a.e. on $[s, t]$, P-a.s., and denote this by $w_1(\cdot) \approx w_2(\cdot)$ on $[s, t]$.

In the deterministic case (i.e., $\sigma = 0$), the class of admissible controls/disturbances is given by

$$\mathcal{U}_d[s, T] = \{u(\cdot)\colon [s, T] \to U \,|\, u(\cdot) \text{ is } \mathcal{B}[s, T]\text{-measurable}\}$$

and

$$\mathcal{W}_d[s, T] = \{w(\cdot)\colon [s, T] \to W \,|\, w(\cdot) \text{ is } \mathcal{B}[s, T]\text{-measurable and (14) has a unique solution } \forall\, u(\cdot) \in \mathcal{U}_d[s, T]\}$$

where $\mathcal{B}[s, T]$ denotes the Borel σ-algebra on $[s, T]$. Also, for any $t \in (s, T]$ and $u_1(\cdot), u_2(\cdot) \in \mathcal{U}_d[s, T]$, $u_1(\cdot) \approx u_2(\cdot)$ on $[s, t]$ if $u_1(\cdot) = u_2(\cdot)$ a.e. on $[s, t]$. (Similarly for disturbances.)

2) Admissible Strategies: The class of *admissible strategies* for player 2 is

$$\Gamma[s, T] = \{\alpha: \mathcal{U}[s, T] \rightarrow \mathcal{W}[s, T] \mid \text{for every } t \in [s, T], u_1(\cdot) \approx u_2(\cdot) \text{ on } [s, t] \Rightarrow \alpha[u_1(\cdot)] \approx \alpha[u_2(\cdot)] \text{ on } [s, t]\}. \tag{18}$$

Similarly, the class of admissible strategies for player 1 is

$$\Delta[s, T] = \{\beta: \mathcal{W}[s, T] \rightarrow \mathcal{U}[s, T] \mid \text{for every } t \in [s, T], w_1(\cdot) \approx w_2(\cdot) \text{ on } [s, t] \Rightarrow \beta[w_1(\cdot)] \approx \beta[w_2(\cdot)] \text{ on } [s, t]\}. \tag{19}$$

In the deterministic case, the admissible strategies of players 1 and 2 (denoted by $\Delta_d[s, T]$ and $\Gamma_d[s, T]$, respectively), are defined in the obvious manner.

3) Upper/Lower Differential Games: The *upper stochastic differential game* associated with (14) and (15) can be stated as follows: Find $(\overline{u}(\cdot), \overline{w}(\cdot)) \in \mathcal{U}[s, T] \times \mathcal{W}[s, T]$ such that

$$V^+(s, x) := J(s, x; \overline{u}(\cdot), \overline{w}(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}[s, T]} \sup_{\alpha[\cdot] \in \Gamma[s, T]} J(s, x; u(\cdot), \alpha[u(\cdot)]). \tag{20}$$

V^+ is commonly referred to as the *upper value* of the stochastic differential game (14) and (15).

The *lower stochastic differential game* associated with (14) and (15) can be stated as follows: Find $(\overline{u}(\cdot), \overline{w}(\cdot)) \in \mathcal{U}[s, T] \times \mathcal{W}[s, T]$ such that

$$V^-(s, x) := J(s, x; \overline{u}(\cdot), \overline{w}(\cdot)) = \sup_{w(\cdot) \in \mathcal{W}[s, T]} \inf_{\beta[\cdot] \in \Delta[s, T]} J(s, x; \beta[w(\cdot)], w(\cdot)). \tag{21}$$

V^- is commonly referred to as the *lower value* of the (stochastic) differential game (14) and (15).

It is well known that $V^- \leq V^+$ on $[0, T] \times \mathbf{R}^n$ (see [4], [20]). On the other hand, the differential game (14) and (15) is said to have *value* if $V := V^+ = V^-$.

Upper and lower deterministic differential games are defined analogously.

4) Isaacs Equations: A summary of basic definitions and results from the theory of viscosity solutions can be found in the Appendix.

Under certain assumptions, it can be shown that V^+ is the unique viscosity solution of the *upper Isaacs equation*

$$\begin{cases} V_t + H^+(t, x, V_x, V_{xx}) = 0, & (t, x) \in [0, T) \times \mathbf{R}^n \\ V(T, x) = g(x) \end{cases} \tag{22}$$

while V^- is the unique viscosity solution of the *lower Isaacs equation*

$$\begin{cases} V_t + H^-(t, x, V_x, V_{xx}) = 0, & (t, x) \in [0, T) \times \mathbf{R}^n \\ V(T, x) = g(x) \end{cases} \tag{23}$$

where

$$H^+(t, x, p, P) = \inf_{u \in U} \sup_{w \in W} \left\{ \frac{1}{2} \operatorname{tr} \sigma(t, x, u, w) \sigma(t, x, u, w)' P + b(t, x, u, w) \cdot p + f(t, x, u, w) \right\}$$

and

$$H^-(t, x, p, P) = \sup_{w \in W} \inf_{u \in U} \left\{ \frac{1}{2} \operatorname{tr} \sigma(t, x, u, w) \sigma(t, x, u, w)' P + b(t, x, u, w) \cdot p + f(t, x, u, w) \right\}.$$

(In the next section, we shall present assumptions which guarantee existence and uniqueness of viscosity solutions for the class of problems which we are studying.) In the deterministic case, the upper and lower Isaacs equations are given by (22) and (23), with $\sigma = 0$. A sufficient condition for the existence of value in (14) and (15) is the so-called *Isaacs* (or min–max) *condition*

$$H^+(t, x, p, P) = H^-(t, x, p, P). \tag{24}$$

Under this assumption, the existence of value follows immediately from the uniqueness of viscosity solutions of (22) and (23). (For further details about the Isaacs equation, the reader is directed to [20] for the stochastic case, and [4] for the deterministic case.)

B. A Class of Differential Games

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t^s\}_{t \geq s}, P)$ be a filtered probability space and $B(\cdot)$ a standard $\mathbf{R}^k$-valued Brownian motion, as discussed in Section III-A. Let $x \in \mathbf{R}^n$ and $\epsilon > 0$ be given and fixed. Consider the following special case of (14):

$$\begin{cases} dx(t) = [b(t, x(t), u(t)) + \sigma(t, x(t), u(t)) w(t)] \, dt \\ \quad + \sqrt{\epsilon} \, \sigma(t, x(t), u(t)) \, dB(t), \qquad t \in [s, T] \\ x(s) = x \end{cases} \tag{25}$$

where the inputs $u(\cdot)$ of player 1 and $w(\cdot)$ of player 2 satisfy $u(\cdot) \in \mathcal{U}[s, T]$ and $w(\cdot) \in \mathcal{W}[s, T]$. We consider cost functionals of the following form:

$$J(s, x; u(\cdot), w(\cdot)) = E\left\{ \ln\left(\int_s^T f(t, x(t), u(t)) \, dt + g(x(T)) \right) - \frac{1}{2} \int_s^T |w(t)|^2 \, dt \right\}. \tag{26}$$

We assume throughout this section that (**A2**) holds. In certain places, we shall replace (**A1**) by the following.

Assumption: (**A1**)′ $U \subseteq \mathbf{R}^m$ and $W \subseteq \mathbf{R}^k$ are compact, and $T > 0$.

In order to study (25) and (26) using dynamic programming, we follow the same procedure that we used for the HARA problem (Section II) by considering the following related differential game. Let $\overline{y} \in (0, C)$ be fixed, where $C > 0$ is the constant in (**A2**), and $y \in (-\overline{y}, \infty)$. Consider the SDE

$$\begin{cases} dx(t) = [b(t, x(t), u(t)) + \sigma(t, x(t), u(t))\, w(t)]\, dt \\ \quad +\sqrt{\epsilon}\, \sigma(t, x(t), u(t))\, dB(t) \\ dy(t) = f(t, x(t), u(t))\, dt, \qquad t \in [s, T] \\ x(s) = x, \quad y(s) = y \end{cases} \tag{27}$$

with cost

$$J(s, x, y;\, u(\cdot), w(\cdot)) = E\left\{ \ln\Big(y(T) + g(x(T))\Big) - \frac{1}{2}\int_s^T |w(t)|^2\, dt \right\}. \tag{28}$$

Using the same argument for (28) as for (11), it can be seen that for every $\epsilon > 0$, there is a constant $K_1 < \infty$, independent of ϵ, such that

$$y + K_1 > y(T) + g(x(T)) > C - \overline{y} > 0$$

for all $(s, x, y) \in [0, T] \times \mathbf{R}^n \times (-\overline{y}, \infty)$ and $(u(\cdot), w(\cdot)) \in \mathcal{U}[s, T] \times \mathcal{W}[s, T]$. In addition, when $W \subseteq \mathbf{R}^k$ is compact by (**A1**)$'$, it follows that there is a constant $K_2 > -\infty$, which is independent of $\epsilon > 0$, such that

$$K_2 < J(s, x, y;\, u(\cdot), w(\cdot)) < \ln(y + K_1) \tag{29}$$

for all $(s, x, y) \in [0, T] \times \mathbf{R}^n \times (-\overline{y}, \infty)$ and $(u(\cdot), w(\cdot)) \in \mathcal{U}[s, T] \times \mathcal{W}[s, T]$. The upper Isaacs equation associated with (27) and (28) is

$$\begin{cases} \psi_t + \inf\limits_{u \in U} \sup\limits_{w \in W} \Big\{ \frac{\epsilon}{2}\, \mathrm{tr}\,[\psi_{xx}\, \sigma\, \sigma'] \\ \qquad\qquad + \psi_x'(b + \sigma\, w) + \psi_y'\, f - \frac{1}{2}\, |w|^2 \Big\} = 0 \\ \quad (t, x, y) \in [0, T) \times \mathbf{R}^n \times (-\overline{y}, \infty) \\ \psi(T, x, y) = \ln\big(y + g(x)\big) \\ \quad (x, y) \in \mathbf{R}^n \times (-\overline{y}, \infty). \end{cases} \tag{30}$$

Theorem III.1: Suppose that (**A1**)$'$ and (**A2**) hold. Then

$$\psi(s, x, y) = \inf_{u(\cdot) \in \mathcal{U}[s, T]} \sup_{\alpha[\cdot] \in \Gamma[s, T]} J(s, x, y;\, u(\cdot), \alpha[u(\cdot)]) \tag{31}$$

is the unique viscosity solution of (30).

