Uncertainty and the Dynamics of Pareto Optimal Allocations

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Abstract

This paper describes the evolution of Pareto optimal allocations in stochastic economies when agents have time non-additive or recursive preferences. Following Lucas and Stokey (1984) and Kan (1995), a non-standard dynamic programming problem in which Pareto weights evolve over time and are endogenous state variables is used to compute Pareto optimal allocations. Pareto weights index allocation rules by a vector of parameters and summarize the effect of past consumption on future allocations.

This paper gives necessary and sufficient conditions for the existence of a steady state at which Pareto weights are time-invariant. Steady states are particularly interesting if they are stable. For deterministic economies, Lucas and Stokey (1984) showed that if agents have *increasing marginal impatience* then interior steady states are stable. In stochastic economies increasing marginal impatience is neither necessary nor sufficient for the stability of interior steady states. Another mechanism, which I call *relative entropy*, in conjunction with marginal impatience determines the stability of steady states. When all agents have the same constant impatience then *increasing* (respectively *decreasing*) relative entropy is sufficient to guarantee the stability (instability) of interior steady states.

Examples are given in which agents have risk-sensitive preferences, as formulated by Hansen and Sargent (1995), and power preferences, as specified by Epstein and Zin (1989).

1 Introduction

When agents have time additive preferences over consumption, the same constant impatience and the same beliefs; Pareto optimal consumption allocations are a time-invariant function of aggregate consumption. Yet, evidence from consumption data suggests that consumption allocations do vary over time in interesting ways. This paper shows that when the assumption of time additive preferences is relaxed Pareto optimal consumption allocation rules typically are time-varying and can display patterns observed in data. A new property of preferences is shown to govern the dynamics of allocations.

Dynamic economic models often assume that agents have preferences over consumption of the form

$$E_{i0} \sum_{t=0}^{\infty} \beta^t u_i(c_{it}) \tag{1}$$

where c_{it} is the consumption of agents of type i at time t, u_i is a one-period utility function (also referred to as a reward function), β is a time discount factor and E_{it} denotes the expected value with respect to the time t information of agents of type i. Agents often are allowed to have different one-period utility functions but are typically assumed to have the same discount factors and the same information. Preferences are time additive since they are written as an additive function of current and future expected one-period utilities. Preferences are also state additive since preferences are linear in probabilities. Optimal allocations are easy to compute (if agents have the same beliefs) since the ratio of the marginal utility of time t consumption for agents of type i and type j is $u_i'(c_{it})/u_j'(c_{jt})$ which does not depend upon previous consumption.

To compute a Pareto optimal allocation when agents have time additive preferences a social planner can maximize a weighted sum of utilities by choice of feasible consumption allocations. The weights are referred to as Pareto weights and index optimal allocations. At an interior Pareto optimal allocation the ratio of the marginal utilities of consumption for any two agents must be constant over time. When agents have the same impatience and the same information, optimal consumption allocations at time t can be written as $c_{it} = A_i(x_t, \theta)$ where x_t is aggregate consumption at time t, θ is a constant time-invariant vector of Pareto weights and A_i is a function which does not depend on t. For a given choice of Pareto weights consumption allocations can be computed using a time-invariant function of aggregate consumption. Consumption allocations are history independent since they do not depend on previous values of aggregate or individual consumption.

Evidence from household consumption and cross country consumption suggests that consumption allocations rules do vary over time. Cochrane (1991) and Hayashi, Altonji, and Kotlikoff (1996) show that consumption allocations are affected by health and employment. Attanasio and Davis (1996) find that changes in consumption parallel changes in income. To account for consumption the standard model (which assumes complete markets, full information and time additive preferences) needs to relax some of its assumptions. Papers by Phelan and Townsend (1991), Atkeson and Lucas (1995) and others have looked at the implications of surrendering perfect information. Papers by Heaton and Lucas (1995), Alvarez and Jermann (1997), and others have looked at the implications of surrendering complete markets. This paper maintains the assumptions of complete markets and full information but surrenders the assumption of time additive preferences.

There are good reasons for surrendering the assumption of time additive preferences. Time-additivity imposes strong restrictions on preferences which prevent agents from having risk preferences over utility. Consider the following two gambles which have been discussed by Duffie and Epstein (1992), Schroder and Skiadas (1997), and others. In gamble A a coin is tossed every period in the future. In periods in which heads obtains the lottery pays ten consumption units. In periods in which tails obtains the lottery pays one consumption unit. In gamble B a coin is tossed only once. If heads then the lottery pays ten consumption units every period in the future. If tails then the lottery pays one consumption unit every period in the future. If agents have time additive preferences then they must be indifferent between the two lotteries. Yet, it seems rational for agents to have preferences over the gambles. An agent might prefer gamble A because he views the distribution of utilities as more favorable under this gamble. A theory of preferences should not *require* agents to be indifferent between these gambles any more than it should require agents to be indifferent between consumption today and consumption tomorrow. The preferences described in this paper allow agents to have preferences over the gambles.[1]

This paper characterizes consumption allocations when agents have time

[1] Other related motivations include: (1) recursive preferences allow agents to have preferences for the early (or the late) resolution of uncertainty (Kreps and Porteus 1978); and (2) recursive preferences allow for a partial separation between risk aversion and the elasticity of intertemporal substitution (Epstein and Zin 1989).

non-additive or recursive preferences of the form

$$U_{it} = u_i\left(c_{it}\right) + \beta h_i^{-1}\left[E_{it} h_i\left(U_{it+1}\right)\right] \tag{2}$$

where h_i is an invertible function, which maps a scalar into a scalar, and U_{it} is the utility of agents of type i at date t. Time non-additive preferences allow agents to be risk averse in future utility in addition to possibly being risk averse in consumption. Optimal allocations are difficult to compute when agents have time non-additive utility since the marginal utility of consumption at date t for agents of type i typically depends non trivially on their consumption between periods zero and t. When h_i is the identity function preferences are time additive and the law of iterated expectations can be used to write preferences in the form of equation (1).

Examples of time non-additive preferences that are discussed in this paper include the power specification studied by Epstein and Zin (1989) and the risk-sensitive specification studied by Hansen and Sargent (1995). The power specification sets $h_i(q_i) = (k_i q_i)^{\omega_i}$ for suitably chosen constant parameters k_i and ω_i. The power specification has been argued by Epstein (1992) to be a flexible device for modeling preferences. The risk-sensitive specification sets $h_i(q_i) = \mathrm{e}^{\sigma_i q_i}$ for a non positive parameter σ_i. The risk-sensitive specification has been argued by Hansen, Sargent, and Tallarini (1997) to be attractive since it can be used to mimic preferences for robustness.

To compute a Pareto optimal allocation when agents have time non-additive preferences a social planner can (as in the case of time additive preferences) maximize a weighted sum of utilities by choice of feasible consumption allocations. The solution typically involves consumption allocation rules which are time-varying and history dependent. Following Lucas and Stokey (1984) and Kan (1995), this paper describes a recursive social planning problem for computing Pareto optimal allocations. The social planning problem is a non standard dynamic programming problem in which Pareto weights evolve over time and are endogenous state variables. Current period Pareto weights can be interpreted as summarizing the effect of the history of aggregate consumption and the initial Pareto weights on current and future consumption allocations. The solution to the social planning problem entails that consumption allocations at time t can be written as

$$\begin{aligned} c_{it} &= A_i\left(x_t, \theta_t\right), \\ \theta_t &= \phi\left(x_{t-1}, \theta_{t-1}, x_t\right) \end{aligned}$$

where A_i and ϕ are functions; and θ_t is a vector of Pareto weights. Pareto weights are a convenient device for modeling the history dependence of consumption allocation rules since they index consumption allocation rules by a vector of parameters. Although Pareto optimal allocations typically entail that Pareto weights evolve over time, optimal allocations are time-consistent and can be decentralized.[2]

To study the implications of Pareto optimality, it is convenient to characterize the optimal law of motion for Pareto weights. This paper gives necessary and sufficient conditions for the existence of a steady state at which Pareto weights (and hence allocation rules) are time-invariant. Steady states are particularly interesting if they are stable. For deterministic economies, Lucas and Stokey (1984) showed that if agents have *increasing marginal impatience* then interior steady states are stable. Increasing marginal impatience implies that when all agents have identical preferences "wealthy" agents discount the future more than "poor" agents. In this case the consumption allocations of wealthy (respectively poor) agents decrease (increase) over time until all agents have identical consumption allocations.

In stochastic economies increasing marginal impatience is neither necessary nor sufficient for the stability of interior steady states. Another mechanism, which will be called *relative entropy*, in conjunction with marginal impatience determines the stability of steady states. Given hypothetical consumption allocations when agents have constant impatience, relative entropy can be measured by the deviation of preferences from time additive preferences. Agents have *increasing* relative entropy if at the Pareto optimal allocations deviations from time additive preferences are increasing in wealth. When all agents have the same constant impatience then increasing relative entropy is sufficient to guarantee the stability of interior steady states. When agents have the same constant impatience then *decreasing* relative entropy is sufficient to guarantee that interior steady states are not stable.

The remainder of this paper is organized as follows. Section 2 describes endowments, preferences and the agent's problem. Section 3 discusses optimal allocations and describes a recursive social planning problem. Section 4 characterizes the dynamics of optimal allocations by looking at steady states, off steady state dynamics, and stability. Section 5 concludes and proposes some extensions.

[2]Ma (1993) has demonstrated the existence of equilibria for a wide range of recursive preferences without making reference to Pareto optima.

2 Economic environment

This section describes endowments, preferences, and the agent's problem. Endowments are exogenous and follow a first-order Markov process. Preferences are assumed to be time non-additive. The agent's problem is to maximize his utility given a lifetime budget constraint and prices.

2.1 Endowments

For simplicity this paper studies an endowment economy in which the aggregate endowment follows a first-order Markov process. The analysis of this paper applies to more general models in which aggregate *consumption* follows a first-order Markov process by replacing assumptions on the "aggregate endowment" with corresponding theorems about "optimal aggregate consumption." For a wide range of technologies, Anderson (1998) shows that optimal aggregate consumption will satisfy these assumptions.

In this paper the aggregate endowment can not be stored between periods and the sum of the consumption of the agents at any date must not exceed the aggregate endowment. The consumption of agents is restricted to be non-negative. Let x_t be the aggregate endowment at time t and c_{it} be the consumption of agents of type i at time t. Consumption must satisfy

$$\sum_{i=1}^{n} c_{it} \leq x_t, \tag{3a}$$

$$c_{it} \geq 0, \qquad \forall i \tag{3b}$$

at all dates $t \geq 0$.

For simplicity this paper assumes the aggregate endowment only takes on values in a finite dimensional set. Though apart from technicalities, the analysis applies when the endowment space is infinite dimensional. Let $\mathcal{X}$ denote the set of possible values for the aggregate endowment at any given date. If the endowment this period is x then the probability that the endowment next period is y will be denoted by $\pi(x, y)$. Let $\pi_i(x, y)$ be the probability that agents of type i believe that y will be the aggregate endowment next period. For most of the economies in this paper I will assume agents have the same (correct) beliefs so that $\pi_i(x, y) = \pi(x, y)$.

I will always make the following assumptions on the space of aggregate endowments and probabilities.

(a) $\mathcal{X}$ has m elements, where m is finite.

(b) All elements of $\mathcal{X}$ are strictly positive and finite.

(c) For all $x, y \in \mathcal{X}$ it is the case that $\pi(x, y) > 0$. For all $x, y \in \mathcal{X}$ and for all i it is the case that $\pi_i(x, y) > 0$.

Part a states that the endowment space is finite dimensional. Part b states that the aggregate endowment is always finite and greater than zero. Part c states that whatever the endowment is this period all endowments can occur next period with positive probability. In addition all agents believe that all endowments can occur with positive probabilities. This assumption guarantees that beliefs are absolutely continuous with respect to the true probabilities and the true probabilities are absolutely continuous with respect to beliefs. These assumptions are stronger than necessary, but make the theorems about dynamics later in this paper less verbose.

To describe information, I take a sample space $(\Omega, \mathcal{F})$ and a filtration $\{\mathcal{F}_t; t \geq 0\}$ of sub σ-fields of $\mathcal{F}$ such that

$$\mathcal{F}_t = \sigma(x_0, x_1, \ldots x_t) \tag{4}$$

is the smallest σ-field for which the aggregate endowment x_s is measurable for every s such that $0 \leq s \leq t$. The information of the agents at date t is $\mathcal{F}_t$. Agents are assumed never to learn about the probabilities $\pi(x, y)$ from realizations of the aggregate endowment.

For simplicity some of the results in this paper will assume:

Assumption 1. *The endowment is i.i.d. over time. For all $x, y, z \in \mathcal{X}$ it is the case that $\pi(x, y) = \pi(z, y)$.*

Assumption 2. *$\mathcal{X}$ contains at least two elements.*

Assumption 2, in conjunction with c above, guarantees that there is uncertainty in the economy since at least two different values of the aggregate endowment can be realized at any date with positive probability.

2.2 Preferences

In most dynamic economic models agents are assumed to have preferences over current period consumption and anticipated future consumption. For convenience, simplicity and tractability preferences are often assumed to be time additive and of the form

$$U_i(x_t, \mathbf{c}_{it}) = E_{it} \sum_{s=t}^{\infty} \beta^{s-t} u_i(c_{is}) \tag{5}$$

where U_i is the utility function for agents of type i, x_t is the time t aggregate endowment, c_{it} is a $\mathcal{F}_t$ measurable function which gives the consumption of agents of type i at time t, and

$$\mathbf{c}_{it} = (c_{it}, c_{it+1}, c_{it+2}, \dots) \tag{6}$$

is a consumption plan for agents of type i for all dates $s \geq t$. Note that c_i and boldface $\mathbf{c}_i$ are different quantities.

As discussed in the introduction there are many drawbacks to time additive utility. In response to these drawbacks one can imagine allowing agents to have arbitrary preferences over current and anticipated future consumption. Instead, I will assume specific functional forms for preferences that are more general than time additive preferences, can allow agents to have preferences over the gambles described in the introduction, and include time additive preferences as a special case. It is quite likely that even these more general preferences preclude some forms of rational behavior. However the preferences described in this paper will lead to a rich set of possible risk-sharing rules. I let agents of type i have preferences $U_i(x, \mathbf{c}_i)$ such that

$$U_i\left(x_t, \mathbf{c}_{it}\right) = U_{it} \tag{7}$$

where U_{it} satisfies the recursion[3] in equation (2). When it is understood that $t = 0$, I will often drop the time subscripts[4] from U_i, x, and $\mathbf{c}_i$.

I now make several definitions and assumptions that will be used throughout this paper. Definition 1 is standard. Definitions 2 thru 6 will be used to formulate preferences.

Definition 1. Let $\Re_{++}$ be all strictly positive real numbers, $\Re_+$ be all nonnegative real numbers, $\Re$ be all real numbers, and $\underline{\Re}$ be all extended real numbers.

Definition 2 (Feasible utilities). Let $\underline{q}_i$ and $\overline{q}_i$ be extended real numbers which are lower and upper bounds for the feasible utilities for agents of type i. Let $\mathcal{Q}_i = [\underline{q}_i, \overline{q}_i]$ be a superset of the space of feasible utilities.

[3]For some specifications of u_i and h_i, which do not satisfy a contraction mapping property, equation 7 may not uniquely define preferences. For these preferences, for a fixed $T > t$ set $U_{iT} = \overline{q}_i$ where $\overline{q}_i$ is an upper bound for the utility of type i agents. Using the consumption plans in $\mathbf{c}_{it}$ consider computing U_{it} by iterating backward on equation 2 from $U_{iT} = \overline{q}_i$. Now let T go to $+\infty$ and compute U_{it} for each T. Define $U_i\left(x_t, \mathbf{c}_{it}\right)$ as the limit of U_{it} as $T \to +\infty$. Ozaki and Streufert (1996) show that such a definition uniquely defines preferences.

[4]When $t = 0$ the expressions U_{i0}, U_i, $U_i\left(x_0, \mathbf{c}_{i0}\right)$ and $U_i\left(x, \mathbf{c}_i\right)$ are interchangeable.

Definition 3 (Reward functions). A *reward function* u_i is a function which maps $\Re_+$ into $\underline{\Re}$. In this paper reward functions are always assumed to be strictly concave, strictly increasing, continuously differentiable on $\Re_{++}$, and such that

$$\begin{aligned} u_i(x_{\max}) &< +\infty, \\ u_i(c_i) &> -\infty, \text{ when } c_i > 0, \\ \lim_{c_i \to 0+} u_i'(c_i) &= +\infty, \end{aligned}$$

where $x_{\max}$ is the maximum value of $\mathcal{X}$.

Definition 4 (Utility allocation functions). A *utility allocation function* is a function which maps $\mathcal{X}$ into $\mathcal{Q}_i$. Let $\mathcal{A}_i$ be the space of all utility allocation functions. The arguments to utility allocation functions are written as subscripts.

Definition 5 (Discount factors). A *time discount factor* β is a real number that is greater than zero and less than one.

Definition 6 (Stochastic aggregators). A *stochastic aggregator* $\mathcal{R}_i$ is an operator which maps $\mathcal{X} \times \mathcal{A}_i$ into $\underline{\Re}$. The first argument to stochastic aggregators is written as a subscript. In this paper stochastic aggregators are assumed to be

(a) continuous and weakly concave in their second arguments.

(b) such that for any $x \in \mathcal{X}$ and for all $q_i, g_i \in \mathcal{A}_i$, $q_i \leq g_i$ implies that $\mathcal{R}_{ix} q_i \leq \mathcal{R}_{ix} g_i$. If q_i and g_i are finite then $q_i \leq g_i$ and $q_i \neq g_i$ implies that $\mathcal{R}_{ix} q_i < \mathcal{R}_{ix} g_i$.

(c) such that if q_i is an allocation function which is constant (that is $q_{iy} = a$ for all y and some constant a) then $\mathcal{R}_{ix} q_i = a$.

The extended real numbers $\underline{q}_i$ and $\overline{q}_i$ referenced in definition 2, which are lower and upper bounds for feasible utility allocations, are defined as

$$\underline{q}_i = (1-\beta)^{-1} u_i(0), \qquad \overline{q}_i = (1-\beta)^{-1} u_i(x_{\max}) \tag{8}$$

where $x_{\max}$ is the maximum possible realization of the endowment (i.e., the maximum element of $\mathcal{X}$). The lower bound is the utility obtained by agents of type i from a consumption plan which entails consuming zero at all current and future dates. The upper bound is the utility obtained by agents of type i from an infeasible consumption plan which entails consuming $x_{\max}$ at all current and future dates.

A reward function is a scalar function which gives the contribution of current period consumption to current period utility. The definition of reward functions given above rules out linear, quadratic and exponential reward functions. The definition includes logarithmic reward function of the form $\log c_i$ and power reward functions of the form $\frac{c_i^{1-\gamma_i}}{1-\gamma_i}$ which will be used frequently in later sections.

For us a utility allocation function gives hypothetical utilities for agents of type i from next period on contingent on next period's aggregate endowment. A stochastic aggregator is a function which maps an aggregate endowment and a utility allocation function into the extended real numbers. For us stochastic aggregators give the contribution of future anticipated utility to current period utility and are taken to be

$$\mathcal{R}_{ix}q_i = h_i^{-1}\left(\int_{y\in\mathcal{X}}\left[\pi_i\left(x,y\right)h_i\left(q_{iy}\right)\right]\right) \tag{9}$$

where $h_i : \Re \to \Re$ is an invertible function with the inverse h_i^{-1}. I assume that on the interior of $\mathcal{Q}_i$ both h_i' and $h_i^{-1\prime}$ are finite and continuously differentiable.[5] The function h_i must be chosen so that all of the requirements for a stochastic aggregator listed in definition 6 are met. Although the analysis in this paper extends to more general stochastic aggregators, I will formally limit the analysis to stochastic aggregators of the form given in equation (9).

