Recursive Preferences and Balanced Growth[1]

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Abstract

We study a class of utility functions that are defined recursively by an aggregator $W(x, y)$ where $u_t = W(c_t, u_{t+1})$. In single-agent economies it is known that a sufficient condition for the existence of a balanced growth path is that utility should be homogenous of degree γ. In the context of a multi-agent economy we show that this restriction implies that *either* a balanced growth equilibrium fails to exist *or* all agents have the same constant discount factor. We suggest a generalization of recursive preferences wherein the intertemporal utility function is time dependent. Within this class we establish that there may exist a balanced growth equilibrium even if agents are different. We give an example of our approach in the international context in which time dependence occurs because countries care about their relative position in the world income distribution.

1 Introduction

Two central features of most modern macroeconomic models are (a) the assumption of infinitely lived families with intertemporally separable utility functions; and (b) an environment that allows for balanced growth. While the first stems primarily from a desire for simplicity, the balanced growth construct originates in the seminal work of Kaldor [4] who stressed that balanced growth provides a good characterization of the long run development experience of the currently industrialized countries. According to the Kaldor growth facts, development paths are characterized by the constancy of growth rates, factor income shares and capital-output ratios. From a modelling standpoint, balanced growth is an attractive feature since it makes otherwise complicated environments simple to analyze.

It is well known that the assumption of time separability of preferences is restrictive since all agents must have the same rate of time preference if a model in this class is to display a non-degenerate wealth distribution. If rates of time preference are different across agents then the most patient family will asymptotically own all of world wealth. A large literature starting with Koopmans [6], Uzawa [10] and more recently Epstein and Hynes [3], and Lucas and Stokey [7] has generalized the choice over intertemporal consumption sequences to the case of recursive preferences in which the rate of time preference is a function of the agent's consumption sequence. Lucas-Stokey [7] and Epstein-Hynes [3] have used these preferences to construct examples of economies for which there exists a non-degenerate asymptotic wealth distribution.

Lucas and Stokey [7] studied an environment in which agents have recursive preferences defined over bounded consumption sequences. Within this environment they proved the existence of a non-degenerate income distribution; but their result does not apply to economies with growth. Boyd [1] generalized recursive utility to a class of preferences defined over unbounded consumption sequences and his work *does* permit the study of environments that permit growth. Using Boyd's results, Dolmas [2] proved that there exists a balanced growth path in the representative agent growth model if there is a recursive representation of preferences, $u_t = W(c_t, u_{t+1})$ for which the aggregator $W(x, y)$ is homogeneous of the form $W(\lambda x, \lambda^{\gamma} y) = \lambda^{\gamma} W(x, y)$.

In this paper we study the conditions under which recursive preferences, balanced growth, and heterogenous preferences (i.e., multiple agents) can coexist. When preferences are described by homothetic utility functions, (a necessary condition for the existence of balanced growth) we show that *either* the asymptotic wealth distribution in a multi-agent economy is degenerate *or* all agents have the same constant discount rate. This result is troubling given that a key motivation for introducing recursive preferences is to permit heterogeneity without inducing a degenerate wealth distribution.

To recover the property of non-degeneracy of the wealth distribution we introduce an exogenous time-dependence into the utility function and show that this formulation permits the coexistence of recursive preferences, preference heterogeneity, and balanced growth. We suggest an application of time dependence in which agents care about their relative position in the world wealth distribution; in this context world wealth enters utility as an external effect. Using this example, we show that our formulation is consistent with a determinate distribution of world wealth in the context of both exogenous and endogenous growth models.

The paper is structured as follows. Sections 2 and 3 describe basic assumptions on aggregator functions and technology sets. These sections are relatively technical and could be skipped by a reader who has some familiarity with the idea of recursive preferences and who is interested in the main economic ideas. Section 4 provides necessary and sufficient conditions for the consistency of general non-separable utility functions with balanced growth paths. This section also contains the intuition behind the main theorem of the paper, proved in Section 5, that balanced growth requires all agents to have the same constant rate of time preference. Sections 6 and 7 provide a generalization of recursive preferences in which we suggest a non autonomous formulation of the utility function and show that this formulation is consistent with balanced growth. Finally, in Section 8 we provide two examples of our approach in which time enters the utility function because agents care about their relative position in the world income distribution. In these examples, the discount factor of an agent becomes lower as he becomes wealthy *relative* to other agents in the world economy. Section 9 provides a few concluding comments.

