Empirical and Policy Performance of a Forward-Looking Monetary Model

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Abstract

In this paper we consider the policy implications of a fully specified dynamic general equilibrium model, developed by Smets and Wouters (2003a). This is a relatively large-scale forward looking model, which was shown to provide a good fit to the data. However there has been little previous work analyzing monetary policy within such a model. We first re-examine the empirical performance of the model. We show that systematically accounting for prior uncertainty leads to substantially different parameter estimates. However many of the qualitative features of the model remain similar under the alternative estimates that we find. We then formulate and analyze optimal policy rules in the model. In addition to a standard, but ad hoc, loss function we derive and justify a utility-based loss function for the model. We determine the optimal equilibrium dynamics under both loss functions and under both sets of parameter estimates. Then we discuss alternative ways of implementing the optimal equilibrium. We find that there is much scope for indeterminacy in the model, and that optimal policy rules developed for one set of estimates may perform poorly if the economy is governed by the other set. We finally turn to the analysis of simple policy rules, finding that for the simple loss function these rules perform relatively well and are robust to different parameter estimates. Overall, our results suggest that the model may be relatively robust in its ability to capture certain aspects of the data. However some caution should be exercised in using the structural estimates, and in particular it may be dangerous to pursue fully optimal policy in such complex models.

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1 Introduction

In recent years, there has been a renewed interest in the study of monetary policy under uncertainty. On the one hand, there has been a large literature studying the effects of uncertainty on policy performance in relatively small macroeconomic models. This includes a number of different papers focusing on backward-looking models without explicit micro-foundations, forward-looking models with explicit micro-foundations, or some combination of the two.[1] At the same time, there have been a number of developments in the study of empirical dynamic general equilibrium models.[2] By incorporating significant frictions in the form of nominal rigidities and adjustment costs, these models have been better able to account for the dynamics observed in the data. However to date there has been relatively little work analyzing policy performance in the context of fully specified empirical dynamic general equilibrium models. The goal of this paper is to assess the usefulness of such a model for monetary policy and to develop policy implications from such a model.

In particular, we focus on a leading new empirical model developed by Smets and Wouters (2003a), henceforth SW. Building on work by Christiano, Eichenbaum, and Evans (2001), SW develop a fully specified equilibrium model for the Euro area that includes a number of frictions and additions which can induce intrinsic persistence in the propagation of shocks. They then estimate the model using Bayesian methods and show that it fits the data as well or better than a conventional atheoretical vector autoregression. But the usefulness and implications of the model for economic policy remain largely unexplored.[3] This is our focus.

We first reconsider the empirical performance of the model, by providing a relatively comprehensive analysis of prior uncertainty and re-estimating the model under a less informative prior than SW employed. We show that this matters substantially for the estimates of the structural parameters. We also discuss some of the complications associated with estimating such large scale models, as we found numerous local minima which confounded many optimization methods. Interestingly, although our parameter estimates differ greatly, the implied time series of the output gap that we find nearly matches that in SW and the qualitative features of many of the impulse responses are similar. These results suggest that the model may be over-parameterized, and that it may be possible to find a smaller model that would fit nearly as well. This is a topic that we plan to study in future work. As a first step toward this, we analyze the sensitivity of different qualitative features of the model to changes in the parameters. We find that in order to account for the variation in the output gap and the response of key variables to a monetary policy shock, the specification of the policy rule and the price stickiness are by far the most important aspects of the model.

We then turn to the implications of the model for monetary policy. Since the model is

[1]This literature is enormous, including the key contributions summarized in Woodford (2003) and the papers in Taylor (1999). Aside from these, some key papers we build on include Fuhrer (2000), Rudebusch (2001), Onatski and Williams (2003), and Levin and Williams (2003).

[2]Again this literature is large, but the key papers we build on include Rotemberg and Woodford (1997), Christiano, Eichenbaum, and Evans (2001), and Smets and Wouters (2003a).

[3]Stock and Watson (2003) perform some policy counterfactuals in the model. Laforte (2003) estimates and studies the performance of different types of policy rules in a similar model.

based on utility maximization, we have a natural welfare criterion on which to judge policy. Following the approach outlined in Woodford (2003), we develop a quadratic approximation to the agent's utility function in the neighborhood of a steady state, providing an explicit model-consistent loss function. We also show that the linear terms in the loss function drop out near the efficient steady state, justifying the use of the linearized model. As is to be expected, this utility-based criterion depends in a crucial way on several of the key estimated structural parameters. For comparison, we also follow much of the literature by considering a simple loss function penalizing inflation and output variability.

Within the context of our estimated model, we then develop and compare policy rules which are optimal for each of the two classes of loss functions. We first focus on deriving the optimal equilibrium dynamics, which in many cases differ rather substantially from the estimated dynamics. We also show that the optimal equilibrium dynamics for the utility-based loss function are substantially different from the simple loss. The dynamics inherent in the model are more fully reflected in the utility-based loss function, and the different components of output are weighted quite differently. In particular, investment fluctuations seem to be influential for the optimal equilibrium.

We then turn to the implementation of the optimal equilibrium via policy rules. We first discuss "implicit" instrument rules as in Giannoni and Woodford (2002), which entails commitment to bring about certain paths for endogenous variables. For the ad hoc loss function, the optimal implicit rule for the interest rate may be stated in terms of lags of the interest rate and lags and (internal) forecasts of the inflation rate and the output gap.[4] We show that the optimal rules imply a substantial and rather complex dynamics, which also suggest that the optimal rule may be best formulated in terms of changes rather than levels of the interest rate. We also develop an explicit instrument rule as in Backus and Driffill (1986) and Currie and Levine (1993) for the optimal equilibrium, in which the policy responds to the state vector and a vector of endogenous multipliers. We verify that one form of such a representation leads to a determinate equilibrium for our estimates (although other forms lead to indeterminacy).

We next examine the effects of uncertainty on monetary policy. While there are many potential sources of uncertainty in the model, we focus on the forms of parameter uncertainty and uncertainty about the model specification that we uncovered in our empirical analysis. Our results showed that our different prior assumptions led to estimates that differed substantially from SW, and we now show how this affects policy recommendations. We re-derive optimal equilibrium dynamics and optimal policy rules using the estimates in SW. We see that these differ rather substantially from the findings under our estimates. Moreover, the optimal rules from one set of estimates lead to indeterminacy when the economy is governed by the other estimates. In the terminology of Levin and Williams (2003) the different estimates are "mutually intolerant" for optimal policy rules, as they may perform poorly when one is matched with the other.

One possibility is that the in the pursuit of fully optimal policy, we may have sacrificed

[4] As we discuss below, this representation is valid within an equilibrium, and hence the forecasts are internal in that they depend on the solution of the equilibrium dynamics.

robustness. Thus we also consider the performance of some simple policy rules in which the interest rate responds to some current variables only. Following Taylor (1993) simple rules have been analyzed in a number of contexts and often have been found to both perform relatively well and to be relatively robust (see the contributions in Taylor (1999) for examples). In our case, we find that there is relatively little degradation in performance by switching from optimal to simple rules. Moreover, the optimal simple rule turns out to be same under each set of estimates, a rather remarkable robustness property. In current work we are looking at the performance of simple rules under the utility-based loss function. The dynamics in the model directly affect the form of this loss, and it seems that we may need to generalize the class of rules we consider in order to insure reasonable performance.

In ongoing work, we are also considering the effects of parameter uncertainty within our benchmark simplified model. Our robustness exercises have shown the importance of the parameter and model specifications by comparing the performance of policy rules optimized under different assumptions. However in each of those cases, we treat the parameters of the model as known by policymakers when they design policy rules. We plan to relax this assumption by incorporating parameter uncertainty as summarized by our estimated posterior distribution. It will be important to see how acknowledging this uncertainty affects policy decisions.

In summary, we find that the model is relatively robust in its ability to capture certain aspects of the data. Many of the qualitative dynamics remain the same under different structural estimates. However some caution should be exercised in using and interpreting the structural estimates, and in particular it may be dangerous to pursue fully optimal policy in such complex models. More robustness may be gained by considering simpler policy rules, but this in turn may depend on the loss function which is used.

2 The Smets-Wouters Model

In an important recent paper, Smets and Wouters (2003a) developed and estimated a dynamic stochastic general equilibrium model for the Euro area. Their paper has garnered much attention because they show that the model fits the data as well as a conventional atheoretical VAR.[5] This suggests that explicitly founded models may have finally reached a crucial stage in empirical work in which they can compete with purely empirical specifications. However, as mentioned in the introduction, the policy implications of the model remain largely unexplored, and is our focus. In this section we briefly describe the key equations of the model, which are derived and discussed in more detail in SW.

Building on work by Christiano, Eichenbaum, and Evans (2001) and Erceg, Henderson, and Levin (2000), the SW model includes a number of frictions and additions which can induce intrinsic persistence in the propagation of shocks. The frictions include sticky prices and sticky wages, both with partial indexation and adjustment costs in investment. The model also allows for habit persistence and variable capacity utilization with utilization

[5]However the model does not fit quite as well as some Bayesian VAR specifications.

costs. Further, in order to empirically confront seven data series in estimation, the model is supplemented with ten structural shocks, six of which are allowed to be temporally dependent. We will focus on the linearized version of the model which we now lay out. Smets and Wouters (2003a) formulate and describe the full nonlinear model as well as deriving the linearization we present here.

The model consists of a continuum of households who value consumption and leisure. The preferences incorporate external habit persistence in consumption and are subject to two types of temporally dependent preference shocks. The households have some degree of market power in the labor market, as they supply differentiated labor in an imperfectly competitive market. Households trade in a complete market to allocate their consumption over time, with the consumption Euler equation summarizing their optimal behavior:

$$C_t = \frac{h}{1+h}C_{t-1} + \frac{1}{1+h}E_tC_{t+1} - \frac{1-h}{(1+h)\sigma_C}(i_t - E_t\pi_{t+1}) + \frac{1-h}{(1+h)\sigma_C}(\epsilon_t^b - E_t\epsilon_{t+1}^b). \qquad (1)$$

All variables are expressed in terms of logarithmic deviations from the steady state. Here C_t is consumption, h is the habit persistence in the additive habit stock, σ_C is the curvature parameter for consumption utility, i_t is the nominal interest rate, π_t is inflation, and ϵ_t^b is a preference shock which scales utility multiplicatively. This preference shock follows an $AR(1)$ process with correlation ρ_b. As an extension of a typical Euler equation, (1) states that the household balances the marginal utility of consumption in successive periods, where now the marginal utility is affected by lagged consumption via the habit stock and is perturbed via the preference shock.

Because of their market power, households are wage setters in the labor market. However they cannot reset their wages every period, but instead face nominal wage rigidity of the Calvo (1983) type in which they can only reset wages with probability $1-\xi_w$. However there is partial indexation, so households that cannot re-optimize have their wages grow at a rate equal to the rate of inflation raised to the power $\gamma_w \in (0,1)$. Firms combine the differentiated labor supplied by individuals into aggregate labor L_t via a Dixit-Stiglitz aggregator with power $1+\lambda_{w,t}$. The power in the aggregator is allowed to vary over time to reflect changes in market power, but is assumed to be i.i.d. around a constant mean: $\lambda_{w,t} = \lambda_w + \eta_t^w$. Households set wages subject to their individual labor demand curves, which arise from the firms' input demands. Optimal wage setting by the household leads to the following evolution of the real wage w_t:

$$\begin{aligned} w_t \; = \; & \frac{\beta}{1+\beta}E_tw_{t+1} + \frac{1}{1+\beta}w_{t-1} + \frac{\beta}{1+\beta}E_t\pi_{t+1} - \frac{1+\beta\gamma_w}{1+\beta}\pi_t + \frac{\gamma_w}{1+\beta}\pi_{t-1} \\ & - \frac{\lambda_w(1-\beta\xi_w)(1-\xi_w)}{(1+\beta)(\lambda_w + (1+\lambda_w)\sigma_L)\xi_w}\left(w_t - \sigma_L L_t - \frac{\sigma_c}{1-h}(C_t - hC_{t-1}) - \epsilon_t^L - \eta_t^w\right). \end{aligned} \qquad (2)$$

The final term captures variation in the current period marginal utility, where $1+\sigma_L$ is the utility parameter for the disutility of labor, β is the subjective discount factor, and ϵ_t^L is a preference shock in labor supply, which is $AR(1)$ with correlation ρ_L. As in usual Calvo pricing models, (2) incorporates forward-looking expectations of future nominal wages, but now includes lagged inflation via the partial indexation.

Households own the capital stock K_t, which they rent to firms at rental rate r_t^k. Households face adjustment costs in investment I_t, where the costs are assumed to be a function of the changes in investment, but subject to a shock ϵ_t^I which is $AR(1)$ with correlation ρ^I. Instead of incurring costs, households may also change the rate of utilization of existing capital which itself entails utilization costs. Around the steady state, the adjustment costs are assumed to be zero and only of second order. Thus the costs do not affect the linearized capital evolution, which is given by:

$$K_t = (1-\tau)K_{t-1} + \tau I_{t-1} \tag{3}$$

where τ is the depreciation rate. The optimal investment decision leads to a linearized Euler equation:

$$I_t = \frac{1}{1+\beta} I_{t-1} + \frac{\beta}{1+\beta} E_t I_{t+1} + \frac{\varphi}{1+\beta} Q_t + \frac{\beta E_t \epsilon_{t+1}^I - \epsilon_t^I}{1+\beta}, \tag{4}$$

where Q_t is the real value of capital and φ is the inverse of the adjustment costs. This equation balances the costs and benefits of investment, with lagged investment and the shocks showing up through the effects of the costs of adjustment. The equilibrium real value of capital itself is determined via a typical asset pricing Euler equation which gives:

$$Q_t = -(i_t - E_t \pi_{t+1}) + \frac{1-\tau}{1-\tau+\bar{r}^k} E_t Q_{t+1} + \frac{\bar{r}^k}{1-\tau+\bar{r}^k} E_t r_{t+1}^k + \eta_t^Q, \tag{5}$$

where here η_t^Q is an i.i.d. equity premium shock (which is not based on the primitives of the model) and $\bar{r}^k$ is the mean real rate of return on capital which is assumed to satisfy $\beta = 1/(1-\tau+\bar{r}^k)$.[6]

On the production side, there are a continuum of intermediate goods producers who are monopolistic competitors. Their products are aggregated into a single final good which is used for consumption and investment via a Dixit-Stiglitz aggregator. As in the labor market, the power in the aggregator is assumed to be stochastic and is i.i.d. around a constant mean: $\lambda_{p,t} = \lambda_p + \eta_t^p$. The intermediate goods producers face fixed costs in production, and are all subject to a common technology shock, ϵ_t^a which is $AR(1)$ with correlation ρ_a. The firms have common constant returns Cobb-Douglas production functions with parameter α, which leads to the implication that the capital-labor ratio is identical across firms. The linearized version of a firm's cost minimization condition expresses the firms' demand for labor:

$$L_t = w_t + (1+\psi) r_t^k + K_{t-1}, \tag{6}$$

where ψ is the inverse of the elasticity of the capital utilization cost function. Because there is no data for the Euro area on hours worked, Smets and Wouters (2003a) model employment

[6] SW interpret the equity premium shock as reflecting variation in the external finance premium, but acknowledge that its appearance here is ad hoc. Our equation (5) corrects a minor typo in SW.

as the labor variable. As an admitted shortcut, they assume that firms face Calvo-type rigidity in adjusting employment (e_t) with unobservable hours (L_t) adjusting as a residual. This leads to the evolution of employment:

$$e_t = \beta e_{t+1} + \frac{(1-\beta\xi_e)(1-\xi_e)}{\xi_e}(L_t - e_t), \tag{7}$$

where ξ_e is the fraction of firms who are able to adjust employment in any given period.

As households do in the labor market, firms also face Calvo-type nominal rigidities in price setting, and the model allows for partial indexation. Each firm may change its price in a given period with probability $1-\xi_p$, but those firms that do not re-optimize see their prices increase by a rate equal to the rate of inflation raised to the power γ_p. The firm's optimal price setting condition leads to a generalized New Keynesian Phillips curve as:

$$\pi_t = \frac{\beta}{1+\beta\gamma_p}E_t\pi_{t+1} + \frac{\gamma_p}{1+\beta\gamma_p}\pi_{t-1} + \frac{(1-\beta\xi_p)(1-\xi_p)}{(1+\beta\gamma_p)\xi_p}(\alpha r_t^k + (1-\alpha)w_t - \epsilon_t^a + \eta_t^p). \tag{8}$$

Here lagged inflation arises due to the indexation, and the final term in the equation represents the contribution of marginal costs.

Equilibrium in the goods market obtains when production equals the demand for goods by households and the government. Fiscal policy is assumed to be Ricardian, and variations in government spending are modelled as an $AR(1)$ shock ϵ_t^G with correlation ρ_G. The linearized goods market equilibrium condition is:

$$Y_t = c_y C_t + g_y \epsilon_t^G + \tau k_y I_t + \bar{r}^k k_y \psi r_t^k = \phi\epsilon_t^a + \phi\alpha K_{t-1} + \phi\alpha\psi r_t^k + \phi(1-\alpha)L_t. \tag{9}$$

The left side of the equation expresses the demand for output, where Y_t is output, c_y, g_y, and k_y are the steady state ratios of consumption, government spending, and capital to output. As we show in Appendix A, our equation (9) corrects a slight error in Smets and Wouters (2003a) due to the capital utilization costs which enters as the final term on the left side. The right side expresses the supply of output via production, where ϕ is 1 plus the share of fixed costs in production.

Finally, Smets and Wouters (2003a) close the model by specifying an empirical monetary policy reaction function. They specify policy in terms of a generalized Taylor-type rule, where the policy authority sets nominal rates in response to inflation and the output gap. To do this, they define a model-consistent output gap as the difference between actual and potential output, where potential output is defined as what would prevail under flexible prices and wages and in the absence of the three "cost-push" shocks ($\eta_t^w, \eta_t^Q, \eta_t^p$) coming from variations in wage and price markups and the equity premium. Thus the model is supplemented with flexible-price versions of (1)-(9) which determine the potential output Y_t^*. Then the policy rule is assumed to take the following form:

$$\begin{aligned} i_t \quad = \quad & \rho i_{t-1} + (1-\rho)\left(\bar{\pi}_t + r_\pi(\pi_{t-1} - \bar{\pi}_t) + r_y(Y_{t-1} - Y_{t-1}^*)\right) \\ & + r_{\Delta\pi}(\pi_t - \pi_{t-1}) + r_{\Delta y}(Y_t - Y_t^* - (Y_{t-1} - Y_{t-1}^*)) + \eta_t^R. \end{aligned} \tag{10}$$

Here $\bar{\pi}_t$ is an $AR(1)$ shock to the inflation objective with correlation ρ_π, while η_t^R is an i.i.d. policy shock. Note that the rule allows for the Taylor (1993) effects, along with interest rate smoothing, and responses to changes in inflation and the output gap.