In this paper preferences will sometimes be written as

$$W_i(x, c_i, q_i) = u_i(c_i) + \beta\mathcal{R}_{ix}q_i \tag{10}$$

instead of as in equation (7). Here u_i is a reward function, x is the scalar current period aggregate endowment, c_i is the scalar consumption of agents of type i in the current period and q_i is a utility allocation function. For U_{it} defined in equation 2 it is the case that $U_{it} = W_i\left(x_t, c_{it}, U_{it+1}\right)$. The specification of preferences allows agents of different types to have different reward functions and different stochastic aggregators, but assumes all agents have the same discount factors.

[5] For any function f of one variable I use the convention that

$$f'(z) = \frac{df(z)}{dz}.$$

This paper will sometimes make the following additional assumptions on preferences.

ASSUMPTION 3. (DISCOUNTING). *For all i the stochastic aggregators satisfy*

$$|\mathcal{R}_{ix}(q_i + a) - \mathcal{R}_{ix} q_i| \leq a$$

for all $q_i \in \mathcal{A}_i$ and all real numbers $a \geq 0$.

ASSUMPTION 4. (DIFFERENTIABILITY OF REWARD FUNCTIONS). *The reward functions u_i are three times differentiable.*

Some of my examples will satisfy assumption 3. Assumption 3 is useful in showing that various operators introduced in the appendix are contraction mappings. Many of the results in this paper will follow without assumption 3. Assumption 4 is satisfied by all of my examples but is not required by most of the general theorems.

2.2.1 Examples

This section discusses the risk-sensitive and Epstein-Zin specification of preferences. The risk-sensitive specification of preferences sets $h_i q_i = \mathrm{e}^{\sigma_i q_i}$. The constant σ_i is referred to as the risk-sensitivity parameter. When $\sigma_i = 0$ preferences are time additive. As σ_i decreases agents become increasingly risk averse in future utility. In this paper when agents have risk-sensitive preferences, I always assume that agents assign the same probabilities to exogenous states. Typically I will also assume that agents have the correct beliefs so that all agents believe that the probability that next period's endowment is y given that this period's endowment is x is $\pi(x, y)$. I always assume that the risk-sensitivity parameter satisfies $\sigma_i \leq 0$. With this assumption it is straightforward to show that risk-sensitive aggregators satisfy the requirements of definition 6. Risk-sensitive aggregators also satisfy assumption 3 since

$$\frac{1}{\sigma_i} \log \int_{y \in \mathcal{X}} \left[\pi(x, y) \, \mathrm{e}^{\sigma_i (q_{iy} + a)}\right] - \frac{1}{\sigma_i} \log \int_{y \in \mathcal{X}} \left[\pi(x, y) \, \mathrm{e}^{\sigma_i q_{iy}}\right] = a.$$

for any $x \in \mathcal{X}$, any $q_i \in \mathcal{A}_i$ and any constant $a \geq 0$.

The dynamics of risk sharing rules depend crucially on both the specification of stochastic aggregators and the specification of reward functions. This paper will emphasize an example in which agents have the same (correct) beliefs, risk-sensitive stochastic aggregators and power reward functions. In this case the preferences of the agents can be written as

$$U_{it} = \frac{{c_{it}}^{1-\gamma_i}}{1-\gamma_i} + \frac{\beta}{\sigma_i} \log E_t \left(\mathrm{e}^{\sigma_i U_{it+1}} \right). \tag{11}$$

By varying σ_i and γ_i we will obtain several different long run implications for risk sharing.

The Epstein-Zin specification of preferences are given by the power specification of h_i in which h_i is given by one of the following two formulas:

$$h_i(q_i) = q_i^{\omega_i}, \tag{12a}$$

$$h_i(q_i) = (-q_i)^{\omega_i}. \tag{12b}$$

Formula 12a is used when reward functions are bounded from below by zero and formula 12b is used when reward functions are bounded from above by zero. When reward functions are bounded from below by zero, the stochastic aggregator implied by equation 12a is concave in future utility when $0 < \omega_i < 1$. When reward functions are bounded above by zero, the stochastic aggregator implied by equation 12b is concave if $\omega_i > 1$.

2.3 The agent's problem

In this section I formulate a problem which it is convenient to view agents as facing. The key concepts of impatience and distorted probabilities, which play a major role in later sections, are introduced.

Let

$$\xi_t = \beta^t \Pi_{s=1}^t \pi(x_{s-1}, x_s).$$

ξ_t is β^t multiplied by the time zero probability that the aggregate endowment sequence $(x_0, x_1, \ldots x_t)$ is realized. (I assume x_0 is known.) ξ_t will be useful below when writing the time zero marginal value of time t consumption.

Let $w_i > 0$ (a real number) be the initial wealth for agents of type i. Let $\{p_t\}_{t=0}^{\infty}$ be a sequence of random variables such that $p_t > 0$ is $\mathcal{F}_t$ measurable and $\xi_t p_t$ is the price of consumption at time t. An agent wants to maximize his utility U_{i0}, which was given in equation (7), by choice of consumption allocations adapted to $\mathcal{F}_t$ such that his lifetime budget constraint

$$E_0 \sum_{t=0}^{\infty} \beta^t p_t c_t = w_i \tag{13}$$

is satisfied.

Define

$$\mathcal{M}_{it} \equiv \left[\frac{\pi_i(x_{t-1}, x_t)}{\pi(x_{t-1}, x_t)}\right] \frac{h'(U_{it})}{h'\left[h_i^{-1} E_{it-1} h_i(U_{it})\right]}, \tag{14}$$

$$\mathcal{M}_i^t \equiv \Pi_{s=1}^t \mathcal{M}_{is}. \tag{15}$$

The marginal utility of time t utility at time zero is given by $\xi_t \mathcal{M}_i^t$. I will refer to $\mathcal{M}_{it}$ and $\mathcal{M}_i^t$ as the *scaled marginal value* of time t utility at time $t-1$ and time zero respectively. First order conditions for the agent's problem include

$$u_i'(c_{i0}) = \mu_i p_t \qquad t = 0, \tag{16}$$

$$u_i'(c_{it}) \mathcal{M}_i^t = \mu_i p_t \qquad t > 0 \tag{17}$$

where μ_i is a Lagrange multiplier on constraint (13). The marginal utility of time t consumption at time zero is $\xi_t u_i'(c_{it}) \mathcal{M}_i^t$. This is just the marginal value of time t utility at time zero multiplied by the marginal value of time t consumption at time t.

If we define $\mu_{it} = \mu_i / \mathcal{M}_i^t$ then the first order conditions for the agent's problem at all dates can be written as

$$u_i'(c_{it}) = \mu_{it} p_t. \tag{18}$$

μ_{it} can be thought of as a Lagrange multiplier for a problem in which at date t an agent maximizes U_{it} by choice of consumption allocations from date t on adapted to $\mathcal{F}_t$ such that his lifetime budget constraint from date t on is satisfied.

Summarizing, it is optimal for the current period marginal utility of current period consumption to be time-varying. By defining the time-varying Lagrange multiplier μ_{it}, the first order conditions can be written in the form of equation (18) at all dates. In later sections I will show that defining time t Pareto weights in an analogous way will be convenient for analyzing heterogeneous agent economies.

2.3.1 An equivalent problem

In this section I show that the agent's problem is equivalent to a problem in which agents have time additive preferences with time-varying discount factors and distorted beliefs. Call the economy described in the previous section the "original" economy and the equivalent economy the "auxiliary" economy.

Define

$$\hat{\beta}_{it} \equiv \beta E_t \mathcal{M}_{it+1}, \tag{19}$$

$$\hat{\pi}_{it}(y) \equiv \pi_i(x_t, y)\left[\frac{h'(q_{iy})}{E_{it}h'(U_{it+1})}\right] \qquad \text{where} \qquad q_{iy} = U_{it+1}(x_0, x_1, \dots x_t, y). \tag{20}$$

Notice in equation 20 I explicitly write the arguments to the $t+1$ random variable U_{it+1} when defining q_{iy}. $U_{it+1}(x_0, x_1, \dots x_t, y)$ is the hypothetical value of time $t+1$ utility if the endowments between periods zero and t are $(x_0, x_1, \dots x_t)$ and time $t+1$ endowment is y. $\hat{\beta}_{it}$ is interpreted as marginal impatience at time t. When agents have time additive preferences marginal impatience is constant and equal to the time discount factor β. For more general economies marginal impatience is often history dependent and has the interpretation of a time t discount factor. $\hat{\pi}_{it}(y)$ can be interpreted as a set of distorted probabilities for next period's endowment. These probabilities will always satisfy $\hat{\pi}_{it}(y) \geq 0$ and $\int_{y\in\mathcal{X}} \hat{\pi}_{it}(y) = 1$.

Consider an auxiliary economy in which:

1. Agents treat $\{\hat{\beta}_{it}\}_{t=0}^{\infty}$ and $\{\hat{\pi}_{it}(y)\}_{t=0}^{\infty}$ for all $y \in \mathcal{X}$ as exogenous random variables. In forming $\hat{\beta}_{it}$ and $\hat{\pi}_{it}(y)$, U_{it} is assumed to follow the law of motion from the solution to the original economy of interest.

2. At time t, agents form probabilities about time $t+1$ states of the economy using $\hat{\pi}_{it}(y)$ and discount next period's utility using $\hat{\beta}_{it}$. At time t they believe that the probability of the aggregate endowment being y at time $t+1$ is y is $\hat{\pi}_{it}(y)$.

3. Agents have time additive preferences of the form

$$E_0^{\hat{\pi}_i} \sum_{t=0}^{\infty} \left[\left(\Pi_{s=0}^{t-1} \hat{\beta}_{is}\right) u_i(c_{it})\right]. \tag{21}$$

   Here $E_0^{\hat{\pi}_i}$ denotes expectation with respect to the probabilities given by $\hat{\pi}_i$. Agents maximize equation 21 subject to their lifetime budget constraint which was given in equation 13.

The solution to the auxiliary economy will coincide with the original economy. This can be shown by comparing the first-order conditions for the two problems.

In order to analyze the dynamics of heterogeneous agent economies it is convenient to introduce the notion of relative entropy.

Definition 7. *Relative entropy* is

$$R_{it} = E_t \log \left(\frac{\pi(x_t, x_{t+1})}{\hat{\pi}_{it}(x_{t+1})} \right). \tag{22}$$

Relative entropy can be interpreted as measuring the distance between the true probabilities and the distorted probabilities that agents use in the auxiliary economy. In computing the expected value in equation 22 the true probabilities are used. Anderson, Hansen, and Sargent (1998) define a similar notion of relative entropy for risk-sensitive economies. Their definition uses distorted probabilities to compute a similar quantity. Blume and Easley (1992) use relative entropy to measure the distances between beliefs.

When agents have risk-sensitive preferences with the same (correct) beliefs

$$\hat{\pi}_{it}(y) = \pi(x_t, y) \left[\frac{\mathrm{e}^{\sigma_i q_{iy}}}{E_t \mathrm{e}^{\sigma_i U_{it+1}}} \right] \qquad \text{where} \qquad q_{iy} = U_{it+1}(x_0, x_1, \ldots x_t, y),$$

$$R_{it} = -\sigma_i \left[E_t U_{it+1} - \frac{1}{\sigma_i} \log E_t \mathrm{e}^{\sigma_i U_{it+1}} \right],$$

and $\hat{\beta}_{it} = \beta$. For risk-sensitive economies marginal impatience is constant over time and always equals β, the time discount factor. The agent's problem is equivalent to a problem in which they have distorted beliefs with the same reward functions and the same time discount factors as in the original risk-sensitive economy. For risk-sensitive agents, relative entropy can be thought of as measuring the deviation of time t preferences (evaluated at the optimal choices) in the original risk-sensitive economy from time additive preferences.

2.3.2 A one-period risk-sensitive problem

For intuition consider a one period risk-sensitive economy in which agents have the correct beliefs and the initial wealth for agents of type i is the real number $w_i > 0$. Let p be a $\mathcal{X}$ measurable function. Consider the problem in which agents of type i maximize

$$\frac{1}{\sigma_i} \log \int_{y \in \mathcal{X}} \pi_y \mathrm{e}^{\sigma_i u_i(c_{iy})}$$

by choice of a $\mathcal{X}$ measurable function c_i subject to

$$\int_{y \in \mathcal{X}} \pi_y c_{iy} p_y = w_i.$$

Here π_y denotes the probability that y is the aggregate endowment. Here I explicitly write the arguments to c_i and p as subscripts. In this problem $\pi_y p$ is interpreted as the price of consumption contingent on the realization of the aggregate endowment. This is a one period problem in which agents start with an initial wealth, face prices they can not influence, and buy state contingent consumption.

For this problem relative entropy can be defined as

$$R_i = -\sigma_i \left[\int_{y \in \mathcal{X}} \pi_y u_i(c_{iy}) - \frac{1}{\sigma_i} \log \int_{y \in \mathcal{X}} \pi_y \mathrm{e}^{\sigma_i u_i(c_{iy})} \right]$$

where c_{iy} is the optimal choice of consumption. This is the natural one-period analog to the relative entropy measure given in the previous section. The following theorem shows that the quantity

$$Q_i(d) = \frac{u_i'(d) u_i'''(d)}{[u_i''(d)]^2} \tag{23}$$

can be used to determine if relative entropy is increasing or decreasing in wealth.

THEOREM 1. *Consider the one-period problem described in this section. Let assumptions 2 and 4 hold. Also assume that agents of type i are risk-sensitive with $\sigma_i < 0$ and that prices are such that for any $y, z \in \mathcal{X}$ such that $y \neq z$ it is the case that $p_y \neq p_z$. If $Q_i(d) > 2$ for all $d > 0$ then relative entropy (in the one period problem) is increasing in wealth for agents of type i. If $Q_i(d) < 2$ for all $d > 0$ then relative entropy is decreasing in wealth. If $Q_i(d) = 2$ for all $d > 0$ then relative entropy is constant in wealth.*

Theorem 1 is proved in the appendix.

When agents have power reward functions of the form $\frac{c_i^{1-\gamma_i}}{1-\gamma_i}$ then

$$Q_i(d) = 1 + \frac{1}{\gamma_i}. \tag{24}$$

Relative entropy is increasing in wealth if $0 < \gamma_i < 1$ and decreasing in wealth if $\gamma_i > 1$. When agents have log rewards of the form $\log c_i$, $Q_i(d) = 2$ and relative entropy does not vary with wealth.

Carroll and Kimball (1996) study the implications of the sign of $Q_i(d)$ in a different context. They point out that for agents with time additive preferences $Q_i(d)$ is equal to the magnitude of prudence divided by the magnitude of risk aversion. They show that properties of $Q_i(d)$ are inherited by value functions in an infinite horizon economy. Later in this paper I show that under additional assumptions the conclusions of theorem 1 apply to relative entropy in infinite horizon problems.

The notion of relative entropy lies behind many of the results that are coming in this paper. It will be the case that at an interior steady state all agents have the same relative entropy. When relative entropy is increasing (or decreasing) in wealth it is possible to characterize the nature of allocations away from the steady state and the asymptotic behavior of allocations.

3 Optimal allocations

This section shows how to compute Pareto optimal allocations when agents have time non-additive preferences. After some preliminary definitions, section 3.1 describes a sequence formulation for optimal allocations, and section 3.2 describes a recursive formulation for optimal allocations. Pareto optima are interesting for their own sake and they also, as section 3.3 discusses, can be used to compute equilibria.

I now define a Pareto optimal allocation. Let $\mathbf{c} = \{\mathbf{c}_i\}_{i=1}^n$ and $\mathbf{d} = \{\mathbf{d}_i\}_{i=1}^n$ be consumption plans for all the agents.

DEFINITION 8. Given the initial endowment x, a feasible allocation $\mathbf{c}$ is a *Pareto optimal allocation* if there is no other feasible allocation $\mathbf{d}$ such that $U_i(x, \mathbf{d}_i) \geq U_i(x, \mathbf{c}_i)$ for all i and $U_i(x, \mathbf{d}_i) > U_i(x, \mathbf{c}_i)$ for some i.

Let there be n types of agents. Define

$$\Delta^n = \left\{ \theta = \{\theta_1, \theta_2, \ldots \theta_n\} \in \Re^n \text{ such that } \theta_i \geq 0 \text{ and } \sum_{i=1}^n \theta_i = 1 \right\}.$$

Elements of Δ^n will be referred to as Pareto weights or as Pareto weight vectors. Let $\theta \in \Delta^n$ and denote a weighted average of the values assigned to the agents for hypothetical consumption allocations as

$$U\left(x, \mathbf{c}\right) = \sum_{i=1}^n \theta_i U_i(x, \mathbf{c}_i).$$

When x is the realization of the time zero endowment, let $\mathcal{C}(x)$ denote the set of all feasible consumption allocations for all types of agents, $\mathbf{c}$, in all dates and states. Feasible consumption allocations must satisfy equations (3a) and (3b) for all t.

Pareto optimal consumption allocations can be computed from the following social planning problem.

PROBLEM 1 (THE PARETO OPTIMAL PROBLEM). Given the initial endowment x and Pareto weights $\theta \in \Delta^n$, maximize $U(x, \mathbf{c})$ by choice of a feasible consumption plan for all agents $\mathbf{c} \in \mathcal{C}(x)$.

The value of the social planner's problem is

$$Q(x, \theta) = \sup_{\mathbf{c} \in \mathcal{C}(x)} U(x, \mathbf{c}). \tag{25}$$

Under assumptions on preferences given in definitions 3 thru 6 the supremum is obtained and the sup in equation 25 can be replaced with a max.

3.1 Sequence formulation

This section informally and heuristically discusses how to compute optimal allocations using a sequence formulation of the social planning problem. To compute a Pareto optimal allocation, one could find the feasible consumption allocations $\mathbf{c}$ which maximize $U(x, \mathbf{c})$. First-order conditions for maximization dictate equating the ratios of the marginal values of consumption for any two types of agents over time. When agents have time non-additive preferences the marginal value of time t consumption at time zero is $u_i'(c_{it})\mathcal{M}_i^t$ times a constant where the constant is the same for all agents and $\mathcal{M}_i^t$ was defined in equation 15. First-order conditions for maximization include

$$\theta_i u_i'(c_{i0}) = \theta_j u_j'(c_{j0}), \qquad \forall i, j, \tag{26a}$$

$$\theta_i u_i'(c_{it})\mathcal{M}_i^t = \theta_j u_j'(c_{jt})\mathcal{M}_j^t \qquad \forall i, j, \qquad t > 1. \tag{26b}$$

At a Pareto optimum the ratio of the marginal values of consumption at any date for any two types of agents, which in this problem is given by

$$\frac{u_i'(c_{it})\mathcal{M}_i^t}{u_j'(c_{jt})\mathcal{M}_j^t},$$

is a time-invariant constant.

When $\left(\mathcal{M}_i^t/\mathcal{M}_j^t\right)$ varies over time Pareto optimal consumption allocation rules vary over time. It is convenient to model time-varying consumption allocation rules with time-varying Pareto weights. Define

$$\theta_{it} \equiv \frac{\theta_i \mathcal{M}_i^t}{\sum_{j=1}^n \theta_j \mathcal{M}_j^t}, \qquad \forall i. \tag{27}$$

With this definition the first-order conditions for time t consumption can be written as

$$\theta_{it} u_i'(c_{it}) = \theta_{jt} u_j'(c_{jt}), \qquad \forall i, j. \tag{28}$$

(Here $\theta_{i0} = \theta_i$ for all i.) Section 3.2 shows that defining $\theta_t = \{\theta_{it}\}_{i=1}^n$ as in equation (27) will allow a recursive dynamic programming formulation of Pareto optimal problems. It will be the case that at date $t > 0$ the social planner's problem can be written as: maximize

$$\sum_{i=1}^{n} \theta_{it} U_{it}$$

by choice of feasible consumption allocations $\{\{c_{is}\}_{i=1}^n\}_{s=t}^\infty$ adapted to information at time s. In light of this, we interpret θ_t as time t Pareto weights. The time t Pareto weights are the Pareto weights that a social planner could use at date t to compute optimal allocations for dates $s \geq t$. The time t Pareto weights depend upon θ and all of the exogenous shocks between dates zero and t.