Proof: Following the arguments in [20], it can be shown that (31) is a viscosity solution of (30). As in the case of Theorem II.1, we can not immediately use the results in [20] to obtain uniqueness since the terminal condition in (30) is unbounded. To get around this, consider the following transformation:

$$\overline{\psi}(t, x, y) = \psi - \ln\langle y \rangle$$

where $\langle y \rangle := \sqrt{1 + |y|^2}$. Following the same arguments as in the proof of Theorem II.1, we conclude that (30) has at most one viscosity solution. ■

It follows from Theorem III.1 that the upper value of the stochastic differential game (27) and (28) is the unique viscosity solution of (30).

The deterministic differential game (25) and (26) [respectively, (27) and (28)] is the special case of (25) and (26) [respectively, (27) and (28)] when $\epsilon = 0$ and the classes of admissible inputs for players 1 and 2 are $\mathcal{U}_d[s, T]$ and $\mathcal{W}_d[s, T]$, respectively. The corresponding upper Isaacs equation is (30) with $\epsilon = 0$. We have the following result.

Theorem III.2: Suppose that (**A1**)$'$ and (**A2**) hold. Then

$$\psi(t, x, y) = \inf_{u(\cdot) \in \mathcal{U}_d[s, T]} \sup_{\alpha[\cdot] \in \Gamma_d[s, T]} J(s, x, y;\, u(\cdot), \alpha[u(\cdot)]) \tag{32}$$

is the unique viscosity solution of the upper Isaacs equation (30) with $\epsilon = 0$.

Proof: Following the same arguments in [4], it can be shown that (32) is a viscosity solution of (30) (with $\epsilon = 0$). Uniqueness is shown using the same techniques as in Theorem III.1. ■

Clearly, (32) is the upper value of the deterministic differential game (27) and (28) with $\epsilon = 0$.

IV. HARA Problems and Differential Games

In this section, we show that (under certain conditions) the HARA problem (9) and (10) is equivalent to the stochastic differential game (27) and (28).

Consider the following transformation:

$$\phi(t, x, y) := \epsilon \ln v(t, x, y) \qquad \forall\, (t, x, y) \in [0, T] \times \mathbf{R}^n \times (-\overline{y}, \infty). \tag{33}$$

Note that by (10)–(12), ϕ is well defined [since $v(t, x, y) > 0$] and

$$-\infty < \ln(C - \overline{y}) < \phi(t, x, y) < \ln(y + C_1) \qquad \forall\, (t, x, y) \in [0, T] \times \mathbf{R}^n \times (-\overline{y}, \infty). \tag{34}$$

Since v is the unique viscosity solution of (13) (Theorem II.1), it follows [from substituting (33) into (13)] that ϕ is the unique viscosity solution of the following PDE:

$$\begin{cases} \phi_t + \inf\limits_{u \in U} \sup\limits_{w \in \mathbf{R}^k} \Big\{ \frac{\epsilon}{2}\, \mathrm{tr}\,[\phi_{xx}\, \sigma\, \sigma'] \\ \qquad\qquad + \phi_x'(b + \sigma\, w) + \phi_y'\, f - \frac{1}{2}\, |w|^2 \Big\} = 0 \\ \quad (t, x, y) \in [0, T) \times \mathbf{R}^n \times (-\overline{y}, \infty) \\ \phi(T, x, y) = \ln\big(y + g(x)\big) \\ \quad (x, y) \in \mathbf{R}^n \times (-\overline{y}, \infty). \end{cases} \tag{35}$$

Although the PDE (35) closely resembles the upper Isaacs equation (30), we are not yet able to use Theorem III.1 to characterize ϕ as the upper value of the stochastic differential game (27) and (28) because the $\sup$ in (35) is over $w \in \mathbf{R}^k$ whereas the $\sup$ in (30) is over $w \in W$, where $W \subseteq \mathbf{R}^k$ is a compact set. Note that the compactness of W is necessary in proving Theorem III.1; see also [20]. In order to establish the equivalence between the HARA problem and the stochastic differential game, we need Proposition IV.1, which gives conditions under which $W \subseteq \mathbf{R}^k$ may be considered as being compact. The proof of Proposition IV.1 depends on the following assumption.

Assumption: (**A3**) $\sigma(t, x, u) = \sigma(t, u)$, and $b(t, x, u)$, $f(t, x, u)$ and $g(x)$ are differentiable in $x, \forall (t, u) \in [0, T] \times U$.

Proposition IV.1: Suppose that $\epsilon > 0$ and Assumptions (**A1**), (**A2**) and (**A3**) hold. Let v be the unique viscosity solution of (30). Suppose that $\phi := \epsilon \ln v$, and $D^{1,2,+}_{t,(x,y)}\phi(s, x, y)$ and $D^{1,2,-}_{t,(x,y)}\phi(s, x, y)$, the super and subdifferentials of ϕ, respectively, be defined as in (94) and (95) in the Appendix. Then, there exists $K < \infty$, independent of $(s, x, y) \in [0, T] \times \mathbf{R}^n \times (-\overline{y}, \infty)$ and $\epsilon > 0$, such that $|p| \le K$, for all $(q, p, P) \in D^{1,2,+}_{t,(x,y)}\phi(s, x, y) \cup D^{1,2,-}_{t,(x,y)}\phi(s, x, y)$.

Proof: Let $\epsilon > 0$ and $(s, x, y) \in [0, T] \times \mathbf{R}^n \times (-\overline{y}, \infty)$ be given and fixed, and $(\overline{x}(\cdot), \overline{y}(\cdot), \overline{u}(\cdot))$ be an admissible triple for (9) with the initial condition (s, x, y). For any $z \in \mathbf{R}^n$, let $(x(\cdot), y(\cdot), \overline{u}(\cdot))$ be an admissible triple for (9) corresponding to the initial condition $(s, x + z, y)$. It follows that

$$x(t) - \overline{x}(t) = \Psi(t)\, z \tag{36}$$

where the $\mathbf{R}^{n\times n}$-valued process $\Psi(\cdot)$ is the solution of the SDE

$$\begin{cases} d\Psi(t) = [b_x(t, \overline{x}(t), \overline{u}(t)) + \Sigma_1(t)]'\, \Psi(t)\, dt, \\ \quad t \in [s, T], \\ \Psi(s) = I \end{cases} \tag{37}$$

and

$$\Sigma_1(t) = \int_0^1 [b_x\Big(t, \overline{x}(t) + \alpha\,(x(t) - \overline{x}(t)), \overline{u}(t)\Big) - b_x(t, \overline{x}(t), \overline{u}(t))]\, d\alpha.$$

In addition

$$\begin{aligned} y(t) - \overline{y}(t) &= \left(\int_s^t [f_x(r, \overline{x}(r), \overline{u}(r)) + \Sigma_2(r)]\, \Psi(r)\, dr\right)' z \\ &= \Phi(t)\, z \end{aligned} \tag{38}$$

where $\Sigma_2(\cdot)$ is defined like $\Sigma_1(\cdot)$, but with f_x replacing b_x. Since b is Lipschitz continuous in x, uniformly in $(t, u) \in [0, T] \times U$, it follows from Gronwall's inequality that there exists a constant $K > 0$, which is independent of (s, x, y) and $\epsilon > 0$, such that

$$|\Phi(t)| \le K, \quad |\Psi(t)| \le K, \qquad \forall\, t \in [s, T], \quad P-\text{a.s.} \tag{39}$$

In particular, the bound (39) (together with the uniform Lipschitz continuity of f in x) implies that

$$|x(T) - \overline{x}(T)| \le K\,|z|, \quad |y(T) - \overline{y}(T)| \le K\,|z|. \tag{40}$$

For any $\delta > 0$, there exists an admissible triple $(\overline{x}(\cdot), \overline{y}(\cdot), \overline{u}(\cdot))$, possibly depending on (s, x, y), ϵ, and δ, such that

$$\phi(s, x, y) \ge J(s, x, y;\, \overline{u}(\cdot)) - \delta$$

where

$$J(s, x, y;\, u(\cdot)) = \epsilon \ln E\left\{\big(y(T) + g(x(T))\big)^{1/\epsilon}\right\}$$

and $(x(\cdot), y(\cdot))$ is the state process obtained from (9) when the input is $u(\cdot)$. Therefore

$$\begin{aligned} &\phi(s, x + z, y) - \phi(s, x, y) \\ &\quad \le J(s, x + z, y;\, \overline{u}(\cdot)) - J(s, x, y;\, \overline{u}(\cdot)) + \delta. \end{aligned} \tag{41}$$