This completes the specification of the model. It is relatively large scale, consisting of the equations (1)-(10) and their flexible price counterparts which determine the behavior of ten endogenous variables (and their flexible price versions) and which are subject to ten different stochastic shocks. We now turn to estimation of the model, which presents some difficulties.

3 Estimation

This section describes the results of our estimation of the Smets and Wouters (2003a) model described above. As in Smets and Wouters (2003a) we estimate the model using Bayesian methods, which in our case is equivalent to maximum likelihood estimation over bounded ranges. We first describe the formulation of our prior, then turn to estimation of the model. Our results suggest that the model may be over-parameterized, so we then describe our simplified model and its estimation.

3.1 Prior Specification

Although our ultimate interest is in the usefulness of the model for policy purposes, it is crucial to know first how sensitive the empirical performance of the model is to different prior assumptions. One option would be to consider a classical estimation method, but the relative size of the model and the number of parameters to be fit made Bayesian estimation almost a necessity. In our case the prior simply delimits the range of possible parameter values over which we maximize the likelihood function. Moreover, while there are few directly comparable studies to Smets and Wouters (2003a), there are a number of studies in the literature which provide insight on some of the key parameters of the model. Thus we formulate our own prior, which is significantly less informative along many dimensions than that used by Smets and Wouters (2003a), but does incorporate knowledge from the literature about the range of reasonable parameter values.

First, we follow Smets and Wouters (2003a) in fixing several parameters throughout. We use the same values they do, which are all relatively standard. This includes setting the wage markup to $\lambda_w = 0.5$, the Cobb-Douglas production parameter to $\alpha = 0.3$, the subjective discount factor to $\beta = 0.99$, and the (quarterly) depreciation rate to $\tau = 0.025$. As Smets and Wouters (2003a) note, most of these parameters govern ratios of different variables in the steady state, but the Euro-area data set which we use is demeaned which precludes calculation of these ratios.

As is common, we set independent priors for each of the parameters which are combined to form the prior for the model. As already mentioned, in each case we take the prior to be uniform over a bounded range, which we viewed as a natural, relatively uninformative prior. Of course, at the edges of the range the prior is dogmatic, but we found it more natural

Parameter	Meaning	Range	Low Value	Source	High Value	Source
φ	Inv. Adj. Cost	0.12-0.28	0.13	ACEL	0.28	CEE
σ_C	Cons. Utility	1-4	Common	-	Common	-
h	Cons. Habit	0.4-0.9	0.57	SW	0.9	BCF
σ_L	Labor Utility	1-3	Common	-	Common	-
ϕ	Fixed cost	1-1.8	Lower bd.	-	1.8	SWUS
ψ	Cap. Util. Cost	2.8-10	2.9	SWUS	10	KR
ξ_w	Calvo wages	0.65-0.85	0.66	RW	0.85	SWUS
ξ_p	Calvo prices	0.4-0.93	0.42	ACEL	0.93	SWUS
ξ_e	Calvo employment	0.4-0.8	0.6	SW	0.6	SW
γ_w	Wage indexation	0-1	Lower bd.	-	Upper bd.	-
γ_p	Price indexation	0-1	Lower bd.	-	Upper bd.	-
r_π	Policy, inflation	1-4	Lower bd.	-	Upper bd.	-
$r_{\Delta\pi}$	Policy, inf. gr.	0-0.2	Lower bd.	-	0.18	SWUS
ρ	Policy, lag interest	0.6-0.99	0.63	S	0.96	SW
r_y	Policy, output gap	0-1	0.04	SWUS	0.98	JR
$r_{\Delta y}$	Policy, out. gap gr.	0-1	0.03	JR	Upper bd.	-

Sources: ACEL=Altig, Christiano, Eichenbaum, and Linde (2002), BCF =Boldrin, Christiano, and Fisher (2001), CEE=Christiano, Eichenbaum, and Evans (2001), JR= Judd and Rudebusch (1998), KR= King and Rebelo (1999), RW=Rotemberg and Woodford (1999), S=Sack (1998), SW=Smets and Wouters (2003a), SWUS=Smets and Wouters (2003b).

Table 1: Prior specification for estimation of structural parameters.

to specify possible ranges the estimates than say to calibrate parameters of a distribution of some different assumed form.[7] We had little prior information about the parameters in the shock processes, so we set quite loose priors. For each of the shocks, we set the prior on the standard deviations to range from zero to 10 times the posterior mode estimated by Smets and Wouters (2003a). For the persistent shocks, we set the range of the prior on the autocorrelation parameters to the entire unit interval. As we discuss below, this matters for the identification of some of the structural shock processes, as in some cases separate persistent and i.i.d. shocks enter additively.

For the structural parameters, we did a brief survey of the related literature, which is summarized in Table 1. We make no claim that this survey is exhaustive, but it serves to delimit the range of reasonable parameter values. As we noted above, the paper by Smets and Wouters (2003a) is the only one that estimates a model of this form on this data set. However recent papers by Christiano, Eichenbaum, and Evans (2001) and Altig, Christiano, Eichenbaum, and Linde (2002) work with similar models on US data, using different estimation/calibration methods than Smets and Wouters. The follow-up paper Smets and Wouters (2003b) considers a slightly modified version of the Euro-area model and fits it to US data. For some of the other parameters, we looked at studies focusing on

[7]See Berger (1985) for a discussion of alternative priors and means of prior elicitation.

real models (King and Rebelo (1999) and Boldrin, Christiano, and Fisher (2001)) as well as papers focusing on monetary policy in smaller models (Rotemberg and Woodford (1997), Judd and Rudebusch (1998), Sack (1998)). We also consulted a number of other papers which we do not list whose estimates fell within the ranges we outlined above. Finally, for some parameters we set ranges with less direct guidance from the literature. Our prior for the utility curvature parameters reflect relatively standard ranges (although they exclude the very high curvature found in Rotemberg and Woodford (1997)). For the price and wage indexation parameters, we included the entire range from no indexation to full indexation. For the parameter governing the Calvo-style adjustment of employment, which does not really have any precedent in the literature, we consider an evenly spaced interval around the point estimate of Smets and Wouters (2003a). Finally, for the policy reaction to inflation we included a large range from the critical value of 1 (hence we respect the "Taylor-principle") to a very high value of 4. Most of the literature estimates coefficients from 1-1.8, which is well within our range.

3.2 Estimating the Model

With the prior specified, we now turn to the estimation of the model. As in Smets and Wouters (2003a), we look for a parameter vector which maximizes the posterior mode, given our prior and the likelihood based on the data. Once again, this is equivalent in our case to simply maximizing the likelihood over the support of our prior. We use the same Euro-area data set, described in Fagan, Henry, and Mestre (2001), which provides quarterly data from 1970-1999, and we use the same detrending procedure (taking separate linear trends out of each variable). Relative to SW, however we use the whole data set, whereas they use the data from the 1970s to "initialize their estimates" and use only the data from 1980 onward for estimation.

While estimation of the model is conceptually straightforward, there are a number of difficulties which we met in practice. To estimate the model, we must maximize the posterior over all the parameters, which is a challenging numerical task. In addition to the 16 structural parameters detailed in Table 1, we also have to find the 10 standard deviation parameters for the stochastic shocks, and the 6 autocorrelations for the persistent shocks, for a total of 32 parameters. Although many numerical optimization methods can in principle handle such a large parameter vector, all of the methods that we tried settled in to local maxima for a range of starting values. We tried the simplex algorithm in Matlab, Chris Sims's algorithms designed to avoid common problems with likelihood functions (available on his web page), and a global search genetic algorithm. The method that produced the greatest value of the posterior was the genetic algorithm, which in turn was initialized after extensive previous search and then locally refined with Sims's algorithm. In particular, we first ran a very long random search algorithm, drawing points from our prior distribution and calculating the posterior. From this, we kept the 40 best points, which we then used these points as the initial population for a genetic algorithm. Once this algorithm appeared to converge, we applied Sims's algorithm to the resulting value, which resulted in further localized improvement

Parameter	Shock	Prior Range	Our Estimate	SW Estimate	SW SE
σ_a	Productivity	0-6	0.343	0.598	0.113
σ_π	Inflation objective	0-1	1.000	0.017	0.008
σ_b	Preference	0-4	0.240	0.336	0.096
σ_G	Govt. spending	0-4	0.354	0.325	0.026
σ_L	Labor supply	0-36	2.351	3.520	1.027
σ_I	Investment	0-1	0.059	0.085	0.030
σ_R	Interest rate	0-1	0.000	0.081	0.023
σ_Q	Equity premium	0-7	7.000	0.604	0.063
σ_p	Price markup	0-2	0.172	0.160	0.016
σ_w	Wage markup	0-3	0.246	0.289	0.027
ρ_a	Productivity	0-1	0.957	0.823	0.065
ρ_π	Inflation objective	0-1	0.582	0.924	0.088
ρ_b	Preference	0-1	0.876	0.855	0.035
ρ_G	Govt. spending	0-1	0.972	0.949	0.029
ρ_L	Labor supply	0-1	0.974	0.889	0.052
ρ_I	Investment	0-1	0.943	0.927	0.022

Table 2: Point estimates for the shock processes, with the σ parameters being standard deviations and the ρ parameters autocorrelations, compared to the estimates of Smets and Wouters (2003a) and their standard errors.

in the likelihood. We examined many other initial conditions and never obtained a result which improved upon this value, but of course we cannot guarantee that we have found a true global maximum.

In Table 2 we report the estimates of the parameters in the exogenous shock processes, with the estimates of Smets and Wouters (2003a) given for comparison.[8] While a number of our estimates appear roughly consistent with theirs, there are also a number of marked differences. The inflation objective shock that we estimate has much more volatile innovations (by a factor of nearly 6), but is much less persistent. Our i.i.d. equity premium shock is also much more volatile, by an order of magnitude. In addition, the i.i.d. policy shock is effectively zeroed out in our estimation. This can be understood by noting that in (10) the "inflation objective" shock $\bar{\pi}_t$ and the "policy" shock η_t^R enter additively. Thus we effectively estimate a combination of the two. Even though it is persistent, it decays rather quickly and is thus more appropriately thought of as a (negative) shock to monetary policy than as a change in the inflation objectives of the central bank.

Less striking but still sizeable are the differences in the labor supply, productivity, and preference shocks, with our estimates less volatile but more persistent in each case. Thus we find that overall the innovations from the shocks with less direct micro-foundations, the equity premium shock and the inflation objective shocks, are much larger under our estimates,

[8]The calculation of standard errors for our estimates is an ongoing task, which is complicated by the fact that our estimates are on a number of boundaries in the parameter space.

Parameter	Meaning	Prior Range	Our Estimate	SW Estimate	SW SE
φ	Inv. Adj. Cost	0.12-0.28	0.152	0.148	0.022
σ_C	Cons. Utility	1-4	2.178	1.353	0.282
h	Cons. Habit	0.4-0.9	0.400	0.573	0.076
σ_L	Labor Utility	1-3	3.000	2.400	0.589
ϕ	Fixed cost	1-1.8	1.800	1.408	0.166
ψ	Cap. Util. Cost	2.8-10	2.800	5.917	2.626
ξ_w	Calvo wages	0.65-0.85	0.704	0.737	0.049
ξ_p	Calvo prices	0.4-0.93	0.930	0.908	0.011
ξ_e	Calvo employment	0.4-0.8	0.400	0.599	0.050
γ_w	Wage indexation	0-1	0.000	0.763	0.188
γ_p	Price indexation	0-1	0.323	0.469	0.103
r_π	Policy, inflation	1-4	4.000	1.684	0.109
$r_{\Delta\pi}$	Policy, inf. gr.	0-0.2	0.181	0.140	0.053
ρ	Policy, lag interest	0.6-0.99	0.962	0.961	0.014
r_y	Policy, output gap	0-1	0.062	0.099	0.041
$r_{\Delta y}$	Policy, out. gap gr.	0-1	0.319	0.159	0.027

Table 3: Point estimates for the structural parameters compared to the estimates of Smets and Wouters (2003a) and their standard errors.

while many of the structural shocks have smaller innovations but are more persistent. Of course the proportion of volatility explained by any of the shocks depends on the structural parameters, which we turn to next.

Table 3 gives the estimates of the structural parameters, with the estimates and standard errors of Smets and Wouters (2003a) again given for comparison. A number of our estimates are on the boundaries of our prior range. To the extent that we set our prior to reflect reasonable ranges of estimates, this is not terribly troubling in itself. But it does suggest that the data may favor some parameter values which may be implausible from an economic viewpoint, and hence are not in the support of our prior. Turning now to the differences with SW, again we find that a number of the parameter estimates are similar but there are also significant differences. For example, the parameters describing agents' preferences are fairly different. We find that the curvature of preferences is greater for both consumption and labor, but that the habit stock parameter is much smaller. These findings may become important when we consider the analysis of monetary policy under a utility-based welfare criterion below. We find relatively large fixed costs, amounting to 80% of the share of production. If we assume that profits are zero in the steady state as in Christiano, Eichenbaum, and Evans (2001), then this amounts to an average price markup factor of $\lambda_p = 0.8$ as well, which is substantial. We also find a relatively small value capital utilization adjustment parameter, which implies relatively large costs as ψ is the inverse elasticity. Although large, both of these estimates are close to the values found by Smets and Wouters (2003b) in their analysis of a similar model for US data. For the wage and price dynamics, our estimates of the

Calvo adjustment parameters are similar, although our employment adjustment parameter is much smaller. However we find much less of a role for indexation, including no indexation of wages. Finally, the policy reaction function that we estimate is substantially different. The inflation response coefficient is very large in absolute terms and more than double what SW estimate, representing a strong response to inflation. Moreover, our estimated output gap response is somewhat smaller, but the response to output gap growth is nearly double that in SW.

Overall, we note that in many cases our estimates are far outside reasonable confidence intervals around the SW estimates, suggesting that the different prior specifications have important effects. However it is a bit difficult to directly gauge the implications of the different parameter estimates on the behavior of the model. We next turn to some implications of the model under the different sets of estimates which makes this more clear.

3.3 Impulse Responses and the Output Gap

Although the estimates of some of the structural parameters are of interest in themselves, the impulse responses of the model show how the different parts of the model interact. We now plot the impulse responses of consumption, inflation, investment, and wages for each of the nine structural shocks. Recall that the policy shock is zeroed out in our estimates. In each case, we show the response under our estimates in a solid line and the estimates of SW in a dashed line. Overall, the qualitative features of the impulse responses are similar under the different sets of estimates. In each case, the variables respond in the same direction in response to a shock and the dynamics are rather similar. However there are some differences in the magnitude and persistence of the response to the different variables.

The most important case which exhibits substantial differences is shown in Figure 1 which plots the impulse responses of the variables to an inflation objective shock. But recall that here we are effectively identifying a different structural shock process than SW, one consistent with a policy shock. As such, the response of the variables under our estimates is roughly consistent with the literature on policy shocks (see Christiano, Eichenbaum, and Evans (1999)). In response to the (expansionary) policy shock, inflation and consumption rise, with the largest effect occurring after roughly four quarters. Investment and wages have more delayed, hump shaped effects. While the estimates of Smets and Wouters (2003a) are orders of magnitude smaller, our results are qualitatively similar to their policy shock estimates.

The other case where our results differ substantially from those in SW concerns the equity premium shock, which is shown in Figure 2. Recall that we estimate the standard deviation of this shock to be more than a factor of ten larger than what SW estimate, which accounts for the larger differences in the magnitude of the responses of the variables to the shock. While it is a bit difficult to see from the figure, the qualitative dynamics are actually similar. An increase in the equity premium increases the demand for investment and decreases consumption, and results in a rise in inflation and real wages.

For all of the other shocks, the impulse responses seem economically reasonable, and the

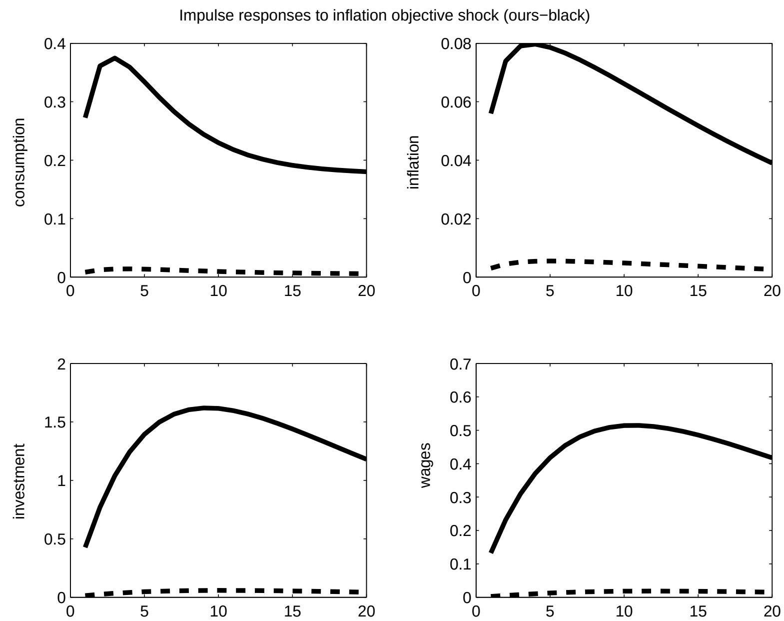

Figure 1: Impulse responses of selected variables to the inflation objective shock under Smets and Wouters's (2003a) estimates (dashed lines) and our estimates (solid lines).