It is important to remember that in a sense there is only one vector of Pareto weights, θ, that applies at all dates. In future dates its effect on consumption allocations depends upon terms like $\mathcal{M}_i^t$. However, by interpreting the Pareto weights as moving over time, we can index time-varying consumption allocation rules by a vector of parameters and obtain a recursive formulation of the social planner's problem.

In practice equation 26b is not convenient for computing Pareto optimal consumption allocations since $\mathcal{M}_i^t$ depends upon the distribution of future utilities. The distribution of future utilities depends upon consumption allocations in future dates. Section 3.2 describes a dynamic programming problem which makes the computation of Pareto optima tractable.

3.2 Dynamic programming formulation

This section shows that Pareto optimal allocations can be computed by solving a non standard dynamic programming problem. The dynamic programming problem has been justified when the reward functions are bounded below in a deterministic setting by Lucas and Stokey (1984) and in a stochastic setting by Kan (1995). The appendix of this paper shows how to extend their analysis when agents have reward functions which are possibly unbounded from below.

Following Lucas and Stokey (1984) and Kan (1995) the social planner's value function Q defined in equation (25) is a fixed point of the functional equation

$$Q(x, \theta) = \max_{\{c_i \in \Re_+, q_i \in \mathcal{A}_i\}_{i=1}^n} \sum_{i=1}^{n} \theta_i W_i(x, c_i, q_i) \tag{29a}$$

where

$$\sum_{i=1}^{n} c_i = x \tag{29b}$$

and for each $y \in \mathcal{X}$

$$0 \leq \min_{\phi \in \Delta^n} \left[Q(y, \phi) - \sum_{i=1}^{n} \phi_i q_{iy} \right]. \tag{29c}$$

Note that in equation (29c) q_{iy} is the continuation value for type i agents when next period's endowment is y. The appendix shows that the function Q defined in equation (25) is the unique bounded, continuous in θ solution to the functional equation. In order to compute a solution to the social planner's problem one could start from an initial bounded and continuous in θ guess of the social planner's value function and iterate until a fixed point Q is found.

This is a non standard dynamic programming in which the social planner chooses current period consumption allocations, continuation values for next period and Pareto weights for next period. The choice of consumption allocations is subject to the feasibility constraint (29b). The choice of continuation values is also subject to the m feasibility constraints listed in equation (29c). The appendix shows that the equations in (29c) are equivalent to the requirement that the utility allocation functions, q_i for all i, are feasible. The Pareto weights that obtain the minimum in constraint (29c) for $y \in \mathcal{X}$ will be the optimal choice of next period's Pareto weights when next period's endowment is y.

This recursive formulation of the social planner's problem is not the only possible recursive formulation. An alternative formulation can be stated which uses continuation values rather than Pareto weights as state variables. See sections 5.11- 5.13 of Stokey and Lucas (1989) for a description of this approach in a deterministic setting.

I now define an interior Pareto weight vector and an analog of the scaled marginal value of time t utility at time $t-1$.

Definition 9. An *interior Pareto weights vector* is a vector of constant Pareto weights θ such that $\theta \in \operatorname{int}(\Delta^n)$.

Define $\mathcal{M}_i : \mathcal{X} \times \mathcal{X} \times \mathcal{A}_i{}^n \to \underline{\Re}$ as

$$\mathcal{M}_{ix}(y, q_i) = \left[\frac{\pi_i(x, y)}{\pi(x, y)}\right] h_i^{-1\prime}\left(\int_{z \in \mathcal{X}} [\pi_i(x, z) h_i(q_{iz})]\right) h_i'(q_{iy}), \tag{30a}$$

$$= \left(\frac{\pi_i(x, y)}{\pi(x, y)}\right) \frac{h_i'(q_{iy})}{h_i'(\mathcal{R}_{ix} q_i)}. \tag{30b}$$

For stochastic aggregators defined in equation (9), $\beta\pi(x, y)\mathcal{M}_{ix}(y, q_i)$ is the marginal value of utility next period if this period's endowment is x, next period's utility is captured by the utility allocation function q_i. I will refer to $\mathcal{M}_{ix}(y, q_i)$ as the *scaled marginal value of future utility.*

When θ is an interior Pareto weight vector, it is straightforward to show that $Q(x, \theta)$ is differentiable with respect to θ and

$$V_i(x, \theta) = \frac{\partial Q(x, \theta)}{\partial \theta_i}. \tag{31}$$

$V_i(x, \theta)$ will be the value remaining for agents of type i from this period on if the Pareto weights are θ and the current period endowment is x. When computing the derivative on the right hand side of equation 31, the requirement that the Pareto weights, $\{\theta_i\}_{i=1}^n$, sum to one is *not* imposed.

When θ is interior first-order conditions for the social planning problem given in equations 29a thru 29c include

$$\theta_i u_i'(c_i) = \theta_j u_j'(c_j), \qquad \forall i, j, \tag{32a}$$

for current period consumption, and for each $y \in \mathcal{X}$

$$\phi_{jy}\theta_i\mathcal{M}_{ix}(y, q_i) = \phi_{iy}\theta_j\mathcal{M}_{jx}(y, q_j), \qquad \forall i, j, \tag{32b}$$

$$q_{iy} = V_i(y, \phi_y), \qquad \forall i. \tag{32c}$$

For each $y \in \mathcal{X}$, $\phi_y = \{\phi_{iy}\}_{i=1}^n$ is normalized so that it is a Pareto weight vector. ϕ_y is the optimal choice of next period's Pareto weight vector when next period's endowment is y.

3.3 Equilibrium

An equilibrium is prices $\{p_t\}_{t=0}^{\infty}$, initial wealths for all agents w_i, and consumption allocations $\mathbf{c}_i$ for all agents such that

1. For each i, given the prices and the agent's initial wealth, the consumption allocations are the solution to the agent's problem described in section 2.3.

2. The aggregate resource constraints

$$\sum_{i=1}^{n} c_{it} \leq x_t, \tag{33a}$$

$$c_{it} \geq 0, \qquad \forall i \tag{33b}$$

are satisfied at all dates $t \geq 0$.

Given a Pareto optimal allocation it is always possible to decentralize the allocation by finding prices and wealths such that if all agents solve the problem described in section 2.3 then their resulting consumption choices are identical to the Pareto optimal allocation. For some fixed i, such choices of prices can be given as

$$p_t = u_i'(c_{it})\mathcal{M}_i^t \tag{34}$$

where c_{it} is the Pareto optimal consumption allocation for agents of type i. Wealth for any type of agents can be defined as the value of their Pareto optimal consumption allocations using the prices p_t.

4 Dynamics

The section studies the dynamics of solutions to the social planner's problem described in section 3. I will be particularly interested in how the Pareto weights (and hence consumption allocation rules) evolve over time. Let $\phi(x, \theta, y)$ be the optimal choice of next period's Pareto weight vector computed from the social planner's problem, if this period's endowment and Pareto weight vector are (x, θ) and next period's endowment is y. Given a realization for the endowments $\{x_t\}_{t=0}^{\infty}$ for all time periods and an initial Pareto weight vector, θ_0, the solution to the social planner's problem implies that the Pareto weight vector evolves as

$$\theta_{t+1} = \phi\left(x_t, \theta_t, x_{t+1}\right).$$

To study dynamics this section studies steady state Pareto weights, the off steady state dynamics of Pareto weights and the stability of steady state Pareto weights.

4.1 Steady states

A steady state is a Pareto weight vector which remains constant over time. This section provides conditions for the existence of steady states. All steady states can be characterized when agents have risk-sensitive preferences. For economies without steady states invariant distributions are an appropriate notion of invariance to examine.

I now formally define steady state Pareto weights.

DEFINITION 10. A *steady state Pareto weight vector* is a vector of constant Pareto weights $\bar{\theta} \in \Delta^n$ such that if $\theta_t = \bar{\theta}$ then it is Pareto optimal to set $\theta_s = \bar{\theta}$ for all $s > t$.

A Pareto weight vector such that all but one type of agents have weights of zero is always a steady state. This section confines its analysis to *interior steady state Pareto weight vectors* in which all types of agents have positive Pareto weights.

In order to provide a simple characterization of steady states, I define $s_{i\theta}(y)$ to be the value remaining for agents of type i if the current period endowment is y and the Pareto weight vector remains fixed at θ for all current and future dates. $s_{i\theta}(y)$ is constructed by letting all future consumption be allocated according to the one-period Pareto optimal consumption allocation rules when the Pareto weight vector is θ. (If θ is an interior Pareto weight vector then the consumption allocations at all futures dates solve equation (32a).)

The next theorem provides a necessary and sufficient condition for a Pareto weight vector to be an interior steady state.

THEOREM 2. *Let assumption 3 hold. Let $\bar{\theta}$ be an interior Pareto weight vector. $\bar{\theta}$ is a steady state Pareto weight vector if and only if*

$$\mathcal{M}_{ix}\left(y, s_{i\bar{\theta}}\right) = \mathcal{M}_{jx}\left(y, s_{j\bar{\theta}}\right) \tag{35}$$

for all i and j and for all $x, y \in \mathcal{X}$.

Theorem 2 is proved in the appendix. Theorem 2 says that in order to check if a Pareto weight vector is a steady state it suffices to look at the scaled marginal values of utility when the Pareto weights remain constant over time. It is straightforward to verify that at a steady state all agents have the same impatience and the same beliefs in the corresponding auxiliary economy described in section 2.2.1. As a consequence, at a steady state agents have the same relative entropy.

When agents have time additive preferences agents scaled marginal values of future utilities are given by

$$\mathcal{M}_{ix}\left(y, q_i\right) = \frac{\pi_i\left(x, y\right)}{\pi\left(x, y\right)}.$$

If agents have the same beliefs ($\pi_i(x,y) = \pi_j(x,y)$ for all i and j) then $\mathcal{M}_{ix}(y,q_i)$ is constant across agents, regardless of q_i. By theorem 2 all Pareto weight vectors are steady states. If agents have heterogeneous beliefs ($\pi_i(x,y) \neq \pi_j(x,y)$ for some i,j,x,y) then by theorem 2 there will be no interior steady states. Summarizing, when agents have time additive preferences all interior Pareto weight vectors are steady states if agents have the same beliefs and no interior Pareto weight vectors are steady states if agents have heterogeneous beliefs.

I now provide a partial characterization of steady states in two examples. The first example shows that there always is at least one steady state when agents have identical preferences and the second example shows that for some economies there are no interior steady states. Both examples are discussed under assumption 3.

EXAMPLE 1. (IDENTICAL PREFERENCES). Let agents have the same beliefs. If agents have identical preferences ($u_i = u_j$ and $h_i = h_j$ for all i and j) then the Pareto weight vector $\bar{\theta}$ which satisfies $\bar{\theta}_i = 1/n$ for all i is an interior steady state. To see this, note that if $\bar{\theta}_i = 1/n$ for all i then $\mathcal{M}_{ix}\left(y, s_{i\bar{\theta}}\right)$ is constant across agents. By theorem 2, $\bar{\theta}$ is a steady state.

EXAMPLE 2. (HETEROGENEOUS PREFERENCES). Let there be two types of agents denoted by $i = 1,2$. Let agents have the same beliefs and let assumptions 1 and 2 hold. Let agents of type one have time additive preferences and agents of type two have recursive preferences, with either $h_2''(z) < 0$ or $h_2''(z) > 0$ for z in the interior of $\mathcal{Q}_2$. The scaled marginal value of future utility for agents of type one is constant over time. The scaled marginal value of future utility for agents of type two is given by equation (30b) and is

$$\mathcal{M}_{2x}(y,q_2) = \frac{h_2'(q_{2y})}{h_2'(\mathcal{R}_{2x}q_2)}.$$

At a steady state, it can not be optimal to give agents of type two a constant utility allocation for all values of next period's endowment. Hence q_{2y} must vary with y. This implies that $h'(q_{2y})$ and $\mathcal{M}_{2x}(y,q_2)$ must vary with y. By theorem 2 there are no interior steady states.

Example 1 shows that if all agents have identical preferences then there always exists a steady state in which agents have identical Pareto weights and receive the same level of utility. Example 2 shows that if there are two types of agents and exactly one type has time additive preferences then there are no interior steady states.

The next two sections look at specific aggregators and provide several examples in which agents have heterogeneous preferences. Steady states are especially interesting if they are stable and section 4.3 will study stability. Even when steady states are not stable they are interesting because, as section 4.2 will show, they provide a bound on the evolution of Pareto weight vectors.

4.1.1 Steady states in risk-sensitive economies

This section gives conditions for the existence and stability of steady state Pareto weight vectors for risk-sensitive economies. Let $c_i(y, \phi)$ be the current period Pareto optimal consumption allocations for agents of type i when the current period endowment is y and the current period Pareto weight vector[6] is ϕ. Define

$$D_{ij}(y, \phi) \equiv \sigma_i u_i [c_i(y, \phi)] - \sigma_j u_j [c_j(y, \phi)] \tag{36}$$

to be a weighted difference of utility obtained by agents of type i and j from current period consumption allocations. The weights are the risk-sensitivity parameters for the agents. For the risk-sensitive economies discussed in this paper properties of $D_{ij}(y, \phi)$ will drive the short and long run behavior of consumption allocation rules. As we will see in later sections there is a close connection between properties of $D_{ij}(y, \phi)$ and relative entropy.

Theorem 3 characterizes all interior steady state Pareto weight vectors for risk-sensitive economies. This theorem specializes and strengthens theorem 2 when agents have risk-sensitive preferences.

THEOREM 3. *Let all agents be risk-sensitive with the same beliefs. A vector of Pareto weights $\bar{\theta}$ is an interior steady state Pareto weight vector if and only if $\bar{\theta}$ is an interior Pareto weight vector and there exist real numbers $\{\{d_{ij}\}_{i=1}^n\}_{j=1}^n$ such that for all $y \in \mathcal{X}$*

$$D_{ij}\left(y, \bar{\theta}\right) = d_{ij}. \tag{37}$$

The proof of theorem 3 is in the appendix. Theorem 3 characterizes steady states in terms of one-period utility allocations and it requires that the real numbers $\{\{d_{ij}\}_{i=1}^n\}_{j=1}^n$ are independent of y. If they are then all agents have the same marginal value of future utility when the Pareto weight vector is $\bar{\theta}$.

[6] If ϕ is an interior Pareto weight vector then $c_i(y, \phi)$ for all i are the solution to the first-order conditions given in equation (32a).

Theorem 3 allows us to analyze the existence and uniqueness of steady state Pareto weight vectors for several example economies. All of the examples are discussed under assumption 2 and the assumption that all agents have the same beliefs. The examples do not require that the endowment is i.i.d..

EXAMPLE 3. (LOG RISK-SENSITIVITY). Consider an economy with agents who have logarithmic reward functions, of the form $u_i(c) = \log(c)$, and risk-sensitive stochastic aggregators. Preferences of this form have been previously studied by Tallarini (1996) in representative agent models. Let there be two types of agents denoted by $i = 1, 2$. The optimal allocation rules for current period consumption are

$$c_i(y, \phi) = \phi_i y$$

when the current period endowment is y and the current period Pareto weight vector is $\phi = [\phi_1 \ \phi_2]$ where $\phi_1 + \phi_2 = 1$. In this example

$$D_{21}(y, \phi) = \sigma_2 \log(\phi_2) - \sigma_1 \log(\phi_1) + (\sigma_2 - \sigma_1) \log(y).$$

Here $D_{21}(y, \phi)$ varies with y if and only if $(\sigma_2 - \sigma_1) \log(y)$ varies with y. Hence by theorem 3 when $\sigma_2 = \sigma_1$ every Pareto weight vector will be a steady state. When $\sigma_2 \neq \sigma_1$ there will be no interior steady states.

EXAMPLE 4. (RISK-SENSITIVE AND NON RISK-SENSITIVE AGENTS). Consider an economy with two types of agents. Let type one agents have time additive preferences and type two agents be risk-sensitive with $\sigma_2 < 0$. Type one agents can be modeled as having degenerate risk-sensitive preferences with $\sigma_1 = 0$. In this case

$$D_{21}(y, \phi) = \sigma_2 u_2 [c_2(y, \phi)].$$

By the first-order condition (32a), $u_2[c_2(y, \phi)]$ must vary with y. Hence by theorem 3, there are are no interior steady states. This example does not require that the endowment is i.i.d.. If the endowment is i.i.d. then this example is a special case of example (2).

EXAMPLE 5. (POWER RISK-SENSITIVITY). Let agents have power reward functions of the form $u_i(c) = \frac{c^{1-\gamma_i}}{1-\gamma_i}$ with $\gamma_i > 0$ and $\gamma_i \neq 1$. Let there be two types of agents, $i = 1, 2$. Consider two cases:

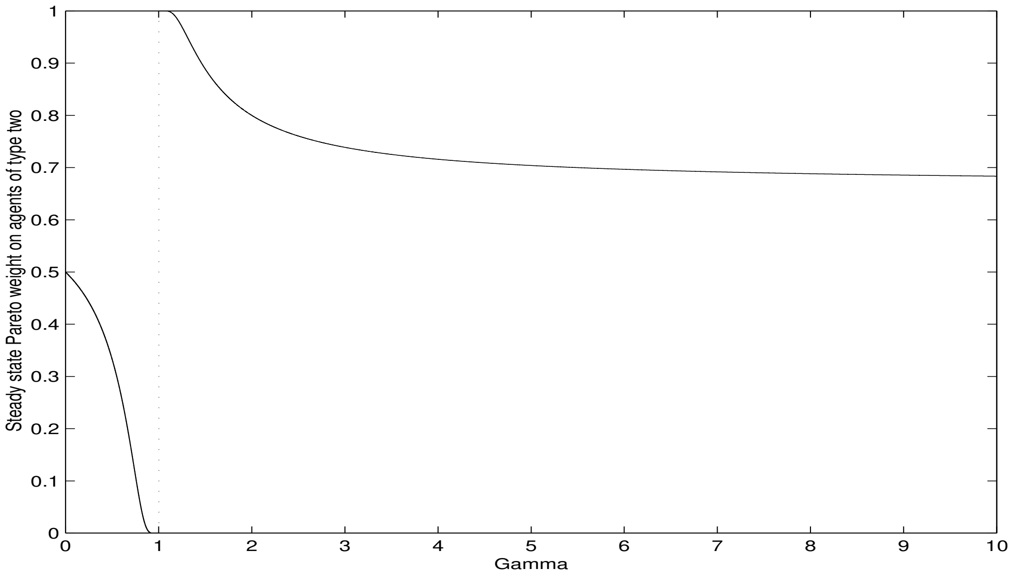

Figure 1: Steady state when agents have the same power preferences
The unique interior steady state Pareto weight on agents of type two as a function of γ when $\sigma_2 = -2$ and $\sigma_1 = -1$. When $\gamma = 1$ (which I take to correspond to log rewards) there are no interior steady states.

1. Same powers. Let all types of agents be risk-sensitive with $\sigma_i < 0$. If all agents have identical power rewards ($\gamma_i \equiv \gamma$ for some γ) then the Pareto weight vector $\bar{\theta}$ which satisfies

$$\frac{\bar{\theta}_2}{\bar{\theta}_1} = \left(\frac{\sigma_1}{\sigma_2}\right)^{\frac{\gamma}{1-\gamma}}$$

and $\bar{\theta}_1 + \bar{\theta}_2 = 1$ is the unique interior steady state Pareto weight vector.