2 Preliminaries

Our approach follows Lucas and Stokey [7] as adapted by Boyd [1]. Lucas and Stokey show how to define a utility function recursively from an aggregator function $W(x, y)$ that is bounded and satisfies certain additional properties. Following their approach, consider the following candidate space of admissible consumption sequences $\mathbf{C} = \{C \in \mathbf{X} \mid \mathbf{X} \subset l^{\infty}, x_t \geq 0\}$ where l^{∞} is the space of bounded sequences with the norm $\|x\| = \sup |x_t|$.[1] Let $\mathbf{B}$ be the space of continuous bounded functions $u : \mathbf{X} \to R$ with the norm

$$\|u\| = \sup_{x \in \mathbf{X}} |u(x)|,$$

and define an operator $T_W : \mathbf{B} \to \mathbf{B}$ as

$$(T_W u)(C) = W(\pi C, u(SC)). \tag{1}$$

In equation (1) π is the projection operator $\pi X = x_1$, and S is the shift operator $SX = (x_2, x_3, ...)$. Lucas and Stokey show that under their assumptions about the properties of W,

[1] We use the convention that uppercase letters represent vectors in R^{∞}, lowercase letters are scalars and boldface letters represent sets.

T_W is a contraction and hence the aggregator W can be used to construct a unique utility function $u(C)$.

For our purposes the approach in Lucas and Stokey is restrictive since we wish to study balanced growth. We need to allow both consumption sequences and aggregator functions $W(x, y)$ to be unbounded. Boyd [1] shows how to apply a modified version of the contraction mapping theorem, *the weighted contraction mapping theorem,* that applies to a much larger class of aggregators than those permitted by Lucas and Stokey. Boyd allows the consumption set to be larger than l^∞ but smaller than R^∞ so that there is an appropriate notion of convergence of sequences and convergence of continuous functions defined on this larger space. The appropriate consumption set, found by bounding the rate of growth of feasible consumption sequences, is

$$\mathbf{X}(g) = \left\{ X \in R^\infty \mid |X|_g < \infty \right\}$$

where $|X|_g = \sup |x_t/g^{t-1}|$ for $g \geq 1$ is the *g-weighted* l^∞ norm. Boyd proposes the following metric on this space.

Definition 1 (Boyd) *Let* $f \in c(\mathbf{A}, \mathbf{B})$, *where* $c(\mathbf{A}, \mathbf{B})$ *is the space of continuous functions from* $\mathbf{A}$ *to* $\mathbf{B}$. *Suppose* $\varphi \in c(\mathbf{A}, \mathbf{B})$ *with* $\mathbf{B} \subset R$ *and* $\varphi > 0$. *The function* f *is* φ-bounded *if the* φ-norm *of* f, $\|f\|_\varphi = \sup\{|f(x)| / \varphi(x)\}$ *is finite. The* φ-norm *turns* $c_\varphi(\mathbf{A}, \mathbf{B}) = \{f \in c(\mathbf{A}, \mathbf{B}) : f$ *is* φ-bounded$\}$ *into a complete metric space.*

In growing economies with a constant returns-to-scale neoclassical technology one can pick g to equal the growth factor of technological progress.

3 Assumptions on Technology and Preferences

In this section we describe a set of assumptions on preferences and technologies that are sufficient to guarantee the existence of an equilibrium in a representative agent economy.

We assume that the net output of the economy is characterized by a function $f(k_t, g^t)$ where $g > 1$ represents the growth factor of technological progress. At date t we assume that output is divided between consumption c_t and next period's capital stock k_{t+1}

$$c_t + k_{t+1} = f\left(k_t, g^t\right), \quad t = 1, 2...$$

3.1 Technology

We assume the following properties of the production function f :

Assumption $P1$	f is continuous
Assumption $P2$	f is strictly concave
Assumption $P3$	f is twice continuously differentiable
Assumption $P4$	f is linearly homogeneous
Assumption $P5$	$\lim_{k\to\infty} f_k(k, a) = 0$
Assumption $P6$	$\lim_{k\to 0} f_k(k, a) = \infty$

Remark 1 *Assumptions $P1 - P4$ are standard properties of neoclassical production functions. Assumptions $P5$ and $P6$ (the Inada conditions) ensure an interior solution.*

We now turn our attention to finding a suitable upper bound on the rate at which consumption sequences are permitted to grow. Define the production correspondence $\mathbf{P}(k)$ to be the set of vectors $K \in R^\infty$ that are feasible given an initial capital stock k.

$$\mathbf{P}(k) = \left\{K \in \mathbf{K} \mid 0 \leq k_{t+1} \leq f\left(k_t, g^t\right), x_1 = k\right\}$$

and let

$$\mathbf{F}(k) = \left\{C \in \mathbf{C} \mid 0 \leq c_t \leq f\left(k_t, g^t\right) - k_{t+1}\right\}$$

be the set of feasible consumption paths. Now define $f^t(k)$ inductively by the initial condition $f^1 = f(k, 1)$ and the recursion $f^t(k) = f(k, g^t) \circ f^{t-1}(k)$. The *path of pure accumulation,* defined as $\{f^t(k)\}_{t=1}^{\infty}$, is the sequence of capital stocks that would be attained if all of society's resources were invested in every period.