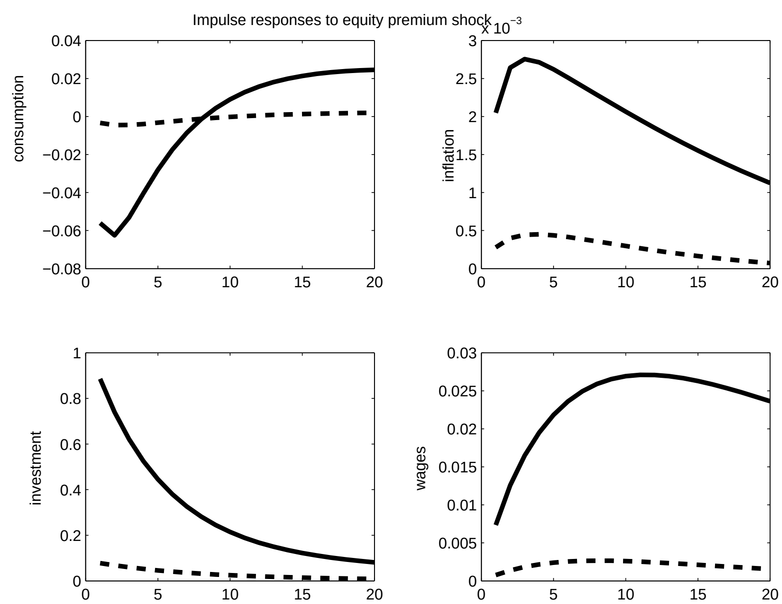

Figure 2: Impulse responses of selected variables to the equity premium shock under Smets and Wouters's (2003a) estimates (dashed lines) and our estimates (solid lines).

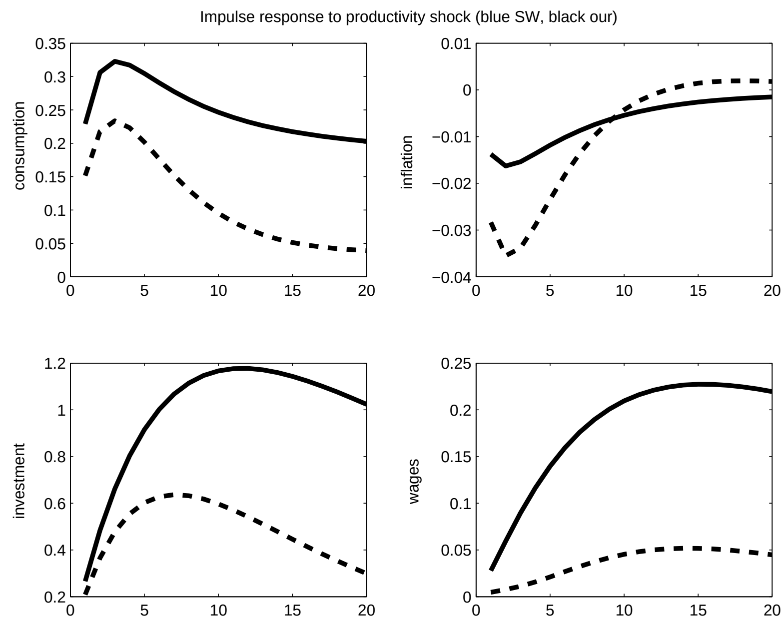

Figure 3: Impulse responses of selected variables to the productivity shock under Smets and Wouters's (2003a) estimates (dashed lines) and our estimates (solid lines).

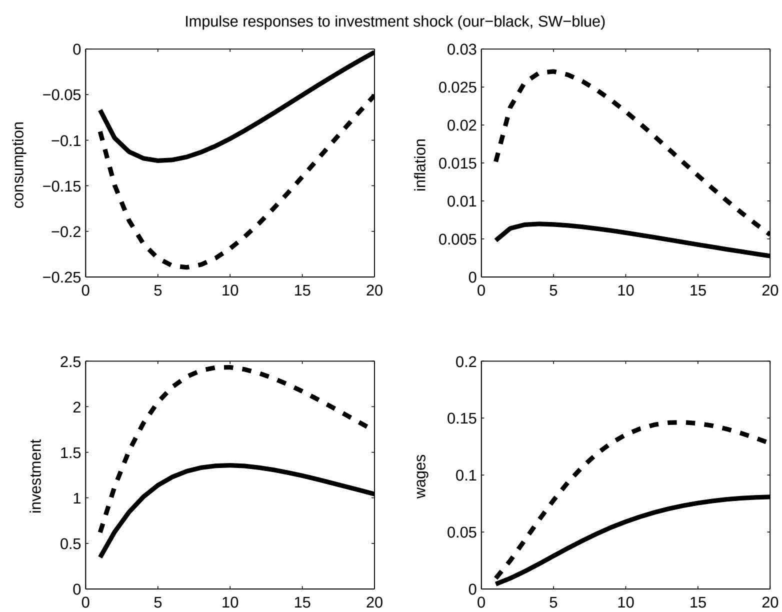

Figure 4: Impulse responses of selected variables to the investment shock under Smets and Wouters's (2003a) estimates (dashed lines) and our estimates (solid lines).

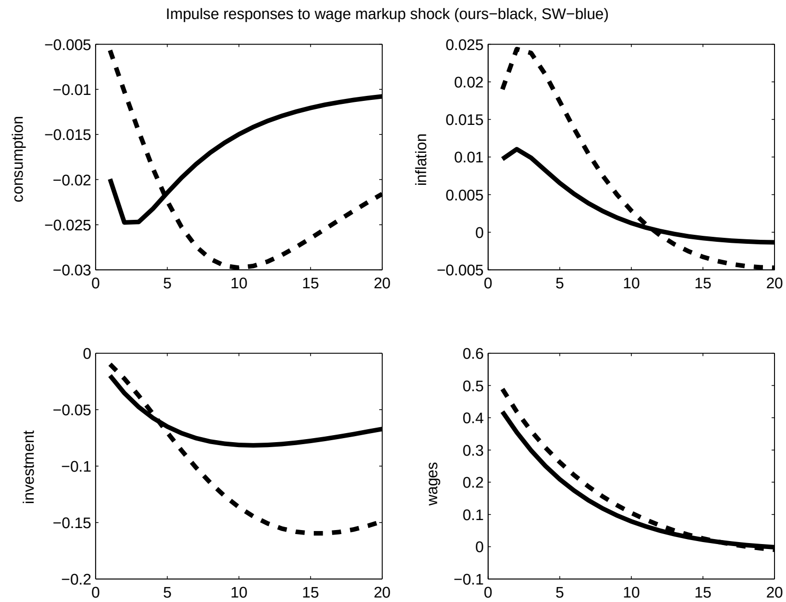

Figure 5: Impulse responses of selected variables to the wage markup shock under Smets and Wouters's (2003a) estimates (dashed lines) and our estimates (solid lines).

differences under the different estimates are not as substantial. In some cases, our estimates give larger and more prolonged responses than SW, but in some cases the reverse happens. For example, Figure 3 plots the impulse responses for a productivity shock, again showing a hump shaped increase in consumption peaking after roughly four quarters, and much more prolonged hump shaped positive responses of investment and real wages. Increased productivity negatively impacts inflation, with the peak again coming after roughly a year. Although the effects are qualitatively similar for our estimates and those of SW, for all variables but inflation our responses are larger and more persistent. Figure 4 shows that in response to an investment shock, investment, inflation, and wages rise, while consumption falls, and in each case the response is hump shaped and relatively persistent. In this case, our estimates are smaller and less persistent than SW.

The impulse responses to fluctuations in the wage markup and price markup are shown in Figures 5 and 6. In each case, the increased markup leads to a fall in consumption and investment and a rise in inflation, as the increased market power feeds through the economy. However real wages increase in response to a wage markup (unsurprisingly), while they fall in response to a price markup (as inflation increases and nominal wages are slow to adjust).

The impulse responses for the preference shock and the labor supply shock are shown in Figures 7 and 8. Here we that for other than consumption, the effects are nearly mirror images of each other. In response to a positive preference shock or negative labor supply shock, investment and wages fall and inflation rises. However consumption rises in response to a preference shock as it increases the utility benefit of current consumption, while a negative labor supply shock makes the consumption/labor tradeoff less attractive, and thus

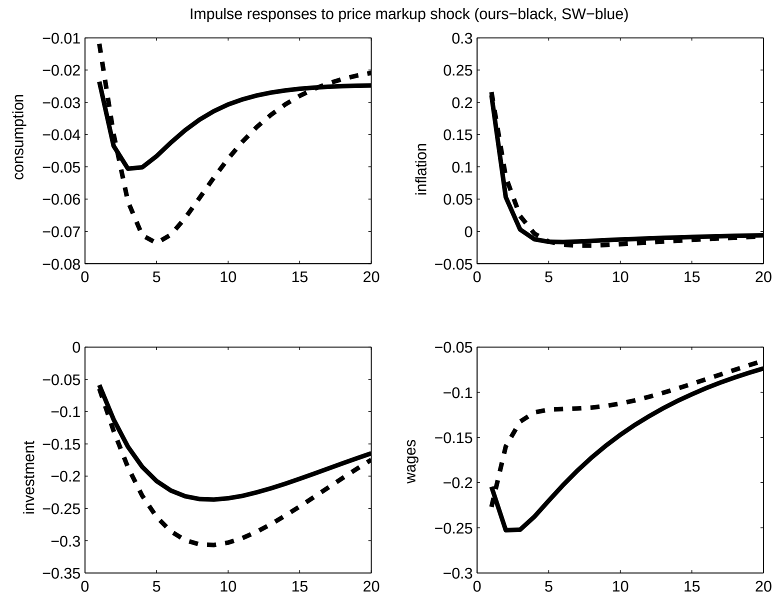

Figure 6: Impulse responses of selected variables to the price markup shock under Smets and Wouters's (2003a) estimates (dashed lines) and our estimates (solid lines).

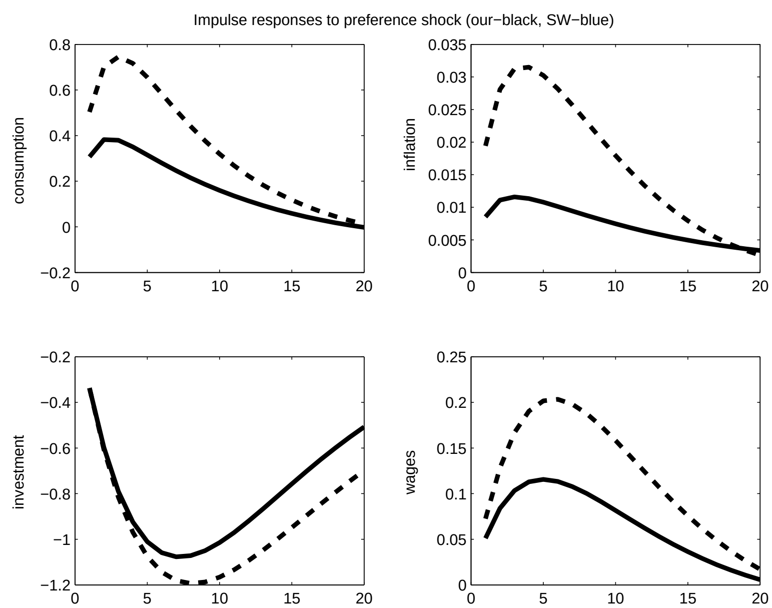

Figure 7: Impulse responses of selected variables to the preference shock under Smets and Wouters's (2003a) estimates (dashed lines) and our estimates (solid lines).

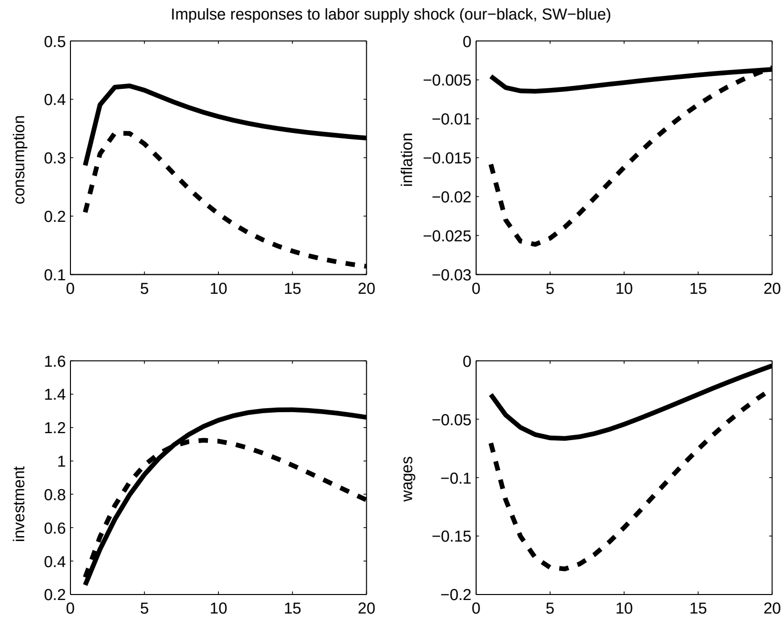

Figure 8: Impulse responses of selected variables to the labor supply shock under Smets and Wouters's (2003a) estimates (dashed lines) and our estimates (solid lines).

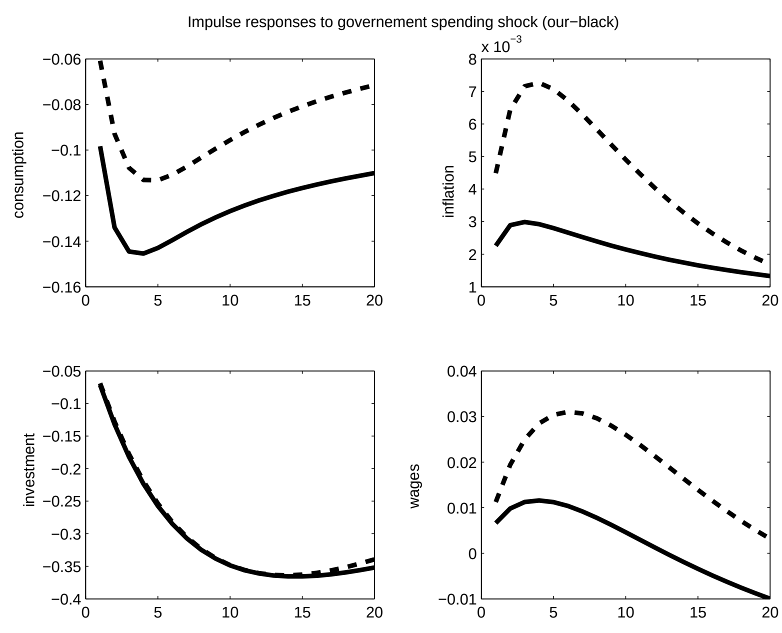

Figure 9: Impulse responses of selected variables to the government spending shock under Smets and Wouters's (2003a) estimates (dashed lines) and our estimates (solid lines).

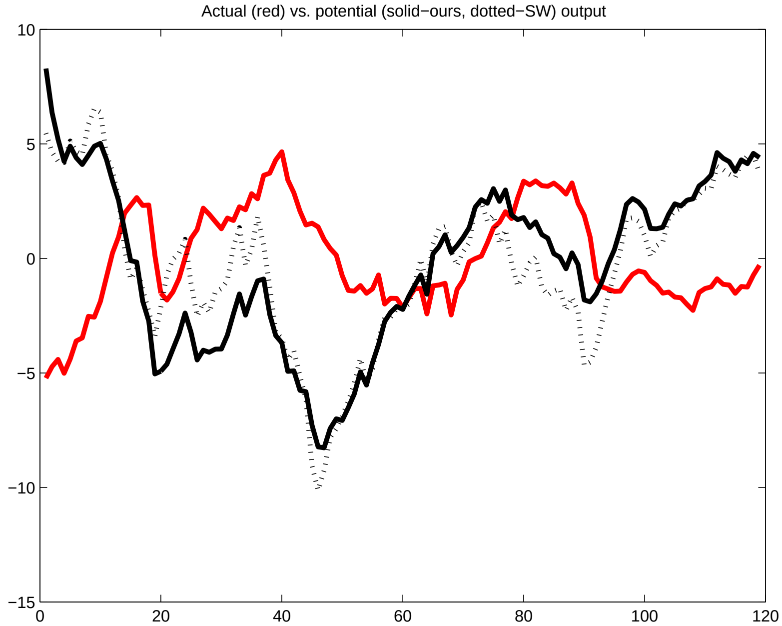

Figure 10: Actual output (red line) and the model-based estimate of potential output under Smets and Wouters's (2003a) estimates (dashed line) and our estimates (solid line).

leads to a fall in consumption. Finally, the responses to a government spending shock are shown in Figure 9. Again the qualitative features are similar, with increases in government spending leading to reductions in consumption and investment, and increases in the real wage and inflation.

One other aspect of the model that is interesting to compare is the time series of the potential output, and hence the output gap, which is implied by the model. In Figure 10 we plot the time series of output along with the level of potential output which is implied by our estimates and those of SW. Again we see that the qualitative features are very similar, with out estimated potential series being slightly smoother than that in SW. In particular, our estimates of potential are consistently lower over the first third of the sample and consistently higher over the second half. This suggest some slight changes in business cycle behavior, most notably implying that the boom early on was more pronounced, while that in the 1980s was much less substantial.

Although we have noted the dimensions along which our results differ from those in Smets and Wouters (2003a), the overall message of our empirical results must be that the main features are quite similar. Even though our point estimates differ substantially for a number of key parameters, the impulse responses suggest that the model behaves similarly in response to exogenous shocks. In addition, the two sets of estimates imply very similar time series for the model-based measure of potential output. These findings suggest that the model may be over-parameterized: different sets of estimates lead to roughly the same empirical conclusions. However, as we see below, the different estimates lead to different policy conclusions. Thus there is some scope for formulating smaller models which may continue

Qualitative Feature	Sum of Squared Level/Difference	Most Influential Parameters	Coordinates of Max Hessian Eigenvector
Potential Output	Levels	ρ, ξ_p, ρ^I	0.96, 0.25, 0.09
	Differences	ρ, ξ_p, r_π	0.99, 0.07, 0.04
Impulse response of	Levels	ρ_π, ξ_p, r_π	0.87, 0.48, 0.08
π to policy shock	Differences	ρ_π, ξ_p, ξ_w	0.72, 0.68, 0.10
Impulse response of	Levels	ρ_π, ρ, r_π	0.96, 0.19, 0.15
Y to policy shock	Differences	ρ_π, ξ_p, ρ	0.68, 0.62, 0.36
Impulse response of	Levels	ρ_π, ξ_p, ρ	0.92, 0.32, 0.18
i to policy shock	Differences	ρ_π, ξ_π, ρ	0.82, 0.53, 0.14

Table 4: The most influential parameters for different qualitative features of the model.

to capture most of the empirical features, while being more stable for policy purposes. This remains a difficult challenge. As a first step in this direction, and of independent interest in its own right, we now analyze the relative importance of different aspects of the model in leading to the different qualitative features.