It is easy to show that

$$\begin{aligned} &J(s, x + z, y;\, \overline{u}(\cdot)) - J(s, x, y;\, \overline{u}(\cdot)) \\ &= \frac{1}{E\left\{[y(T) + g(x(T))]^{1/\epsilon}\right\}} \\ &\quad \times E\left\{\frac{[y(T) + g(x(T))]^{1/\epsilon}}{[y(T) + g(x(T))]} \times \begin{bmatrix} g_x(\overline{x}(T)) \\ 1 \end{bmatrix}' \begin{bmatrix} x(T) - \overline{x}(T) \\ y(T) - \overline{y}(T) \end{bmatrix}\right\} \\ &\quad + o\left(|x(T) - \overline{x}(T)| + |y(T) - \overline{y}(T)|\right). \end{aligned} \tag{42}$$

Since

$$\overline{y}(T) + g(\overline{x}(T)) = y + \int_s^T f(t, \overline{x}(t), \overline{u}(t))\, dt + g(\overline{x}(T))$$

it follows from (**A2**) that

$$\begin{aligned} 0 < \frac{1}{y + K} \le \frac{1}{y(T) + g(x(T))} \le \frac{1}{C - \overline{y}} \\ \forall\, y \in (-\overline{y}, \infty) \end{aligned} \tag{43}$$

where $C > 0$ is the constant in (**A2**) and $0 < K < \infty$. Note that C and K are independent of ϵ, z, (s, x, y) and $\overline{u}(\cdot)$. Substituting (36) and (38) into (42), and noting (39), (43), and the uniform Lipschitz continuity of g, it follows that:

$$\begin{aligned} &\phi(s, x + z, y) - \phi(s, x, y) \\ &\quad \le J(s, x + z, y;\, \overline{u}(\cdot)) - J(s, x, y;\, \overline{u}(\cdot)) + \delta \\ &\quad \le K_1\,|z| + o(|z|) + \delta \end{aligned} \tag{44}$$

for some constant $K_1 < \infty$, which is independent of $\overline{u}(\cdot)$, (s, x, y), z, $\epsilon \in (0, 1)$ and $\delta > 0$. Since this is true for all $\delta > 0$, we can let $\delta \to 0$ in (44). In a similar way, the reverse inequality can be obtained. Hence, it follows from (44) and (102) (see Appendix) that $|p| \le K$ for all $p \in \partial_x\phi(s, x, y)$, where $\partial_x\phi$ denotes the partial generalized gradient of ϕ with respect to x; see the remarks following Definition VIII.3 in the Appendix. Our result is then an immediate consequence of

$$\begin{aligned} &(q, p, P) \in D^{1,2+}_{t,(x,y)}\phi(t, x, y) \cup D^{1,2-}_{t,(x,y)}\phi(t, x, y) \\ &\quad \Rightarrow p \in \partial_x\, \phi(t, x, y). \end{aligned}$$

[This follows from (99), (100), and (103)]. ■

Theorem IV.1: Suppose that Assumptions (**A1**), (**A2**) and (**A3**) hold. Let $\epsilon > 0$ be fixed, and v be the unique viscosity solution of (13). Then $\phi := \epsilon \ln v$ is the unique viscosity solution of (35). Moreover, there exists a compact subset

$W \subseteq \mathbf{R}^k$ which is independent of $\epsilon > 0$ such that ϕ has the representation

$$\phi(s, t, x, y) = \inf_{u(\cdot)\in\mathcal{U}[s,T]} \sup_{\alpha[\cdot]\in\Gamma[s,T]} J(s, x, y; u(\cdot), \alpha[u(\cdot)]) \tag{45}$$

where $\mathcal{U}[s, T]$ is defined by (16), $\Gamma[s, T]$ by (18) via (17), and

$$J\big(s, x, y; u(\cdot), w(\cdot)\big) = E\left\{\ln[y(T) + g\big(x(T)\big)] - \frac{1}{2}\int_s^T |w(t)|^2\,dt\right\}$$

with $\big(x(\cdot), y(\cdot)\big)$ being the solution of (27) associated with $\big(u(\cdot), w(\cdot)\big)$.

Proof: It is immediate from the derivation of (35) that $\phi = \epsilon \ln v$ is the unique viscosity solution of (35). Since ϕ is a viscosity super-solution of (35), it follows that for every $(q, p, P) \in D_{t,x}^{1,2,+}\phi(t, x, y)$ where $p = (p_1, p_2) \in \mathbf{R}^n \times \mathbf{R}$, we have

$$q + \inf_{u\in U} \sup_{w\in\mathbf{R}^k} \left\{\frac{\epsilon}{2}\,\mathrm{tr}[P\,\sigma\,\sigma'] \right. \tag{46}$$

$$\left. + p_1'(b + \sigma\,w) + p_2\,f - \frac{1}{2}|w|^2\right\} \geq 0. \tag{47}$$

Moreover, the maximizing $w \in \mathbf{R}^k$ in (47) is given by

$$w = p_1'\,\sigma(t, u). \tag{48}$$

Since $\sigma(t, u)$ and p_1 are uniformly bounded on $[0, T] \times U$ and $[0, T] \times \mathbf{R}^n \times (-\overline{y}, \infty)$ respectively [see (**A2**) and Proposition IV.1], it follows that

$$w = p_1'\,\sigma(t, u) \in \overline{W} \quad \forall (t, x, y) \in [0, T] \times \mathbf{R}^n \times (-\overline{y}, \infty), \qquad u \in U$$

for some compact $\overline{W} \subseteq \mathbf{R}^k$, which is independent of ϵ. Therefore, we may replace $w \in \mathbf{R}^k$ by $w \in W$ in (47), for any compact $W \subseteq \mathbf{R}^k$ (with $\overline{W} \subseteq W$), and the maximizing w in (47) will still be given by (48) and $\phi = \epsilon \ln v$ is still a super-solution of (47). It is easy to see that the same argument applies for the case of sub-solutions and hence, under the constraint $w \in W$, $\phi = \epsilon \ln v$ is still the unique viscosity solution of (35). However, under the constraint that $w \in W$, it follows from Theorem III.1 that the unique viscosity solution of (35) is the upper value of a stochastic differential game, as defined by the right hand side of (45). Hence, we have equality in (45). ■

Remark IV.1: It is clear from the proof of Theorem IV.1 that we may choose W to be a closed ball in $\mathbf{R}^k$, centered at the origin, of sufficiently large radius; that is, $W = B(0, R)$, for some $R > 0$. In particular, this guarantees that $0 \in W$, which simplifies some of the analysis in Section VI-B.

V. Asymptotic Analysis

For the remainder of this paper, for any given $\epsilon > 0$, we shall denote the value function of the HARA problem (9) and (10), as defined by (12), by $v^{(\epsilon)}$, and $\phi^{(\epsilon)} := \epsilon \ln v^{(\epsilon)}$. A similar comment applies to the upper value $\psi^{(\epsilon)}$ of the differential game (27) and (28) when $\epsilon > 0$. In the deterministic case of (27) and (28) with $\epsilon = 0$, we continue to use ψ.

We have shown that under Assumptions (**A1**), (**A2**), and (**A3**) with $\epsilon > 0$, the HARA problem (9) and (10) is equivalent to a stochastic differential game of the form (27) and (28); see Theorem IV.1. In this section, we study the asymptotic properties of $\phi^{(\epsilon)}$ and $\psi^{(\epsilon)}$ as $\epsilon \to 0$. We prove that $\psi^{(\epsilon)} \to \psi$ and hence [under (**A1**), (**A2**) and (**A3**)] $\phi^{(\epsilon)} \to \psi$ when $\epsilon \to 0$. This reveals a relationship between the HARA problem and the deterministic differential game.

Our convergence proof follows the general methods of Barles and Perthame [2]. In particular, the notion of solution that is used in this approach is the generalized definition of a discontinuous viscosity solution. This is required since the functions (49) and (51) below are only semi-continuous in general. In addition, the proof uses a comparison theorem for semi-continuous viscosity sub- and super-solutions. The definition of a discontinuous viscosity solution is quite similar to that of a continuous solution. The reader should refer to [19, Ch. VII] for a detailed account of the Barles and Perthame method. The definition of a discontinuous viscosity solution as well as the comparison theorem for semi-continuous sub and supersolutions can also be found there.

We begin with the following asymptotic result for $\psi^{(\epsilon)}$.

Proposition V.1: Suppose that (**A1**)$'$ and (**A2**) hold. Let $\psi^{(\epsilon)}$ and ψ be the upper values of the stochastic and deterministic cases of the differential game (27) and (28), respectively. Then

$$\lim_{\epsilon\downarrow 0} \psi^{(\epsilon)}(s, x, y) = \psi(s, x, y) \quad \forall (s, x, y) \in [0, T] \times \mathbf{R}^n \times (-\overline{y}, \infty)$$

uniformly on compact subsets.