2. Different powers. Let either $\sigma_1 < 0$ or $\sigma_2 < 0$. If agents have different power preferences ($\gamma_1 \neq \gamma_2$) then there are no interior steady state Pareto weight vectors.

The appendix justifies the results in this example.

When agents have identical power preferences, the unique steady state Pareto weight vector will be such that $\bar{\theta}_1 = \bar{\theta}_2 = 1/2$ only if $\sigma_1 = \sigma_2$. If agents of one type are less risk-sensitive[7] then in the steady state they

[7] Agent i is less risk-sensitive than agent j if $|\sigma_i| < |\sigma_j|$.

receive more consumption (in every date-state) when $0 < \gamma < 1$ and less when $\gamma > 1$. If $|\sigma_1| < |\sigma_2|$ then as γ approaches one from the left the steady state ratio of Pareto weights, $\bar{\theta}_2/\bar{\theta}_1$, approaches zero. As γ approaches one from the right the steady state ratio of Pareto weights tends towards $+\infty$. Figure 1 graphs the steady state Pareto weight on agents of type two as a function of γ when agents of type one are less risk-sensitive.

4.1.2 Steady states in Epstein-Zin economies

THEOREM 4. *Let there be two types of agents who have the same beliefs and the power aggregators described by Epstein and Zin (1989). Assume both agents either have reward functions which are bounded below by zero in which case $h_i(q_i) = q_i^\omega$ and $0 < \omega < 1$, or both agents have reward functions which are bounded above by zero in which case $h_i(q_i) = (-q_i)^\omega$ and $\omega > 1$. (I assume ω is the same for both agents.) An interior Pareto weight vector $\bar{\theta}$ is a steady state Pareto weight vector if and only if*

$$u_1\left(c_1\left(y, \bar{\theta}\right)\right) = g u_2\left(c_2\left(y, \bar{\theta}\right)\right) \tag{38}$$

for all $y \in \mathcal{X}$ and for some constant $g > 0$.

The proof of theorem 4 is in the appendix. The proof is more than a specialization of theorem 2 since preferences do not satisfy assumption 3. Theorem 4 can be applied when agents have identical power reward functions since one-period consumption allocation rules are linear in the aggregate endowment and for any interior Pareto weight vector $\bar{\theta}$ there exists a g such that equation 38 is satisfied. In this case all Pareto weight vectors are steady states.

4.1.3 Invariant distributions

For some economies when agents have heterogeneous preferences there will be no interior steady states. For these economies at all interior Pareto weight vectors the marginal values of future utilities of the agents differ at some values of the aggregate endowment.. The existence of invariant distributions is of interest.

DEFINITION 11. An *invariant distribution* is a probability measure ζ such that if ζ is the probability measure over the Pareto weight vector at date t then ζ is also the probability measure over the Pareto weight vector at all dates $s > t$.

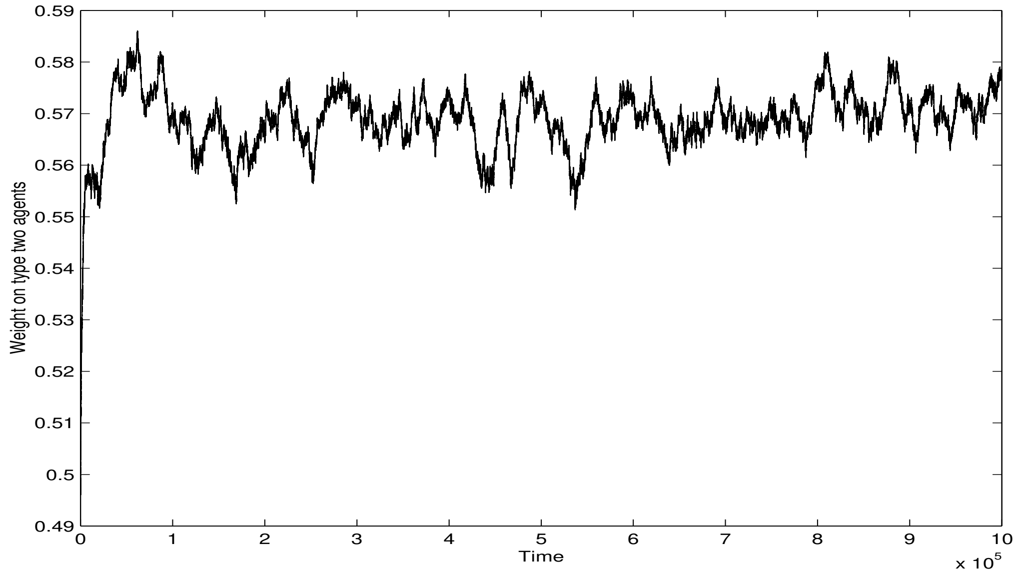

Figure 2: Evolution when there is an invariant distribution
An example of the evolution of the Pareto weight on type two agents when agents have power reward functions with heterogeneous γ's. The Pareto weight on type two agents is graphed for 10×10^5 periods.

Steady state Pareto weights are degenerate invariant distributions which put all of their probability on a single Pareto weight vector.

Consider an economy in which there are two types of agents with risk-sensitive preferences and heterogeneous power reward functions. Let

$$\gamma_1 = 0.5, \qquad \gamma_2 = 0.9, \qquad \beta = 0.95. \tag{39}$$

Assume the risk-sensitivity parameters for both type of agents are -1. Assume that the aggregate endowment is i.i.d. and takes on 11 values in the range $[0.6, 1.4]$, each with equal probability. Figure 2 graphs the evolution of the Pareto weight on agents of type 2 for this economy. The initial Pareto weight is 0.5 but quickly moves toward the interval $[0.55, 0.59]$, in which it stays for at least 10×10^5 periods.

4.2 Off steady state dynamics

This section studies optimal allocations away from the steady state and gives examples of allocations when agents have risk-sensitive preferences. Sufficient conditions are given under which the evolution of Pareto weights are bounded by steady states. Sufficient conditions are also given to guarantee that the choice of next period's Pareto weight, on a given type of agent, is monotonic in this period's Pareto weight and in next period's endowment.

In order to prove the results in this section, I will make the following additional assumption.

Assumption 5. *For all i and for all z in the interior of $\mathcal{Q}_i$ the functions $h_i(z)$ are twice continuously differentiable with*

$$h_i'(z) h_i''(z) < 0. \tag{40}$$

The risk-sensitive aggregator satisfies this assumption when $\sigma_i < 0$ since

$$\left[\frac{d\mathrm{e}^{\sigma_i z}}{dz}\right]\left[\frac{d^2\mathrm{e}^{\sigma_i z}}{dz^2}\right] = {\sigma_i}^3 \mathrm{e}^{2\sigma_i z} < 0.$$

The Epstein-Zin aggregator satisfies this assumption since

$$\left[\frac{d\left[(k_i z)^{\omega_i}\right]}{dz}\right]\left[\frac{d^2\left[(k_i z)^{\omega_i}\right]}{dz^2}\right] = k_i^3 \omega_i^2 (\omega_i - 1)\,(k_i z)^{2\omega_i - 3}.$$

When utility is bounded below by zero it is the case that $0 < \omega_i < 1$ and $k_i = 1$. When utility is bounded above by zero it is the case that $\omega_i > 1$ and

$k_i = -1$. In both cases assumption 5 is satisfied since $k_i^3(\omega_i - 1) < 0$ in both cases.

The following theorem shows that the optimal choice of next period's Pareto weight on a given type of agents is increasing in this period's weight.

THEOREM 5. *Let assumption 5 hold and let there be two types of agents with the same beliefs. Let ν and θ be Pareto weight vectors. Let the scalar $\phi_i(x,\theta,y)$ be the optimal choice of next period's Pareto weight for type i agents if this period's endowment and Pareto weight vector are (x,θ) and next period's endowment is y. If $\nu_i > \theta_i$ then $\phi_i(x,\nu,y) > \phi_i(x,\theta,y)$ for all $x, y \in \mathcal{X}$.*

The proof of theorem 5 is in the appendix. The following corollary is an immediate consequence of theorem 5.

COROLLARY 1 (NO CROSSING PROPERTY). *Let assumption 5 hold and let there be two types of agents with the same beliefs. The ratio of the Pareto weights for the agents can never cross a steady state ratio of Pareto weights.*

If $\bar{\theta}$ is a steady state Pareto weight vector, it follows that if $\theta_t = \bar{\theta}$ then $\theta_{t+1} = \bar{\theta}$ in every possible state at date $t+1$. Theorem 5 shows that if $\theta_{2t} > \bar{\theta}_2$ then $\theta_{2t+1} > \bar{\theta}_2$ in every possible state at date $t+1$. This implies that the ratio of the Pareto weights for the agents can never cross a steady state ratio of Pareto weights. This "no crossing" property of steady state Pareto ratios will play a role in the proofs of stability.

Theorem 5 gave conditions under which next period's Pareto weights are monotonic in this period's Pareto weights. I now look at conditions under which next period's Pareto weights are monotonic in next period's aggregate endowment when agents have risk-sensitive preferences. First, consider an example in which there are two types of agents who have identical power reward functions and identical risk-sensitivity parameters. Let γ be the parameter for the reward functions and σ the risk-sensitivity parameter. Let

$$\sigma = -1.0, \qquad \beta = 0.95, \tag{41}$$

and assume that the aggregate endowment is i.i.d. and takes on 11 values in the range $[0.6, 1.4]$, each with equal probability. Figures 3 and 4 graph the choice of next period's Pareto weight on agents of type two for $\gamma = 0.5$ and $\gamma = 2.0$ when this period's Pareto weight is 0.75.

Figure 3 shows that the Pareto weight on the agents who are better off this period is increasing in next period's endowment when $\gamma = 2$ and

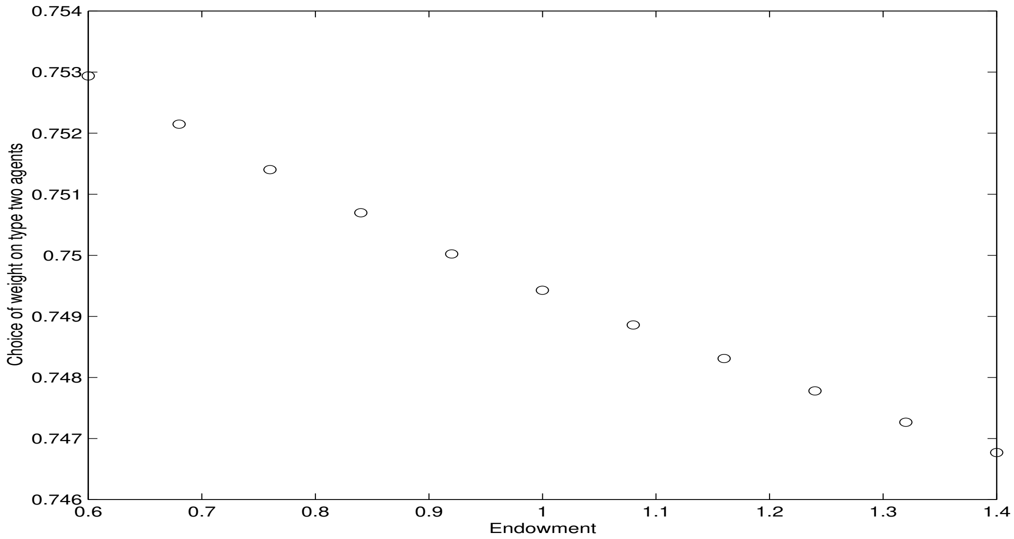

Figure 3: Optimal choices when $\gamma = 0.5$
Choice of next period's Pareto weight on agents of type two as a function of next period's endowment when $\gamma = 0.5$.

decreasing in next period's endowment when $\gamma = 0.5$. Sufficient conditions can be given to guarantee monotonicity of Pareto weights in the aggregate endowment when agents have risk-sensitive preferences. The following theorem uses the function $D_{ij}(y, \phi)$ defined in equation (36).

THEOREM 6. *Consider an economy with two types of risk-sensitive agents who have the same beliefs and who are denoted by $i = 1, 2$. Let assumptions 1 and 2 hold. Let this period's Pareto weight vector be interior. Let $\Gamma \subseteq \operatorname{int}(\Delta^n)$ be the possible values for next period's Pareto weight vector.*

1. *If for any given $\phi \in \Gamma$, $D_{21}(y, \phi)$ is a strictly increasing function of y then the optimal choice of next period's Pareto weight for agents of type two is a strictly increasing function of next period's endowment y.*

2. *If for any given $\phi \in \Gamma$, $D_{21}(y, \phi)$ is a strictly decreasing function of y then the optimal choice of next period's Pareto weight for agents of type two is a strictly decreasing function of next period's endowment y.*

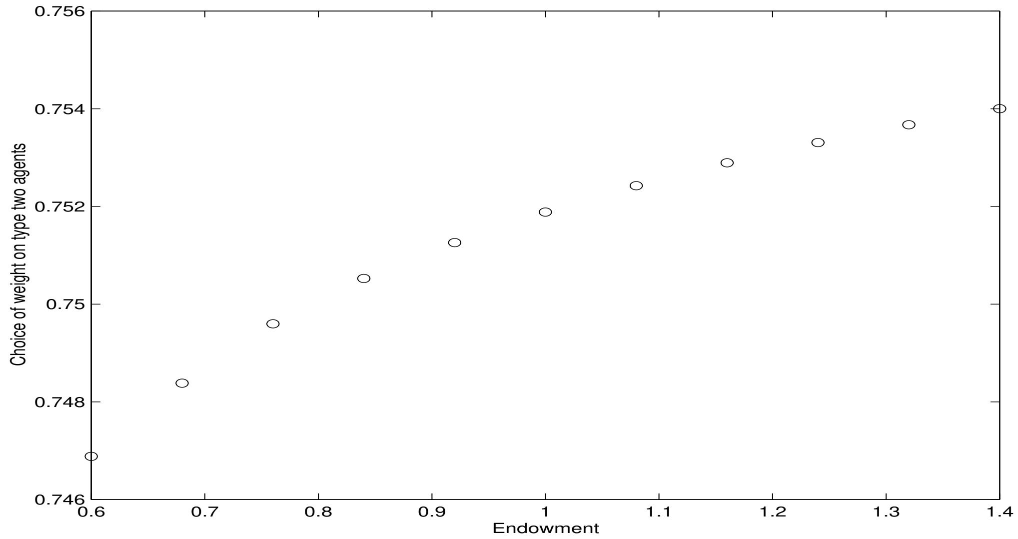

Figure 4: Optimal choices when $\gamma = 2.0$
Choice of next period's Pareto weight on agents of type two as a function of next period's endowment when $\gamma = 2.0$.

The proof of theorem 6 is in the appendix.

Consider the quantity $Q_i(d)$ which was defined in equation 24 in terms of u_i. Define two properties of $Q_i(d)$.

DEFINITION 12. Property IRE holds if $Q_i(d) > 2$ for all $d > 0$ and all i.

DEFINITION 13. Property DRE holds if $Q_i(d) < 2$ for all $d > 0$ and all i.

Theorem 1 showed that if IRE holds then relative entropy is increasing in wealth in the one-period problem and that if DRE holds then relative entropy is decreasing in wealth in the one-period problem. The following theorem shows that the same conditions on $Q_i(d)$ guarantee that relative entropy is decreasing or increasing in an infinite horizon economy and implies conditions in theorem 6.

THEOREM 7. *Consider an economy with two types of risk-sensitive agents who have the same (correct) beliefs and who are denoted by $i = 1, 2$. Let assumptions 1, 2 and 4 hold. If IRE and $\sigma_i < 0$ for both agents then*

(a) *In the infinite horizon economy relative entropy for a given type of agents (as defined in definition 7) at the optimal allocations is increasing in his Pareto weight.*

(b) If there is a unique interior steady state $\bar{\theta}$ then for any fixed Pareto weight vector ϕ such that $1 > \phi_2 > \bar{\theta}_2$ case two of theorem 6 is met and the optimal choice of next period's Pareto weight for agents of type two is a strictly decreasing function of next period's endowment y.

If DRE then

(a) In the infinite horizon economy relative entropy for a given type of agents at the optimal allocations is decreasing in his Pareto weight.

(b) If there is a unique interior steady state $\bar{\theta}$ then for any fixed Pareto weight vector ϕ such that $1 > \phi_2 > \bar{\theta}_2$ case one of theorem 6 is met and the optimal choice of next period's Pareto weight for agents of type two is a strictly increasing function of next period's endowment y.

To illustrate the usefulness of theorems 6 and 7, I consider several examples that are discussed under the assumptions made for theorem 6.

EXAMPLE 6. (LOG RISK-SENSITIVITY). Consider the environment described in example (3). Recall that for this environment

$$D_{21}(y, \phi) = \sigma_2 \log(\phi_2) - \sigma_1 \log(\phi_1) + (\sigma_2 - \sigma_1)\log(y).$$

$D_{21}(y, \phi)$ is increasing in y when $\sigma_2 > \sigma_1$ and decreasing in y when $\sigma_2 < \sigma_1$. Hence by theorem 6 the optimal choice of next period's Pareto weight for agents of type two is a strictly increasing function of next period's endowment y when $\sigma_2 > \sigma_1$ and decreasing function of y when $\sigma_2 < \sigma_1$.

EXAMPLE 7. (RISK-SENSITIVE AND NON RISK-SENSITIVE AGENTS). Consider the environment described in example (4). Recall that

$$D_{21}(y, \phi) = \sigma_2 u_2 \left[c_2(y, \phi)\right].$$

From definition 3 it follows that $D_{21}(y, \phi)$ is always a decreasing function of y since $u_2\left[c_2(y, \phi)\right]$ is always an increasing function of y. Hence by theorem 6 the choice of next period's Pareto weight for agents of type two is a strictly decreasing function of next period's endowment y.

EXAMPLE 8. (SAME POWER RISK-SENSITIVITY). Consider the environment described in example 5 in which agents have the same powers, $\gamma_1 = \gamma_2 \equiv \gamma$. Assume that agents have the correct beliefs. Recall from example

5 there is a unique interior steady state Pareto weight vector in this environment. From the discussion in section 2.3.2, we know that IRE holds if $0 < \gamma < 1$ and DRE holds if $\gamma > 1$. Now theorem 6 can be applied. Assume $\theta_2 > \bar{\theta}_2$. If $0 < \gamma < 1$ then the optimal choice of next period's Pareto weight for agents of type two is a strictly decreasing function of next period's endowment y. If $\gamma > 1$ then the optimal choice of next period's Pareto weight for agents of type two is a strictly increasing function of next period's endowment y.

In examples 6 and 7 the Pareto weights for the agents who are less risk-sensitive increase when the aggregate endowment is high and decrease when the aggregate endowment is low. This suggests that the Pareto weights may be moving over time in order that the agents who are more risk-sensitive receive a smoother consumption allocation than they would in an economy without risk-sensitivity in which agents have the same rewards and time discount factors. However theorem 7 and example 8 show that what determines who smoothes consumption more is who has a higher relative entropy not who has a lower (more negative) risk-sensitivity parameter.

4.3 Stability

This section studies the stability of steady states. An expression for the expected value of future Pareto ratios and sufficient conditions for the stability of Pareto weights are given. The conditions can be interpreted by considering the auxiliary economy described in section 2.3.1. Several examples will be given in which agents have risk-sensitive preferences.

Definition 14. The n-dimensional random variables $\{\theta_t\}_{t=0}^{\infty}$ converge with probability one to the random vector θ^* if

$$\text{prob}\left[\lim_{t\to\infty} \theta_t = \theta^*\right] = 1$$

where prob denotes the (true) probability with respect to information available at time zero.