3.4 Qualitative Features of the Model

Given the size and complexity of the model, it is difficult to determine directly which parts of the model are important for different aspects of our results. In this section we provide one means of getting at this issue by seeing which parameters of the model are most influential in changing some of the key qualitative features of the model. Our results are summarized in Table 4. The table was constructed in the following way. First, we listed several qualitative features of the model which we felt were important to explain. Because of its interest in characterizing business cycles and hence for policy purposes, we consider first the time series of potential output. Additionally, much of the literature has focused on the response of different variables to a monetary policy shock. In particular, the paper by Christiano, Eichenbaum, and Evans (2001) which forms much of the basis of the Smets and Wouters (2003a) model explicitly focuses on the effects of a policy shock. Thus we also analyze the impulse responses of inflation, output, and the nominal interest rate to a policy shock (which recall in our case is captured by the "inflation objective" shock $\bar{\pi}_t$).

For each of these qualitative features, we take the values from our estimates as the baseline values. Then we formulate a criterion which measures how the feature changes when we change the parameters, looking at the sum of squared differences in the levels and the differences, where we look at the entire potential time series and the first 20 quarters of the impulse responses. We then look at the eigenvector corresponding to the maximum eigenvalue of Hessian matrix of each criterion. This measures direction of the steepest rate of increase in the criterion around our estimates.[9] The table then lists the three most influential

[9]More explicitly, say $\{I_t^\pi(\theta^*)\}$ gives the impulse response of inflation to a policy shock for $t = 1, \ldots, 20$ for our parameter vector θ^*. Then we analyze the sum of squared differences in levels L and differences D

parameters in each case, along with the values of their coordinates in the unit eigenvector.

The table shows that for each of these qualitative features, a few key parameters turn out to be very influential. For the estimated potential output time series, the interest rate smoothing coefficient ρ in the policy rule (10) is by far the most influential parameter. This suggests that the estimate of potential output may be very sensitive to the *a priori* specification of the policy rule. The Calvo price stickiness parameter ξ_p also matters somewhat for potential output. For the impulse responses to the policy shock, the persistence of the shock ρ_π is clearly and unsurprisingly very influential. However the price stickiness parameter is also quite important, particulary for the inflation response, and the interest rate smoothing parameter ρ is again of significant importance, particularly for the output and interest rate responses. We also see that response to inflation in the policy rule r_π is of some importance as well. This suggests that although the model has a number of frictions and sources of stickiness, for these features of the model the price stickiness and policy rule specifications are of the most importance.

4 Policy Analysis

We now turn from the empirical aspects of the model to its use for policy analysis. In particular, we first describe the loss functions that we apply, considering both a simple and standard but ad hoc version, as well as deriving and justifying a utility-based welfare criterion. Then we analyze the optimal equilibrium, initially without reference to any particular policy rules, then discussing how to implement the equilibrium. We then analyze the robustness of the optimal policy rules under alternative parameter estimates, and finally discuss the performance of simple policy rules in this environment.

4.1 The Loss Functions

An obvious prerequisite for policy analysis is a specification of the policy objectives. We suppose that policymakers minimize an intertemporal loss function which is the discounted sum of all future and current one period losses. We consider two different specifications of the period loss. We first consider a standard, but ad hoc one period loss given by:

$$\Lambda_t = \pi_t^2 + \Lambda_y (Y_t - Y_t^*)^2 + \Lambda_i i_t^2, \tag{11}$$

where as above Y_t^* is the potential output and Λ_y and Λ_i are positive weights. Note that as in Woodford (2003) the loss function penalizes variation in the level of the interest rate, but

as:

$$L(\Delta) = \sum_{t=1}^{20} \left[I_t^\pi(\theta^* + \Delta) - I_t^\pi(\theta^*)\right]^2, \; D(\Delta) = \sum_{t=1}^{20} \left[(I_t^\pi(\theta^* + \Delta) - I_{t-1}^\pi(\theta^* + \Delta)) - (I_t^\pi(\theta^*) - I_{t-1}^\pi(\theta^*))\right]^2.$$

Then we construct the Hessian matrices $\left.\frac{\partial^2 L(\Delta)}{\partial\Delta\partial\Delta'}\right|_{\Delta=0}$ and $\left.\frac{\partial^2 D(\Delta)}{\partial\Delta\partial\Delta'}\right|_{\Delta=0}$.

that an interest rate smoothing objective is not assumed. We consider a range of different possibilities for the relative weights on inflation, output gap, and interest rate variability. We will abuse notation and refer to different loss functions by their ordered pair of weights, thus we denote $\Lambda = (\Lambda_y, \Lambda_i)$.

While loss functions of this type are common in the literature, and have micro-foundations in some simpler models, it is natural in our setting to also consider the loss function consistent with agents' preferences. Thus we now deduce a quadratic approximation to the welfare of the agents in our setting. While the discussion and results in Woodford (2003) make this conceptually straightforward, there are a number of complications involved due to complexity of the model. Parts of our analysis mix results in Woodford (2003), Amato and Laubach (2002), and Erceg, Henderson, and Levin (2000). However none of these sources consider welfare analysis in a model with variable capital, which causes us some complications. Following the directives of Woodford (2003), we derive a quadratic expansion the utility function in a way that insures that linear terms may be ignored. This allows us to accurately evaluate welfare using simply the linearized model we presented in Section 2. In our case, this implies that we substitute out for consumption and labor in terms of a number of different variables. Because of the complexity of the model however, the "output gap" which plays a prominent role in the loss functions considered in the literature is not readily present here.

For our derivation of the welfare function, we need to revert to the original nonlinear model. Hence for this section only we use "unhatted" letters (i.e. L_t) to denote the levels of variables, while hatted variables (i.e. $\hat{L}_t$) denote logarithmic deviations from the steady state. The social welfare function then reflects the agents' preferences and is taken to be:

$$E \sum_{t=0}^{\infty} \beta^t [U_t - V_t],$$

where we define:

$$U_t = \frac{\epsilon_t^B}{1-\sigma_C}(C_t - hC_{t-1})^{1-\sigma_C} \tag{12}$$

$$V_t = \frac{\epsilon_t^B \epsilon_t^L}{1+\sigma_L} \int_0^1 L_t(i)^{1+\sigma_L} di. \tag{13}$$

Here $L_t(i)$ represents the different types of labor supplied by households. As in the literature, the social welfare function reflects the dispersion in labor supply but not consumption because of the risk-sharing properties of the market structure.

We follow Rotemberg and Woodford (1997) and Erceg, Henderson, and Levin (2000) in taking the discounted unconditional utility as our criterion, as opposed to the conditional utility studied by Woodford (2003) and others. Kim, Kim, Schaumburg, and Sims (2003) and others have noted some of the potential difficulties with unconditional welfare analysis, the main fault being that it ignores transitional dynamics. However we only analyze policies with second order welfare effects (so that our expansion is valid), and thus any transitions are negligible. Moreover, the unconditional utility provides for a cleaner comparison with the *ad hoc* loss function in (11).

As Woodford (2003) emphasizes, it is important to carefully choose the variables in which to expand utility. In order for a quadratic approximation to be valid with a linearized model, the first derivatives of the utility function must be negligible in the neighborhood of the steady state considered. Thus we substitute out for consumption and labor and analyze the first order approximation. First, note that via the capital evolution equation we can write investment I_t as an implicit function $I_t = I(K_t, K_{t-1}, \epsilon_t^I, I_{t-1})$:

$$K_t - (1-\tau)K_{t-1} = \left[1 - S\left(\epsilon_t^I I_t / I_{t-1}\right)\right] I_t,$$

where S is the adjustment cost function satisfying $S(1) = S'(1) = 0$ and $S''(1) = 1/\varphi$. Then considering the utility from consumption (12) first, we use the goods market equilibrium condition to solve for consumption:

$$C_t = Y_t - I(K_t, K_{t-1}, \epsilon_t^I, I_{t-1}) - \Psi(z_t)K_{t-1} - G_t. \tag{14}$$

Here Ψ is the utilization adjustment cost function satisfying $\Psi'(1)/Psi''(1) = \psi$, and G_t is the level of government spending. Thus (14) defines consumption at date t as function of a number of other variables. Similarly, we use the production function of an individual firm indexed by j to substitute out for labor as:

$$L_t(j) = \left(\frac{Y_t(j) + \Phi}{\epsilon_t^a (z_t K_{t-1}(j))^\alpha}\right)^{\frac{1}{1-\alpha}}. \tag{15}$$

Here $(Y_t(j), K_t(j), L_t(j))$ are the firm's output, installed capital, and labor demand, and Φ represents the fixed costs in production.

Thus by using (14) and (15) we are able to express the welfare function in terms of a number of different variables. We then proceed with a second order expansion of the utility function in terms of these new variables. In Appendix B we show that to first order, the only variables which are relevant for policy are (K_t, Y_t, r_t^k). Moreover we derive the explicit expression for the first order coefficients:

$$\begin{aligned} E\sum_{t=0}^{\infty}\beta^t[U_t - V_t] \approx E\sum_{t=0}^{\infty}\beta^t \bar{Y} U_C (1-\beta h)\Bigg\{\left(1 - \frac{MC}{P(1+\lambda_w)}\right)\hat{Y}_t \\ + k_y\left[\beta\left(1 - \tau + \frac{\bar{r}^k}{1+\lambda_w}\right) - 1\right]\hat{K}_t - k_y\psi\bar{r}^k\left(1 - \frac{1}{1+\lambda_w}\right)\hat{r}_t^k\Bigg\} + \text{t.i.p.} \end{aligned} \tag{16}$$

where MC/P is the real marginal cost and "t.i.p." represents constants and terms independent of policy. Now we note that if we are near the efficient steady state, $MC \approx P$ and $\lambda_w \approx 0$ as market power is small in both the goods and labor markets. This implies that all of the coefficients on these first order terms are approximately zero, and hence can be ignored in the welfare analysis.

This leaves us with a quadratic objective function which leads to valid analysis simply using the linearized model. The full expression for the quadratic objective is given in equation (43) in Appendix B. We show there that it can be represented in the form:

$$U_t - V_t \approx \frac{1}{2}\begin{bmatrix} \hat{S}_t \\ \hat{S}_{t-1} \end{bmatrix}' \Lambda_{SS} \begin{bmatrix} \hat{S}_t \\ \hat{S}_{t-1} \end{bmatrix} + \Lambda_\pi(\hat{\pi}_t - \gamma_p \hat{\pi}_{t-1})^2 + \Lambda_w \left(\hat{w}_t - \hat{w}_{t-1} + \hat{\pi}_t - \gamma_w \hat{\pi}_{t-1}\right)^2$$

where we collect a number of real variables and shocks in the vector S_t:

$$S_t = (Y_t, K_t, K_{t-1}, z_t, G_t, \epsilon_t^I, I_{t-1}, \epsilon_t^a, \epsilon_t^b, \epsilon_t^L).$$

Explicit expressions for $\Lambda_{SS}, \Lambda_\pi$ and Λ_w are given in the appendix. Thus the loss captures second order effects through the derivatives of the consumption function (14) and the labor demand function (15) as well as terms in price inflation and wage inflation. As in Woodford (2003) price inflation affects welfare via the dispersion of output across firms, while as in Erceg, Henderson, and Levin (2000) and Woodford (2003) wage inflation enters via the dispersion of labor supply across households. The lags of the real variables S_{t-1} enter due to the habit persistence in consumption. While the loss function is large and includes nonzero weights on many terms, for our estimates most of the weight is put on fluctuations in inflation, real wages, the capital stock, and the covariances between inflation and wages. In particular, normalizing the weight on the variance of inflation to one, the loss function is approximately:

$$E\left[\pi_t^2 + 0.21K_{t-1}^2 - 0.51\pi_t\pi_{t-1} + 0.24(w_t + \pi_t)(w_t - w_{t-1})\right],$$

where the additional terms are at least one order of magnitude smaller. Clearly there are alternative ways of expressing the loss in terms of the state variables, but this suggests that wage and price inflation receive large weights, and their dynamics clearly matter as well. Less clear from this expression but of direct importance as well is that the utility based loss weights differently fluctuations in investment and consumption, with investment in particular receiving substantial weight. These features are absent in the simple loss function (11) discussed above.

4.2 Optimal Equilibrium

4.2.1 Formulation

With preferences specified, we now turn to the determination of the optimal equilibrium. In the analysis to follow we focus first on the simple, standard loss function (11) but we also consider the implications of the utility-based loss function. For this analysis, it is convenient to represent the model in the form:

$$\begin{aligned} \begin{bmatrix} x_{1t+1} \\ \Gamma E_t x_{2t+1} \end{bmatrix} &= A \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} + i_t + \begin{bmatrix} C\eta_{t+1} \\ 0 \end{bmatrix}, \text{ or :} \\ \hat{\Gamma} E_t x_{t+1} &= Ax_t + Bi_t + \hat{C}\eta_{t+1}, \end{aligned} \tag{17}$$

where in the second line defines $\hat{\Gamma}$ and B. Here x_t is the state vector, which is partitioned into (x_{1t}, x_{2t}) so that x_{1t} is a n_1-dimensional vector of predetermined variables in the sense of Klein (2000). That is, the prediction error $x_{1t+1} - E_t x_{1t+1}$ is an exogenously given martingale difference process and the initial value x_{10} is given. x_{2t} is a n_2-dimensional vector of non-predetermined variables, i_t is the nominal interest rate, and η_t are serially uncorrelated shocks having unit covariance matrix.

Since our goal is the analysis of optimal policy, we drop the empirical policy reaction equation (10) from the model and represent the remaining system (1)-(9) in the above form. The optimal policy for models (17) with Γ equal to identity matrix was studied in Backus and Driffill (1986). Svensson and Woodford (2003) generalize the analysis to allow for a singular Γ matrix and unobservable variables.

To compute the optimal equilibrium, we rewrite the loss (11) in a more general form:

$$\Lambda_t = x_t' \Lambda_x x_t + 2 i_t' \Lambda_{ix} x_t + i_t' \Lambda_i i_t.$$

Then we formulate the Lagrangian associated with the loss and (17):

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \left\{ x_t' \Lambda_x x_t + 2 i_t' \Lambda_{ix} x_t + i_t' \Lambda_i i_t + 2 \lambda_{t+1}' \left(A x_t + B i_t - \hat{\Gamma} E_t x_{t+1} \right) \right\}.$$

Taking the first order conditions and solving the resulting system we find, assuming that the solution exists and is unique, that:

$$\begin{bmatrix} x_{1t+1} \\ \lambda_{2t+1} \end{bmatrix} = F_1 \begin{bmatrix} x_{1t} \\ \lambda_{2t} \end{bmatrix} + S \eta_{t+1}, \quad \lambda_{20} = 0 \tag{18}$$

$$\begin{bmatrix} x_{2t} \\ i_t \\ \lambda_{1t} \end{bmatrix} = F_2 \begin{bmatrix} x_{1t} \\ \lambda_{2t} \end{bmatrix} \tag{19}$$

where F_1 and F_2 are matrices of constant coefficients and λ_{1t} and λ_{2t} are the sub-vectors of the vector of Lagrange multipliers corresponding to the first n_1 and last n_2 equations of the system (17) respectively. An algorithm for computing F_1 and F_2 is given, for example, in Söderlind (1999). These equations characterize the optimal equilibrium dynamics of the economy under a yet-to-be-specified optimal policy rule.

As was stressed by Backus and Driffill (1986) and Currie and Levine (1993), and more recently by Woodford (2003), there may exist many different forms of policy rules resulting in the same equilibrium dynamics of the economy. However, we can compute the optimal loss without actually specifying the form of the optimal policy rule. Indeed, equation (19) shows that in equilibrium the non-predetermined variables, x_{2t}, can be expressed as functions of the predetermined variables and multipliers, which we stack as $X_t = (x_{1t}', \lambda_{2t}')'$. Hence the optimal equilibrium loss can be reformulated as quadratic function of these variables only. Formally, the one period loss can be expressed as:

$$\Lambda_t = X_t' \Lambda_X X_t,$$

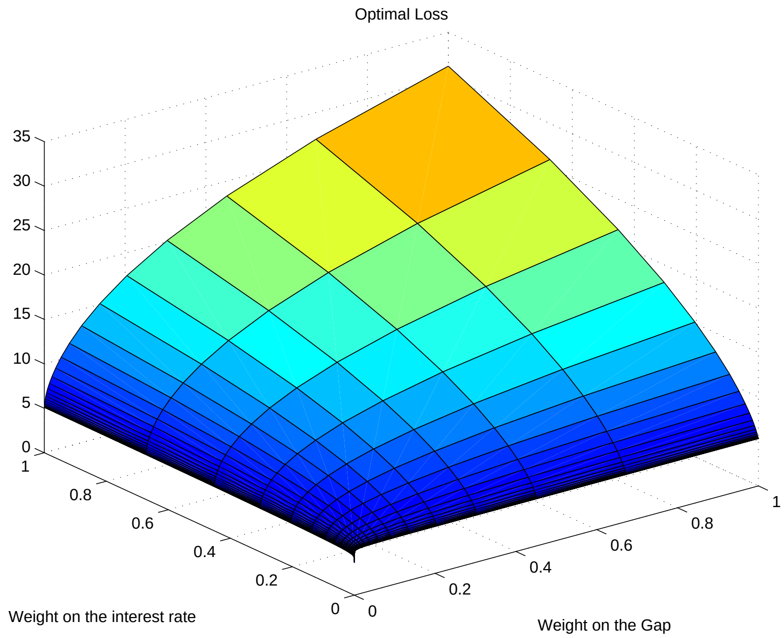

Figure 11: The optimal loss function versus the weights $\Lambda = (\Lambda_y, \Lambda_i)$ on output gap and interest rate variability.

for an appropriate choice of the matrix Λ_X. This loss and equation (18) constitute a standard purely backward-looking system. Therefore the loss can be computed using the standard formula:

$$E_0 \sum_{t=0}^{\infty} \beta^t \Lambda_t = X_0' V X_0 + \Omega, \tag{20}$$

where the matrices V and Ω satisfy:

$$\begin{aligned} V &= \beta F_1' V F_1 + \Lambda_X \\ \Omega &= \frac{\beta}{1-\beta} \mathrm{tr}\left(\mathrm{S'VS}\right). \end{aligned}$$

Since we focus on unconditional losses, the first term in (20) drops out.