For the class of constant returns to scale neoclassical production functions that satisfy the Inada conditions $P5$ and $P6$, $f^t(k)$ converges to $g^t f(\bar{k}, 1)$ where $\bar{k}$ is defined by the equation $\bar{k} = f(\bar{k}, 1)/g$. In this case

$$\lim_{t\to\infty}\left[\frac{f^{\,t}(k)}{g^t} = \bar{k} < \infty\right],$$

and there is no loss to restricting feasible consumption and capital sequences to lie in $\mathbf{X}(g)$, the set of sequences in R^{∞} for which x_t/g^t is bounded.

3.2 Preferences

We define a class of aggregators $W : \mathbf{X}(g)\times\mathbf{Y} \to \mathbf{Y}$ that satisfy the following properties.

Assumption $U1$ $\quad W(0,0) = 0$

Assumption $U2$ $\quad W$ is continuous and increasing in both arguments

Assumption $U3$ $\quad 0 \leq W(x,0) \leq A(1 + x^{\eta}), \quad A, \eta > 0$

Assumption $U4$ $\quad W$ is continuous and φ-bounded for $\varphi(X) = 1 + |X|_g^{\eta}$

Assumption $U5$ $\quad |W(x,y) - W(x,y')| \leq \delta|y - y'|$ for all $x \in \mathbf{X}(g)$, $y, y' \in \mathbf{Y}$ where $\delta g^{\eta} < 1$

Assumption $U6$ $\quad (T_W^N y)(X)$ is concave in X for all N and all constants $y \in \mathbf{Y}$.

Assumptions $U2$ and $U5$ correspond to $W1$ and $W2$ in Boyd ([1] page 330). Assumption $U1$ restricts us to aggregators that are bounded below and $U3$ allows us to define a natural concept of distance using the definition of φ-boundedness. Specifically, this assumption allows us to define a function φ:

$$\varphi(X) = 1 + |X|_g^{\eta}$$

where $|X|_g$ is the $g-$weighted l^{∞} norm.

Given assumptions $U1 - U5$ it follows from Boyd's Continuous Existence Theorem ([1] page 333) that there exists a unique $u \in c_\varphi$ such that $W(\pi X, u(SX)) = u(X)$. Moreover, from Boyd's Lemma 1 ([1] page 331), assumption $U6$ guarantees the concavity of u.

4 Planning Optima and Balanced Growth

In this section we solve a planning problem and ask: What restrictions must we place on preferences and technology for the solution to this problem to be consistent with balanced growth?

Consider a representative agent economy in which the agent's preferences are represented by an aggregator W that satisfies $U1 - U6$. A planning optimum for this economy solves the following problem.

Problem 1

$$\max u(C) = W(\pi C, u(SC)) \tag{2}$$

subject to:

$$c_t + k_{t+1} = f\left(k_t, g^t\right), \quad t = 1, 2... \tag{3}$$

$$k_0 = \bar{k}. \tag{4}$$

The solution to this problem is characterized by the following set of necessary and sufficient conditions;

$$W_c(c_t, u_{t+1}) - W_u(c_t, u_{t+1}) W_c(c_{t+1}, u_{t+2}) f_K\left(k_{t+1}, g^{t+1}\right) = 0, \quad t = 1, 2.. \tag{5}$$

$$\lim_{T \to \infty} u_T(C) k_T = 0, \tag{6}$$

$$\text{where} \quad u_T(C) \equiv W_u(c_1, u(SC)) W_u\left(c_2 u\left(S^2 C\right)\right) ... W_u\left(c_{T-1}, u\left(S^{T-1} C\right)\right) W_c\left(c_T, u\left(S^T C\right)\right). \tag{7}$$

We next turn to the definition of balanced growth.

Definition 2 *A balanced growth path is a set of sequences* $\{K, C, Y\}$ *for which there exists a triple* $\{k, c, y\}$ *such that if* $k_0 = k$ *then*

$$k_t = kg^t, \quad c_t = cg^t, \quad y_t = f\left(k_t, g^t\right) = yg^t.$$

Balanced growth is an important assumption because it is a relatively accurate characterization of most industrialized economies. A fair amount is known about technologies and preferences that are consistent with balanced growth for the case in which preferences over intertemporal consumption sequences are time separable. Swan [9] proved that the existence of a balanced growth path implies that exogenous technical progress must be labor augmenting while King, Plosser and Rebelo [5] showed, for the case of additively time separable preferences, that the utility function must be homogeneous of some degree γ in consumption. The following lemma extends the results of King-Plosser-Rebelo to the case of a recursive utility function.