I will refer to a steady state Pareto weight vector $\bar{\theta}$ as *stable* if the Pareto weights $\{\theta_t\}_{t=0}^{\infty}$ converge with probability one to $\bar{\theta}$ from any initial interior Pareto weight vector θ_0. Likewise, I will refer to a steady state Pareto weight vector $\bar{\theta}$ as *not stable* if the Pareto weights $\{\theta_t\}_{t=0}^{\infty}$ do not converge with probability one to $\bar{\theta}$ from any initial interior Pareto weight vector θ_0.

In order to prove convergence of Pareto weights I will use the martingale convergence theorem, which was first proved by Doob (1953). Although most advanced textbooks on probability theory contain discussions of the martingale convergence theorem, I state it here as a convenience for the reader. First I define a martingale and a supermartingale.

DEFINITION 15. If $E_t\lambda_{t+1} = \lambda_t$ for all t then λ_t is a *martingale.*

DEFINITION 16. If $E_t\lambda_{t+1} \leq \lambda_t$ for all t then λ_t is a *supermartingale.*

THEOREM 8. (MARTINGALE CONVERGENCE THEOREM). *If $\{\lambda_t\}_{t=0}^{\infty}$ is a supermartingale and $\lambda_t \geq b$ for all t where b is finite then λ_t converges with probability one to a finite random variable.*

Since all martingales are supermartingales theorem 8 applies to martingales as well. Its conclusion tells us that supermartingales bounded from below converge to a random variable. The following lemma gives additional sufficient conditions for supermartingales to converge to the scalar zero with probability one.

LEMMA 1. *If*

1. *There is a probability $\underline{\pi}$ such that for any $\lambda_t > 0$ there exists a $d > 0$ such that*

$$prob_t\left(|\lambda_{t+1} - \lambda_t| \geq d\right) \geq \underline{\pi} > 0. \tag{42}$$

   For any constants $\underline{\lambda}$ and $\overline{\lambda}$, if λ_t satisfies $+\infty > \overline{\lambda} \geq \lambda_t \geq \underline{\lambda} > 0$ then the constant d can be chosen independent of λ_t.

2. *$E_t\lambda_{t+1} \leq \lambda_t$ and $\lambda_t \geq 0$.*

3. *The time t probability of future values of λ_t depends only on the value of λ_t.*

then $\{\lambda_t\}$ converges with probability one to zero.

The proof of lemma 1 is in the appendix.

For many specifications of preferences, it can be shown that Pareto ratios are supermartingales that satisfy the hypotheses of lemma 1. Define

$$\mathcal{D}_i\left(x,\theta,y\right) \equiv \mathcal{M}_{ix}\left(y,q_i\right) \qquad \text{where} \qquad q_{iz} = V_i\left(z,\phi\left(x,\theta,z\right)\right). \tag{43}$$

Here q_i is the optimal utility allocation function. $\mathcal{D}_i\left(x,\theta,y\right)$ gives the marginal value of utility for agents of type i when this period's endowment is x, this period's Pareto weight is θ and next period's endowment is y. $\mathcal{D}_i\left(x,\theta,y\right)$ is a version of $\mathcal{M}_{ix}\left(y,q_i\right)$ evaluated at the optimal choice of q_i as determined form the social planing problem. The following theorem gives the one period ahead conditional expectation of Pareto ratios.

LEMMA 2. *Let*

$$g_{ij}(x,\theta) \equiv \int_{y\in\mathcal{X}} \left[\pi(x,y)\frac{\mathcal{D}_i(x,\theta,y)}{\mathcal{D}_j(x,\theta,y)}\right]. \tag{44}$$

If the time t Pareto weight vector is interior then the expected value of the time $t+1$ Pareto ratios are given by

$$E_t\left[\frac{\theta_{it+1}}{\theta_{jt+1}}\right] = g_{ij}(x_t,\theta_t)\frac{\theta_{it}}{\theta_{jt}}, \qquad \forall i,j. \tag{45}$$

The proof of lemma 2 is in the appendix. Lemma 2 will be useful in giving conditions under which the hypotheses of lemma 1 are partially satisfied. Additional arguments are needed to show that hypothesis one of lemma 1 is satisfied. Consider the following example.

EXAMPLE 9. (HETEROGENEOUS BELIEFS). Consider an economy in which agents have time additive utility and heterogeneous beliefs. Let the endowment satisfy assumption 1. Here $g_{ij}(x,\theta)$ from lemma 2 can be written as

$$g_{ij}(x,\theta) = \int_{y\in\mathcal{X}} \left[\pi(x,y)\frac{\pi_i(x,y)}{\pi_j(x,y)}\right].$$

Assume that agents of type j use the correct probabilities about the aggregate endowment when forming their preferences and agents of type i use incorrect probabilities about the aggregate endowment. Then

$$g_{ij} = \sum_{y\in\mathcal{X}} \pi_i(x,y) = 1$$

and hence

$$E_t\left[\frac{\theta_{it+1}}{\theta_{jt+1}}\right] = \frac{\theta_{it}}{\theta_{jt}}.$$

It is straightforward to show that hypothesis (1) of lemma 1 is satisfied. Hence lemma 1 guarantees that if $\theta_{j0} > 0$ then as t goes to ∞, θ_{it}/θ_{jt} converges with probability one to zero. In the long run agents of type i will not be allocated any aggregate consumption. The limiting consumption for agents of type j depends upon the beliefs of the other agents.

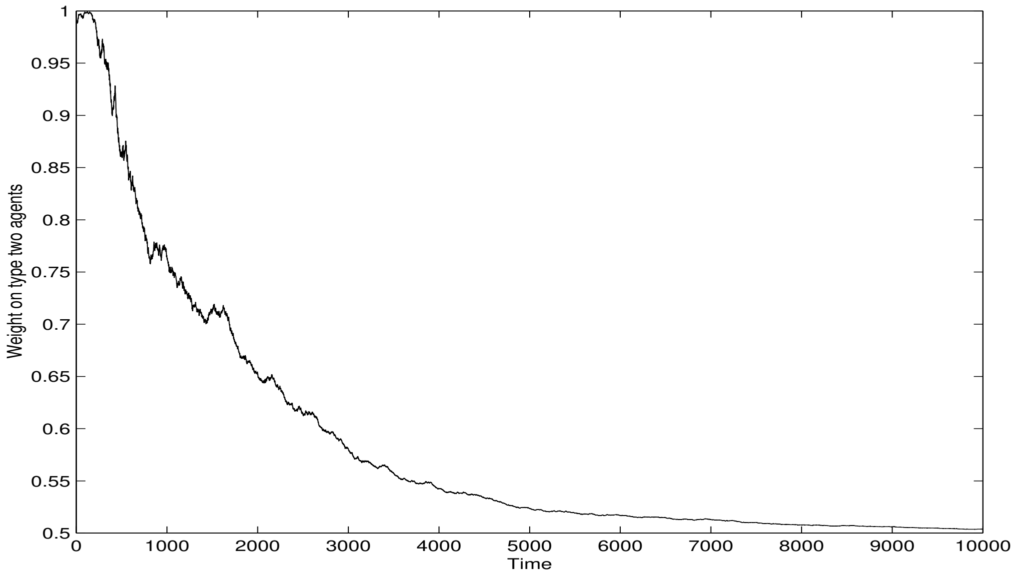

Figure 5: Evolution when $\gamma = 0.5$
Evolution of the Pareto weight on agents of type two when $\gamma = 0.5$ for the economy specified at the beginning of section 4.2.

4.3.1 Stability in risk-sensitive economies

This section looks as stability in risk-sensitive economies when there are only two types of agents, $i = 1, 2$ and both agents have the same beliefs. Figures 5 and 6 graph the evolution of the Pareto weight on agents of type two for two different specifications of γ. We see that the steady state appears to be stable when $\gamma = 0.5$ and not stable when $\gamma = 2.0$. To analyze stability analytically it will be convenient to look at the expected value of next period's Pareto ratio giving this period's Pareto ratio.

LEMMA 3. *If all agents are risk-sensitive with the same (correct) beliefs then*

$$E_t\left[\frac{\theta_{it+1}}{\theta_{jt+1}}\right] = \frac{\theta_{it}}{\theta_{jt}} - \frac{\operatorname{cov}_t\left(e^{\sigma_j V_j(x_{t+1},\theta_{t+1})}, \frac{\theta_{it+1}}{\theta_{jt+1}}\right)}{E_t\left[e^{\sigma_j V_j(x_{t+1},\theta_{t+1})}\right]} \tag{46}$$

and

$$E_t \log\left[\frac{\theta_{it+1}}{\theta_{jt+1}}\right] = \log\left[\frac{\theta_{it}}{\theta_{jt}}\right] - R_{it} + R_{jt} \tag{47}$$

where relative entropy R_{kt} for all k was defined in definition 7.

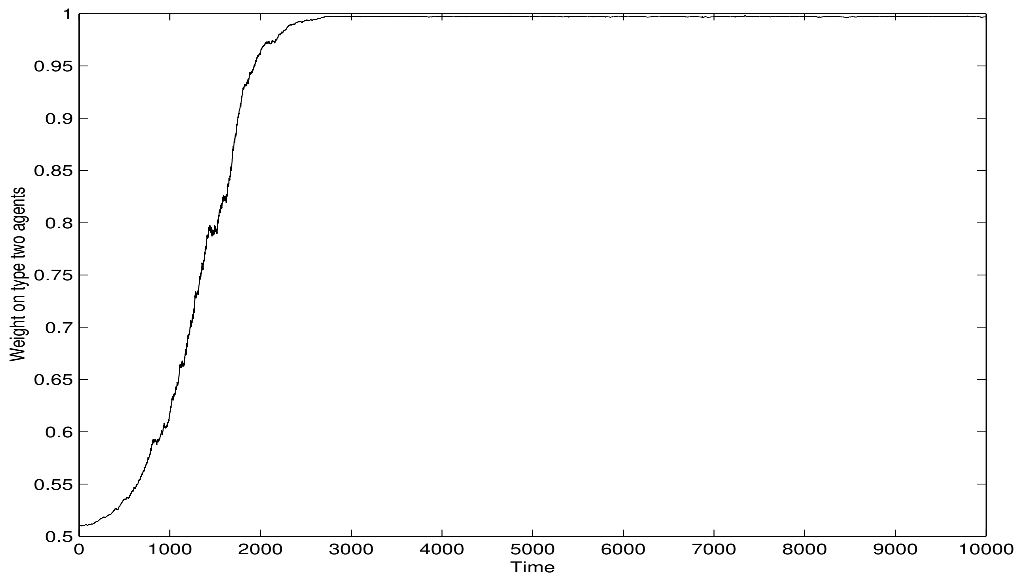

Figure 6: Evolution when $\gamma = 2.0$
Evolution of the Pareto weight on agents of type two when $\gamma = 2.0$ for the economy specified at the beginning of section 4.2.

The proof of lemma 3 is in the appendix.

Consider three cases:

1. $R_{it} > R_{jt}$. In this case the preferences of agents of type i deviate more from time additive preferences than do the preferences of agents of type j. In the auxiliary economy the relative entropy of type i agent's beliefs is larger than the relative entropy of type j agent's beliefs. It follows that

$$E_t \log \left[\theta_{it+1}/\theta_{jt+1}\right] < \log \left[\frac{\theta_{it}}{\theta_{jt}}\right].$$

2. $R_{it} < R_{jt}$. In this case the preferences of agents of type i deviate less from time additive preferences than do the preferences of agents of type j. In the auxiliary economy the relative entropy of type i agent's beliefs is smaller than the relative entropy of type j agent's beliefs. It follows that

$$E_t \log \left[\theta_{it+1}/\theta_{jt+1}\right] > \log \left[\frac{\theta_{it}}{\theta_{jt}}\right].$$

3. $R_{it} = R_{jt}$. In this case the preferences of agents of type i and type j deviate the same amount from time additive preferences. In the auxiliary economy the beliefs of both types of agents have the same relative entropy. It follows that

$$E_t \log \left[\theta_{it+1}/\theta_{jt+1}\right] = \log \left[\frac{\theta_{it}}{\theta_{jt}}\right].$$

The stability results in this section for risk-sensitive economies can be interpreted in terms of relative entropy. Assume there is a unique interior steady state. At a steady state agents have the same relative entropy. In an economy with two types of agents, if the relative entropy for all agents are increasing in their Pareto weight then the interior steady state is stable. If relative entropy for all agents is decreasing in their Pareto weight then the interior steady state is not stable. In a decentralized version of the economy the words Pareto weight could be replaced with wealth in the previous two sentences.

The proofs of the stability results discussed in the preceding paragraph will work by generating conditions under which the covariance term in equation (46) can be signed or alternatively by generating conditions when one type of agents has a larger relative entropy. The following theorem is the main result of this subsection, and gives sufficient conditions for a steady state Pareto weight vector to be stable or to be not stable.

THEOREM 9. *Consider an economy with two types of risk-sensitive agents who have the same (correct) beliefs and who are denoted by $i = 1, 2$. Let assumptions 1 and 2 hold. Let $\bar{\theta}$ be an interior steady state Pareto weight vector. Recall the function $D_{ij}(y, \phi)$ defined in equation* (36).

1. *If for any fixed Pareto weight vector ϕ such that $1 > \phi_2 > \bar{\theta}_2$ it is the case that $D_{21}(y, \phi)$ is a strictly decreasing function of y and for any fixed Pareto weight vector ϕ such that $0 < \phi_2 < \bar{\theta}_2$ it is the case that $D_{21}(y, \phi)$ is a strictly increasing function of y then $\bar{\theta}$ is stable.*

2. *If for any fixed Pareto weight vector ϕ such that $1 > \phi_2 > \bar{\theta}_2$ it is the case that $D_{21}(y, \phi)$ is a strictly increasing function of y and for any fixed Pareto weight vector ϕ such that $0 < \phi_2 < \bar{\theta}_2$ it is the case that $D_{21}(y, \phi)$ is a strictly decreasing function of y then $\bar{\theta}$ is not stable.*

The proof of theorem 9 is in the appendix. Although in case (2) above I only claim that the steady state is not stable, numerical evidence suggests that with probability one (or at least close to probability one), when there is a unique interior steady state and the initial Pareto weights are not equal to the interior steady state, the Pareto weights will diverge so that in the long run one type of agents has a Pareto weight of one and consumes all of the aggregate endowment.

Using theorems 7 and 9 conditions which are easier to interpret can be stated that guarantee stability.

COROLLARY 2. *Consider an economy with two types of risk-sensitive agents who have the same (correct) beliefs and who are denoted by $i = 1, 2$. Let $\sigma_i < 0$ for both agents. Let assumptions 1,2 and 4 hold. Let $\bar{\theta}$ be an interior steady state Pareto weight vector. If IRE then $\bar{\theta}$ is stable and if DRE then $\bar{\theta}$ is not stable.*

I now consider several risk-sensitive examples. The first example shows that when agents have identical power reward functions the unique interior steady state will be stable when the power is less than one and not stable when the power is greater than one.

EXAMPLE 10. (SAME POWER RISK-SENSITIVITY). Consider the environment described in examples 5 and 8, under the assumptions made for example 8. Recall that for this economy there is a unique interior steady state Pareto weight vector, $\bar{\theta}$. From the discussion in section 2.3.2, we know that IRE holds if $0 < \gamma < 1$ and DRE holds if $\gamma > 1$. It follows from corollary 2 that $\bar{\theta}$ is stable when $0 < \gamma < 1$ and not stable when $\gamma > 1$.

EXAMPLE 11. (RISK-SENSITIVE AND NON RISK-SENSITIVE AGENTS). Consider the environment described in examples 4 and 7 under the assumptions made for example 7. In addition assume that agents have the correct beliefs. Recall that for this economy there are no interior steady state Pareto weight vectors. In lemma 3 (letting agents of type one play the role of agents of type j and letting agents of type two play the role of agents of type i) the covariance term is zero since $\sigma_1 = 0$. Hence the Pareto ratios are a nonnegative martingale:

$$E_t \left[\frac{\theta_{2t+1}}{\theta_{1t+1}} \right] = \frac{\theta_{2t}}{\theta_{1t}}.$$

It can be shown that hypothesis one of lemma 1 is met so that the Pareto ratios converge to zero over time.

The conclusion of example 11 is a strong result since it shows that for any reward functions that satisfy week assumptions the Pareto ratios are a martingale and converge to zero. There is no requirement that agents have the same rewards. In the long run the agents with time additive preferences are allocated all of the aggregate endowment. For this result (and the other results in this section) it is essential that agents have the correct beliefs about the evolution of the aggregate endowment and that all agents use the same discount factor β.

When steady states are not stable why are the Pareto weights (and hence consumption allocations) converging to zero for some agents? One interpretation is that under their beliefs their consumption allocations are not converging to zero. If agents have the distorted beliefs specified in the auxiliary economy then all agents believe that in the long run their consumption will not be converging to zero. In a decentralized economy agents purchase consumption for all future date-states at the initial date. Agents purchase consumption in states they believe are likely to occur. For dates far into the future agents with distorted beliefs purchase most of their consumption in a group of states that they think will occur with probability one. In fact this group of states occurs with probability zero. In states that occur with high probability agents with distorted beliefs purchase very little consumption as a consequence with probability one their consumption tends toward zero.

5 Conclusions

This paper described the implications Pareto optimal allocation rules in stochastic economies when some or all agents have time non-additive or recursive preferences. The central implications are

1. Allocation rules typically vary over time and are history dependent, except at steady states.

2. Steady states provide bounds on the evolution of Pareto weights.

3. In risk-sensitive economies with unique interior steady states and two types of agents if $Q_i(d) < 2$ (respectively $Q_i(d) > 2$) for all i then

   (a) The choice of next periods Pareto weight on the agents whose initial weight is higher than their steady state weight is increasing (decreasing) in next period's endowment. In other words agents who are initially richer, than their steady state wealth, smooth consumption less (more) than in an economy without risk-sensitivity in which agents have the same rewards and time discount factors.

(b) Agents have decreasing (increasing) relative entropy at the optimal allocations.

(c) The unique interior steady state is not stable (stable).

Although I only proved the stability results when there are two types of agents, the results extend to economies in which there are n types of agents. In particular if $Q_i(d) > 2$ for all i then interior steady states are stable and if $Q_i(d) < 2$ for all i then the group of agents whose Pareto weight is initially the highest will in the long run dominate the economy.

For risk-sensitive economies stability was characterized in terms of relative entropy or the deviations of preferences from time additive preferences. For economies with agents who have more general forms of recursive utility impatience also matters. When agents have identical constant relative entropies then the stability analysis of Lucas and Stokey (1984) can be imitated to show that if impatience is increasing in wealth then interior steady states are stable. For other economies in which both relative entropy and impatience vary with wealth, stability depends upon the relative magnitudes of the two quantities.

A Dynamic Programming Problem

Appendix A formally justifies the dynamic programming formulation of the social planning problem described in section 3. Appendix B contains the proofs for the results described in the other sections.

The contribution of this appendix to the literature is a formal treatment of heterogenous agent problems in which agents have reward functions that are possibly unbounded from below. The assumptions, results and pattern of discussion in this section follow Lucas and Stokey (1984) and Kan (1995) closely. Let

$$\mathcal{U}(x) = \{s \in \mathcal{R}^n : s_i = U_i(x, \mathbf{c}_i) \text{ for all } i \text{ where } \mathbf{c} \in \mathcal{C}(x)\}$$

be the set of feasible remaining utilities for the agents. The value of the social planning problem can be written as

$$Q\left(x, \theta\right) = \sup_{\mathbf{c} \in \mathcal{C}(x)} U\left(x, \mathbf{c}\right) = \sup_{s \in \mathcal{U}(x)} \sum_{i=1}^{n} \theta_i s_i. \tag{48}$$

If $\theta \in \operatorname{int}\left(\Delta^n\right)$ the solution to the social planning problem will be Pareto optimal.