4.2.2 Results for the Simple Loss Function

For a range of different relative weights, we computed the optimal loss and the corresponding loss implied by the estimated policy reaction function from (10). Using the simple loss from (11), we focused on a 900 point logarithmically-spaced grid over the weights $\Lambda \in [10^{-5}, 1] \times [10^{-5}, 1]$.[10] We started the grid from small positive values instead to insure against problems that might be caused by the "singularity" of the loss function. The optimal loss

[10]Since for the optimal rule the monetary policy and inflation target shocks are assumed to be zero, for the purpose of compatibility, we set these shocks to zero for the suboptimal rule, too.

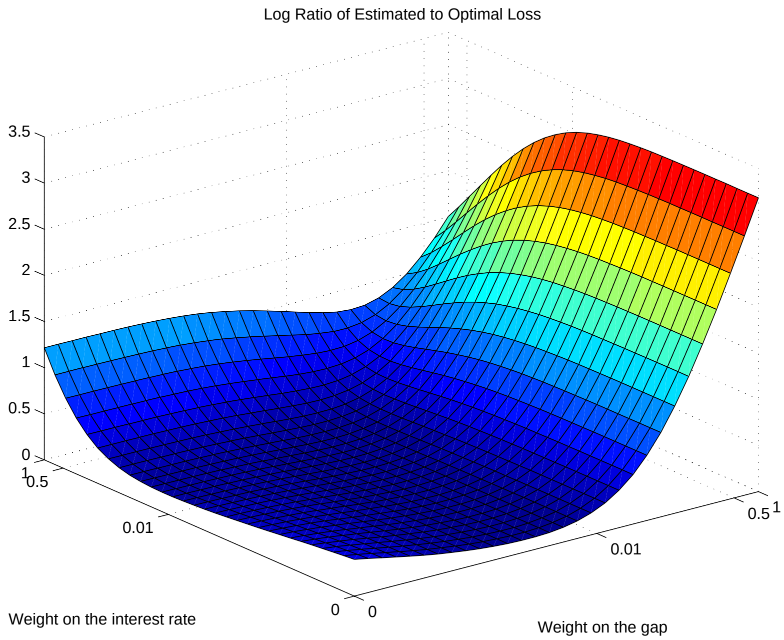

Figure 12: Log of the ratio of the loss under the estimated policy rule to the loss under the optimal rule.

is shown in Figure 11. We see that increasing weight on the output stabilization and, at the same time, penalizing the variability of the interest rate instrument increases the value of the loss dramatically. In particular, the loss rises from about 5 when the variability of the output gap and the interest rate are of no concern ($\Lambda \approx (0,0)$) to about 32 when there are equal weights on the inflation, the output gap, and the interest rate in the loss function ($\Lambda = \bar{\Lambda} \equiv (1,1)$). Loosely speaking, it is a difficult task for policymakers to stabilize inflation and the output gap with a stable interest rate. On the other hand, it is possible to control both inflation and the gap very effectively if there is no concern about instrument stability. Similarly, it is possible to keep inflation stable without interest rate variability, but at the cost of increased output gap variability.

Figure 12 reports the logarithm of the ratio of the "estimated" loss to the optimal loss. The estimated rule becomes extremely suboptimal for the loss function with equal weights on the inflation and output gap variability and no concern about the variability of the interest rate ($\Lambda \approx (1,0)$). Interestingly, the estimated rule is only slightly less efficient than the optimal rule when policy makers care almost exclusively about inflation. The minimum ratio on our grid of 1.14 is achieved for weights we denote $\Lambda = \Lambda^* \equiv (0.0017, 0.0039)$. We can think of at least two reasons why the estimated rule is nearly optimal for Λ^*. First, it may be that European policy makers indeed have cared mostly about inflation and are not concerned much about the output gap or interest rate variability. In this case, our results suggest that the policy makers are using a policy that is nearly the best for controlling inflation. Alternatively, it may be that the interest rate simply does not have much effect on inflation in the SW model. In this case, any rule policymakers would use would result in

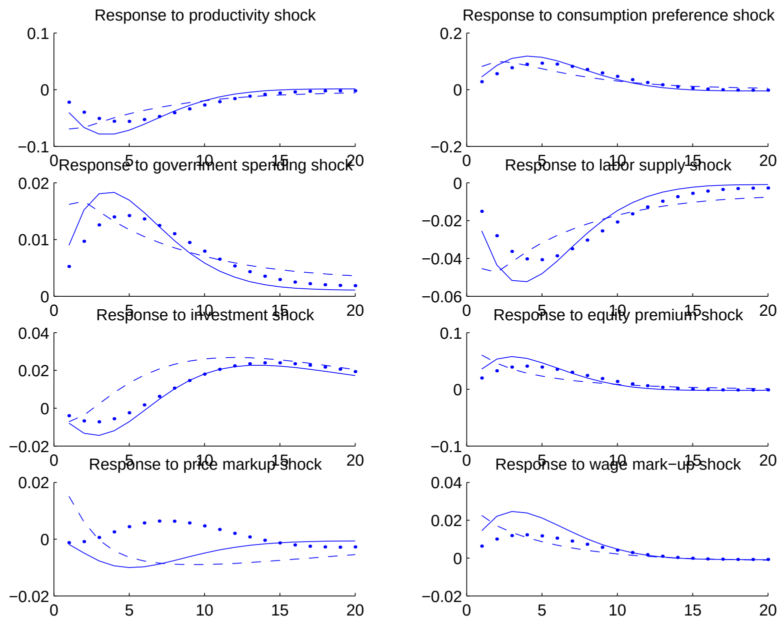

Figure 13: Impulse responses of the interest rate under the estimated policy rule (dashed lines) and the optimal policies for loss functions $\bar{\Lambda}$ (solid lines) which puts equal weights on all terms and Λ^* (dotted lines) which puts low weight on output gap and interest rate variability.

nearly the same inflation variability. In particular, the inflation component of the loss would stay roughly constant for any weights in the loss function. According to our calculations, the inflation component of the optimal loss is 4.78 for Λ^*, which puts very low weight on the output gap and interest rate variability. It is equal to 5.56, only about 16% larger, when instead the loss function is $\bar{\Lambda}$, which places equal weights on all terms. This suggests that, at least for the range of policies which are optimal for the losses we consider, policymakers can do relatively little to stabilize inflation in this model.

In order to further distinguish the effects policy, it is interesting to compare the impulse responses of the interest rate to different exogenous shocks under the optimal and the estimated policy rules. This gives an indication of the relative weight the different rules put on different sources of variability in the economy. The impulses for the rules which are optimal with losses $\bar{\Lambda}$ and Λ^* are reported in Figure 13 along with the impulses for the estimated policy rule. We see that with the exception of the response to price markup shock, all estimated impulse responses are similar to the optimal ones. Moreover, again apart from the price markup shock, the impulses for the optimal policies are qualitatively similar for the two different loss functions. However the optimal policy for $\bar{\Lambda}$ is more aggressive, as the interest rate reaction is larger in absolute value for nearly all of the shocks. We also note that the estimated impulses are somewhat closer to the optimal responses for $\bar{\Lambda}$, even though the estimated rule performs better in terms of losses for Λ^*. This suggests that although the estimated rule and optimal rule for Λ^* are close in terms of losses, they may result in fairly different dynamics. We will see this further below.

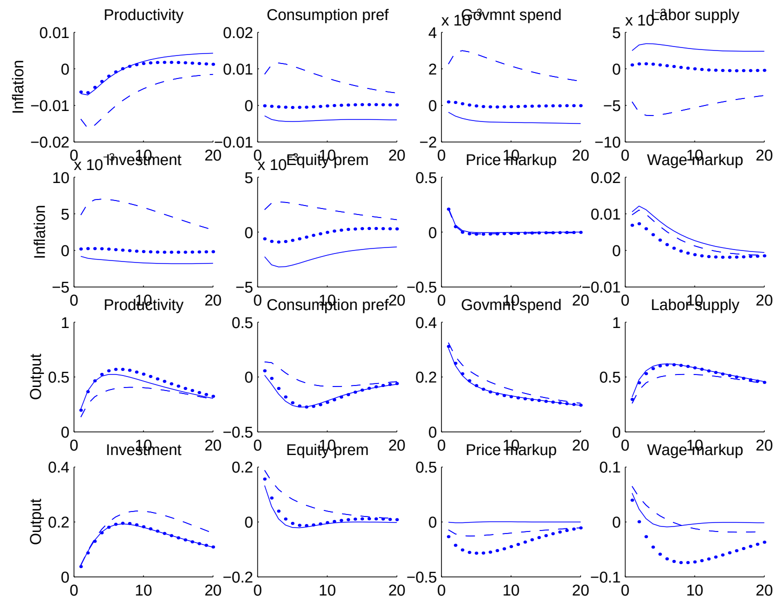

Figure 14: Impulse responses of output and inflation under the estimated policy rule (dashed lines) and the optimal policies for loss functions $\bar{\Lambda}$ (solid lines) which puts equal weights on all terms and Λ^* (dotted lines) which puts low weight on output gap and interest rate variability.

Figure 14 reports the impulse responses of inflation and output to the exogenous shocks. Again we consider the dynamics under the estimated rule and under the rules which are optimal for $\bar{\Lambda}$ and Λ^*. There are several interesting observations to be made. First, the estimated responses of inflation to the shocks seem to be larger than the optimal responses under both almost zero weights on the gap and interest rate stabilization and equal weights on inflation, the gap, and the interest rate. This lends further support to the view that the "near optimality" of the estimated rule for Λ^* does not tell anything about actual weights on the gap and interest rate stabilization that policy makers might have. Second, smaller output fluctuations under the optimal policy corresponding to $\bar{\Lambda}$ are achieved by making output relatively insensitive to price markup and wage markup shocks. This result is not surprising, as these shocks only affect the output gap, which get near zero weight under Λ^*. The estimated responses of output to the price markup and wage markup impulses are close to the optimal ones for $\bar{\Lambda}$. This suggests that the policy makers might care about the output after all and the "near optimality" of the estimated rule for Λ^* is spurious.

4.2.3 Results for the Utility-Based Loss Function

We now turn to the analysis of the optimal equilibrium for the utility-based loss function. This is still ongoing work, and as such the results in this section should be viewed as suggestive but not definitive. However they do give an idea of the type of results which may be obtained with a more exhaustive analysis. Figure 15 shows the impulse responses of a number of key variables under the estimated policy rule and under the optimal policy in

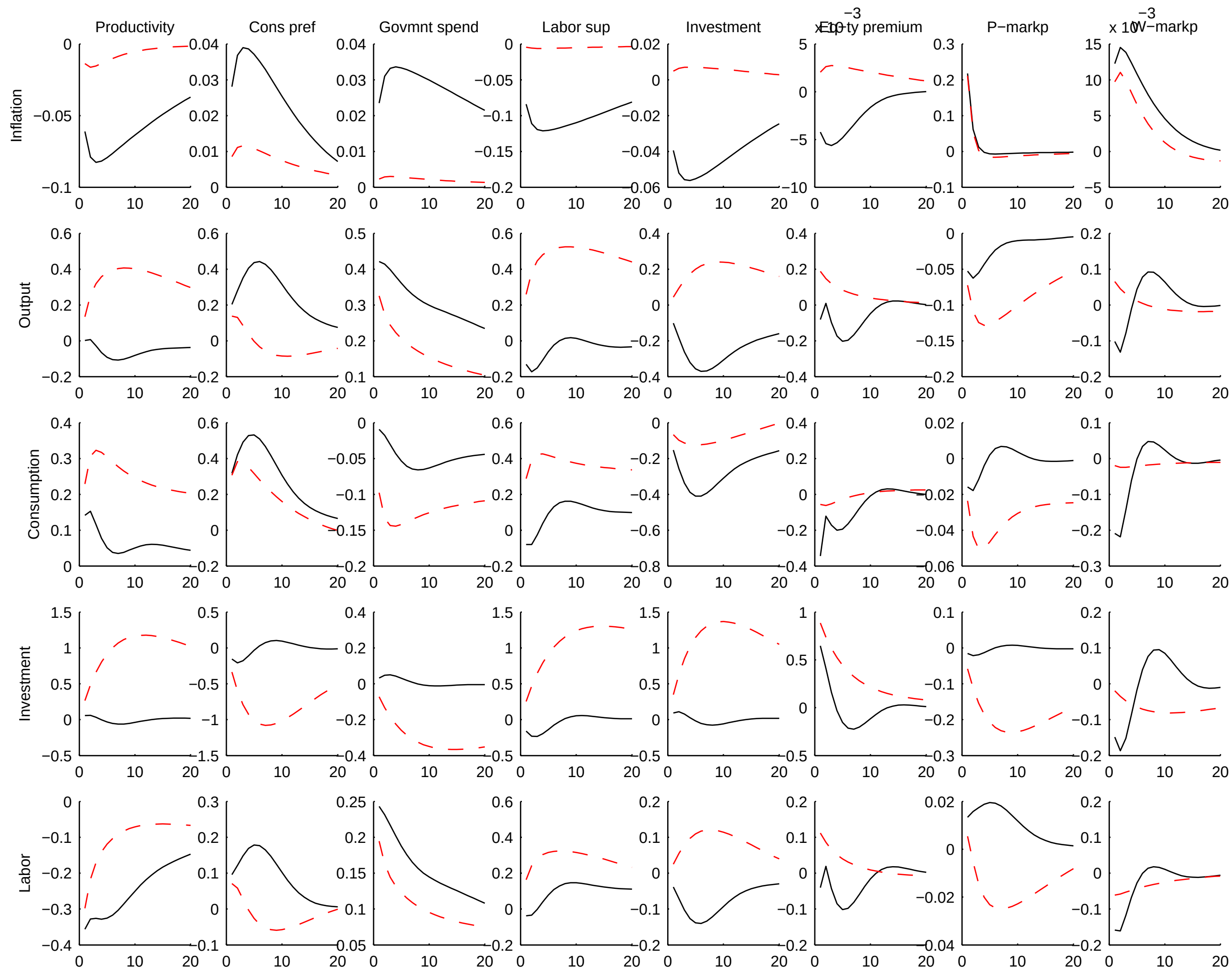

Figure 15: Impulse responses of selected variables under the estimated policy rule (dashed lines) and the optimal policy for the utility-based loss function.

the case. Here we see that optimal policy for the utility-based loss leads to substantially different dynamics than under the estimated rule. As this suggests, the estimated rule is grossly sub-optimal in this case, resulting in a loss roughly fifty times that obtained in the optimal equilibrium.

Relative to the case of the simple loss from Figure 14 above, we focus on a larger number of variables because the utility-based criterion weights the components of output differently which explains some of the sources of the differences. In particular, one dominant feature of Figure 15 is that the response of investment to a number of different shocks is severely dampened in the optimal equilibrium. This is most apparent for the productivity, labor supply, and investment shocks. This suggests that fluctuations in investment receive a significant weight in the loss function, much more so than in the simple loss function. The differences in the dynamics of some of the other variables seem to take up the slack as residuals, doing whatever necessary to stabilize investment. As we noted, these results are preliminary and we are pursuing extensions of them as well as trying to better understand the forces which lead to such different outcomes.

4.3 Optimal Rules

So far we have analyzed the optimal equilibrium without specifying policy rules that implement it. In this section, we compute and describe such optimal rules. We will consider two types of rules: explicit instrument rules and implicit rules. By an explicit instrument rule we mean a plan for setting the interest rate as a function of available data. By implicit rule we mean a commitment to bring about paths for endogenous variables that satisfy a "criterion equation." The criterion equation can be chosen by the policy makers so that their model of the economy taken together with the equation imply the optimal equilibrium discussed in the previous section. The equation may exclude the interest rate in which case the corresponding rule is called pure targeting rule.[11]

To the extent that implicit rules do not describe policy makers' behavior outside equilibrium, saying that such rules implement the equilibrium means no more than that they select a particular equilibrium and communicate this choice to the public. An implicit rule will correspond to a specific plan for interest rate setting only if the criterion equation can be solved for the interest rate, and in addition if all of the forecasts entering the equation are understood as external forecasts *available* to policy makers. The external forecasts should be contrasted with internal ones, based on solving the model of the economy taken together with the criterion equation. The internal forecast is an equilibrium concept, and hence a plan for interest rate policy based on such forecasts is only well defined inside the equilibrium.

Below we will first compute an optimal implicit rule stated exclusively in terms of the observed and projected paths of target variables, as described in Giannoni and Woodford (2002) (GW henceforth). This rule will have an additional useful property that it remains optimal when the correlation structure of the exogenous shocks changes. After that, we will compute an optimal explicit instrument rule. Although this rule will not remain optimal

[11] See Woodford (2003), p.543 for discussion of implicit rules.

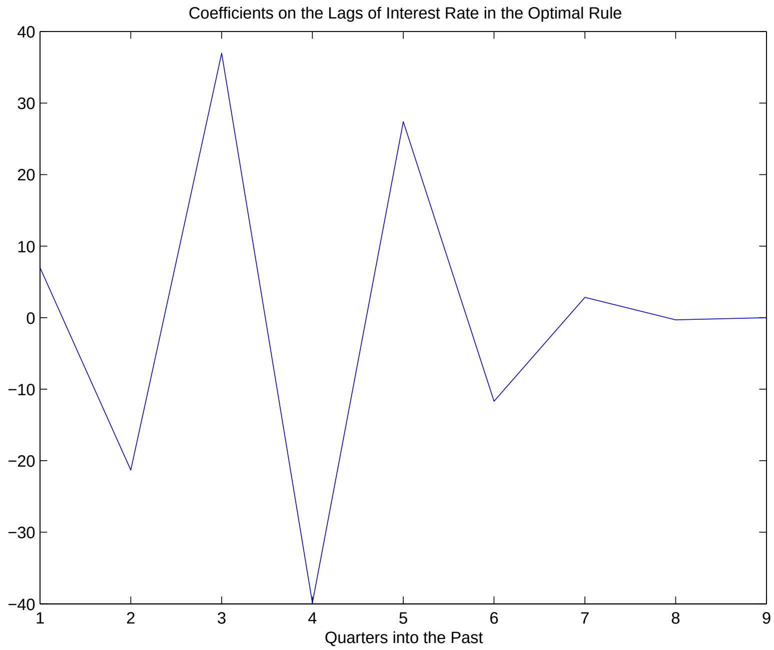

Figure 16: Coefficients on lagged interest rates in the optimal rule.

when the nature of the shocks changes, it will be defined simply as a function of available data and hence will instruct policy makers' actions even outside the equilibrium. It is in this sense that we say that an explicit instrument rule actually implements the optimal equilibrium rather than serves as an implicit selection device.