Lemma 1 *For a utility function* $u(C)$ *to display a constant marginal rate of substitution along a balanced growth path it is necessary and sufficient that the function can be written as a monotonically increasing function of a linearly homogeneous function. In this case we say that the function* $u(C)$ *is homothetic.*

Remark 2 *Homotheticity of preferences is often defined in terms of preference orderings. A preference ordering is homothetic if* $C_a \gtrapprox C_b$ *implies* $\lambda C_a \gtrapprox \lambda C_b$ *for all* $\lambda > 0$.

Proof. Suppose $u(C)$ is homothetic. By our definition of homotheticity, u is a monotone increasing transformation of a linearly homogenous function. Hence, let

$$u(C) = F[v(C)], \qquad F_v > 0$$

where $v(C)$ is linearly homogenous in the consumption sequence C. The marginal rate of substitution is defined by the expression

$$\beta(C) = \frac{\partial u/\partial c_2}{\partial u/\partial c_1} = \frac{u_2(C)}{u_1(C)}.$$

Since $u(C)$ is homothetic, $\frac{u_2(C)}{u_1(C)} = \frac{v_2(C)}{v_1(C)}$, and since $v(C)$ is linearly homogeneous, $v_1(C)$ and $v_2(C)$ are each homogenous of degree zero in C. Hence,

$$\beta(\lambda C) = \frac{u_2(\lambda C)}{u_1(\lambda C)} = \frac{v_2(\lambda C)}{v_1(\lambda C)} = \frac{v_2(C)}{v_1(C)} = \beta(C).$$

Along a balanced growth path $SC = gC$ where S is the shift operator and g is the growth factor. Hence zero degree homogeneity implies $\beta(SC) = \beta(gC) = \beta(C)$ which establishes that homotheticity of $u(C)$ is sufficient for the marginal rate of substitution to be constant along a balanced growth path.

To establish necessity, suppose that the marginal rate of substitution is time invariant, then

$$\beta(C) = \beta(SC) = \beta(gC).$$

Hence consistency with balanced growth implies that the marginal rate of substitution is homogeneous of degree 0. Since utility is invariant to monotonically increasing transformations we consider the class of functions $u(C) = F(v(C))$ where $F_v(v): R \to R$ is monotonically increasing and $v(C)$ is quasiconcave in C. Then writing the marginal rate of substitution in terms of the partial derivatives of $u(C)$ leads to the expression,

$$\beta(\lambda C) = \frac{F'(v(\lambda C))\, v_2(\lambda C)}{F'(v(\lambda C))\, v_1(\lambda C)}.$$

Suppose that $v_1(C)$ and $v_2(C)$ are not homogeneous of degree 0 in C. Then either (1) $\beta(\lambda C) \neq \beta(C)$ which is a contradiction or (2) $v_2(\lambda C)$ and $v_1(\lambda C)$ have a common factorization of the form $v_2(\lambda C) = f(\lambda C)\, w_2(\lambda C)$, $v_1(\lambda C) = f(\lambda C)\, w_1(\lambda C)$ where $w_2(\lambda C)$ and $w_1(\lambda C)$ are each homogenous of degree 0. But in this case we can let $w = v$ through an appropriate choice of F. Hence $u(C)$ is homothetic. $\blacksquare$

Lemma 1 establishes that prefernces will be homothetic (the marginal rate of substitution will be constant along a balanced growth path) if and only if the utility function is homothetic. Lemma 1 has strong implications. Homotheticity means that the rate of time preference is independent of consumption along any consumption sequence that grows at a constant rate. This property implies not only that the marginal rate of substitution is constant along a balanced growth but it is equal to the *same constant* along any two paths that grow at the same rate g. In other words, the rate of time preference along balanced growth paths is a primitive property of preferences. An agent with these preferences, placed in a small open economy, would not in general have a balanced consumption path unless the world interest factor happened to equal his discount factor.

Homotheticity is restrictive but it is necessary if we are to construct models in which the equilibria are consistent with balanced growth. A subclass of homothetic functions is the class of time-separable homogenous utility functions studied by King-Plosser-Rebelo [5]. A larger subclass is that of homogenous functions that are not necessarily separable. The following theorem, due to Dolmas, shows how to construct such functions using a class of recursive aggregators that display a certain homogeneity property. It also establishes the connection between aggregators of this class and the marginal rate of substitution along balanced growth paths.