The following lemmas follow results in Lucas and Stokey (1984) and Kan (1995) and form the basis for the justification of the dynamic programming problem. The proof of lemmas 5, 7 and 8 are very similar to proofs in Lucas and Stokey (1984) and are not given in this paper.

Lemma 4. *$Q\left(x, \theta\right)$ is bounded.*

Proof. Consider the allocation (which is not feasible) of assigning all the agents the maximum possible value of the aggregate endowment each period no matter what the realization of the aggregate endowment. Given Pareto weights, θ, the value of this allocation for the social planner's problem is $\sum_{i=1}^{n} \theta_i \overline{q}_i$ where $\overline{q}_i$ was defined in equation (8). This is finite since under our assumptions $u_i(x_{\max})$ is finite for all i and $0 < \beta < 1$. An upper bound for the social planner's value function is given by $\max_i \overline{q}_i$. Consider the feasible allocation of assigning all the agents $1/n$ of the lowest possible value of the aggregate endowment in every current and future state. The value obtained by agents of type i is

$$\hat{q}_i = (1 - \beta)^{-1} u_i \left(n^{-1} x_{\min}\right)$$

where $x_{\min} > 0$ is the minimum possible realization of the endowment. Given Pareto weights, θ, the value of this allocation for the social planner's problem is $\sum_{i=1}^{n} \theta_i \hat{q}_i$. This is finite since $u_i\left(n^{-1}x_{\min}\right)$ is finite for all i. A lower bound for the social planner's value function is given by $\min_i \hat{q}_i$. $\square$

LEMMA 5. *For each $x \in \mathcal{X}$, $\mathcal{U}(x)$ is convex, closed and exhibits free-disposal in the sense that if $s \in \mathcal{U}(x)$ and $\underline{q}_i \leq s_i' < s_i$ for all i then $s' \in \mathcal{U}(x)$.*

LEMMA 6. *$Q(x, \theta)$ is strictly convex in θ and continuous in θ.*

Proof. Let s^a and s^b be the optimal value allocations for all the agents when the Pareto weights are $\theta^a \in \Delta^n$ and $\theta^b \in \Delta^n$ respectively. (s^a and s^b are n dimensional vectors whose ith elements are the utility obtained by type i agents in the Pareto optimal allocation.) Let $\theta^a \neq \theta^b$. For some $\vartheta \in (0,1)$ let s^c be the optimal value allocation when the Pareto weights are $(1-\vartheta)\theta^a + \vartheta\theta^b$. Decompose s^z where $z = a, b, c$ as

$$s^z = u^z + v^z.$$

Let u^z be the value obtained from the current period consumption allocation and let v^z be the value obtained from anticipated future consumption. Assume

$$\sum_{i=1}^{n} \left[(1-\vartheta)\theta_i^a + \vartheta\theta_i^b\right] v_i^c > (1-\vartheta) \sum_{i=1}^{n} \theta_i^a v_i^a + \vartheta \sum_{i=1}^{n} \theta_i^b v_i^b. \tag{49}$$

In order for this inequality to hold it must be the case that at least one of the following two equations holds:

$$\sum_{i=1}^{n} \theta_i^a v_i^c > \sum_{i=1}^{n} \theta_i^a v_i^a, \tag{50}$$

$$\sum_{i=1}^{n} \theta_i^b v_i^c > \sum_{i=1}^{n} \theta_i^b v_i^b. \tag{51}$$

Note that v^c is a feasible allocation. If equation (50) holds then v^a is not the optimal allocation when the Pareto weights are θ^a. Likewise, if equation (51) holds then v^b is not the optimal allocation when the Pareto weights are θ^b. Contradiction. Conclude that the greater than sign in equation (49) must be replaced with a less than or equal sign:

$$\sum_{i=1}^{n} \left[(1-\vartheta)\theta_i^a + \vartheta\theta_i^b\right] v_i^c \leq (1-\vartheta) \sum_{i=1}^{n} \theta_i^a v_i^a + \vartheta \sum_{i=1}^{n} \theta_i^b v_i^b. \tag{52}$$

Let

$$S(x,\theta) = \max_{\{c_i \geq 0\}_{i=1}^{n}} \sum_{i=1}^{n} \theta_i u_i\left(c_i\right)$$

such that $\sum_{i=1}^{n} c_i = x$. The definition of a reward function given in definition 3 implies that $S(x,\theta)$ is strictly convex in θ. Hence

$$\sum_{i=1}^{n} \left[(1-\vartheta)\theta_i^a + \vartheta\theta_i^b\right] u_i^c < (1-\vartheta) \sum_{i=1}^{n} \theta_i^a u_i^a + \vartheta \sum_{i=1}^{n} \theta_i^b u_i^b \tag{53}$$

and (adding the left and right sides of equation (53) to the left and right sides of equation (52))

$$\sum_{i=1}^{n} \left[(1-\vartheta)\theta_i^a + \vartheta\theta_i^b\right] s_i^c < (1-\vartheta) \sum_{i=1}^{n} \theta_i^a s_i^a + \vartheta \sum_{i=1}^{n} \theta_i^b s_i^b.$$

Hence $Q\left(x,\theta\right)$ is strictly convex in θ.

Now, since by lemma 4 $Q\left(x,\theta\right)$ is bounded, the convexity of $Q\left(x,\theta\right)$ guarantees that $Q\left(x,\theta\right)$ is continuous in θ. (See corollary 10.1.1 on page 83 of Rockafellar (1970).) $\square$

LEMMA 7. *$s \in \mathcal{U}(x)$ if and only if $s_i \geq \underline{q}_i$ for all i and*

$$Q\left(x,\theta\right) \geq \sum_{i=1}^{n} \theta_i s_i \tag{54}$$

for all $\theta \in \Delta^n$.

LEMMA 8. *The optimal utility allocations $s(x,\theta) \in \mathcal{U}(x)$ are continuous in θ.*

Lemma 4 shows that the social planner's value function is bounded from below even though the reward functions may be unbounded from below. It will never be optimal for the social planner to assign $-\infty$ utility to agents of type i, unless the Pareto weight for agents of type i is zero.[8]

Lucas and Stokey (1984) also showed that the set of feasible utilities, $\mathcal{U}(x)$, is compact. When the reward functions are unbounded from below the set of feasible utilities will not be compact. However for any $\theta \in \operatorname{int}\left(\Delta^n\right)$

[8] When the Pareto weight for some types of agents is zero, I interpret the social planner's problem as: maximize the value given to the agents who have positive Pareto weights.

a compact subset of $\mathcal{U}(x)$ can be selected which contains the optimal allocations of utilities. In light of this and lemma 5, the sup's in equation (48) can be replaced with max's.

Lemma 7 allows us to characterize feasible allocations in terms of the social planner's value function $Q(x, \theta)$ and will be the essential ingredient in formulating a dynamic programming problem. It is easy to see that the inequality in equation (54) is a necessary condition for feasibility. Assume s is a feasible allocation and $Q(x, \theta) < \sum_{i=1}^{n} \theta_i s_i$ for some Pareto weights θ. When the Pareto weights are θ then $Q(x, \theta)$ could not be the maximum value of the social planner's problem since s is a feasible allocation which gives the social planner more value. This contradicts the fact that $Q(x, \theta)$ is the social planner's value function. To prove the converse use lemma 5 and a separating hyperplane theorem.

I now formulate a dynamic programming problem for computing the solution to the social planner's problem. Let F be the Banach space of bounded functions $f : \mathcal{X} \times \Delta^n \to \Re$ that are continuous in their second argument with the norm

$$\| f \| = \sup_{x \in \mathcal{X}, \theta \in \Delta^n} |f(x, \theta)|.$$

Define the operator $T : F \to F$ by

$$(TQ)(x, \theta) = \max_{\{c_i \in \Re_+, q_i \in \mathcal{A}_i\}_{i=1}^{n}} \sum_{i=1}^{n} \theta_i W_i(x, c_i, q_i) \tag{55a}$$

where

$$\sum_{i=1}^{n} c_i = x \tag{55b}$$

and for each $y \in \mathcal{X}$

$$0 \leq \min_{\phi \in \Delta^n} \left[Q(y, \phi) - \sum_{i=1}^{n} \phi_i q_{iy} \right]. \tag{55c}$$

I now show the relationship between fixed points of T and the social planner's value function $Q(x, \theta)$ given in equation (48).

THEOREM 10. *If assumption 3 holds then*

1. *The operator T defined in equations* (55a) *thru* (55c) *is a contraction mapping.*

2. *The function Q defined in equation* (48) *is the unique bounded, continuous in θ fixed point of T.*

The proof is similar to the proof given in Lucas and Stokey (1984) for a deterministic economy. When the reward functions for some agents are unbounded from below T is still a contraction mapping when assumption 3 holds since Blackwell's sufficient conditions for a contraction hold. The discounting argument of Lucas and Stokey (1984) applies and the social planner's value function is bounded by lemma 4.

When assumption 3 does not hold then one can sometimes use other properties of the functions h_i to establish that T is a contraction mapping. If T is not a contraction mapping then T may have multiple fixed points. In this case typically the social planner's value function can be computed by iterating on T starting from an upper bound for the social planner's value function.

B Proofs

This part of the appendix contains the proofs for all the sections other than section 3. The proofs for that section were given in the previous appendix.

B.1 Proof of theorem 1

In the one-period problem relative entropy can be written as

$$R_i = -\sigma_i \left[Eu_i(c_i) - \frac{1}{\sigma_i} \log E e^{\sigma_i u_i(c_i)} \right] \tag{56}$$

where for any function f defined on $\mathcal{X}$

$$Ef = \int_{y \in \mathcal{X}} \pi_y f_y. \tag{57}$$

In this proof wealth will be denoted by z. I assume z is monotonically increasing in "wealth" w_i defined in section 2.3.2. If relative entropy is decreasing, increasing or constant in wealth indexed by z then it must also be decreasing, increasing or constant in wealth indexed by w_i. (In this proof, unless explicitly stated otherwise, wealth refers to z and not w_i.)

In equation 56, c_i is viewed implicitly as a function of both y and z. I assume agents have some wealth so that they can purchase at least ϵ units of consumption in at least one state where ϵ is an arbitrarily small positive number. Differentiating relative entropy with respect to wealth yields

$$\frac{dR_i}{dz} = -\sigma_i \left[Eu_i'(c_i)\, c_i' - \frac{E\left[\mathrm{e}^{\sigma_i u_i(c_i)} u_i'(c_i)\, c_i'\right]}{E\mathrm{e}^{\sigma_i u_i(c_i)}} \right] \tag{58}$$

$$= \sigma_i \mathrm{cov}\left(\frac{\mathrm{e}^{\sigma_i u_i(c_i)}}{E\mathrm{e}^{\sigma_i u_i(c_i)}}, u_i'(c_i)\, c_i' \right) \tag{59}$$

where c_i' is the function on $\mathcal{X}$ which is the derivative of c_i with respect to z. Recall that by assumption $\sigma_i < 0$. To show that relative entropy is decreasing, increasing or constant in wealth it is sufficient to show that the covariance term in equation 59 is negative, positive or zero for all z of interest.

For notational simplicity assume that the prices p_y are a decreasing function of y. If they were not then the problem could be reformulated so that prices are written as a decreasing function of some other variable since for any $y, z \in \mathcal{X}$ it is the case that $p_y \neq p_z$. Since p_y is decreasing in y, by optimality it is straightforward to show that the agent's optimal choices of consumption c_i are increasing in y.

To sign the covariance term it will be useful to derive a convenient expression for $u_i'(c_i)\, c_i'$. I will obtain such an expression by differentiating first-order conditions. The first-order conditions for the agents one-period problem include

$$\left[\frac{\mathrm{e}^{\sigma_i u_i(c_i)}}{E\mathrm{e}^{\sigma_i u_i(c_i)}} \right] u_i'(c_i) = \mu_i p_i, \tag{60}$$

where μ_i is a Lagrange multiplier on the agent's budget constraint. Here p_i implicitly depends on y and c_i implicitly depends on both y and z. The first-order condition in equation 60 is for a generic y and there are m (where m is the dimension of $\mathcal{X}$) first-order conditions indexed by y. Formally define wealth as

$$z = -\log \mu_i - \log E\mathrm{e}^{\sigma_i u_i(c_i)}. \tag{61}$$

As promised z is monotonically increasing in w_i since both μ_i and $E\mathrm{e}^{\sigma_i u_i(c_i)}$ are monotonically decreasing in w_i. Take logarithms of both sides of the first order conditions and substitute in the definition of z to obtain

$$\sigma_i u_i(c_i) + \log u_i'(c_i) = -z + \log p_i. \tag{62}$$

Differentiating both sides of this equation with respect to z yields

$$\sigma_i u_i'(c_i)\, c_i' + \frac{u_i''(c_i)}{u_i'(c_i)} c_i' = -1 \tag{63}$$

which implies

$$u_i'(c_i)\, c_i' = B_i(c_i)\,, \tag{64}$$

$$B_i(d) \equiv -\left(\frac{1}{\sigma_i + \frac{u_i''(d)}{\left[u_i'(d)\right]^2}} \right) \tag{65}$$

where equation 65 defines the function B_i.

I will sign the covariance term in equation 59 by showing that its arguments are either increasing, decreasing or constant in y. The first argument to the covariance term, $\frac{\mathrm{e}^{\sigma_i u_i(c_i)}}{E\mathrm{e}^{\sigma_i u_i(c_i)}}$, is always decreasing in y since c_i is increasing in y. So, to show that the covariance term in equation 59 is (positive, negative, zero) it is sufficient to show that $B_i(d)$ is (decreasing, increasing, constant) in d. Now differentiate $B_i(d)$ with respect to d to obtain

$$\frac{dB_i(d)}{dd} = \left(\frac{1}{\sigma_i + \frac{u_i''(d)}{\left[u_i'(d)\right]^2}} \right)^2 \left(\frac{u_i'''(d)}{\left[u_i'(d)\right]^2} - \frac{2\left[u_i''(d)\right]^2}{\left[u_i'(d)\right]^3} \right) \tag{66}$$

$$= u_i'(d) \left(\frac{u_i''(d)}{\sigma_i \left[u_i'(d)\right]^2 + u_i''(d)} \right)^2 \left(\frac{u_i'''(d)\, u_i'(d)}{\left[u_i''(d)\right]^2} - 2 \right). \tag{67}$$

Note that $u_i'(d)$ is always positive and the term which is squared is always positive. The last term determines the sign of $\frac{dB_i(d)}{dd}$ and whether or not $B_i(d)$ is increasing, decreasing or constant. If for all $d > 0$

$$\frac{u_i'''(d)\, u_i'(d)}{\left[u_i''(d)\right]^2} > 2 \tag{68}$$

then $B_i(d)$ is an increasing function of d. If for all $d > 0$

$$\frac{u_i'''(d)\, u_i'(d)}{\left[u_i''(d)\right]^2} < 2 \tag{69}$$

then $B_i(d)$ is a decreasing function of d. If for all $d > 0$

$$\frac{u_i'''(d)\, u_i'(d)}{\left[u_i''(d)\right]^2} = 2 \tag{70}$$

then $B_i(d)$ is a constant function of d. So the covariance term is (positive, negative, zero) if $Q_i(d)$ is (less than, greater than, equal to) two for all $d > 0$. Since $\sigma_i < 0$ relative entropy is (decreasing, increasing, constant) in wealth if $Q_i(d)$ is (less than, greater than, equal to) two for all $d > 0$. The preceding statement holds for wealth as measured by z and wealth as measured by w_i. (Again I assume that agents have some wealth so that this theorem says nothing about relative entropy as w_i moves from zero to a small positive number.)

B.2 Proof of theorem 2

Let $\bar{\theta}$ be an interior steady state. Then by definition $s_{i\bar{\theta}}(y)$ gives the value remaining for agents of type i when the current period endowment is y. Assume $\mathcal{M}_{ix}\left(y, s_{i\bar{\theta}}\right) \neq \mathcal{M}_{jx}\left(y, s_{j\bar{\theta}}\right)$ for some x, y, i, j. Then for this x, y, i, j it is the case that

$$\frac{\bar{\theta}_i \mathcal{M}_{ix}\left(y, s_{i\bar{\theta}}\right)}{\bar{\theta}_j \mathcal{M}_{jx}\left(y, s_{j\bar{\theta}}\right)} \neq \frac{\bar{\theta}_i}{\bar{\theta}_j}$$

which in conjunction with the first-order condition 32b implies

$$\frac{\phi_{iy}}{\phi_{jy}} \neq \frac{\bar{\theta}_i}{\bar{\theta}_j}.$$

Hence it is not optimal to set the Pareto weights at $\bar{\theta}$ when this period's Pareto weights are $\bar{\theta}$, this period's endowment is x and next period's endowment is y. Contradiction. Conclude that equation (35) must hold for all i, j, x, y.

Now I prove the converse. Assume there exists interior Pareto weights $\bar{\theta}$ such that equation (35) holds for all i, j, x, y. Define a function

$$Q_0\left(y, \phi\right) = \sum_{i=1}^{n} \phi_i s_{i\phi}(y). \tag{71}$$

From the definition of reward functions it is straightforward to show that $Q_0\left(y, \phi\right)$ is strictly convex and continuous in ϕ. Define the following property of functions.

DEFINITION 17. Let f be a function which maps $\mathcal{X} \times \Delta^n$ into the real numbers. Property ISS holds if for all $y \in \mathcal{X}$

$$f(y, \phi) \geq \sum_{i=1}^{n} \phi_i s_{i\bar{\theta}}(y), \qquad \forall \phi \in \Delta^n, \tag{72}$$

$$f(y, \bar{\theta}) = \sum_{i=1}^{n} \bar{\theta}_i s_{i\bar{\theta}}(y), \tag{73}$$

and $f(y, \phi)$ is strictly convex and continuous in ϕ.

The previous remarks showed that Q_0 has property ISS.

The strategy of this proof is to show that

(a) $Q_k = T^k Q_0$ for all k and $Q = \lim_{k \to \infty} Q_k$ have property ISS.

(b) If Q has property ISS then the optimal choice of next period's Pareto weights is $\bar{\theta}$ when this period's Pareto weights are $\bar{\theta}$.

I will first prove part b. If Q has property ISS then note that since $\mathcal{M}_{ix}\left(y, s_{i\bar{\theta}}\right) = \mathcal{M}_{jx}\left(y, s_{j\bar{\theta}}\right)$ a solution to the first-order conditions 32b and 32c is $\phi = \bar{\theta}$ when this period's Pareto weight vector is $\bar{\theta}$. This must be the optimal choice of Pareto weights since no other Pareto weight vectors could solve the first-order conditions.

To prove part a, I first assume that Q_k has property ISS and then show that $Q_{k+1} = TQ_k$ has property ISS. When computing TQ_k note that the optimal choice of next period's Pareto weight vectors, computed in constraint 55c, must be $\bar{\theta}$ when this period's Pareto weight vector is $\bar{\theta}$ by an argument similar to the argument given in the previous paragraph. So

$$Q_{k+1}\left(y, \bar{\theta}\right) = \sum_{i=1}^{n} \bar{\theta}_i u_i \left[c_i\left(y, \bar{\theta}\right)\right] + \beta \sum_{i=1}^{n} \bar{\theta}_i h_i^{-1} \left\{ \int_{z \in \mathcal{X}} \left[\pi\left(y, z\right) h_i \left\{s_{i\bar{\theta}}(z)\right\}\right] \right\} \tag{74}$$

$$= \sum_{i=1}^{n} \bar{\theta}_i s_{i\bar{\theta}}(y). \tag{75}$$

By optimality it must be the case that

$$Q_{k+1}\left(y, \phi\right) \geq \sum_{i=1}^{n} \phi_i s_{i\bar{\theta}}(y), \qquad \forall \phi \in \Delta^n. \tag{76}$$

If equation 76 was false for some y and ϕ then since $s_{i\bar{\theta}}(y)$ describes an allocation which satisfies constraint 55c, there exists an allocation, namely

$s_{i\bar{\theta}}(y)$, which would give the social planner more value than Q_{k+1}. In addition Q_{k+1} is strictly convex and continuous in ϕ by an argument similar to the proof of lemma 6. Hence Q_{k+1} satisfies property ISS.