4.3.1 Implicit Rules

For the case of our ad hoc loss, the target variables are inflation, the output gap and the interest rate. Hence, as explained in GW, an implicit rule that we are about to compute will describe the instrument, i_t, in terms of the observed and projected paths of inflation, the output gap and the interest rate. Precisely, we will compute parameters α_s, γ_{jr}, and δ_{jr} of the criterion equation:

$$i_t = \alpha_1 i_{t-1} + ... + \alpha_k i_{t-k} + \sum_{j=0}^{d} \sum_{r=0}^{\infty} E_{t-j} \left(\gamma_{jr} \pi_{t-j+r} + \delta_{jr} \left(Y_{t-j+r} - Y^*_{t-j+r} \right) \right). \tag{21}$$

Optimality of the rule will be understood from the timeless perspective as defined in GW. Note that we have considered discounted unconditional losses, the rules optimal from the timeless perspective will result exactly in the losses computed above.

Figures 16, 17, and 18 show the coefficients α_s, γ_{jr}, and δ_{jr} of the optimal criterion when the loss parameters are $\Lambda = (0.5, 0.5)$.[12] We see that the coefficients on the lagged interest

[12]We thank Marc Giannoni for providing us with Matlab code.

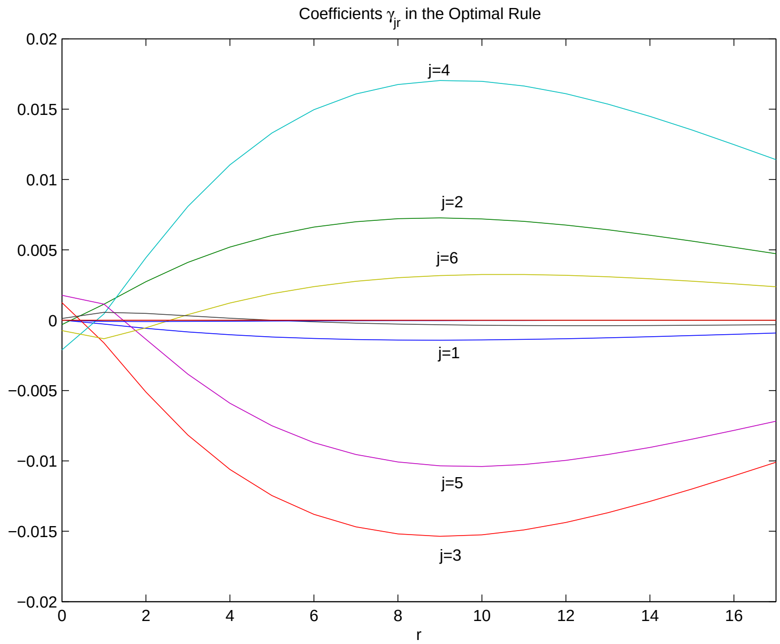

Figure 17: Coefficients on inflation in the optimal rule.

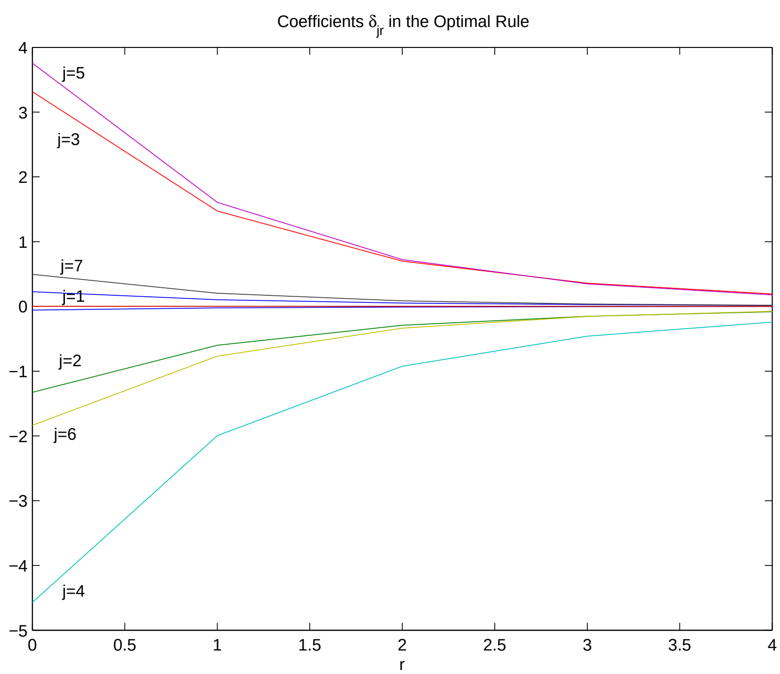

Figure 18: Coefficients on the output gap in the optimal rule.

rate in the optimal criterion are very large in absolute value and fluctuate in sign. The size of the coefficients and the number of lags getting non-zero coefficients are considerably larger than those obtained by GW for optimal rules for their study of simpler New Keynesian models. However, the fact that the coefficients are of such large sizes retains some flavor of the argument of Woodford (2003) that the optimal rules for forward-looking models often show "super-inertial" behavior of the interest rate. In fact, the polynomial $1-\alpha_1 L - ... - \alpha_k L$ is very close to $(1-L)^7$ and at least one of its roots close to unity is inside the unit circle, which can be thought of as a definition of super-inertial behavior.

The largest weight on inflation in the optimal rule corresponds to the previous year's forecasts of the next year's inflation. The largest weight on the output gap corresponds to the previous year's forecasts of the current output gap. However, positive weights on forecasts made at one quarter are balanced by negative weights on forecasts made in the following quarter. We are inclined to interpret such fluctuating weights as indicating that the optimal rule would be more naturally formulated in higher order differences of the interest rate and inflation and output gap forecasts. A rule formulated in differences that, when reformulated in levels, resembles the above optimal rule is:

$$\Delta^7 i_t = \Delta^5 \sum_{r=0}^{\infty} E_t \left(\gamma_{1r} \pi_{t+r} + \delta_{1r} \left(Y_{t+r} - Y^*_{t+r} \right) \right),$$

where the coefficients are:

r	1	2	3	4	5	6	7	8	9	10	11	12	13	14	...
$\gamma_{1r} * 10^2$	0	.03	.06	.08	.1	.12	.13	.14	.14	.14	.14	.14	.13	.12	...
$\delta_{1r} * 10^2$	-22.6	-10.4	-5.1	-2.8	-1.5	-.8	-.3	.02	.3	.5	.7	.8	.9	.9	...

4.3.2 Explicit rule

In this paper we assume that policy makers have full information about all variables and shocks entering the model. More specifically, values of all contemporaneous shocks, predetermined variables, and non-predetermined variables are assumed to be observed before the instrument setting decision is made.[13] Although such an assumption lacks realism, we consider it as describing a benchmark case against which future analysis can be contrasted.

There exist many different explicit instrument rules consistent with the optimal equilibrium. One such rule expresses the nominal interest rate as a function of past and present shocks hitting the economy. It can be read from the impulse response graphs reported on Figure 13. Unfortunately, such a policy rule will result in indeterminate equilibrium in SW model and we are interested only in those rules corresponding to the unique equilibrium which is optimal.

Another possibility suggested by Backus and Driffill (1986) is to specify the policy rule in terms of two equations: the equation from system (19) that describes the policy instrument,

[13] As long as policy makers commit to a rule for instrument setting, the private sector can decide upon the values of the non-predetermined variables before the interest rate is actually set by correctly anticipating how it will be set.

and the equation from system (18) that describes the dynamics of the Lagrange multiplier corresponding to the non-predetermined variables. Backus and Driffill (1986) describe conditions under which such a policy rule results in the unique rational expectations equilibrium, which is necessarily optimal. Unfortunately, it turns out that the Backus and Driffil rule results in indeterminacy in the model. Instead of checking that the Backus and Driffil conditions are violated, we verified the indeterminacy directly by augmenting the model with the Backus and Driffil rule and showing that there exist many equilibria for such an augmented system.

Still another way to specify the optimal policy rule would be to express the interest rate as a function of current predetermined variables, current non-predetermined variables and current Lagrange multipliers corresponding to the non-predetermined variables. Backus and Driffill (1986) suggest this alternative representation of the optimal policy rule as naturally following from the analysis of Currie and Levine (1993). They explain that such a representation will correspond to the unique equilibrium under weaker conditions than their preferred specification discussed above.

To obtain the alternative representation of the policy, use equation (19) to express the Lagrange multiplier corresponding to the predetermined variables as

$$\lambda_{1,t} = \phi_x x_{1,t} + \phi_\lambda \lambda_{2,t}$$

and substitute this expression to the first order conditions corresponding to i_t and $x_{2,t}$. From the obtained equations, find i_t and $\lambda_{2,t+1}$ as functions of $x_{1,t}, x_{2,t}$ and $\lambda_{2,t}$:

$$\begin{aligned} i_t &= -\Lambda_i^{-1} B_2' \left(\varphi_{x1} x_{1,t} + \varphi_{x2} x_{2,t} + \varphi_\lambda \lambda_{2,t}\right) && (22) \\ \lambda_{2,t+1} &= \varphi_{x1} x_{1,t} + \varphi_{x2} x_{2,t} + \varphi_\lambda \lambda_{2,t} && (23) \\ \lambda_{2,0} &= 0 && (24) \end{aligned}$$

where

$$\begin{aligned} \varphi_{x1} &= -\left(A_{12}' \phi_\lambda + A_{22}'\right)^{-1} \left(\Lambda_{x21} + A_{12}' \phi_x A_{11}\right) \\ \varphi_{x2} &= -\left(A_{12}' \phi_\lambda + A_{22}'\right)^{-1} \left(\Lambda_{x22} + A_{12}' \phi_x A_{12}\right) \\ \varphi_\lambda &= \left(A_{12}' \phi_\lambda + A_{22}'\right)^{-1} \frac{\Gamma'}{\beta} \\ A &= \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \ \Lambda_x = \begin{pmatrix} \Lambda_{x11} & \Lambda_{x12} \\ \Lambda_{x21} & \Lambda_{x22} \end{pmatrix}, \ \Lambda_{ix} = 0 \end{aligned}$$

and the partitions are conformable with the division of the vector x_t to predetermined and non-predetermined variables $x_{1,t}$ and $x_{2,t}$ respectively.[14]

We checked that the policy rule (22)-(24) indeed results in the unique rational expectations equilibrium for the model and hence implements the optimal equilibrium. We chose not to report here the coefficients of the rule because the coefficient vector has very large dimension and is hard to interpret.

[14]The policy rule (22)-(24) is optimal from the point of view of time 0 and not from the timeless perspective.

In summary, both the implicit and explicit rules result in substantial dynamics in policy and seem difficult to imagine applying in practice. The implicit rules have complicated dynamics and respond to forecasts well into the future as well as actions well into the past. The explicit rules are also large and difficult to interpret, and many natural specifications lead to indeterminate equilibria. These findings already suggest that these optimal rules are not likely to be robust in practice as they require highly detailed policy specifications. We now turn to robustness of a different sort, by considering the effects of different structural estimates.

4.4 Robustness analysis

As we discussed in our empirical analysis, based on different prior assumptions and a slightly different estimation method, we arrived at rather different parameter estimates than Smets and Wouters (2003a). We now study the implications of these different estimates for policy. We will first recompute the optimal equilibrium under the SW estimates and compare it to what we found above. After that, we will compute optimal implicit and explicit policy rules corresponding to the SW estimates. Finally, we will analyze the performance of the optimal rules based on our estimates, but assuming that the true model corresponds to the SW estimates and vice versa.

The optimal loss surface for the SW estimates on the space of the loss parameters $\Lambda \in [0,1] \times [0,1]$ is qualitatively very similar to that in Figure 11. Quantitatively, the optimal loss under the SW estimates is about the same as the optimal loss under our estimates when $\Lambda \approx (0,0)$ but is much larger (91 vs. 32) when $\Lambda = (1,1)$. The ratio of the optimal loss to that under the estimated policy rule is very similar to that shown in Figure 12. The minimum of the ratio on our grid is achieved at $\Lambda = (0.0012, 0.0189)$ (compared to $\Lambda^* = (0.0017, 0.0039)$ for our estimates), which still gives inflation the predominant weight. Figure 19 shows the optimal impulse responses of inflation and the output gap under our estimates (solid line) and the SW estimates (dashed line). In both cases the parameters of the loss function were taken to be $\Lambda = (0.5, 0.5)$. We see that many of the responses are similar, with the exception of the equity premium shock which recall has much larger volatility under our estimates. We see that the response of output to productivity and labor supply shocks is dampened initially but drawn out under our estimates.

Figures 20 - 22 show the coefficients α_s, γ_{jr}, and δ_{jr} of the optimal implicit rule (21) under the SW estimates. The coefficients on the inflation are very different from those in the optimal rule corresponding to our estimates. As before, the optimal rule resembles a rule formulated in differences

$$\Delta^7 i_t = \Delta^5 \sum_{r=0}^{\infty} E_t \left(\gamma_{1r} \pi_{t+r} + \delta_{1r} \left(Y_{t+r} - Y_{t+r}^* \right) \right),$$

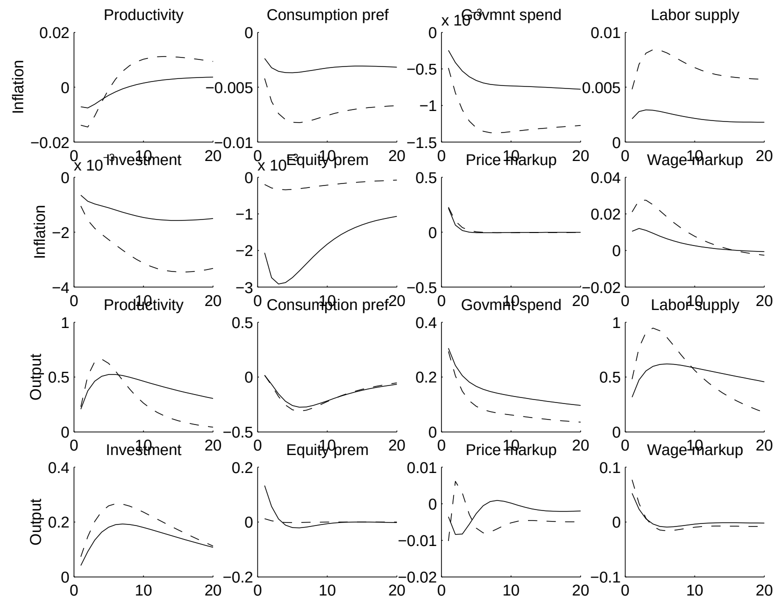

Figure 19: Optimal impulse responses of inflation and the output gap under our estimates (solid line) and the SW estimates (dashed line).

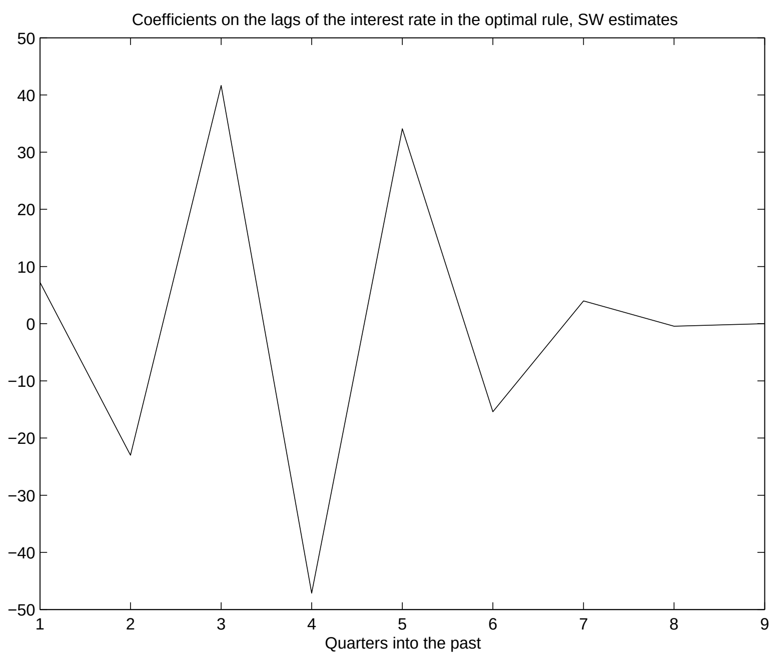

Figure 20: Coefficients on lagged interest rates in the optimal rule under Smets and Wouters's (2003a) estimates.

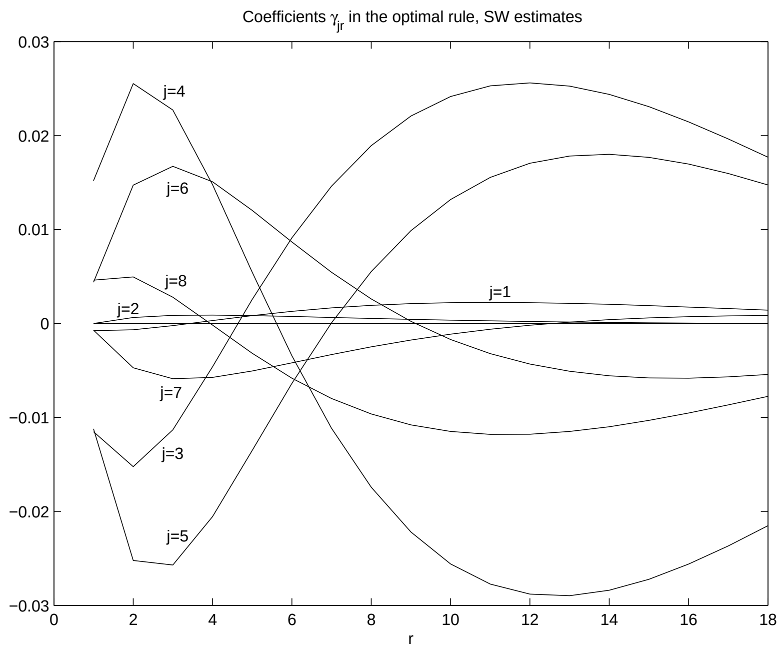

Figure 21: Coefficients on inflation in the optimal rule under Smets and Wouters's (2003a) estimates.