Theorem 1 (Dolmas) *Suppose W satisfies assumptions $U1-U5$ and is such that $W(\lambda x, \lambda^{\gamma} y) = \lambda^{\gamma} W(x, y)$ for all x and y and all $\lambda > 0$, for some γ. Then, the recursive utility function u exists and is homogeneous of degree γ. If W is also once-differentiable, the marginal rate of substitution exists and is a constant along a balanced growth path .*

Homogenous aggregators of this type have some useful properties that were noted by Dolmas [2]. In particular, if the utility function is constructed from a recursive aggregator and $u(C)$ is homogeneous of degree γ, then the aggregator $W(c_t, u_{t+1})$ has the following properties

$$W(\lambda c_t, \lambda^\gamma u_{t+1}) = \lambda^\gamma W(c_t, u_{t+1}), \tag{8}$$

$$W_c(\lambda c_t, \lambda^\gamma u_{t+1}) = \lambda^{\gamma-1} W_c(c_t, u_{t+1}), \tag{9}$$

$$W_u(\lambda c_t, \lambda^\gamma u_{t+1}) = W_u(c_t, u_{t+1}). \tag{10}$$

Moreover, the homogeneity of utility of degree γ implies that $u(\lambda c_t, \lambda c_{t+1}, ...) = \lambda^\gamma u(c_t, c_{t+1}, ...)$. Since $u_t = u(c_t, c_{t+1}, ...)$, along any balanced growth path it follows that utilities in adjacent periods are related by the expression, $u_{t+1} = g^\gamma u_t$.

Theorem 1 allows us to construct examples from homogenous aggregators for which we can solve explicitly for the variables u, c, and k that solve the social planning problem along a balanced growth path. To find such a solution, first, define the function $\phi(c)$ by the expression

$$\phi(c) = W(c, g^\gamma \phi(c))$$

where $\phi(c)$ is the utility associated with the balanced growth consumption sequence $c_t = g^t c$. Now define the marginal rate of substitution as follows;

$$\beta(C, g) = \frac{\partial u / \partial c_{t+1}}{\partial u / \partial c_t} = \frac{W_u(c_t, u_{t+1}) W_c(c_{t+1}, u_{t+2})}{W_c(c_t, u_{t+1})}. \tag{11}$$

Along a balanced growth path, using homogeneity properties (8-10) this expression simplifies to the following expression,

$$\beta(C, g) = g^{\gamma-1} W_u(c, g^\gamma \phi(c)).$$

To solve for the solution to the planning problem, along a balanced growth path, one solves three equations in three unknowns: These are (1) the Euler equation; (2) the resource constraint and (3) the recursive definition of utility. Beginning with the Euler equation, the homogeneity properties (8-10) imply that

$$g^{\gamma-1} W_u(c, g^\gamma \phi(c)) = \beta(g) = [f_K(k, 1)]^{-1}. \tag{12}$$

where k is the ratio of capital to the productivity trend. We have used the fact that $f_K(k_{t+1}, g^{t+1})$ is homogeneous of degree 0 to write it as $f_K(k, 1)$. We define $\beta(g)$ to be the function $\beta(C, g)$ evaluated along any balanced growth sequence $C = \{c, cg, cg^2..\}$. Equation (12) determines k as a function of $\beta(g)$. To find c, use the resource constraint

$$c + gk = f(k, 1). \tag{13}$$

Finally, u is defined by the expression,

$$u = \phi(c). \tag{14}$$

We should note that the solution to Equations (12)–(14) does not necessarily solve the social planning problem, since it ignores the initial condition (4). If the initial capital stock, $\bar{k}$ happens to equal the value of k that solves the social planning optimum then the social planning solution will be on the balanced growth path at all points in time. If the initial condition is anything else, the social planning optimum may or may not converge to the balanced growth path. A proof of local convergence require us to put more structure on preferences.

5 An Economy with Two Agents

In this section we extend our analysis to an environment with two agents with preferences represented by utility functions $u^i : \mathbf{X}(g) \to R$ for $i = \{1, 2\}$ each of which is defined by an aggregator

$$u^i\left(C^i\right) = W^i\left(\pi C^i, u^i(SC^i)\right). \tag{15}$$

Consider the following social planning problem:

Problem 2

$$\max_{C^1, C^2} u = \lambda u^1\left(C^1\right) + (1 - \lambda) u^2\left(C^2\right),$$

$$\begin{aligned} k_{t+1} &= f\left(k_t, g^t\right) - c_t^1 - c_t^2 \quad t = 1, 2... \\ k_0 &= \bar{k}, \end{aligned}$$

where the technology $f\left(k, g^t\right)$ *satisfies properties* $P1 - P6$ *and the utility functions* $u^i\left(C^i\right)$, $i = 1, 2$, *are generated by aggregators that satisfy assumptions* $U1 - U6$.