If the operator T, defined in appendix A, is not a contraction mapping then the second part of this proof does not follow since $Q = \lim_{k\to\infty} Q_k$ may not be the social planner's value function since T may have multiple fixed points.

B.3 Proof of theorem 3

Let $\bar{\theta}$ be an interior Pareto weight vector. Define

$$\Gamma_i\left(x,\bar{\theta}\right) = \int_{z\in\mathcal{X}} \left[\pi_i\left(x,z\right) \mathrm{e}^{\sigma_i s_{i\bar{\theta}}(z)}\right] \tag{77}$$

for all i and x. I will prove the result by showing that the following statements

(a) $\bar{\theta}$ is a steady state.

(b) $\mathcal{M}_{ix}\left(y, s_{i\bar{\theta}}\right) = \mathcal{M}_{jx}\left(y, s_{j\bar{\theta}}\right)$ for all x, y, i and j.

(c) It is the case that

$$\frac{\mathrm{e}^{\sigma_i s_{i\bar{\theta}}(y)}}{\Gamma_i\left(x,\bar{\theta}\right)} = \frac{\mathrm{e}^{\sigma_j s_{j\bar{\theta}}(y)}}{\Gamma_j\left(x,\bar{\theta}\right)} \tag{78}$$

for all x, y, i and j.

(d) There exist real numbers $\{\{G_{ij}\}_{i=1}^n\}_{j=1}^n$ such that

$$\sigma_i s_{i\bar{\theta}}(y) - \sigma_j s_{j\bar{\theta}}(y) = G_{ij} \tag{79}$$

for all y, i and j.

(e) There exist real numbers $\{\{D_{ij}\}_{i=1}^n\}_{j=1}^n$ such that

$$\sigma_i u_i\left[c_i\left(y,\bar{\theta}\right)\right] - \sigma_j u_j\left[c_j\left(y,\bar{\theta}\right)\right] = D_{ij} \tag{80}$$

for all y, i and j.

are equivalent when agents have risk-sensitive preferences with the same beliefs. Theorem 2 guarantees that claim a is equivalent to claim b. When agents are risk-sensitive

$$\mathcal{M}_{ix}\left(y, s_{i\bar{\theta}}\right) = \left[\frac{\pi_i\left(x,y\right)}{\pi\left(x,y\right)}\right] \frac{\mathrm{e}^{\sigma_i s_{i\bar{\theta}}(y)}}{\Gamma_i\left(x,\bar{\theta}\right)}. \tag{81}$$

Since agents have the same beliefs, $\pi_i(x,y) = \pi_j(x,y)$ for all i, j, x, and y. Hence claim b is equivalent to claim c.

I now show that claim c is equivalent to d. First I show that c implies d. Claim c implies that

$$\frac{e^{\sigma_i s_{i\bar{\theta}}(y)}}{e^{\sigma_j s_{j\bar{\theta}}(y)}} = \frac{\Gamma_i\left(x, \bar{\theta}\right)}{\Gamma_j\left(x, \bar{\theta}\right)}. \tag{82}$$

Since $\bar{\theta}$ is interior all of the terms in the numerator and denominator of equation 82 are strictly positive. The left hand side of equation 82 does not depend on x and the right hand side does not depend on y. The only way the equation 82 can be satisfied for all x and y is if both the left and right hand sides are equal to a constant for all y and all x. Taking logarithms of both sides equation 82 shows that claim d holds with the constant

$$G_{ij} = \log\left(\frac{\Gamma_i\left(x, \bar{\theta}\right)}{\Gamma_j\left(x, \bar{\theta}\right)}\right). \tag{83}$$

Second, I show that claim d implies claim c. Claim d implies that

$$e^{\sigma_i s_{i\bar{\theta}}(y)} = e^{G_{ij}} e^{\sigma_j s_{j\bar{\theta}}(y)}, \tag{84}$$

$$\Gamma_i\left(x, \bar{\theta}\right) = e^{G_{ij}} \Gamma_j\left(x, \bar{\theta}\right). \tag{85}$$

Dividing the left and right hand sides of equation 84 by the left and right hand sides of equation 85 shows that claim c holds.

Claim d implies

$$\sigma_i u_i\left[c_i\left(y, \bar{\theta}\right)\right] + \beta \log \Gamma_i\left(y, \bar{\theta}\right) - \sigma_j u_j\left[c_j\left(y, \bar{\theta}\right)\right] - \beta \log \Gamma_j\left(y, \bar{\theta}\right) = G_{ij}, \tag{86}$$

$$\log \Gamma_i\left(y, \bar{\theta}\right) - \log \Gamma_j\left(y, \bar{\theta}\right) = G_{ij}. \tag{87}$$

Equation 86 follows from equation 79 since

$$s_{i\bar{\theta}}(y) = u_i\left[c_i\left(y, \bar{\theta}\right)\right] + \frac{\beta}{\sigma_i} \log \Gamma_i\left(y, \bar{\theta}\right). \tag{88}$$

Equation 87 follows from the definition of Γ_i and since $\sigma_i s_{i\bar{\theta}}(z) - \sigma_j s_{j\bar{\theta}}(z) = G_{ij}$ for all z. Combining equations 86 and 87 yields

$$\sigma_i u_i\left[c_i\left(y, \bar{\theta}\right)\right] - \sigma_j u_j\left[c_j\left(y, \bar{\theta}\right)\right] = (1-\beta)\, G_{ij}. \tag{89}$$

Thus claim e holds with $D_{ij} = (1-\beta)\, G_{ij}$. This argument can be reversed to show that claim e implies claim d.

B.4 Justification of example 5

First consider the case in which $\gamma_1 = \gamma_2$. Let $\gamma \equiv \gamma_1 = \gamma_2$. The current period Pareto optimal consumption allocations are

$$c_i(y,\phi) = \left[\frac{\phi_i^{\frac{1}{\gamma}}}{\phi_1^{\frac{1}{\gamma}} + \phi_2^{\frac{1}{\gamma}}}\right] y$$

for $i = 1, 2$. Hence

$$D_{21}(y,\phi) = \left[\frac{\sigma_2 \phi_2^{\frac{1-\gamma}{\gamma}} - \sigma_1 \phi_1^{\frac{1-\gamma}{\gamma}}}{(1-\gamma)\left(\phi_1^{\frac{1}{\gamma}} + \phi_2^{\frac{1}{\gamma}}\right)^{1-\gamma}}\right] y^{1-\gamma}.$$

$D_{21}(y,\phi)$ will equal a constant for all $y \in \mathcal{X}$ if and only if

$$\sigma_2 \phi_2^{\frac{1-\gamma}{\gamma}} - \sigma_1 \phi_1^{\frac{1-\gamma}{\gamma}}$$

is equal to zero. The last equation will equal a zero if and only if $\phi = \bar{\theta}$ (where $\bar{\theta}$ is as defined in the statement of this example). Hence, by theorem 3, $\bar{\theta}$ is the unique interior steady state Pareto weight.

Let $\gamma_1 \neq \gamma_2$. Assume $\sigma_1 < 0$ and $\sigma_2 < 0$. If they are not and $\sigma_1 \leq 0$ and $\sigma_2 \leq 0$ then the results of previous examples apply to show that there are no interior steady states. Let $c_1(y,\phi)$ and $c_2(y,\phi) = y - c_1(y,\phi)$ be the consumption allocation rules for the agents. At a steady state there must exist a ϕ which solves the equations

$$\sigma_1 \frac{c_1(y,\phi)^{1-\gamma_1}}{1-\gamma_1} - \sigma_2 \frac{c_2(y,\phi)^{1-\gamma_2}}{1-\gamma_2} = d, \tag{90}$$

$$\phi_1 c_1(y,\phi)^{-\gamma_1} = \phi_2 c_2(y,\phi)^{-\gamma_2} \tag{91}$$

for some constant d. Equation (90) must hold by theorem 3 and equation (91) must hold by the first-order condition (32a).

Solving equation (91) for $c_1(y,\phi)$ and substituting the result into equation (90) yields

$$\sigma_1 \left(\frac{\phi_2}{\phi_1}\right)^{-\frac{1-\gamma_1}{\gamma_1}} \frac{c_2(y,\phi)^{\frac{\gamma_2}{\gamma_1}(1-\gamma_1)}}{1-\gamma_1} - \sigma_2 \frac{c_2(y,\phi)^{1-\gamma_2}}{1-\gamma_2} = d. \tag{92}$$

The left side of equation (92) can equal a constant for all $y \in \mathcal{X}$ if and only if one of the following conditions is met

(a) $\gamma_1 = \gamma_2$.

(b) $c_2(y, \phi) = 0$ for all $y \in \mathcal{X}$.

(c) $c_2(y, \phi) = 1$ for all $y \in \mathcal{X}$.

To show that these are the only possibilities differentiate both sides of equation 92 with respect to $c_2(y, \phi)$ to obtain

$$k_1 c_2(y, \phi)^{\frac{\gamma_2}{\gamma_1} - \gamma_2 - 1} - k_2 c_2(y, \phi)^{-\gamma_2} = 0. \tag{93}$$

where k_1 and k_2 are non-zero constants. Unless one of conditions a thru c is met there is at most one value of $c_2(y, \phi)$ at which equation 93 is satisfied. I now show that these cases do not hold. By assumption $\gamma_1 \neq \gamma_2$ so case a does not hold. If ϕ is an interior Pareto weight vector then b and c can not hold since by optimality, under the assumptions on u_i given in definition 3, $c_2(y, \phi)$ is increasing in y.

B.5 Proof of theorem 4

For simplicity I only insider the case when reward functions are bounded below by zero. Let $\bar{\theta}$ be an interior Pareto weight vector. Define

$$\Gamma_i\left(x, \bar{\theta}\right) = \left(\int_{z \in \mathcal{X}} \left[\pi_i(x, z)\, s_{i\bar{\theta}}(z)^{\omega}\right]\right)^{1 - \frac{1}{\omega}} \tag{94}$$

for all i and x. I will prove the results by showing that the following statements are all equivalent when agents have Epstein-Zin aggregators with the same beliefs.

(a) $\bar{\theta}$ is a steady state.

(b) $\mathcal{M}_{1x}\left(y, s_{1\bar{\theta}}\right) = \mathcal{M}_{2x}\left(y, s_{2\bar{\theta}}\right)$ for all x and y.

(c) It is the case that

$$\frac{s_{1\bar{\theta}}(y)^{\omega - 1}}{\Gamma_1\left(x, \bar{\theta}\right)} = \frac{s_{2\bar{\theta}}(y)^{\omega - 1}}{\Gamma_2\left(x, \bar{\theta}\right)} \tag{95}$$

for all x and y.

(d) There exists a real number M such that

$$s_{1\bar{\theta}}(y) = M s_{2\bar{\theta}}(y) \tag{96}$$

for all y.

(e) There exists a real number g such that

$$u_1 \left[c_1 \left(y, \bar{\theta}\right)\right] = g u_2 \left[c_2 \left(y, \bar{\theta}\right)\right] \tag{97}$$

for all y.

The fact that claim a implies claim b follows from arguments given in the proof of theorem 2. Since the Epstein-Zin aggregators do not satisfy assumption 3 more arguments are needed to show that claim b implies a. I now sketch these arguments. When agents have Epstein-Zin aggregators for any $k > 0$ it is the case that

$$h_i^{-1} \int_{z \in \mathcal{X}} \left[\pi_i \left(y, z\right) h_i(kz)\right] = k h_i^{-1} \int_{z \in \mathcal{X}} \left[\pi_i \left(y, z\right) h_i(z)\right]. \tag{98}$$

Using this it can be shown that the operator T defined in appendix A is a contraction mapping. Hence the proof of theorem 2 can be adapted to this example to show that claim b implies a.

Claim b is equivalent to claim c since when agents have Epstein-Zin aggregators with the same powers since

$$\mathcal{M}_{ix} \left(y, s_{i\bar{\theta}}\right) = \left[\frac{\pi_i \left(x, y\right)}{\pi \left(x, y\right)}\right] \frac{s_{i\bar{\theta}}(y)^{\omega - 1}}{\Gamma_i \left(x, \bar{\theta}\right)}. \tag{99}$$

Here agents have the same beliefs so that $\pi_i \left(x, y\right) = \pi_j \left(x, y\right)$.

Claim c implies claim d since equation 95 implies

$$\left(\frac{s_{1\bar{\theta}}(y)}{s_{2\bar{\theta}}(y)}\right)^{\omega - 1} = \frac{\Gamma_1 \left(x, \bar{\theta}\right)}{\Gamma_2 \left(x, \bar{\theta}\right)}. \tag{100}$$

The left hand side of equation 100 does not depend on x and the right hand side does not depend on y. The only way the equation 100 can be satisfied for all x and y is if both the left and right hand sides are equal to a constant for all y and all x. Thus

$$s_{1\bar{\theta}}(y) = M s_{2\bar{\theta}}(y) \tag{101}$$

where

$$M = \frac{\Gamma_1 \left(x, \bar{\theta}\right)}{\Gamma_2 \left(x, \bar{\theta}\right)}. \tag{102}$$

This argument can be reversed to show that claim d implies claim c.

To show that claim d is equivalent to e write out

$$s_{i\bar{\theta}}(y) = u_i\left[c_i\left(y,\bar{\theta}\right)\right] + \beta\left(\int_{z\in\mathcal{X}}\left[\pi\left(y,z\right)s_{i\bar{\theta}}(z)^{\omega}\right]\right)^{\frac{1}{\omega}}. \tag{103}$$

This follows from the definition of $s_{i\bar{\theta}}(y)$. Claim d implies

$$\begin{aligned} &u_1\left[c_1\left(y,\bar{\theta}\right)\right] - Mu_2\left[c_2\left(y,\bar{\theta}\right)\right] = \\ &\quad M\beta\left(\int_{z\in\mathcal{X}}\left[\pi\left(y,z\right)s_{2\bar{\theta}}(z)^{\omega}\right]\right)^{\frac{1}{\omega}} - \beta\left(\int_{z\in\mathcal{X}}\left[\pi\left(y,z\right)s_{1\bar{\theta}}(z)^{\omega}\right]\right)^{\frac{1}{\omega}}. \end{aligned} \tag{104}$$

The right hand side of this equation must be zero since $s_{1\bar{\theta}}(z) = Ms_{2\bar{\theta}}(z)$ for all z. Hence claim d follows with $g = M$. This argument can be reversed to show that claim e implies claim d.

B.6 Proof of theorem 5

Since by assumption $h_i'(z)h_i''(z) < 0$ it must be the case that either

(i) $h_i'(z) < 0$ and $h_i''(z) > 0$ or

(ii) $h_i'(z) > 0$ and $h_i''(z) < 0$

for all z in the interior of $\mathcal{Q}_i$. If either condition (i) or (ii) does not hold then the implied stochastic aggregator would not satisfy the requirements in definition 6. In particular $\mathcal{R}_{ix}q_i$ would not be increasing for all finite utility allocation functions q_i.

Let λ_a and λ_b be two different Pareto ratios such that $\lambda_b > \lambda_a$. (Here λ_a and λ_b are ratios of the Pareto weights for agents of type two to the Pareto weights of type one.) Let α_a and α_b be the optimal choice of next period's Pareto ratios for a particular value of next period's endowment, y, when this period's Pareto ratios are λ_a and λ_b respectively. To prove theorem 5 it will suffice to show that $\alpha_b > \alpha_a$.

Let this period's endowment be x. First-order conditions when the this period's Pareto ratios are λ_a and λ_b are respectively

$$\alpha_a = \lambda_a \frac{h_2'\left(q_{2y}^a\right)}{h_1'\left(q_{1y}^a\right)}\frac{h_1'\left(\mathcal{R}_{1x}q_1^a\right)}{h_2'\left(\mathcal{R}_{2x}q_2^a\right)}, \tag{105a}$$

$$\alpha_b = \lambda_b \frac{h_2'\left(q_{2y}^b\right)}{h_1'\left(q_{1y}^b\right)}\frac{h_1'\left(\mathcal{R}_{1x}q_1^b\right)}{h_2'\left(\mathcal{R}_{2x}q_2^b\right)}, \tag{105b}$$

where q_1^c and q_2^c for $c = a, b$ are the optimal utility allocation functions. If $\lambda_b > \lambda_a$ then $\mathcal{R}_{2x} q_2^b > \mathcal{R}_{2x} q_2^a$ and $\mathcal{R}_{1x} q_1^b < \mathcal{R}_{1x} q_1^a$. Under either case (i) or (ii) listed above it follows that

$$\left| \frac{h_1'\left(\mathcal{R}_{1x} q_1^b\right)}{h_2'\left(\mathcal{R}_{2x} q_2^b\right)} \right| > \left| \frac{h_1'\left(\mathcal{R}_{1x} q_1^a\right)}{h_2'\left(\mathcal{R}_{2x} q_2^a\right)} \right|$$

since in both cases $|h_i'(q_i)|$ is decreasing in q_i. Assume that $\alpha_b \leq \alpha_a$. Then it follows by optimality that $q_{2y}^b \leq q_{2y}^a$ and $q_{1y}^b \geq q_{1y}^a$ for all y. Hence

$$\left| \frac{h_2'\left(q_{2y}^b\right)}{h_1'\left(q_{1y}^b\right)} \right| \geq \left| \frac{h_2'\left(q_{2y}^a\right)}{h_1'\left(q_{1y}^a\right)} \right| .$$

But this along with equations (105a) and (105b) implies that

$$\lambda_b \frac{h_2'\left(q_{2y}^b\right)}{h_1'\left(q_{1y}^b\right)} \frac{h_1'\left(\mathcal{R}_{1x} q_1^b\right)}{h_2'\left(\mathcal{R}_{2x} q_2^b\right)} > \lambda_a \frac{h_2'\left(q_{2y}^a\right)}{h_1'\left(q_{1y}^a\right)} \frac{h_1'\left(\mathcal{R}_{1x} q_1^a\right)}{h_2'\left(\mathcal{R}_{2x} q_2^a\right)}$$

which in turn implies that $\alpha_b > \alpha_a$ by equations 105a and 105b. Contradiction. (I have shown that if it is assumed that $\alpha_b \leq \alpha_a$ then it follows that $\alpha_b > \alpha_a$.) Conclude that $\alpha_b > \alpha_a$.

B.7 Proof of theorem 6

I will prove part (2). The proof of part (1) is analogous. Define

$$\begin{aligned} S(y, \phi) &\equiv \sigma_2 V_2(y, \phi) - \sigma_1 V_1(y, \phi), \\ &= D_{21}(y, \phi) + \beta \log \Omega_2(y, \phi) - \beta \log \Omega_1(y, \phi) \end{aligned}$$

where

$$\Omega_i(y, \phi) = \int_{z \in \mathcal{X}} \left[\pi(y, z) \, \mathrm{e}^{\sigma_i V_i(z, \phi)} \right] . \tag{106}$$

$S(y, \phi)$ is a weighted difference of value allocations. If for a fixed ϕ, $D_{21}(y, \phi)$ is a strictly decreasing function of y then

(i) For a fixed ϕ, $S(y, \phi)$ is strictly decreasing in y. Since the endowment is i.i.d., the term $\Omega_i(y, \phi)$ does not vary with y. So for a given ϕ, $S(y, \phi)$ is equal to $D_{21}(y, \phi)$ plus a constant. $S(y, \phi)$ is strictly decreasing in y since $D_{21}(y, \phi)$ is strictly decreasing in y.