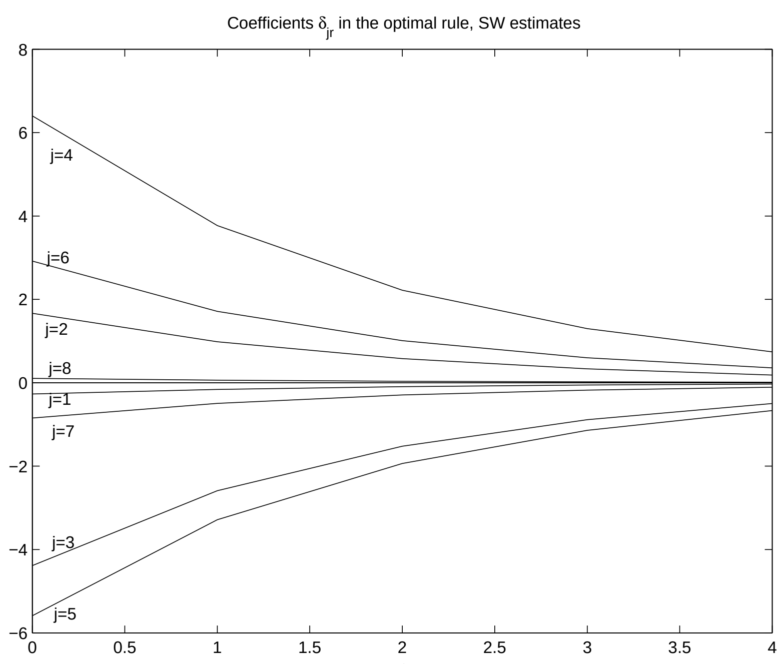

Figure 22: Coefficients on inflation in the optimal rule under Smets and Wouters's (2003a) estimates.

where:

r	1	2	3	4	5	6	7	8	9	10	11	12	13	14	...
$\gamma_{1r} * 10^2$	-.08	-.07	-.02	.03	.08	.13	.17	.19	.21	.22	.22	.22	.22	.20	...
$\delta_{1r} * 10^2$	-27.0	-16.0	-9.4	-5.4	-2.9	-1.4	-.4	.3	.8	1	1.2	1.3	1.4	1.3	...

Compared to our estimates, this exhibits larger "fluctuations": the weights are more negative early on and increase to larger values for projections further in the future.

Similar to the case of our estimates, an explicit instrument rule formulated in terms of shocks only and the Backus and Driffill rule do not result in a unique equilibrium under the SW estimates. We computed the optimal rule representing the interest rate as a function of current predetermined variables, current non-predetermined variables and current Lagrange multipliers corresponding to the non-predetermined variables, which does result in the unique (optimal) equilibrium. We choose not to report the coefficients of the rule here because interpreting these coefficients is difficult.

Neither the implicit rules nor the explicit instrument rules computed above are robust to the change in the estimates. Implicit rules are not even well defined for such a change. As we explained above, assuming that the forecasts in the optimal criterion equation are internal, optimal implicit rules are just a selection device used by policy makers to communicate to public the equilibrium chosen. If policy makers mistakenly think that the true model corresponds to SW estimates when in fact it correspond to our estimates (or vice versa), then the mismatch causes the selection mechanism to break down. More precisely, since the actual workings of the economy differs from the policy makers' model, the equilibrium corresponding to the true model and the policy equation differs from that contemplated by the policy makers. The economic outcome of such a situation will be determined not by the policy rule criterion equation, but by the actual operational procedure followed by the central bank, about which the implicit rule is silent. The explicit instrument rules computed above result in indeterminacy under the alternative estimates of the parameters.

The fact that the optimal rules computed above are not robust to the uncertainty about estimates of the model's parameters suggests that to obtain robust policy rules one may need to sacrifice full optimality. In the next section, we study the performance of simple explicit instrument rules under the alternative estimates of the model.

4.5 Simple Rules

We have seen above that the optimal policy rules in the model are quite complex and difficult to summarize, and we have just seen that they fail to be robust to different parameter estimates. However in many cases, simple policy rules which react to only a few current variables are nearly as efficient in terms of performance and are more robust.[15] Hence this is a natural approach to consider here.

For simplicity, we focus on the case of the ad-hoc loss function (11) with coefficients of 0.5 each on the output gap and the interest rate, and we analyze simple Taylor-type policy

[15]Many of the contributions in Taylor (1999) arrived at this conclusion .

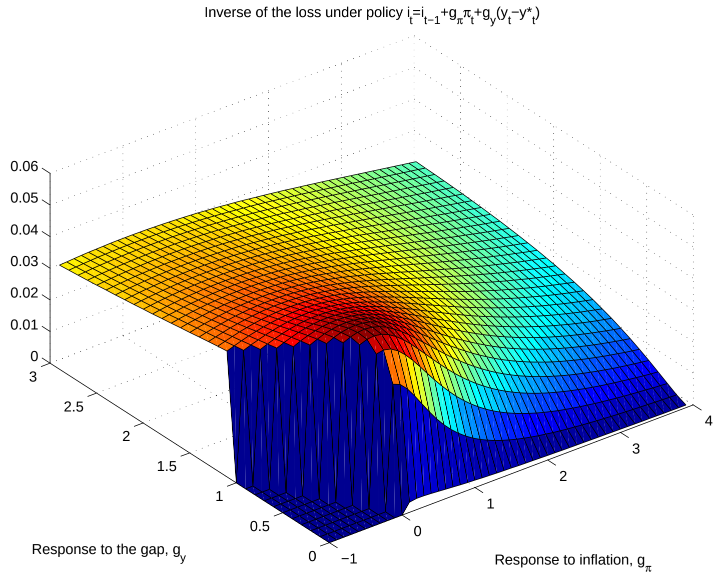

Figure 23: Inverse of the loss function for simple rules.

rules of the form:

$$i_t = \rho i_{t-1} + r_\pi \pi_t + r_y (Y_t - Y_t^*).$$

We analyzed these rules over a grid on $\rho \in [0.5, 1.5]$, $r_\pi \in [-1 : 3]$, and $r_y in [0, 2]$. We experimented with the grid and found that this one because includes the rule minimizing the loss. The performance of simple rules with the utility-based loss is a topic we are currently pursuing. With the utility-based loss, the dynamics of the model enter more directly and as we've seen above investment dynamics receive a large weight. This suggests that a different form of simple rule would likely need to be used in order to insure reasonable performance.

For our estimates of the model, the best rule on the grid was:

$$i_t = i_{t-1} + 0.4(Y_t - Y_t^*).$$

A surface plot of the *inverse* of the loss corresponding to the rules with $\rho = 1$ and r_π, r_y running through the grid is given in Figure 23. Interestingly, we find that the optimal simple rule is formulated in differences, consistent with the suggestions of our earlier results on fully optimal rules. Surprisingly, the optimal rule does not respond to inflation at all, setting changes in interest rates solely in response to the output gap. The nominal anchor in the policy rule is entirely the nominal interest rate. Of equal importance, the loss under this rule is equal to 19.51, which is only 6% higher than the optimal loss of 18.34. Thus even though the model is sufficiently complex and has significant dynamics, simple rules are able to adequately stabilize inflation, the output gap, and interest rates.

Interestingly, this exact same optimal simple rule remains optimal (on our grid) when we adopt Smets and Wouters' estimates of the model's coefficients. In such a case, the corresponding loss is equal to 52.33, which is again about 6% larger than the optimal loss of

49.59. Thus simple rules are truly robust here. We can think of the two sets of parameter estimates as very particular competing reference models in the sense of Levin and Williams (2003). The fact that the optimal rule is the same under each case means that it is also the minimax rule for this type of model uncertainty. Thus although the models are "mutually intolerant" for optimal rules, they are "highly tolerant" for simple rules. It was rather surprising to us to find out that such a simple rule performs only marginally worse than the optimal rule, which are necessarily very complicated merely because of the size of the model. It was even more surprising to learn that the rule is the best for two rather different sets of parameter estimates.

5 Conclusion

The basic goal of this project was to study the usefulness of a fully specified forward looking model for monetary policy purposes. We were impressed by the empirical performance of Smets and Wouters (2003a) but were unsure of the importance of their empirical success for policy. The answer to this question depends on the exact uses of models in policy analysis. As a description of the data, we find that the model is a qualified success. By re-estimating the model imposing less prior information, we arrived at different point estimates, but many of the qualitative features of the model remained unchanged. However as a practical guide for policy, the results are less reassuring. Optimal policy in the model results in significant dynamics which may be difficult to implement, and is not robust to alternative parameter estimates. More promising results were found by sacrificing full optimality in favor of simple rules. In ongoing work we are completing our analysis with a loss function which more accurately reflects agents' welfare, as well as more carefully considering the effects of parameter uncertainty. It will be important to know whether the promising results we have found thus far will be upheld.

Appendix

A Correcting the Equilibrium Condition

In this Appendix we derive the corrected version of the goods market equilibrium condition we discussed above. We start with the goods market equilibrium condition given in equation (27) of Smets and Wouters (2003a):

$$Y_t = C_t + G_t + I_t + \psi(z_t)K_{t-1} \tag{25}$$

Linearizing this gives (suppressing time subscripts and using bars for steady states and hats for log deviations):

$$\bar{Y}\hat{Y} = \bar{C}\hat{C} + \bar{G}\hat{G} + \bar{I}\hat{I} + \psi'\bar{K}\hat{z}.$$

In a steady state everybody is identical so from the production function given in Smets and Wouters (2003a) equation (21) we also have the relation:

$$Y_t = \epsilon_t^a (z_t K_{t-1})^\alpha L_t^{1-\alpha} - \Phi$$

Define $F(x, L) = x^\alpha L^{1-\alpha}$, where $x = zK$. Note that F is CRS and in steady state $\bar{Y} + \Phi = F$. Then linearizing gives:

$$\begin{aligned}
\bar{Y}\hat{Y} &= F\hat{\epsilon}^a + F_x \bar{x}\left(\hat{K} + \hat{z}\right) + F_L \bar{L}\hat{L} \\
&= (\bar{Y} + \Phi)\left(\hat{\epsilon}^a + \alpha\hat{K} + \alpha\hat{z} + (1-\alpha)\hat{L}\right)
\end{aligned}$$

Equating the two expressions for $\hat{Y}$ then gives:

$$\hat{Y} = \frac{\bar{C}}{\bar{Y}}\hat{C} + \frac{\bar{G}}{\bar{Y}}\hat{G} + \frac{\bar{I}}{\bar{Y}}\hat{I} + \psi' \frac{\bar{K}}{\bar{Y}}\hat{z} = \frac{(\bar{Y} + \Phi)}{\bar{Y}}\left(\hat{\epsilon}^a + \alpha\hat{K} + \alpha\hat{z} + (1-\alpha)\hat{L}\right).$$

Now using the relations $\psi' = \bar{r}^k$, $\hat{z} = \psi\hat{r}^k$, $\hat{G} = \epsilon^G$ and $\bar{I} = \tau\bar{K}$, and defining c_y, k_y and g_y as in the paper this simplifies to:

$$\hat{Y} = c_y\hat{C} + g_y\epsilon^G + \tau k_y\hat{I} + \bar{r}^k k_y \psi\hat{r}^k = \phi\left(\hat{\epsilon}^a + \alpha\hat{K} + \alpha\psi\hat{r}^k + (1-\alpha)\hat{L}\right).$$

The right side agrees with Smets and Wouters (2003a), and the left side agrees with Christiano, Eichenbaum, and Evans (2001).

B The Utility-Based Loss Function

In this appendix we derive the quadratic approximation to the utility function, which we discussed in the text. We first consider the utility from consumption (12), using (14) to solve out for consumption. This equation defines consumption at date t as function of a number of other variables. To simplify this define a composite state vector $X_t = (Y_t, K_t, K_{t-1}, z_t, G_t, \epsilon_t^I, I_{t-1})$, and thus $C_t = C(X_t)$. Note that in a deterministic steady state by assumption the level and derivative of the adjustment costs drop out in the steady state: $S(1) = S'(1) = 0$. Thus (ϵ_t^I, I_{t-1}) only have second order effects on consumption.

We now state the general form of the expansion of utility, then provide some simplification. In what follows, let $\tilde{X}_t = X_t - \bar{X}$ and $\hat{X}_t = \log X_t - \log\bar{X}$. Then, recognizing the role of the habit persistence, the second order expansion of the utility function (12) takes the form:

$$\begin{aligned}
&U_t \approx U_C C_X\left(\tilde{X}_t - h\tilde{X}_{t-1} + \tilde{\epsilon}_t^B\tilde{X}_t - h\tilde{\epsilon}_t^B\tilde{X}_{t-1}\right) + \\
&\frac{1}{2}\begin{bmatrix}\tilde{X}_t \\ \tilde{X}_{t-1}\end{bmatrix}'\left\{U_{CC}\begin{bmatrix}C_X C_X' & -hC_X C_X' \\ -hC_X C_X' & h^2 C_X C_X'\end{bmatrix} + U_C\begin{bmatrix}C_{XX'} & 0 \\ 0 & -hC_{XX'}\end{bmatrix}\right\}\begin{bmatrix}\tilde{X}_t \\ \tilde{X}_{t-1}\end{bmatrix} + \text{t.i.p.} \\
&= U_C\hat{C}_X\left(\hat{X}_t - h\hat{X}_{t-1} + \hat{\epsilon}_t^B\hat{X}_t - h\hat{\epsilon}_t^B\hat{X}_{t-1}\right) + \frac{1}{2}\begin{bmatrix}\hat{X}_t \\ \hat{X}_{t-1}\end{bmatrix}'\Lambda_{XX}\begin{bmatrix}\hat{X}_t \\ \hat{X}_{t-1}\end{bmatrix} + \text{t.i.p.} \qquad (26)
\end{aligned}$$

where we define the weighting matrix:

$$\Lambda_{XX} = \left[\begin{array}{cc} U_{CC}\hat{C}_X\hat{C}'_X + U_C\left(\hat{C}_{XX'} + [\hat{C}_X]\right) & -hU_{CC}\hat{C}_X\hat{C}'_X \\ -hU_{CC}\hat{C}_X\hat{C}'_X & h^2U_{CC}\hat{C}_X\hat{C}'_X - hU_C\left(\hat{C}_{XX'} + [\hat{C}_X]\right) \end{array} \right].$$

Here all the coefficients are evaluated at the deterministic steady state, and we follow Woodford (2003) in writing "t.i.p." for constants and terms which are independent of policy. The second equality in (26) standardizes the coefficients and writes the expression in terms of the log deviations, and we use the notation $[C_X]$ in the definition of Λ_{XX} for a diagonal matrix formed from the vector C_X.

We now provide a more explicit expression for the first order terms in (26). It is vital that the linear terms in the welfare function be negligible in order for our welfare function to be valid. Recall that we noted above that the first order expansion only depends on $(Y_t, K_t, K_{t-1}, z_t, G_t)$. The first order term in G_t doesn't affect policy, so it can be ignored here. In addition, since we are considering unconditional utility, we can consider solely (Y_t, K_t, z_t) provided we consider the effects of K_t on future consumption through X_{t+1}. More explicitly, we have to first order:

$$\begin{aligned} &E[\ldots U_t + \beta U_{t+1} + \beta^2 U_{t+1} + \ldots] = \\ &\ldots U_C C_X E\left[\tilde{X}_t - h\tilde{X}_{t-1} + \beta(\tilde{X}_{t+1} - h\tilde{X}_t) + \beta^2(\tilde{X}_{t+2} - h\tilde{X}_{t+1})\right]\ldots = \\ &\ldots U_C(1-\beta h)E\left[\tilde{Y}_t + (\beta(1-\tau)-1)\tilde{K}_t - \psi'(1)\bar{K}\tilde{z}_t\right] + \text{t.i.p.}\ldots \end{aligned}$$

Here the final line uses (14) to compute C_X and groups together terms. This last expression can be simplified slightly. First, we use the notation $\hat{X}_t$ for logarithmic deviations from the steady state and $k_y = \bar{K}/\bar{Y}$ for the steady state capital/output ratio. Then note that from the first order condition for utilization $\hat{z}_t = \psi\hat{r}^k_t$ and $\psi'(1) = \bar{r}^k$. Then we have the first order expansion:

$$E\sum_{t=0}^{\infty}\beta^t U_t \approx E\sum_{t=0}^{\infty}\beta^t\bar{Y}U_C(1-\beta h)\left[\hat{Y}_t + k_y(\beta(1-\tau)-1)\hat{K}_t - k_y\bar{r}^k\psi\hat{r}^k_t\right] + \text{t.i.p.} \qquad (27)$$

Then we want to expand the disutility from labor V_t. We first expand the integrand, which gives the following:

$$\frac{\epsilon^B_t\epsilon^L_t}{1+\sigma_L}L_t(i)^{1+\sigma_L} \quad\approx\quad \bar{L}^{1+\sigma_L}\left[(1+\hat{\epsilon}^B_t+\hat{\epsilon}^L_t)\hat{L}_t(i) + \frac{1}{2}(\sigma_L+1)\hat{L}_t(i)^2\right] + \text{t.i.p.} \qquad (28)$$

To deal with this expression, we must evaluate the cross-sectional dispersion of labor supply across households. But recall that:

$$L_t = \left[\int_0^1 L_t(i)^{\frac{1}{1+\lambda_{w,t}}}di\right]^{1+\lambda_{w,t}}$$

Note that to second order, we can expand this integral as:

$$\hat{L}_t \approx E_i \hat{L}_t(i) + \frac{1}{2(1+\lambda_w)} V_i \hat{L}_t(i). \tag{29}$$

Here we use the notation $E_i \hat{L}_t(i) = \int_0^1 \hat{L}_t(i) di$ for the cross-sectional mean and similarly for the cross-sectional variance V_i. Note that the fluctuations in $\lambda_{w,t}$ are also of higher order for this integral and so don't affect the result. Using (29) we can rewrite the integral of (28) as:

$$\begin{aligned} (1+\hat{\epsilon}_t^B + \hat{\epsilon}_t^L) E_i \hat{L}_t(i) + \frac{1}{2}(\sigma_L + 1) E_i \hat{L}_t(i)^2 &= \\ (1+\hat{\epsilon}_t^B + \hat{\epsilon}_t^L)\hat{L}_t + \frac{1}{2}(\sigma_L+1)\hat{L}_t^2 &+ \frac{1}{2}\left(\sigma_L + 1 - \frac{1}{1+\lambda_w}\right) V_i \hat{L}_t(i) \end{aligned} \tag{30}$$

Then notice that the labor demand curve gives:

$$V_i \hat{L}_t(i) = \left(\frac{1+\lambda_w}{\lambda_w}\right)^2 V_i \hat{W}_t(i), \tag{31}$$

where $\hat{W}_t(i)$ is the log of the nominal wage for household i. Again, the fluctuations in markup $\lambda_{w,t}$ will be of higher order and can be ignored here.