Proof: Define

$$\overline{\psi}(t, x, y) = \limsup_{\epsilon\downarrow 0,\, s\to t,\, p\to x,\, q\to y} \psi^{(\epsilon)}(s, p, q) \tag{49}$$

for all $(t, x, y) \in [0, T] \times \mathbf{R}^n \times (-\overline{y}, \infty)$. In view of (29), $\psi^{(\epsilon)}$ is uniformly bounded when (s, x, y) and $\epsilon > 0$ belong to compact subsets. Therefore, $\overline{\psi}$ is well defined and upper semicontinuous. We now show that $\overline{\psi}$ is a viscosity subsolution of (30) (with $\epsilon = 0$). Let $\varphi \in C^\infty\big([0, T] \times \mathbf{R}^n \times (-\overline{y}, \infty)\big)$. Suppose that $\overline{\psi} - \varphi$ has a local maximum over $[0, T] \times \mathbf{R}^n \times (-\overline{y}, \infty)$ at $(t^0, x^0, y^0) \in (0, T) \times \mathbf{R}^n \times (-\overline{y}, \infty)$. Then, there exists a subsequence $(t^\epsilon, x^\epsilon, y^\epsilon) \in (0, T) \times \mathbf{R}^n \times (-\overline{y}, \infty)$ (indexed by $\epsilon > 0$) such that (see [19])

1) $\psi^{(\epsilon)} - \varphi$ has a local maximum at $(t^\epsilon, x^\epsilon, y^\epsilon)$;
2) $\psi^{(\epsilon)}(t^\epsilon, x^\epsilon, y^\epsilon) \to \overline{\psi}(t^0, x^0, y^0)$ as $\epsilon \downarrow 0$;
3) $(t^\epsilon, x^\epsilon, y^\epsilon) \to (t^0, x^0, y^0)$ as $\epsilon \downarrow 0$.

Since $\psi^{(\epsilon)}$ is a viscosity subsolution of (30) (Theorem III.1), it follows that for every $\epsilon > 0$ [denoting $\varphi_t^\epsilon = \varphi_t(t^\epsilon, x^\epsilon, y^\epsilon)$ etc.]

$$\varphi_t^\epsilon + \inf_{u\in U} \sup_{w\in W} \left\{\frac{\epsilon}{2}\,\mathrm{tr}[\varphi_{xx}^\epsilon\,\sigma\,\sigma'] + \varphi_x^{\epsilon\,\prime}(b + \sigma\,w) + \varphi_y^\epsilon\,f - \frac{1}{2}\,|w|^2\right\} \geq 0. \tag{50}$$

Letting $\epsilon \downarrow 0$, it follows from (51) and the continuous differentiability of φ that

$$\varphi_t^0 + \inf_{u\in U} \sup_{w\in W} \left\{ \varphi_x^{0\prime}(b+\sigma w) + \varphi_y^0 f - \frac{1}{2}|w|^2 \right\} \geq 0$$

at (t^0, x^0, y^0). Clearly, $\overline{\psi}(T, x, y) = \ln\big(y+g(x)\big)$. Therefore $\overline{\psi}$ is an upper-semi-continuous viscosity sub-solution of (30) (with $\epsilon = 0$); see [19, Ch. VII.4] for a generalization of the definition of subsolution that applies to discontinuous functions. Similarly, it can be shown that if $\underline{\psi}$ is defined by

$$\underline{\psi}(t, x, y) = \liminf_{\epsilon\downarrow 0,\, s\to t,\, p\to x,\, q\to y} \psi^{(\epsilon)}(s, p, q) \tag{51}$$

then $\underline{\psi}$ is a lower semicontinuous viscosity super-solution of (30) (with $\epsilon = 0$); see [19, Ch. VII.4]. By the definition of $\overline{\psi}$ and $\underline{\psi}$, it follows that $\underline{\psi} \leq \overline{\psi}$. On the other hand, the comparison theorem for discontinuous viscosity sub and supersolutions (see [19, Ch. VII.8]) implies that $\overline{\psi} \leq \underline{\psi}$. Therefore, $\psi^0 = \overline{\psi} = \underline{\psi}$ is a continuous viscosity solution of (30) (with $\epsilon = 0$) and

$$\lim_{\epsilon\downarrow 0} \psi^{(\epsilon)}(t, x, y) = \psi^0(t, x, y)$$

uniformly on compact subsets. Since the upper-value ψ of the deterministic case of the differential game (27) and (28) is the unique viscosity solution of (30) with $\epsilon = 0$ (see Theorem III.2), it follows that $\psi^0 = \psi$. ∎

The following result relates the HARA problem (9) and (10) and the deterministic case of the differential game (27) and (28).

Theorem V.1: Suppose that (**A1**), (**A2**) and (**A3**) hold. For every $\epsilon > 0$, let $v^{(\epsilon)}$ be the value function of the HARA problem (9) and (10), $\phi^{(\epsilon)} := \epsilon \ln v^{(\epsilon)}$ and $W \subseteq \mathbf{R}^k$ be the compact set from Theorem IV.1. Suppose that ψ is the upper value of the deterministic differential game (27) and (28) associated with W. Then, ψ and $\phi^{(\epsilon)}$ are the unique viscosity solutions of (30) (with $\epsilon = 0$) and (35), respectively. Moreover

$$\lim_{\epsilon\downarrow 0} \phi^{(\epsilon)}(t, x, y) = \psi(t, x, y)$$
$$\forall\, (t, x, y) \in [0, T] \times \mathbf{R}^n \times (-\overline{y}, \infty)$$

uniformly on compact subsets.

Proof: It follows from Theorems III.2 and IV.1 that ψ and $\phi^{(\epsilon)}$ are the unique viscosity solutions of (30) (with $\epsilon = 0$) and (35), respectively. By Theorems III.1 and IV.1, $\phi^{(\epsilon)}$ is also the unique viscosity solution of (30) with $W \subseteq \mathbf{R}^k$ being the ϵ-independent compact set determined by Theorem IV.1. The convergence result then follows from Proposition V.1. ∎

In [22], there is a similar asymptotic result for the exponential risk-sensitive cost. However, stronger assumptions than Assumptions (**A1**), (**A2**), and (**A3**) are made. In particular, nondegeneracy of σ is assumed, which guarantees the existence of classical solutions to the HJB equation associated with the ϵ-parametrized risk-sensitive problems, whereas Assumptions (**A1**), (**A2**) and (**A3**) only guarantee the existence of viscosity solutions of (13). As can be seen from the proof of Theorem V.1, having classical solutions is not a fundamental requirement for the result to hold.

VI. Comparisons

A. HARA Controllers

In Section II, we introduced the HARA utility function as an alternative to the exponential utility approach to finding robust controllers. Intuitively, we expect the HARA approach to give robust controllers that have superior performance characteristics to exponential controllers. A conclusive proof of this statement remains an open problem. In this section, we obtain some relationships between the value function of the HARA problem and that of the exponential problem. Using this result, a weaker statement about the performance of HARA controllers and exponential controllers can be made.

Suppose once again that the system dynamics are given by the following SDE:

$$\begin{cases} dx(t) = f\big(t, x(t), u(t)\big)\, dt \\ \quad +\sqrt{\epsilon}\,\sigma\big(t, x(t), u(t)\big)\, dB(t), \qquad t \in [s, T], \\ x(s) = x \end{cases} \tag{52}$$

and that the performance of a particular control $u(\cdot) \in \mathcal{U}[s, T]$ is measured by the following cost functional:

$$J\big(s, x;\, u(\cdot)\big) = E\left\{ \int_s^T f\big(t, x(t), u(t)\big)\, dt + g\big(x(T)\big) \right\}. \tag{53}$$

Let $\epsilon > 0$ be given. The exponential risk-sensitive performance measure associated with the system (52) for a given $u(\cdot) \in \mathcal{U}[s, T]$ is

$$J_E\big(s, x;\, u(\cdot)\big) = \epsilon \ln E\left\{ \exp \frac{1}{\epsilon} \left(\int_s^T f\big(t, x(t), u(t)\big)\, dt + g\big(x(T)\big) \right) \right\} \tag{54}$$

and the associated value function is defined by

$$W^{(\epsilon)}(s, x) = \inf_{u(\cdot)\in\mathcal{U}[s, T]} J_E\big(s, x;\, u(\cdot)\big), \quad \forall\, (s, x) \in [0, T] \times \mathbf{R}^n. \tag{55}$$

One method of obtaining robust controllers for the system (52) [with the performance measure (53)] is to solve the exponential risk-sensitive problem (52), (54) for the optimal exponential risk-sensitive controller $u_E^*(\cdot)$, and to use $u_E^*(\cdot)$ in the system (52). The performance of (52) with $u_E^*(\cdot)$ is measured by $J\big(s, x;\, u_E^*(\cdot)\big)$.

On the other hand, an immediate consequence of Jensen's inequality [8], [30] is the following:

$$J\big(s, x;\, u(\cdot)\big) \leq J_E\big(s, x;\, u(\cdot)\big), \qquad \forall\, u(\cdot) \in \mathcal{U}[s, T]. \tag{56}$$

The inequality (56) can be viewed as a performance guarantee: For any controller $u(\cdot) \in \mathcal{U}[s, T]$, the performance associated with $u(\cdot)$ [which is measured by the value of $J\big(s, x;\, u(\cdot)\big)$] is bounded above by the exponential cost $J_E\big(s, x;\, u(\cdot)\big)$ associated with $u(\cdot)$. In particular, the optimal exponential control

$u_E^*(\cdot)$ minimizes the family of upper bounds $J_E(s, x;\ u(\cdot))$ over $\mathcal{U}[s, T]$.