(ii) For a fixed y, $S(y, \phi)$ is strictly decreasing in ϕ_2. This follows since $V_2(y, \phi)$ is increasing in ϕ_2 and $V_1(y, \phi)$ is decreasing in ϕ_2. (Since there are two types of agents as ϕ_2 increases it must be the case that $\phi_1 = 1 - \phi_2$ decreases.) This follows from optimality.

Consider two possible realizations of the endowment, a and b such that $b > a$. Let ϕ_a and ϕ_b be the optimal choices of the next period's Pareto weights when next period's endowments are a and b respectively. Define

$$\lambda_a \equiv \frac{\phi_{2a}}{\phi_{1a}} \qquad \lambda_b \equiv \frac{\phi_{2b}}{\phi_{1b}}$$

to be the ratio of the optimal choices of Pareto weights. The first-order conditions include equations of the form

$$\lambda_c = h\mathrm{e}^{S(c, \phi_c)}$$

for $c = a$ and for $c = b$. h is a constant which is the same if $c = a$ or $c = b$.

Dividing the left and right sides of the condition for b by the left and right sides of the condition for a yields the necessary condition:

$$\frac{\lambda_b}{\lambda_a} = \mathrm{e}^{S(b, \phi_b) - S(a, \phi_a)}. \tag{107}$$

By property (i) and the fact $b > a$,

$$1 > \mathrm{e}^{S(b, \phi_a) - S(a, \phi_a)}. \tag{108}$$

Hence it could not be optimal to set $\lambda_b = \lambda_a$. (If it were optimal to set $\lambda_b = \lambda_a$ then it follows from equation (107) that equation (108) would have to be satisfied with equality.) If $\lambda_b > \lambda_a$ then equation (107) could not be satisfied since the left side is greater than one and the right side is less than one since by property (ii) above

$$1 > \mathrm{e}^{S(b, \phi_a) - S(a, \phi_a)} > \mathrm{e}^{S(b, \phi_b) - S(a, \phi_a)}.$$

(This follows since $\lambda_b > \lambda_a$ implies $\phi_{2b} > \phi_{2a}$ which implies $S(b, \phi_b) < S(b, \phi_a)$.) Contradiction. Conclude $\lambda_b < \lambda_a$. I have shown that if $b > a$ then $\lambda_b < \lambda_a$.

B.8 Proof of theorem 7

I first prove that conditions on $Q_i(d)$ can be given which guarantee that the optimal choices of next period's Pareto weights are monotonic in the aggregate endowment.

Recall the weighted utility difference term $D_{21}(y,\phi)$ defined earlier. Differentiate $D_{21}(y,\phi)$ with respect to y to obtain[9]

$$\frac{dD_{21}(y,\phi)}{dy} = \sigma_2 u_2'[c_2(y,\phi)]\frac{\partial c_2(y,\phi)}{\partial y} - \sigma_1 u_1'[c_1(y,\phi)]\frac{\partial c_1(y,\phi)}{\partial y} \tag{111a}$$

$$= \psi(y,\phi)[\sigma_2 P_2(c_2) - \sigma_1 P_1(c_1)], \tag{111b}$$

where

$$\Psi(y,\phi) \equiv \frac{u_1''[c_1(y,\phi)]\,u_2''[c_2(y,\phi)]}{u_1''[c_1(y,\phi)]\,u_2'[c_2(y,\phi)] + u_2''[c_2(y,\phi)]\,u_1'[c_1(y,\phi)]},$$

$$P_i(d) \equiv \frac{[u_i'(d)]^2}{u_i''(d)}.$$

Strictly speaking this derivative is nonsense since I defined the endowments to be members of the discrete set $\mathcal{X}$. However since $D_{21}(y,\phi)$ is differentiable (when it is assumed for purposes of this argument that the the endowment can be any positive real number) we can use its derivative to tell when it is (increasing, decreasing, constant) in y and when the conditions of theorem 6 are met.

At any interior steady state $\frac{dD_{21}(y,\phi)}{dy} = 0$. If $\frac{dD_{21}(y,\phi)}{dy}$ can be signed away from the steady state then theorem 6 can be applied to show that the Pareto weights are monotonic in the endowment. Note that $\psi(y,\phi)$ is always negative. By assumption σ_i is also negative. The sign of $\frac{dD_{21}(y,\phi)}{dy}$ depends on the magnitude of the terms $P_i(c_i)$ defined earlier. If $P_2(c_2)$ $(>,<,=)$ $P_1(c_1)$ then $\frac{dD_{21}(y,\phi)}{dy}$ is (positive, negative, zero).

Differentiating P_i with respect to d yields

$$\frac{dP(d)}{dd} = u'(d)\left(2 - \frac{u'(d)u'''(d)}{[u''(d)]^2}\right) \tag{112}$$

$$= u'(d)[2 - Q_i(d)] \tag{113}$$

[9]Equation 111b is obtained by writing the first order condition 32a as

$$(1-\phi)\,u_1'[c_1(y,\phi)] = \phi u_2'[c_2(y,\phi)] \tag{109}$$

when the endowment is y and the Pareto weight is ϕ. Differentiating equation 109 with respect to y yields

$$(1-\phi)\,u_1''[c_1(y,\phi)]\frac{\partial c_1(y,\phi)}{\partial y} = \phi u_2''[c_2(y,\phi)]\frac{\partial c_2(y,\phi)}{\partial y} \tag{110}$$

Solving 109 for $u_1'[c_1(y,\phi)]$ and 110 for $\frac{\partial c_1(y,\phi)}{\partial y}$ and substituting the resulting expressions into 111a yields 111b.

where $Q_i(d)$ was defined in equation 24. Let $\bar{\theta}$ be a interior steady state. If $Q_i(d)$ $(>,<,=)$ 2 then

1. $D_{21}(y,\phi)$ is a (decreasing, increasing, constant) function of y when $\phi_2 > \bar{\theta}$ and a (increasing, decreasing, constant) function of y when $\phi_2 < \bar{\theta}$.

2. When $\phi_2 > \bar{\theta}_2$ the choice of next period's Pareto weight for agents of type two is a strictly (decreasing, increasing, constant) function of next period's endowment y.

3. When $\phi_2 < \bar{\theta}_2$ the choice of next period's Pareto weight for agents of type two is a strictly (increasing,decreasing, constant) function of next period's endowment y.

These claims follow by applying theorem 6 when case 1 holds.

I now sketch a proof that shows that conditions on $Q_i(d)$ can be given which guarantee that the relative entropy is decreasing, increasing or constant for some agents. In this proof I write relative entropy as

$$R_i(x,\theta) = -\sigma_i \left[\int_{y\in\mathcal{X}} \left[\pi\left(x,y\right) u_i\left[c_i\left[y,\phi\left(x,\theta,y\right)\right]\right]\right] - \frac{1}{\sigma_i}\log\Gamma_i\left(x,\theta\right) \right] \quad (114)$$

where

$$\Gamma_i\left(x,\theta\right) = \int_{z\in\mathcal{X}} \left[\pi\left(x,z\right) \mathrm{e}^{\sigma_i V_i(z,\phi(x,\theta,z))}\right]. \quad (115)$$

I will consider the case in which $Q_i(d) > 2$. The proofs for other cases are analogous. I break the sketch into a series of steps. Let $\bar{\theta}$ be an interior steady state.

1. $Q_i(d) > 2$.

2. $S(y,\phi)$ (defined in the proof of theorem 6) is decreasing in y when $\phi_2 > \bar{\theta}$ and increasing when $\phi_2 < \bar{\theta}$.

3. Let $\phi_2 > \bar{\theta}$. As ϕ_2 increases $S(y,\phi)$ decreases faster in y. Let $\phi_2 < \bar{\theta}$. As ϕ_2 decreases $S(y,\phi)$ increases faster in y.

4. Relative entropy $R_i(x,\theta)$ is an increasing function of one's own Pareto weight.

B.9 Proof of lemma 1

Since $E_t \lambda_{t+1} \leq \lambda_t$ and $\lambda_t \geq 0$, the martingale convergence theorem guarantees that $\{\lambda_t\}_{t=0}^{\infty}$ converges with probability one to a finite random variable, λ^*.

Let $\lambda_t(\omega)$ denote the sample path for the Pareto ratios for a given realization of the aggregate endowments for all period's $t = 0, 1, \ldots$ (here ω indexes realizations of the aggregate endowment). Let Ω be the set of all sample paths. By Egoroff's theorem, see Davidson (1994) page 283, for every $\delta > 0$ there exists a set $F(\delta) \subseteq \Omega$ with $\text{prob}\,[F(\delta)] \geq 1 - \delta$ such that for $\omega \in F(\delta)$

$$\lim_{t \to \infty} \lambda_t(\omega) = \lambda^*(\omega)$$

uniformly on $F(\delta)$. For every $\delta > 0$ and every $\epsilon > 0$ there exists a $T(\delta, \epsilon)$ such that for $t > T(\delta, \epsilon)$

$$|\lambda_t(\omega) - \lambda^*(\omega)| < \epsilon \tag{116}$$

for all $\omega \in F(\delta)$. For given δ and ϵ it follows that

$$|\lambda_{t+1}(\omega) - \lambda_t(\omega)| < 2\epsilon \tag{117}$$

for $t > T(\delta, \epsilon)$, for all $\omega \in F(\delta)$.

Choose some $z > 0$ and some $\delta > 0$. Let d be the number defined in hypothesis one of this lemma corresponding to value of $\underline{\lambda}$ given by z and the value of $\overline{\lambda}$ given by some finite upper bound for λ^*. Choose $\epsilon > 0$ such that $\epsilon < d/2$ and $\epsilon < z$. Define

$$S_t = \left\{\omega \in F(\delta) : \overline{\lambda} > \lambda_t(\omega) > z\right\}, \tag{118a}$$

$$N_t = \left\{\omega \in \Omega : \overline{\lambda} > \lambda_t(\omega) > z\right\}, \tag{118b}$$

$$\Delta_t = \{\omega \in \Omega : |\lambda_{t+1} - \lambda_t| < 2\epsilon\}. \tag{118c}$$

Let $\text{prob}\,[S_t] = p_t$. By equation (117), we know that for $t > T(\delta, \epsilon)$

$$\text{prob}\,[N_t \text{ and } \Delta_t] \geq p_t. \tag{119}$$

But for $t > T(\delta, \epsilon)$ we also know

$$\text{prob}\,[N_t \text{ and } \Delta_t] \leq (p_t + \delta)(1 - \underline{\pi}). \tag{120}$$

This follows since $\text{prob}\,[N_t] \leq p_t + \delta$ and conditioned on $\lambda_t > z$ the probability that $|\lambda_{t+1} - \lambda_t| < 2\epsilon$ is less than or equal to $1 - \underline{\pi}$ (recall that $\underline{\pi}$ was defined in the statement of this lemma). Equations (119) and (120) imply that

$$p_t \leq \delta \left(\frac{1-\underline{\pi}}{\underline{\pi}} \right). \tag{121}$$

The arguments of this paragraph can be repeated for any $\delta > 0$. Hence equation (121) holds for all $\delta > 0$. Egoroff's theorem can again be invoked to show that $\{\lambda_t\}$ converges with probability one to a random variable which is less than or equal to z. Since the arguments of this paragraph can be repeated for all $z > 0$ it must be that $\{\lambda_t\}$ converges with probability one to zero.

B.10 Proof of lemma 2

The first-order conditions for the social planner's problem can be written as

$$\theta_{it+1}\theta_{jt}\mathcal{D}_j\left(x_t, \theta_t, x_{t+1}\right) = \theta_{it}\theta_{jt+1}\mathcal{D}_i\left(x_t, \theta_t, x_{t+1}\right). \tag{122}$$

Equation (45) can be obtained by dividing both sides of equation (122) by $\theta_{jt}\theta_{jt+1}\mathcal{D}_j\left(x_t, \theta_t, x_{t+1}\right)$ and taking expected values conditioned on t of both sides of the resulting equation.

B.11 Proof of lemma 3

The first-order conditions for the social planner's problem when agents are risk-sensitive with the same (correct) beliefs can be written as

$$\theta_{it+1}\theta_{jt}\mathcal{D}_j\left(x_t, \theta_t, x_{t+1}\right) = \theta_{it}\theta_{jt+1}\mathcal{D}_i\left(x_t, \theta_t, x_{t+1}\right) \tag{123}$$

where

$$\mathcal{D}_i\left(x, \theta, y\right) = \frac{\mathrm{e}^{\sigma_i V_i[y, \phi(x,\theta,y)]}}{\int_{z \in \mathcal{X}} \left[\pi\left(x, z\right) \mathrm{e}^{\sigma_i V_i(z, \phi(x,\theta,z))}\right]}.$$

Recall that $V_i\left(y, \phi\right)$ was defined in equation (31). Equation (46) can be obtained by dividing both sides of equation (123) by $\theta_{jt}\theta_{it}$ and taking expected values conditioned on t of both sides of the resulting equation. Now use the identity that for any random variables y and z it is the case that $E\left(yz\right) = \text{cov}\left(y, z\right) + E\left(y\right) E\left(z\right)$.

To obtain equation 47 take logarithms of both sides of equation 123 and then take expected values conditioned on t of both sides of the resulting equation. Finally, substitute the formula for relative entropy given in definition 7.

B.12 Proof of theorem 9

First, I will prove part (1) when $1 > \theta_2 > \bar{\theta}_2$. Assume $D_{21}(y, \phi)$ is a strictly decreasing function of y. This assumption coupled with the assumption that there is an interior steady state guarantees that $\sigma_1 < 0$ and $\sigma_2 < 0$. The proof uses the following three facts:

(i) The optimal choice of next period's Pareto ratio ϕ_{2y}/ϕ_{1y} a is strictly decreasing function of y. (This follows from theorem 6 and the assumption that $D_{21}(y, \phi)$ is a strictly decreasing function of y.)

(ii) For a fixed ϕ, $V_1(y, \phi)$ is a strictly increasing function of y. Since $\sigma_1 < 0$ it follows that $e^{\sigma_1 V_1(y,\phi)}$ is a strictly decreasing function of y.

(iii) For a fixed y, $V_1(y, \phi)$ is a strictly decreasing function of ϕ_2. Since $\sigma_2 < 0$ it follows that $e^{\sigma_1 V_1(y,\phi)}$ is a strictly increasing function of ϕ_2.

It follows from (i) thru (iii) that $e^{V_1(y,\phi_y)}$ is a strictly decreasing of y. (Here $V_1(y, \phi_y)$ is evaluated at the optimal choice of Pareto weights ϕ_y when next period's endowment is y.) Since part (i) showed that ϕ_{2y}/ϕ_{1y} is also a strictly decreasing function of y it must be the case that

$$\frac{\mathrm{cov}_t\left(e^{\sigma_1 V_1(x_{t+1},\theta_{t+1})}, \frac{\theta_{2t+1}}{\theta_{1t+1}}\right)}{E_t\left[e^{\sigma_1 V_1(x_{t+1},\theta_{t+1})}\right]} > 0$$

for all x_t and all finite $\frac{\theta_{2t}}{\theta_{1t}} > \bar{\theta}_2/\bar{\theta}_1$. Let k be[10]

$$k = \min_{\underline{\lambda} \leq \frac{\theta_{2t}}{\theta_{1t}} \leq \overline{\lambda}} \left[\frac{\mathrm{cov}_t\left(e^{\sigma_1 V_1(x_{t+1},\theta_{t+1})}, \frac{\theta_{2t+1}}{\theta_{1t+1}}\right)}{E_t\left[e^{\sigma_1 V_1(x_{t+1},\theta_{t+1})}\right]}\right] \tag{124}$$

for some $\underline{\lambda}$ and $\overline{\lambda}$ such that $+\infty > \overline{\lambda} > \underline{\lambda} > \bar{\theta}_2/\bar{\theta}_1$. The minimum is obtained since the function minimized is continuous and the constraint set is compact. Evidently $k > 0$. Now lemma 3 can be used to guarantee that

$$E_t\left[\frac{\theta_{2t+1}}{\theta_{1t+1}}\right] \leq \frac{\theta_{2t}}{\theta_{1t}} - k \tag{125}$$

when $\underline{\lambda} \leq \frac{\theta_{2t}}{\theta_{1t}} \leq \overline{\lambda}$. At time t there must be some positive probability that the time $t+1$ Pareto ratios $\theta_{2t+1}/\theta_{1t+1}$ are less than or equal to $(\theta_{2t}/\theta_{1t} - k)$

[10] In equation (124), $\mathrm{cov}_t(w, z)$ means the time t covariance of w and z conditioned on the value of θ_{2t}/θ_{1t}.

when $\underline{\lambda} \leq \frac{\theta_{2t}}{\theta_{1t}} \leq \overline{\lambda}$. (If there was not some positive probability then equation (125) would be false.) Let

$$\underline{\pi} = \min_x \pi(x, x).$$

All of the m endowments at a given date must occur with probability greater than or equal to $\underline{\pi}$. Hence the probability that the time $t+1$ Pareto ratios satisfy $\theta_{2t+1}/\theta_{1t+1} \leq \theta_{2t}/\theta_{1t} - k$ must be greater than or equal to $\underline{\pi}$ when $\underline{\lambda} \leq \frac{\theta_{2t}}{\theta_{1t}} \leq \overline{\lambda}$.

Note that by corollary 1 the Pareto ratios $\{\theta_{2t}/\theta_{1t}\}$ are bounded below by $\bar{\theta}_2/\bar{\theta}_1$. All of the hypotheses of lemma 1 are met to guarantee that $\{\theta_{2t}/\theta_{1t}\}$ converges with probability one to $\bar{\theta}_2/\bar{\theta}_1$. (Lemma 1 can be used to show that $\left\{\theta_{2t}/\theta_{1t} - \bar{\theta}_2/\bar{\theta}_1\right\}$ converges with probability one to zero, which implies that $\{\theta_{2t}/\theta_{1t}\}$ converges with probability one to $\left\{\bar{\theta}_2/\bar{\theta}_1\right\}$.)

The proof of part (1) when $0 < \theta_2 < \bar{\theta}_2$ is analogous and can be formulated by switching the roles of the agents in the argument above.

I now sketch the proof of part (2) when $0 < \theta_2 < \bar{\theta}_2$. In this case using arguments similar to the proof above for case (1) it can be shown that

$$\frac{\operatorname{cov}_t\left(e^{\sigma_1 V_1(x_{t+1}, \theta_{t+1})}, \frac{\theta_{2t+1}}{\theta_{1t+1}}\right)}{E_t\left[e^{\sigma_1 V_1(x_{t+1}, \theta_{t+1})}\right]} > 0$$

and

$$E_t\left[\frac{\theta_{2t+1}}{\theta_{1t+1}}\right] < \frac{\theta_{2t}}{\theta_{1t}}. \tag{126}$$

Hypothesis one of lemma 1 is not met in this case since the constant d can not be chosen independent of $\underline{\lambda}$ and $\overline{\lambda}$. However, $\{\theta_{2t}/\theta_{1t}\}$ is bounded below by zero and bounded above by $\bar{\theta}_2/\bar{\theta}_1$ (this was shown in corollary 1). The martingale convergence theorem guarantees that the Pareto ratios converge to a finite random variable λ^*. It is straightforward to show that λ^* can take on at most two values, zero and $\bar{\theta}_2/\bar{\theta}_1$. It must be the case

$$\operatorname{prob}_t\left[\lambda^* = 0\right] \geq 1 - \frac{\theta_{2t}/\theta_{1t}}{\bar{\theta}_2/\bar{\theta}_1} > 0.$$

(This follows from equation (126).) This provides a bound for the probability that the Pareto ratios converge to zero. Since the Pareto ratios could converge to $\bar{\theta}_2/\bar{\theta}_1$ with probability of at most $\frac{\theta_{2t}/\theta_{1t}}{\bar{\theta}_2/\bar{\theta}_1}$, the steady state Pareto weight vector $\bar{\theta}$ is unstable.

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