This is a concave programming problem and the necessary and sufficient conditions for a solution are given by the equations

$$W_c^1\left(c_t^1, u_{t+1}^1\right) = W_u^1\left(c_t^1, u_{t+1}^1\right) W_c^1\left(c_{t+1}^1, u_{t+2}^1\right) f_K\left(k_{t+1}, g^{t+1}\right), \quad t = 1, 2.. \tag{16}$$

$$\lambda W_c^1\left(c_t^1, u_{t+1}^1\right) = (1 - \lambda) W_c^2\left(c_t^2, u_{t+1}^2\right), \quad t = 1, 2.. \tag{17}$$

$$W_u^1\left(c_t^1, u_{t+1}^1\right) = W_u^2\left(c_t^2, u_{t+1}^2\right), \quad t = 1, 2.. \tag{18}$$

$$\lim_{T\to\infty} \left(\lambda u_T^1\left(C^1\right) + (1 - \lambda) u_T^2\left(C^2\right)\right) k_T = 0, \tag{19}$$

together with the initial condition

$$k_0 = \bar{k}. \tag{20}$$

Since this economy is neoclassical, the first and second welfare theorems hold. It follows that the equations that characterize the solution to Problem 2 are a subset of the equations that define a competitive equilibrium and for any social welfare weight λ there will exists a distribution of initial resources such that the social planning optimum can be decentralized as a competitive equilibrium.

Definition 3 *A balanced growth path for the two-person economy is a set of sequences* $\{K, C^1, C^2, Y\}$ *and a* $4-$*tuple* $\{k, c^1, c^2, y\}$ *such that*

$$k_t = kg^t, \quad c_t^1 = c^1 g^t, \quad c_t^2 = c^2 g^t \quad y_t = f\left(k_t, g^t\right) = yg^t.$$

We will say that preferences and technologies are consistent with a balanced growth path if there exist balanced growth sequences that satisfy the first order conditions for a planning optimum (16–19).

We will be interested in heterogenous agent economies that are consistent with the existence of a balanced growth path. To be interesting economies, we will also require that the income distribution is non-degenerate. In economies without growth, Lucas and Stokey [7] have shown that the existence of an endogenous income distribution requires that the agents' discount rates must depend in a non-trivial way on their consumption sequences. They define a condition (increasing marginal impatience) as the assumption that

$$W^i_{uc}\left(c, \phi\left(c\right)\right) < 0,$$

where $\phi\left(c\right) = u\left(C\right)$, and C is the constant sequence $C = \{c, c, ...\}$. In words, increasing marginal impatience means that a representative agent, when faced with alternative constant consumption sequences, will have a lower discount factor (smaller value of $W_u\left(c, \phi\left(c\right)\right)$) the higher is his consumption path. As agents get richer, in this sense, they become more impatient.

An analogous condition for a balanced growth economy would require that, faced with two alternative balanced growth paths, C_1 and C_2 defined as

$$\begin{aligned} C_1 &= \{c_1, c_1 g, ...\}, \\ C_2 &= \{c_2, c_2 g, ...\}, \end{aligned}$$

that

$$W_u\left(c_1, g^\gamma \phi\left(c_1\right)\right) < W_u\left(c_2, g^\gamma \phi\left(c_2\right)\right),$$

whenever $c_1 > c_2$. This condition would require that as agents get richer, in the sense of moving to strictly higher balanced growth paths, they become more impatient. But Lemma 1 establishes that $W_u\left(c, g^\gamma \phi\left(c\right)\right)$ is independent of c. It follows that the assumption of increasing marginal impatience is inconsistent with balanced growth.

The following theorem establishes that balanced growth economies require that all agents be alike in a very strong sense.

Theorem 2 *Consider a two-person economy with a technology that satisfies properties $P1-P6$. Let the preferences of the two agents be constructed from aggregator functions that satisfy properties $U1-U6$. These preferences are consistent with balanced growth if and only if all, agents have the same constant rate of time preference, that is, if*

$$W_u^1\left(c^1, g^\gamma \phi\left(c^1\right)\right) = W_u^2\left(c^2, g^\gamma \phi\left(c^2\right)\right) = \beta\left(g\right),$$

where the functions $W^1\left(c^2, g^\gamma \phi\left(c^1\right)\right)$ and $W^2\left(c^1, g^\gamma \phi\left(c^2\right)\right)$, satisfy the homogeneity property defined in Theorem 1.