Similarly, if we define

$$J_H(s, x;\ u(\cdot)) = \left(E\left\{\int_s^T f(t, x(t), u(t))\,dt + g(x(T))\right\}^{1/\epsilon}\right)^{\epsilon} \tag{57}$$

then for every $u(\cdot) \in \mathcal{U}[s, T]$, we have the inequality:

$$J(s, x;\ u(\cdot)) \le J_H(s, x;\ u(\cdot)). \tag{58}$$

The optimal HARA controller $u_H^*(\cdot)$ minimizes the upper bound $J_H(s, x;\ u(\cdot))$ over $u(\cdot) \in \mathcal{U}[s, T]$.

Intuitively, we expect the optimal HARA controller $u_H^*(\cdot)$ to result in better performance for the system (52) [as measured by (53)] than the optimal exponential controller $u_E^*(\cdot)$; that is, we expect $J(s, x;\ u_H^*(\cdot)) < J(s, x;\ u_E^*(\cdot))$. Unfortunately, we have not been able to establish this statement conclusively. However, we have been able to obtain some related, though weaker, relationships between the (risk-neutral) performance measure (53), the value function of the HARA problem and the value function of the exponential problem.

Consider the following transformation of $v^{(\epsilon)}$:

$$V^{(\epsilon)}(t, x, y) = \left[v^{(\epsilon)}(t, x, y)\right]^{\epsilon} \quad \forall\,(t, x, y) \in [0, T] \times \mathbf{R}^n \times (-\overline{y}, \infty),\ \epsilon > 0. \tag{59}$$

Adopting the convention

$$V^{(\epsilon)}(t, x) := V^{(\epsilon)}(t, x, 0), \qquad \forall\,(t, x) \in [0, T] \times \mathbf{R}^n$$

it is clear from the definition of $v^{(\epsilon)}$ that, for any $\epsilon > 0$, we have

$$V^{(\epsilon)}(t, x) = \inf_{u(\cdot)\in\mathcal{U}[s, T]} J_H(t, x;\ u(\cdot)) \quad \forall\,(t, x) \in [0, T] \times \mathbf{R}^n \tag{60}$$

where $J_H(t, x;\ u(\cdot))$ is defined by (57). The following result shows the relationship between $V^{(\epsilon)}$ and $W^{(\epsilon)}$.

Theorem VI.1: Suppose that (**A1**) and (**A2**) hold. Let $\epsilon > 0$ and $(s, x) \in [0, T] \times \mathbf{R}^n$ be given and fixed. If, for every admissible pair $(x(\cdot), u(\cdot))$, we have

$$\int_s^T f(t, x(t), u(t))\,dt + g(x(T)) \ge 1 - \epsilon, \quad P-\text{a.s.} \tag{61}$$

then

$$V^{(\epsilon)}(s, x) \le W^{(\epsilon)}(s, x). \tag{62}$$

Remark VI.1: The condition (61) is simply for the sake of convenience and Theorem VI.1 can be extended to cover the case when (61) (and nonnegativity/uniform positivity of f and g) is not satisfied; see also Remark II.1. In particular, uniform boundedness (**A2**) implies that (61) is satisfied by $\overline{f} := f + k$ and $\overline{g} := g + k$ for sufficiently large $k \ge 0$. Denoting

$$\overline{V}^{(\epsilon)}(t, x) := \inf_{u(\cdot)\in\mathcal{U}[s, T]} \times \left(E\left\{\int_s^T \overline{f}(t, x(t), u(t))\,dt + \overline{g}(x(T))\right\}^{1/\epsilon}\right)^{\epsilon} - k(T+1) \tag{63}$$

one can easily see that $\overline{V}^{(\epsilon)}(s, x)$ is the least upper bound associated with the HARA problem (63) for the *original* performance measure (53), while $W^{(\epsilon)}(s, x)$, as defined by (55), is the least upper bound for the original performance measure associated with the exponential approach. Theorem VI.1 can now be applied from which we conclude that $\overline{V}^{(\epsilon)}(s, x) \le W^{(\epsilon)}(s, x)$. Therefore, the HARA approach and the inequality (62) can be extended to the general case when (61) is not satisfied.

Remark VI.2: The following is a simple example of a situation where the comparison (62) holds with strict inequality. Let $\epsilon = 1$, $m = 1$, $n = 1$, $b(t, x, u) = Ax + Bu$, $\sigma(t, x, u) = 1$, $f(t, x, u) = (1/2)\,Qx^2 + (1/2)\,|u|^2$ and $g(x) = (1/2)\,Hx^2$ where $Q, H \ge 0$. It can be shown that

$$V^{(\epsilon)}(t, x) = \frac{1}{2}\,P(t)\,x^2 + \frac{1}{2}\int_t^T P(s)\,ds$$

$$W^{(\epsilon)}(t, x) = \frac{1}{2}\,Z(t)\,x^2 + \frac{1}{2}\int_t^T Z(s)\,ds$$

where P and Z are the unique solutions of the Riccati equations

$$\begin{cases} \dot{P} + PA + AP - PBB'P + Q = 0, & t \in [0, T] \\ P(T) = H, \end{cases}$$

$$\begin{cases} \dot{Z} + ZA + AZ - Z(BB' - I)Z + Q = 0, & t \in [0, T], \\ Z(T) = H \end{cases}$$

respectively. (See [23] for a derivation of W^{ϵ}. On the other hand, since $\epsilon = 1$, the HARA problem coincides with the risk-neutral LQ problem, a solution of which can be found in [30]). When $A = -1/2$, $B = 1$, $Q = 0$, $R = 1$, $H = 1$ and $T = 1$, these equations can be solved explicitly to give

$$P(t) = \frac{e^{t-1}}{2 - e^{t-1}}, \quad Z(t) = e^{t-1}, \qquad t \in [0, 1].$$

Clearly, $0 < P(t) < Z(t)$ for all $t \in [0, 1)$ and it follows immediately that $V^{(\epsilon)}(t, x) < W^{(\epsilon)}(t, x)$ for all $(t, x) \in [0, 1) \times \mathbf{R}$.

The following results are used in the proof of Theorem VI.1.

Lemma VI.1: Let $\delta > 0$ be given and fixed. Suppose that g: $[\delta, \infty) \to \mathbf{R}$ is twice differentiable and f: $[\delta, \infty) \to \infty$ is given by

$$f(x) = \exp\{g(x)\}. \tag{64}$$

Then, f is convex if and only if

$$\big(g_x(x)\big)^2 + g_{xx}(x) \geq 0 \qquad \forall\, x \in [\delta,\, \infty). \tag{65}$$

Proof: f is convex if and only if

$$f_{xx}(x) = \exp\{g(x)\} \cdot \Big[\big(g_x(x)\big)^2 + g_{xx}(x)\Big] \geq 0 \quad \forall\, x \in [\delta,\, \infty).$$

■

Lemma VI.2: Let $\epsilon \in (0,\, 1]$ be given and fixed. Then f: $[(1-\epsilon)^{1/\epsilon},\, \infty) \to \mathbf{R}$ where

$$f(x) = \exp\left\{\frac{1}{\epsilon}\, x^\epsilon\right\}$$

is convex.

Proof: Let $g(x) = (1/\epsilon)\, x^\epsilon$. Since

$$\big(g_x(x)\big)^2 + g_{xx}(x) = x^{\epsilon-1}(x^\epsilon + \epsilon - 1) \geq 0 \quad \forall\, x \in [(1-\epsilon)^{1/\epsilon},\, \infty)$$

it follows from Lemma VI.1 that $f(\cdot)$ is convex. ■

The proof of Theorem VI.1 is as follows.

Proof: For any given admissible pair $\big(x(\cdot),\, u(\cdot)\big)$, define the random variable

$$z := \left(\int_s^T f\big(t,\, x(t),\, u(t)\big)\, dt + g\big(x(T)\big)\right)^{1/\epsilon}.$$

We begin by considering the case $\epsilon \in (0,\, 1]$. Then by (61), it follows that $z \geq (1-\epsilon)^{1/\epsilon}$, $P-$ a.s. By Lemma VI.2, $f(x) = \exp\{(1/\epsilon)\, x^\epsilon\}$ is convex on $[(1-\epsilon)^{1/\epsilon},\, \infty)$ and, hence, by Jensen's inequality, we have

$$\exp\left\{\frac{1}{\epsilon}\,(Ez)^\epsilon\right\} \leq E\exp\left\{\frac{1}{\epsilon}\, z^\epsilon\right\} \tag{66}$$

or, equivalently

$$\left[E\left(\int_s^T f\big(t,\, x(t),\, u(t)\big)\, dt + g\big(x(T)\big)\right)^{1/\epsilon}\right]^\epsilon \leq \epsilon \ln E \exp\frac{1}{\epsilon}\left(\int_s^T f\big(t,\, x(t),\, u(t)\big)\, dt + g\big(x(T)\big)\right). \tag{67}$$