As in Erceg, Henderson, and Levin (2000) we note that the evolution of the cross-sectional variance of the nominal wage can be determined from the wage equation:

$$V_i \hat{W}_t(i) = \xi_w V_i \hat{W}_{t-1}(i) + \frac{\xi_w}{1-\xi_w}\omega_t^2, \tag{32}$$

where ω_t is a term reflecting the increased dispersion of those households that don't have the chance to re-optimize. With the partial indexation it is given by:

$$\begin{aligned} \omega_t &= \log W_t - \log W_{t-1} - \gamma_w(\log P_{t-1} - \log P_{t-2}) \\ &= \hat{w}_t - \hat{w}_{t-1} + \hat{\pi}_t - \gamma_w \hat{\pi}_{t-1}. \end{aligned}$$

Note that the loss function is additive in $V_i \hat{W}_t(i)$. Thus, as in Woodford (2003), we solve (32) forward to obtain the contribution to the current discounted loss as:

$$V_i \hat{W}_t(i) = \frac{\xi_w}{(1-\xi_w)(1-\beta\xi_w)}\omega_t^2, \tag{33}$$

where we ignore the constant from the initial condition. Then, combining (31) and (33), we can re-write the final term from (30) as:

$$\frac{1}{2}\left(\sigma_L + 1 - \frac{1}{1+\lambda_w}\right) V_i \hat{L}_t(i) = \frac{(1+\lambda_w)[\sigma_L(1+\lambda_w)+\lambda_w]\xi_w}{2\lambda_w^2(1-\xi_w)(1-\beta\xi_w)} \left(\hat{w}_t - \hat{w}_{t-1} + \hat{\pi}_t - \gamma_w \hat{\pi}_{t-1}\right)^2 \tag{34}$$

Then collecting terms from (30) and (34), we have the expansion of V_t:

$$\begin{aligned} V_t &\approx \bar{L}^{1+\sigma_L}\Bigg[(1+\hat{\epsilon}_t^B+\hat{\epsilon}_t^L)\hat{L}_t+\frac{1}{2}(\sigma_L+1)\hat{L}_t^2 \\ &+\frac{(1+\lambda_w)[\sigma_L(1+\lambda_w)+\lambda_w]\xi_w}{2\lambda_w^2(1-\xi_w)(1-\beta\xi_w)}\left(\hat{w}_t-\hat{w}_{t-1}+\hat{\pi}_t-\gamma_w\hat{\pi}_{t-1}\right)^2\Bigg]+\text{t.i.p.} \end{aligned} \tag{35}$$

Now we seek to substitute out for the terms in (35) in the composite labor L_t. In particular, we use the production function to write the demand for labor by firm j from (15) above. Defining the vector $Z_t(j) = (Y_t(j), K_{t-1}(j), \epsilon_t^a, z_t)$, we see that (15) defines a function $L_t(j) = L(Z_t(j))$, and so $L_t = \int_0^1 L(Z_t(j))dj$. Expanding the integrand first, we have to second order:

$$\begin{aligned} L_t(j) &\approx L_Z\tilde{Z}_t(j)+\frac{1}{2}\tilde{Z}_t(j)'L_{ZZ'}\tilde{Z}_t(j)+\text{t.i.p., so:} \\ \hat{L}_t(j) &= \hat{L}_Z\hat{Z}_t(j)+\frac{1}{2}\hat{Z}_t(j)'\left[\hat{L}_{ZZ'}+[\hat{L}_Z]\right]\hat{Z}_t(j)+\text{t.i.p.} \end{aligned} \tag{36}$$

where the second line normalizes the coefficients of the derivatives and again uses the notation $[L_Z]$ for the diagonal matrix formed from the vector L_Z. Then we can expand the integral to second order as:

$$\hat{L}_t \approx E_j\hat{L}_t(j)+\frac{1}{2}V_j\hat{L}_t(j). \tag{37}$$

Then combining (36) and (37) and letting $\hat{Z}_t = E_j\hat{Z}_t(j) = (\hat{Y}_t, \hat{K}_{t-1}, \hat{\epsilon}_t^a, \hat{z}_t)$ we have:

$$\begin{aligned} \hat{L}_t &= \hat{L}_Z\hat{Z}_t-\frac{1}{2}\hat{Z}_t'\hat{L}_Z\hat{L}_Z'\hat{Z}_t+\frac{1}{2}E_j\left(\hat{Z}_t(j)'\left[\hat{L}_{ZZ'}+[\hat{L}_Z]+\hat{L}_Z\hat{L}_Z'\right]Z_t(j)\right) \\ &= \hat{L}_Z\hat{Z}_t+\frac{1}{2}\hat{Z}_t'\left[\hat{L}_{ZZ'}+[\hat{L}_Z]\right]\hat{Z}_t+\frac{1}{2}(\hat{L}_{YY}+\hat{L}_Y^2+\hat{L}_Y)V_j\hat{Y}_t(j) \\ &+(\hat{L}_{YK}+\hat{L}_Y\hat{L}_K)\text{cov}_j(\hat{Y}_t(j),\hat{K}_t(j))+\frac{1}{2}(\hat{L}_{KK}+\hat{L}_K^2+\hat{L}_K)V_j\hat{K}_{t-1}(j). \end{aligned} \tag{38}$$

To deal with the last couple of terms in (38), we must evaluate the dispersion of output and installed capital across firms. First, recall that cost minimization for an individual firm implies:

$$\frac{W_tL_t(j)}{R_t^kK_t(j)} = \frac{1-\alpha}{\alpha}, \tag{39}$$

where $R_t^k = r_t^kP_t$ is the nominal return on capital. Thus all firms choose the same capital labor ratio. Then note that, as in the derivation of the equilibrium condition above we have:

$$\hat{Y}_t(j) = \phi\left(\hat{\epsilon}_t^a+\hat{K}_t(j)+\alpha\hat{z}_t+(1-\alpha)[\hat{L}_t(j)-\hat{K}_t(j)]\right). \tag{40}$$

In (40) we group together the log capital/labor ratio, as it is constant across firms. Thus (40) implies:

$$V_j\hat{Y}_t(j) = \phi^2V_j\hat{K}_t(j),\ \text{cov}_j(\hat{Y}_t(j),\hat{K}_t(j)) = \phi V_j\hat{K}_t(j). \tag{41}$$

Then we evaluate the output dispersion. To do so, we make the substitution parallel to the case of labor dispersion above to relate output dispersion and price dispersion:

$$V_j \hat{Y}_t(j) = \left(\frac{1+\lambda_p}{\lambda_p}\right)^2 V_j \hat{P}_t(j),$$

where $\hat{P}_t(j)$ is the log of the price for firm j. Again, the fluctuations in markup $\lambda_{p,t}$ will be of higher order and can be ignored here. Calculations directly parallel to those above for the wage dispersion give the current contribution of price dispersion to the loss as:

$$V_j \hat{P}_t(j) = \frac{\xi_p}{(1-\xi_p)(1-\beta\xi_p)}(\hat{\pi}_t - \gamma_p \hat{\pi}_{t-1})^2$$

Finally then, combining terms and substituting into (35) we have the second order expansion:

$$V_t \approx \bar{L}^{1+\sigma_L}\left\{(1+\hat{\epsilon}_t^B+\hat{\epsilon}_t^L)\hat{L}_Z\hat{Z}_t + \frac{1}{2}\hat{Z}_t'\left[(\sigma_L+1)\hat{L}_Z\hat{L}_Z' + \hat{L}_{ZZ'} + [\hat{L}_Z]\right]\hat{Z}_t + \text{t.i.p.} + \right.$$
$$\left. \frac{(1+\lambda_p)^2\xi_p}{2\lambda_p^2(1-\xi_p)(1-\beta\xi_p)}\Gamma_L(\hat{\pi}_t-\gamma_p\hat{\pi}_{t-1})^2 + \frac{(1+\lambda_w)[\sigma_L(1+\lambda_w)+\lambda_w]\xi_w}{2\lambda_w^2(1-\xi_w)(1-\beta\xi_w)}(\hat{w}_t-\hat{w}_{t-1}+\hat{\pi}_t-\gamma_w\hat{\pi}_{t-1})^2\right\}$$

where we define the weighting term Γ_L as:

$$\Gamma_L = \hat{L}_{YY} + \hat{L}_Y^2 + \hat{L}_Y + \frac{1}{\phi}(\hat{L}_{YK} + \hat{L}_Y\hat{L}_K) + \frac{1}{\phi^2}(\hat{L}_{KK} + \hat{L}_K^2 + \hat{L}_K).$$

Next, we explicitly evaluate the first order term in V_t in order to combine it with that in U_t in order to assess the validity of our expansion. Note that the first order term is $\bar{L}^{1+\sigma_L}\hat{L}_Z\hat{Z}_t$, and that to first order we can ignore the component in ϵ_t^a as it is independent of policy. Then note that with sticky wages and habit persistence we have the steady state relation from the wage equation:

$$V_L = (1-\beta h)U_C \frac{W}{P(1+\lambda_w)}$$

Also note that the cost minimization condition (39) implies that real marginal cost can be written:

$$\frac{MC}{P} = \frac{1}{1-\alpha}\frac{W}{P}\left(\frac{(1-\alpha)W}{\alpha R^k}\right)^{-\alpha} = \frac{1}{1-\alpha}\frac{W}{P}\left(\frac{\bar{K}}{\bar{L}}\right)^{-\alpha}$$

Next, we evaluate the derivatives L_Z from (15) at the steady state:

$$\begin{aligned} L_{Y_t(j)} &= \frac{1}{1-\alpha}\left(\frac{\bar{K}}{\bar{L}}\right)^{-\alpha} \\ L_{K_{t-1}(j)} &= -\frac{\alpha}{1-\alpha}\left(\frac{\bar{K}}{\bar{L}}\right)^{-1} \\ L_{z_t} &= -\frac{\alpha}{1-\alpha}\left(\frac{\bar{K}}{\bar{L}}\right)^{-1}\bar{K} \end{aligned}$$

Note that in the efficient steady state, $P = MC$. Also recall that the subjective discount factor satisfies:

$$\beta = \frac{1}{1 - \tau + \bar{r}^k}$$

Then we expand in (K_t, Y_t, z_t) as we did for the marginal utility of consumption. Then we collect terms and write the first order component of (42) explicitly as:

$$\begin{aligned} \ldots V_t + \beta V_{t+1} \ldots &\approx \ldots V_L \left[L_{Y_{t}(j)} \tilde{Y}_t + \beta L_{K_{t-1}(j)} \tilde{K}_t + L_{z_t} \tilde{z}_t \right] + \text{t.i.p.} \ldots \\ &= \ldots \bar{Y} \frac{(1 - \beta h) U_C}{1 + \lambda_w} \left[\frac{MC}{P} \hat{Y}_t - \beta k_y \bar{r}^k \hat{K}_t - k_y \bar{r}^k \psi \hat{r}_t^k \right] + \text{t.i.p.} \ldots \quad (42) \end{aligned}$$

where the second line uses the preceding derivations and the fact that $\bar{r}^k = R^k/P$. Finally, combining (27) and (42) we get the expression in (16) in the text.

In summary, by combining (26) and (42) we can write the utility function in terms of one large quadratic function. First, define now the expanded state vector:

$$S_t = (Y_t, K_t, K_{t-1}, z_t, G_t, \epsilon_t^I, I_{t-1}, \epsilon_t^a),$$

which combines the vector X_t from the utility of consumption and Z_t from the disutility of labor. (Note that this is a slightly different definition of S_t than in the text.) Clearly we can now write the derivatives with respect to either X or Z in terms of S by simply adding zeros appropriately. Then we can write the combined second order expansion to the utility function as:

$$\begin{aligned} U_t - V_t \approx [\bar{U}\hat{C}_S - \bar{V}\hat{L}_S]\hat{\epsilon}_t^B \hat{S}_t - h\bar{U}\hat{C}_S \hat{\epsilon}_t^B \hat{S}_{t-1} - \bar{V}\hat{L}_S \hat{\epsilon}_t^L \hat{S}_t + \frac{1}{2} \begin{bmatrix} \hat{S}_t \\ \hat{S}_{t-1} \end{bmatrix}' \Lambda_{SS} \begin{bmatrix} \hat{S}_t \\ \hat{S}_{t-1} \end{bmatrix} - \quad (43) \\ \frac{\bar{V}(1+\lambda_p)^2 \xi_p}{2\lambda_p^2 (1-\xi_p)(1-\beta\xi_p)} \Gamma_L (\hat{\pi}_t - \gamma_p \hat{\pi}_{t-1})^2 - \frac{\bar{V}(1+\lambda_w)[\sigma_L(1+\lambda_w) + \lambda_w]\xi_w}{2\lambda_w^2 (1-\xi_w)(1-\beta\xi_w)} (\hat{w}_t - \hat{w}_{t-1} + \hat{\pi}_t - \gamma_w \hat{\pi}_{t-1})^2 \end{aligned}$$

plus terms independent of policy. Here we note that $\bar{V} = \bar{L}^{1+\sigma_L}, \bar{U} = \bar{C}^{1+\sigma_C}$, and Λ_{SS} is derived from Λ_{XX} and is given by the following:

$$\bar{U} \begin{bmatrix} -\sigma_C \hat{C}_S \hat{C}_S' + \hat{C}_{SS'} + [\hat{C}_S] - \frac{\bar{V}}{\bar{U}} \left[(1+\sigma_L)\hat{L}_S \hat{L}_S' + \hat{L}_{SS'} + [\hat{L}_S] \right] & h\sigma_C \hat{C}_S \hat{C}_S' \\ h\sigma_C \hat{C}_S \hat{C}_S' & -h \left(h\sigma_C \hat{C}_S \hat{C}_S' + \hat{C}_{SS'} + [\hat{C}_S] \right) \end{bmatrix}$$

References

Altig, D., L. J. Christiano, M. Eichenbaum, and J. Linde (2002). Technology shocks and aggregate fluctuations. Mimeo.

Amato, J. D. and T. Laubach (2002). Implications of habit formation for optimal monetary policy. BIS Working Paper No. 121.

Backus, D. K. and J. Driffill (1986). The consistency of optimal policy in stochastic rational expectations models. Centre for Economic Policy Research Discussion Paper No. 124.

Berger, J. O. (1985). *Statistical Decision Theory and Bayesian Analysis* (Second ed.). New York: Springer-Verlag.

Boldrin, M., L. J. Christiano, and J. D. M. Fisher (2001). Habit persistence, asset returns, and the business cycle. *American Economic Review 91*, 149–66.

Calvo, G. (1983). Staggered prices in a utility-maximizing framework. *Journal of Monetary Economics A12*, 383–398.

Christiano, L., M. Eichenbaum, and C. Evans (1999). Monetary policy shocks: What have we learned and to what end? In J. Taylor and M. Woodford (Eds.), *Handbook of Macroeconomics*. Amsterdam: North-Holland.

Christiano, L. J., M. Eichenbaum, and C. L. Evans (2001). Nominal rigidities and the dynamic effects of a shock to monetary policy. Mimeo.

Currie, D. and P. Levine (1993). *Rules, Reputation and Macroeconomic Policy Coordination*. Cambridge: Cambridge University Press.

Erceg, C. J., D. W. Henderson, and A. T. Levin (2000). Optimal monetary policy with staggered wage and price contracts. *Journal of Monetary Economics 46*, 281–313.

Fagan, G., J. Henry, and R. Mestre (2001). An area wide model (AWM) for the Euro area. ECB Working Paper No. 42.

Fuhrer, J. C. (2000). Habit formation in consumption and its implications for monetary-policy models. *American Economic Review 90*, 67–90.

Giannoni, M. P. and M. Woodford (2002). Optimal interest rate rules: I. General theory. Mimeo, Princeton University.

Judd, J. P. and G. D. Rudebusch (1998). Taylor's rule and the Fed: 1970-1997. *Federal Reserve Bank of San Francisco Economic Review 0*, 3–16.

Kim, J., S. Kim, E. Schaumburg, and C. Sims (2003). Calculating and using second order accurate solutions of discrete time dynamic equilibrium models. Mimeo.

King, R. G. and S. T. Rebelo (1999). Resuscitating Real Business Cycles. In J. Taylor and M. Woodford (Eds.), *Handbook of Macroeconomics*, pp. 927–1007. Amsterdam: North-Holland.

Klein, P. (2000). Using the generalized Schur form to solve a multivariate linear rational expectations model. *Journal of Economics Dynamics and Control 24*, 1405–1423.

Laforte, J.-P. (2003). Comparing monetary policy rules in an estimated equilibrium model of the US economy. Mimeo.

Levin, A. T. and J. C. Williams (2003). Robust monetary policy with competing reference models. *Journal of Monetary Economics 50*, 945–975.

Onatski, A. and N. Williams (2003). Modeling model uncertainty. *Journal of the European Economic Association 1*, 1087–1122.

Rotemberg, J. J. and M. Woodford (1997). An optimization-based econometric framework for the evaluation of monetary policy. In B. S. Bernanke and J. J. Rotemberg (Eds.), *NBER Macroeconomics Annual 1997*, pp. 297–346. Cambridge: MIT Press for the NBER.

Rotemberg, J. J. and M. Woodford (1999). Interest rate rules in an estimated stick price model. See Taylor (1999).

Rudebusch, G. D. (2001). Is the Fed too timid? Monetary policy in an uncertain world. *Review of Economics and Statistics 83*, 203–217.

Sack, B. (1998). Uncertainty, learning, and gradual monetary policy. Finance and Economics Discussion Series Working Paper No. 1998-34, Board of Governors of the Federal Reserve System.

Smets, F. and R. Wouters (2003a). An estimated dynamic stochastic general equilibrium model of the Euro area. *Journal of the European Economic Association 1*, 1123 – 1175.

Smets, F. and R. Wouters (2003b). Shocks and frictions in US business cycles: A Bayesian DSGE approach. Mimeo.

Söderlind, P. (1999). Solution and estimation of RE macromodels with optimal policy. *Journal of Economics Dynamics and Control 43*, 813–23.

Stock, J. H. and M. W. Watson (2003). Has the business cycle changed? Evidence and explanations. Mimeo.

Svensson, L. E. O. and M. Woodford (2003). Indicator variables for optimal policy. *Journal of Monetary Economics 50*, 691–720.

Taylor, J. B. (1993). Discretion versus policy rules in practice. *Carnegie-Rochester Conference Series on Public Policy 39*, 195–214.

Taylor, J. B. (Ed.) (1999). *Monetary Policy Rules*. Chicago: University of Chicago Press for the NBER.

Woodford, M. D. (2003). *Interest and Prices: Foundations of a Theory of Monetary Policy*. Princeton: Princeton University Press.