Proof. In the one agent economy, consistency of preferences with balanced growth means that the marginal rate of substitution must be constant and equal to the marginal rate of transformation. For the two-person economy this condition must hold for each agent:

$$\left[f_K\left(k,1\right)\right]^{-1} = g^{\gamma-1} W_u^1\left(c^1, g^\gamma \phi^1\left(c^1\right)\right) = g^{\gamma-1} W_u^2\left(c^2, g^\gamma \phi^2\left(c^2\right)\right). \tag{21}$$

Lemma 1 implies that $g^{\gamma-1} W_u^1\left(c^1, g^\gamma \phi^1\left(c^1\right)\right) = \beta^1\left(g\right)$ and $g^{\gamma-1} W_u^2\left(c^2, g^\gamma \phi^2\left(c^2\right)\right) = \beta^2\left(g\right)$ are independent of the initial level of consumption along any two balanced growth paths with the same growth factor g. It follows that a balanced growth equilibrium cannot exist unless the discount factors of the two agents are equal. ■

The implication of theorem 2 is that the assumption of recursivity adds nothing of interest to the study of the income distribution beyond the case of additively separable preferences. Economies populated by agents with recursive homogeneous preferences like this do have non-trivial wealth distributions, but these distributions are not endogenous in the sense of Lucas-Stokey. Just as in the case of additively separable preferences, the distribution of wealth in these economies is a function of initial conditions.

6 Time Dependent Preferences

In this section we suggest a way around this dilemma by introducing an *exogenous* time-dependent factor into preferences. There are a number of interpretations to our analysis. In Section 7 we pursue an example in which time enters as a result of a preference externality because households care about their relative position in the income distribution.

We begin by expanding the commodity space. Consider aggregator functions W^i : $\mathbf{X}(g)^2 \times R \rightarrow R$ and let W^i satisfy properties $U1 - U6$ where $W(x, y)$ refers to a function in which x has two elements; the first represents consumption and the second represents an exogenous sequence $A = \{a_1, a_2, a_3 ...\}$ where $a_t = g^t$. The existence theorems of Boyd and Dolmas do not restrict c to be a scalar, hence one can appeal to these theorems to assert that there is a unique well defined solution $u^i(C^i, A)$ to the functional equation

$$u^i(C^i, A) = W^i\left(\pi C^i, \pi A, u^i(SC^i, SA)\right), \tag{22}$$

where $\pi A = a_1$, $SA = (a_2, a_3, ...)$. The social planner's problem in the expanded economy is defined in the same way as Problem 2 and since A is exogenous, the optimality conditions given by (16–19) continue to hold.

The homogeneity properties (8–10) have the following analogs in the economy where utility depends on time;

$$W(\lambda c_t, \lambda a_t, \lambda^\gamma u_{t+1}) = \lambda^\gamma W(c_t, a_t, u_{t+1}), \tag{23}$$

$$W_c\left(\lambda c_t, \lambda a_t, \lambda^\gamma u_{t+1}\right) = \lambda^{\gamma-1} W_c\left(c_t, a_t, u_{t+1}\right), \tag{24}$$

$$W_u\left(\lambda c_t, \lambda a_t, \lambda^\gamma u_{t+1}\right) = W_u\left(c_t, a_t, u_{t+1}\right). \tag{25}$$

In order to find conditions under which there exists an equilibrium with a non-trivial income distribution we need to define an analog to the property of increasing marginal impatience that holds in an economy with growth. We proceed as follows. Define the growth weighted variables

$$\tilde{k}_t = \frac{k_t}{g^t}, \quad \tilde{y}_t = \frac{y_t}{g^t},$$
$$\tilde{c}^1_t = \frac{c^1_t}{g^t}, \quad \tilde{c}^2_t = \frac{c^2_t}{g^t},$$
$$\tilde{u}^1_t = \frac{u^1_t}{g^{\gamma t}}, \quad \tilde{u}^2_t = \frac{u^2_t}{g^{\gamma t}}.$$

Now write the first order conditions for an optimum, in terms of these variables;

$$g^{1-\gamma} W^1_c \left(\tilde{c}^1_t, 1, g^\gamma \tilde{u}^1_{t+1}\right) = W^1_u \left(\tilde{c}^1_t, 1, g^\gamma \tilde{u}^1_{t+1}\right) W^1_c \left(\tilde{c}^1_{t+1}, 1, g^\gamma \tilde{u}^1_{t+2}\right) f_K \left(\tilde{k}_{t+1}, 1\right), \quad t = 1, 2... \tag{26}$$

$$\lambda W^1_c \left(\tilde{c}^1_t, 1, g^\gamma \tilde{u}^1_{t+1}\right) = (1-\lambda) W^2_c \left(\tilde{c}^2_t, 1, g^\gamma \tilde{u}^2_{t+1}\right) \quad t = 1, 2... \tag{27}$$

$$W^1_u \left(\tilde{c}^1_t, 1, g^\gamma \tilde{u}^1_{t+1}\right) = W^2_u \left(\tilde{c}^2_t, 1, g^\gamma \tilde{u}^2_{t+1}\right) \quad t = 1, 2.... \tag{28}$$

$$\lim_{T\to\infty} \left(\lambda u^1_T \left(C^1, A\right) + (1-\lambda) u^2_T \left(C^2, A\right)\right) k_T = 0, \tag{29}$$

and write the initial condition as

$$\tilde{k}_0 = \bar{k}. \tag{30}$$

The variables $\tilde{c}^i$ and $\tilde{u}^i$ are constant along a balanced growth path and they are related to each other by the functions $\phi^i(\tilde{c})$, $i = 1, 2$ where

$$\phi^i\left(\tilde{c}^i\right) = W^i\left(\tilde{c}^i, 1, g^\gamma \phi^i\left(\tilde{c}^i\right)\right)$$

is the utility of agent i, weighted by $g^{\gamma t}$, along the balanced growth path.