We obtain (62) by taking $\inf_{u(\cdot)\in\mathcal{U}[s,\,t]}$ on both sides of (67) and noting the definitions (55) and (60). On the other hand, if $\epsilon > 1$, then $f(x) = \exp\{(1/\epsilon)\, x^\epsilon\}$ is convex on $x \in [0,\, \infty)$ and $z \geq 0$; see (**A2**). Therefore, we can apply Jensen's inequality again to obtain (66), (67) and our result (62). ■

B. Deterministic Differential Games

In this section, we turn our attention to the deterministic system

$$\begin{cases} \dot{x}(t) = b\big(t,\, x(t),\, u(t)\big), & t \in [s,\, T], \\ x(s) = x \end{cases} \tag{68}$$

with performance measure

$$J\big(s,\, x;\, u(\cdot)\big) = \int_s^T f\big(t,\, x(t),\, u(t)\big)\, dt + g\big(x(T)\big). \tag{69}$$

Consider the following system

$$\begin{cases} \dot{x}(t) = b\big(t,\, x(t),\, u(t)\big) \\ \quad +\sigma\big(t,\, x(t),\, u(t)\big)\, w(t), & t \in [s,\, T], \\ x(s) = x. \end{cases} \tag{70}$$

In H_∞ control, robust controllers are obtained by solving the upper deterministic differential game associated with the system (70) and the cost

$$J\big(s,\, x;\, u(\cdot),\, w(\cdot)\big) = \int_s^T f\big(t,\, x(t),\, u(t)\big)\, dt + g\big(x(T)\big) - \frac{1}{2}\int_s^T |w(t)|^2\, dt. \tag{71}$$

In Section III-B, we introduced a deterministic differential game with cost

$$J\big(s,\, x;\, u(\cdot),\, w(\cdot)\big) = \ln\left(\int_s^T f\big(t,\, x(t),\, u(t)\big)\, dt + g\big(x(T)\big)\right) - \frac{1}{2}\int_s^T |w(t)|^2\, dt \tag{72}$$

and dynamics (70). The main difference between (71) and (72) is the introduction of the $\ln(\cdot)$ term. To get a feel for the role that this term plays, consider a situation where the first player chooses an input $u(\cdot)$ and applies this input to (70) and (71) and then to (70), (72). In the first case, the opponent chooses an input $w(\cdot)$ to maximize the cost (71) corresponding to this $u(\cdot)$. This corresponds to the standard approach to robust control. In the second situation, the $\ln(\cdot)$ term acts like a weight between the two components of the cost. In particular, the $\ln(\cdot)$ term reduces the importance of the first component relative to second component in the optimization of the opponent player. For this reason, one expects the input $w(\cdot)$ of the opponent chosen by maximizing (72) will be less "aggressive" than the input of the opponent player chosen by maximizing (71). That is, the first player in (72) is dealing with a more conservative opponent than the first player in (71). For this reason, we expect the controller $u_1(\cdot)$ obtained by solving (70), (72) to be robust, but at the same time, to have better performance [as measured by (69)] than the controller $u_2(\cdot)$ obtained by solving (70) and (71).

In Theorem IV.1, it is shown that under (**A1**), (**A2**), and (**A3**), for every $\epsilon > 0$, the HARA problem is related to a differential game of the form (70), (72), in which the input of player 2 is restricted to $W_1 = B(0,\, R_1)$, an ϵ-independent closed ball in

$\mathbf{R}^k$ of radius $R_1 > 0$; see Remark VI.1. Similarly, it is shown in [22] that the exponential-of-an-integral problem is related to a differential game of the form (70) and (71) in which, as in the HARA case, the input of player 2 may be restricted to a closed, ϵ-independent ball in $\mathbf{R}^k$, $W_2 = B(0, R_2)$. Throughout this section, we shall assume that $W = B(0, R)$ for both (70) and (71) and (70), (72), where $\infty > R \geq \max\{R_1, R_2\}$ is fixed.

Suppose that $u(\cdot) \in \mathcal{U}_d[s, T]$ is given. Since the strategy $\alpha[\cdot]$ which satisfies $\alpha[u(\cdot)] = 0$ is admissible, we have the following inequality:

$$\int_s^T f\big(t, x(t), u(t)\big)\, dt + g\big(x(T)\big) \leq \sup_{\alpha[\cdot]\in\Gamma_d[s,T]} \left\{ \int_s^T f\big(t, x(t), u(t)\big)\, dt + g\big(x(T)\big) - \frac{1}{2}\int_s^T |\alpha[u(t)]|^2\, dt \right\} \tag{73}$$

where $\big(x(\cdot), u(\cdot)\big)$ on the left-hand side of (73) is an admissible pair for (68), and $(x(\cdot), u(\cdot), \alpha[u(\cdot)])$ on the right-hand side of (73) is an admissible 3-tuple for (70).

For (73), it can be seen that for any controller $u(\cdot) \in \mathcal{U}_d[s, T]$

$$J_1\big(s, x;\ u(\cdot)\big) := \sup_{\alpha[\cdot]\in\Gamma_d[s,T]} \left\{ \int_s^T f\big(t, x(t), u(t)\big)\, dt + g\big(x(T)\big) - \frac{1}{2}\int_s^T |\alpha[u(t)]|^2\, dt \right\}$$

is an upper bound on the performance of the system (68) under $u(\cdot)$, where performance is measured by $J\big(s, x;\ u(\cdot)\big)$. Clearly, the right-hand side in (73) is minimized by

$$\begin{aligned}\theta(s, x) &:= \inf_{u(\cdot)\in\mathcal{U}_d[s,T]} J_1\big(s, x;\ u(\cdot)\big)\\ &= \inf_{u(\cdot)\in\mathcal{U}_d[s,T]} \sup_{\alpha[\cdot]\in\Gamma_d[s,T]} \times \left\{ \int_s^T f\big(t, x(t), u(t)\big)\, dt + g\big(x(T)\big) - \frac{1}{2}\int_s^T |\alpha[u(t)]|^2\, dt \right\}\end{aligned} \tag{74}$$

the upper value of (70) and (71).

Similarly, it is easy to see that

$$\int_s^T f\big(t, x(t), u(t)\big)\, dt + g\big(x(T)\big) \leq \sup_{\alpha[\cdot]\in\Gamma_d[s,T]} \times \exp\left\{ \ln\left(\int_s^T f\big(t, x(t), u(t)\big)\, dt + g\big(x(T)\big)\right) - \frac{1}{2}\int_s^T |\alpha[u(t)]|^2\, dt \right\} \tag{75}$$

where, as before, $\big(x(\cdot), u(\cdot)\big)$ on the left-hand side of (75) is an admissible pair for (68), while $(x(\cdot), u(\cdot), \alpha[u(\cdot)])$ on the right-hand side of (75) is admissible for (70); that is

$$J_2\big(s, x;\ u(\cdot)\big) := \sup_{\alpha[\cdot]\in\Gamma_d[s,T]} \exp\left\{ \ln\left(\int_s^T f\big(t, x(t), u(t)\big)\, dt + g\big(x(T)\big)\right) - \frac{1}{2}\int_s^T |\alpha[u(t)]|^2\, dt \right\}$$

is an upper bound on $J\big(s, x;\ u(\cdot)\big)$, which is minimized, over $u(\cdot) \in \mathcal{U}_d[s, T]$, by

$$\begin{aligned}\Psi(s, x) &:= \inf_{u(\cdot)\in\mathcal{U}_d[s,T]} J_2\big(s, x;\ u(\cdot)\big)\\ &= \inf_{u(\cdot)\in\mathcal{U}_d[s,T]} \sup_{\alpha[\cdot]\in\Gamma_d[s,T]} \times \exp\left\{ \ln\left(\int_s^T f\big(t, x(t), u(t)\big)\, dt + g\big(x(T)\big)\right) - \frac{1}{2}\int_s^T |\alpha[u(t)]|^2\, dt \right\}\end{aligned} \tag{76}$$

the upper value of (70) and (72). Clearly

$$\Psi(s, x) = \exp \psi(s, x, 0)$$

where $\psi(s, x, y)$ is the upper value of the deterministic case of the differential game (27) and (28), as defined by (32).

Our next result shows the relationship between θ and Ψ.

Theorem VI.2: Suppose (**A1**), (**A2**) and (**A3**) hold. Let $(s, x) \in [0, T] \times \mathbf{R}^n$ be given, $W = B(0, R)$ as discussed above, $\theta(s, x)$ the upper value of (70) and (71) as defined by (74) and $\Psi(s, x)$, the upper value of (70), (72) as defined by (76). If $g(x) \geq 1$ for every $x \in \mathbf{R}^n$, then

$$\Psi(s, x) \leq \theta(s, x).$$

Remark VI.3: As in the case of Theorem VI.1, the assumption that $f(x) \geq 0$ [from (**A2**)] and $g(x) \geq 1$ in Theorem VI.2 is simply for convenience. It is straightforward to extend this result to the case when f and g are only bounded; see Remark VI.1.

Proof: By Theorem V.1 and (59), we have:

$$\begin{aligned}V^{(\epsilon)}(s, x) := V^{(\epsilon)}(s, x, 0) &= \left[v^{(\epsilon)}(s, x, 0)\right]^{\epsilon}\\ &= \exp \phi^{(\epsilon)}(s, x, 0)\end{aligned}$$

and

$$\lim_{\epsilon\downarrow 0} V^{(\epsilon)}(s, x) = \exp \psi(s, x, 0) = \Psi(s, x).$$

In [26], it is shown that under the conditions of the theorem

$$\lim_{\epsilon\downarrow 0} W^{(\epsilon)}(s, x) = \theta(s, x)$$

where $W^{(\epsilon)}$ is defined by (55). Since $V^{(\epsilon)}(s, x) \leq W^{(\epsilon)}(s, x)$, see Theorem VI.1, it follows that $\Psi(s, x) \leq \theta(s, x)$. ∎

VII. LQR Control with HARA Utility

In this section, we study an LQR problem with HARA utility. This problem is interesting because it reveals an interesting relationship between robustness and uncertainty.

1) HARA Problem: Consider the following linear-quadratic problem with HARA utility (we assume throughout that $H > 0$)

$$\begin{cases} \min_{u(\cdot)\in\mathcal{U}[s,T]} J\big(s, x;\, u(\cdot)\big) = E\left\{\frac{1}{2}Hx(T)^2\right\}^{1/\epsilon} \\ dx(t) = \left[\overline{A}(t)\,x(t) + \overline{B}(t)\,u(t)\right]\,dt \\ \quad + \sum_{j=1}^{k} \overline{D}_j(t)u(t)dB^j(t), \qquad t \in [s, T] \\ x(s) = x \end{cases} \tag{77}$$

where $B(t) \equiv \big(B^1(t),\ldots,B^k(t)\big)'$ is the Brownian motion. Throughout this section, we shall assume that $x(\cdot)$ is scalar valued and $u(\cdot)$ is $\mathbf{R}^m$-valued, for $m \geq 1$. Define $\overline{D}(t)' = \big(\overline{D}_1(t)',\ldots,\overline{D}_k(t)'\big)$.We shall assume that $\overline{D}(t)'\,\overline{D}(t) > 0$ for all $t \in [s, T)$, and that $\epsilon \in (0, 2)$ is given and fixed. In particular, it should be noted that the model (77) (with scalar state, but a multivariable control) is one that arises quite frequently in financial applications; see [16], [18], [32] for an example of this.

Suppose that $\overline{D} = 0$. Then, (77) is a deterministic problem with a nonnegative infimal cost and hence is well posed. However, this infimal cost in general can not be achieved. For example, when $\big(\overline{A}(\cdot), \overline{B}(\cdot)\big)$ is controllable, the infimal cost is zero, but is not achievable. In this situation, we must be satisfied with near-optimal controls. (For a discussion on near-optimal controls for infinite-time singular LQR problems, refer to the paper [29].) In particular, we can make the cost arbitrarily close to 0 by choosing a "sufficiently large" control.

When $\overline{D} \neq 0$, (77) is a well-posed stochastic problem, and the optimal cost is achieved by a unique optimal control. In this case, the value function involves the solution of the so-called *stochastic Riccati equation*, the properties of which are studied in [9]. The fundamental difference between the case $\overline{D} = 0$ and $\overline{D} \neq 0$ is the role that the uncertainty plays. The reader is directed to [9] for a deeper discussion of this and other related issues.

The HJB equation associated with (77) is

$$\begin{cases} v_t + \inf_{u\in\mathbf{R}^m} \left\{\frac{1}{2}\,u'\,\overline{D}'\,v_{xx}\,\overline{D}\,u \right. \\ \qquad\qquad \left. +v_x'\,\left(\overline{A}\,x + \overline{B}\,u\right)\right\} = 0 \\ (t, x) \in [s, T] \times \mathbf{R} \\ v(T, x) = \left(\frac{1}{2}\,H\,x^2\right)^{1/\epsilon}, \qquad x \in \mathbf{R}. \end{cases} \tag{78}$$

It is easy to show that the unique solution of (78) [and, hence, the value function associated with (77)] is

$$v(t, x) = \left(\frac{1}{2}\,P(t)\,x^2\right)^{1/\epsilon} \tag{79}$$

where $P(\cdot)$ is the unique (positive) solution of the stochastic Riccati equation:[2]

$$\begin{cases} \dot{P} + P\overline{A} + \overline{A}P - P\,\overline{B}\left[\left(\frac{2}{\epsilon} - 1\right)\,\overline{D}'P\overline{D}\right]^{-1}\overline{B}P = 0, \\ P(T) = H. \end{cases} \tag{80}$$

Note that (80) is a linear ODE, hence, it must admit a solution. The optimal feedback control for (77) is

$$u(t) = -\left(\left(\frac{2}{\epsilon} - 1\right)\overline{D}(t)'\overline{D}(t)\right)^{-1}\overline{B}(t)'x(t). \tag{81}$$

We also note that for the LQR problem with *exponential* utility, it seems that there is no closed-form expression for the optimal control when the control appears in the diffusion (which is the case in many finance applications). In fact, for this particular problem, the issue of existence and uniqueness of solutions of the associated HJB equation is still an open question.

2) Stochastic LQR Problem: Consider the following stochastic LQR problem:

$$\begin{cases} \min_{u(\cdot)\in\mathcal{U}[s,T]} J\big(s, x;\, u(\cdot)\big) = E\left\{\frac{1}{2}Hx(T)^2\right\} \\ dx(t) = \left[\overline{A}(t)\,x(t) + \overline{B}(t)\,u(t)\right]\,dt \\ \quad +\sqrt{\frac{2}{\epsilon} - 1}\,\sum_{j=1}^{k} \overline{D}_j(t)u(t)dB^j(t), \qquad t \in [s, T], \\ x(s) = x. \end{cases} \tag{82}$$

It is easy to show (see [9]) that

$$V(t, x) = \frac{1}{2}\,P(t)\,x^2 \tag{83}$$

is the value function associated with (82), where $P(\cdot)$ is determined by (80), and (81) is the optimal control. This shows an equivalence between the HARA problem (77) and the stochastic LQR problem (82). In particular, the ϵ-parameter in the cost (77) has been transferred to the state equation (82).

3) Discussion: The equivalence between (77) and (82) shows an interesting relationship between uncertainty and robustness. Suppose that under "normal" conditions, the dynamics are given by the system

$$\begin{cases} dx(t) = \left[\overline{A}(t)\,x(t) + \overline{B}(t)\,u(t)\right]\,dt \\ \quad + \sum_{j=1}^{k} \overline{D}_j(t)u(t)dB^j(t), \qquad t \in [s, T], \\ x(s) = x. \end{cases} \tag{84}$$

In addition, suppose that the *performance* of any given controller $u(\cdot)$ is measured by the cost functional:

$$J\big(s, x;\, u(\cdot)\big) = E\left\{\frac{1}{2}\,Hx(T)^2\right\}, \qquad H > 0. \tag{85}$$

As stated in the Introduction, the cost functional (85) is used to evaluate the *performance* of any given controller, irrespective of how this controller was obtained. [Of course, controllers obtained by using different control methodologies, while not being

[2]Although the stochastic Riccati equation (80) is clearly deterministic, it is actually a special case of a Riccati-type backward stochastic differential equation that is introduced in [9], and is deterministic only under the assumptions of this paper.

optimal according to the measure (85), may have other advantages, such as robustness, that the optimal controller for (84) and (85) may not have.] Note that the optimal controller associated with (84) and (85) is given by

$$u(t) = -\left(\overline{D}(t)'\overline{D}(t)\right)^{-1}\overline{B}(t)'x(t). \tag{86}$$

Suppose that we wish to find a controller $u(\cdot)$ for the system (84) that is more robust (or risk-averse) than the optimal controller (86). One method for finding such a controller is to solve the HARA problem (77) corresponding to some $\epsilon \in (0, 1)$. [That is, to use the optimal HARA controller (81), with $\epsilon \in (0, 1)$, in the system (84)]. On the other hand, the equivalence between (77) and (82) shows that the (robust) HARA controller (81) corresponding to $\epsilon \in (0, 1)$ is also the optimal controller for (82) with the same value of ϵ. Moreover, since $\sqrt{(2/\epsilon) - 1} > 1$ when $\epsilon \in (0, 1)$, finding this robust HARA controller corresponds to solving a standard LQR problem [since (82) is of the same form as (84) and (85)] but with a higher noise intensity. It should also be noted that greater robustness corresponds to a controller of smaller magnitude [i.e., decreasing $\epsilon \in (0, 1)$ corresponds to the magnitude of $u(t)$ in (81) decreasing].

When $\epsilon \in (1, 2)$, the optimal controller (81) for the HARA problem is risk-seeking. The relationship between the HARA problem (77) and the LQR problem (82) shows that a risk-seeking controller for the system (84) under the criterion (85) is obtained by solving an LQR problem (82) with a smaller noise intensity [since $\sqrt{(2/\epsilon) - 1} < 1$ when $\epsilon \in (1, 2)$]. Note also that risk-seeking controllers have a larger magnitude than the optimal controller (86).

VIII. Conclusion

In this paper, we have studied some of the properties of optimal HARA controllers. Our study of the HARA problem was motivated by the belief that in addition to being robust, optimal HARA controllers are less conservative than optimal exponential controllers. We have shown that the HARA problem is equivalent to a certain stochastic differential game, different from the one commonly encountered in the robust control literature, and have studied the asymptotic properties of both the HARA problem and the associated game problem. One feature of this differential game is that it involves a logarithmic term which acts like a weight for the different components of the cost. We have argued that this weighting has the effect of making the opposing (disturbance) player less aggressive; that is, the controller in the logarithmic-weighted game (i.e., the HARA controller) is designed in a "less than worst case" environment. This is consistent with the intuition that HARA controllers, while being robust, may perform better than exponential controllers (which are designed in a "worst case environment") when applied to a system operating under normal situations. While conclusive theoretical justification of this intuition remains an important open question, certain related, though weaker, comparisons between the value functions of the HARA exponential problems have been obtained in this paper. Another important issue that we have not addressed relates to the robustness/disturbance attenuation properties of the HARA controller, especially when compared to those of the exponential controller; see [17] for related analysis for the exponential problem. As an example, we examined a particular linear-quadratic case of the HARA problem. For this problem, we showed that finding a robust controller for a certain class of linear systems with a quadratic terminal cost is equivalent to solving a linear-quadratic problem of the *same* form, *but with a larger noise intensity*. This shows an interesting relationship between robustness and uncertainty.

Appendix

We present here some basic definitions and results from the theory of viscosity solutions and nonsmooth analysis which are referred to in this paper. For a detailed discussion of viscosity solutions, the reader is referred to [11] and [12], as well as [1], [19], and [30]. For a discussion of nonsmooth analysis, we recommend [10]. A proof of the relationship between sub/superdifferentials and Clarke's generalized gradient can be found in [30] and [31].

A. Viscosity Solutions

Consider the following nonlinear, scalar, first-order PDE

$$\begin{cases} v_t + H(t, x, v_x) = 0, & (t, x) \in [0, T) \times \mathbf{R}^n \\ v(T, x) = g(x) \end{cases} \tag{87}$$

[a special case of which is (30) with $\epsilon = 0$], and the nonlinear, scalar second-order PDE

$$\begin{cases} v_t + H(t, x, v_x, v_{xx}) = 0, & (t, x) \in [0, T) \times \mathbf{R}^n \\ v(T, x) = g(x). \end{cases} \tag{88}$$

It is well known that the upper/lower Isaacs equations (22) and (23), which are special cases of (88), do not, in general, have classical (smooth) solutions. A generalized concept of solution, called a viscosity solution, is introduced in [12]. The main result in [12] is that under certain mild conditions, there exists a unique viscosity solution of (87). In the second-order case, uniqueness is proven in [21], [23]. The definition of a viscosity solution of the first-order PDE (87) is as follows.

Definition VIII.1: Let $v \in C([0, T] \times \mathbf{R}^n)$ and $(t_0, x_0) \in (0, T) \times \mathbf{R}^n$. Then the first-order superdifferential of v at (t_0, x_0) is given by

$$\begin{aligned} D_{t,x}^{1,+}v(t_0, x_0) &= \{(\psi_t(t_0, x_0), \psi_x(t_0, x_0)) \mid \psi \in C^\infty((0, T) \times \mathbf{R}^n) \\ &\quad \text{and } v - \psi \text{ has a local maximum at } (t_0, x_0)\} \end{aligned} \tag{89}$$

and the first order subdifferential by

$$\begin{aligned} D_{t,x}^{1,-}v(t_0, x_0) &= \{(\psi_t(t_0, x_0), \psi_x(t_0, x_0)) \mid \psi \in C^\infty((0, T) \times \mathbf{R}^n) \\ &\quad \text{and } v - \psi \text{ has a local minimum at } (t_0, x_0)\}. \end{aligned} \tag{90}$$

Moreover, v is a viscosity solution of (87) if

$$v(T, x) = g(x) \qquad \forall x \in \mathbf{R}^n \tag{91}$$

and

$$q + H(t, x, p) \geq 0 \qquad \forall (q, p) \in D_{t,x}^{1,+} v(t, x) \quad (92)$$

$$q + H(t, x, p) \leq 0 \qquad \forall (q, p) \in D_{t,x}^{1,+} v(t, x) \quad (93)$$

for all $(t, x) \in [0, T) \times \mathbf{R}^n$.

In particular, v is called a *viscosity subsolution* if it satisfies (91) and (92), and a *viscosity supersolution* if it satisfies (91) and (93). Also, for any $v \in C([0, T] \times \mathbf{R}^n)$, we can define *partial super/subdifferentials* of v with respect to x at (t_0, x_0) [which we denote by $D_x^{1,+} v(t, x)$ and $D_x^{1,-} v(t, x)$, respectively] by keeping $t = t_0$ fixed, and calculating the super/subdifferentials of $v(t_0, x)$ in the x variable.

For the second-order case, we have the following.

Definition VIII.2: Let $v \in C([0, T] \times \mathbf{R}^n)$ and $(t_0, x_0) \in (0, T) \times \mathbf{R}^n$. Then the second-order superdifferential of v at (t_0, x_0) is defined by

$$\begin{aligned} D_{t,x}^{1,2,+} v(t_0, x_0) &= \{(\psi_t(t_0, x_0), \psi_x(t_0, x_0), \psi_{xx}(t_0, x_0)) \\ &\quad \mid \psi \in C^\infty((0, T) \times \mathbf{R}^n) \text{ and } v - \psi \\ &\quad \text{has a local maximum at } (t_0, x_0)\} \end{aligned} \quad (94)$$

and the second order sub-differential of v is defined by

$$\begin{aligned} D_{t,x}^{1,2,-} v(t_0, x_0) &= \{(\psi_t(t_0, x_0), \psi_x(t_0, x_0), \psi_{xx}(t_0, x_0)) \\ &\quad \mid \psi \in C^\infty((0, T) \times \mathbf{R}^n) \text{ and } v - \psi \\ &\quad \text{has a local minimum at } (t_0, x_0)\}. \end{aligned} \quad (95)$$

Moreover, v is a viscosity solution of (88) if

$$v(T, x) = g(x) \qquad \forall x \in \mathbf{R}^n \quad (96)$$

and

$$q + H(t, x, p, P) \geq 0 \qquad \forall (q, p, P) \in D_{t,x}^{1,2,+} v(t, x) \quad (97)$$

$$q + H(t, x, p, P) \leq 0 \qquad \forall (q, p P) \in D_{t,x}^{1,2,-} v(t, x) \quad (98)$$

for all $(t, x) \in [0, T) \times \mathbf{R}^n$.

As in the first-order case, v is called a *viscosity subsolution* of (88) if (96) and (97) are satisfied, and a *viscosity supersolution* if (96) and (98) are satisfied. Clearly

$$\begin{aligned} (q, p, P) &\in D_{t,x}^{1,2,+} v(t, x) \\ &\Rightarrow (q, p) \in D_{t,x}^{1,+} v(t, x) \\ &\Rightarrow p \in D_x^{1,+} v(t, x) \end{aligned} \quad (99)$$

and

$$\begin{aligned} (q, p, P) &\in D_{t,x}^{1,2,-} v(t, x) \\ &\Rightarrow (q, p) \in D_{t,x}^{1,-} v(t, x) \\ &\Rightarrow p \in D_x^{1,-} v(t, x). \end{aligned} \quad (100)$$

B. Nonsmooth Analysis

The following results from nonsmooth analysis are used in our proof of Proposition IV.1. For an in depth discussion, we recommend the book [10].

We begin with a definition of the *generalized gradient*.

Definition VIII.3 (Generalized Gradient): Let $f: \mathbf{R}^n \to \mathbf{R}$ be a locally Lipschitz function. The generalized gradient of f at $x \in \mathbf{R}^n$ is

$$\partial f(x) = \left\{ \xi \in \mathbf{R}^n \,\middle|\, \langle \xi, y \rangle \leq \limsup_{z \to x, h \downarrow 0} \frac{f(z + h y) - f(z)}{|h|} \right\}. \quad (101)$$

If $f: \mathbf{R}^n \times U \to \mathbf{R}$ for some subset U of $\mathbf{R}^m$, then the *partial generalized gradient* of f at $(\overline{x}, \overline{u}) \in \mathbf{R}^n \times U$ is obtained by fixing $u = \overline{u}$, and calculating the generalized gradient by treating $f(x, \overline{u})$ as a function of x.

An alternative characterization of ∂f is obtained from the following well known result: If $f: \mathbf{R}^n \to \mathbf{R}$ is Lipschitz, then f is differentiable almost everywhere (Rademacher's Theorem [10]). Let Ω_f denote the set of all points at which f is not differentiable. Then we have the following result. (See [10, Th. 2.5.1, p. 63]).

Theorem VIII.1: Let f satisfy the conditions in Definition VIII.3 and suppose that S is any set of Lebesgue measure 0. Then

$$\partial f(x) = \text{co} \left\{ \lim_{i \to \infty} \nabla f(x_i) \,\middle|\, x_i \to x,\, x_i \notin S,\, x_i \notin \Omega_f \right\} \quad (102)$$

where co denotes the convex hull.

The following result is used in the proof of Proposition IV.1. (See [30] and [31]).

Proposition VIII.1: If v is locally Lipschitz in (t, x), then

$$D_x^{1,+} v(t, x) \cup D_x^{1,-} v(t, x) \subseteq \partial_x v(t, x). \quad (103)$$

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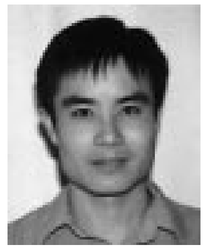

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