By assumption, $u(C, A)$ is homogeneous in C and A. This assumption implies that the functions W^i_u are homogenous of degree 0 in C and A; that is,

$$W^i_u\left(\lambda c^i, \lambda a, \lambda^\gamma u^i\right) = W^i_u\left(c^i, a, u^i\right).$$

This *does not* imply zero degree homogeneity in $\tilde{C}$. The marginal rate of substitution along a balanced growth path is described by the function;

$$\beta(\tilde{c}^i, g) = g^{\gamma-1} W_u^i \left(\tilde{c}^i, 1, \phi^i \left(\tilde{c}^i\right)\right),$$

which *is not* homogeneous of degree 0 in $\tilde{c}$. It follows the marginal rate of substitution may be different along growth paths for which consumption grows at the same rate g. The agent's rate of time preference varies across different constant growth paths and is ordered by the *levels* of the paths. By including the exogenous sequence A in preferences, we are able to maintain consistency with balanced growth *and* allow discount factors to vary along a balanced growth path.

7 Time Dependence and Balanced Growth

We now introduce an additional assumption that allows us to demonstrate the consistency of balanced growth with an endogenous income distribution in the sense of Lucas and Stokey in a two agent economy.

Definition 4 (Time Preference Variation) *There exist numbers c^1 and c^2 such that*

$$W_u^1 \left(c^1, 1, g^{\gamma} \phi \left(c^1\right)\right) = W_u^2 \left(c^2, 1, g^{\gamma} \phi \left(c^2\right)\right).$$

To demonstrate existence of an equilibrium, rates of time preference along the balanced growth must vary sufficiently with consumption such that there exist consumption sequences for which the discount rates of different agents are equated. That is the role of Definition 4. The following theorem illustrates that if Definition 4 holds, there will exist a social planning problem for which the solution is a balanced growth path.

Theorem 3 *Consider a two-agent economy in which the technology satisfies condition $P1-P6$ and the preferences of each agent are constructed from two different aggregator functions*

$W^i(c,a,u)$, $i=1,2$, that satisfy assumptions $U1-U6$. Assume further that preferences satisfy the Definition 4 (time preference variation). Then there exists an initial value k_0 and a welfare weight λ such that the solution to the Social Planning Problem 2 is a balanced growth path $\{k,y,c^1,c^2\}$.

Proof. We first show that the first-order conditions (26–29) are consistent with the existence of a balanced growth path. A balanced growth path is defined by the numbers $\{k,y,c^1,c^2\}$ where $k_t=g^tk, y_t=g^ty, c_t^1=g^tc^1$, and $c_t^2=g^tc^2$. From the homogeneity of utility in C and A, there exist numbers u^1 and u^2 such that $u_t^1=g^{\gamma t}u^1$ and $u_t^2=g^{\gamma t}u^2$. Define the functions $\phi^1(\tilde{c}^1)$ and $\phi^2(\tilde{c}^2)$ as follows,

$$\phi^i\left(\tilde{c}^i\right)=W^i\left(\tilde{c}^i,1,g^\gamma\phi^i\left(\tilde{c}^i\right)\right),\quad i=1,2.$$

The first order condition (28), evaluated along a balanced growth path, requires that the following equation should hold;

$$W_u^1\left(\tilde{c}^1,1,g^\gamma\phi^1\left(\tilde{c}^1\right)\right)=W_u^2\left(\tilde{c}^2,1,g^\gamma\phi^i\left(\tilde{c}^2\right)\right). \tag{31}$$

Let $\tilde{c}^1$ and $\tilde{c}^2$ be two different numbers that satisfy Equation (31). The existence of two such numbers follows from Definition (4). Define the welfare weight λ by the expression

$$\lambda W_c^1\left(\tilde{c}^1,1,g^\gamma\phi^1\left(\tilde{c}^1\right)\right)=(1-\lambda)\,W_c^2\left(\tilde{c}^2,1,g^\gamma\phi^1\left(\tilde{c}^2\right)\right), \tag{32}$$

and define $\tilde{k}$ to be the unique solution to the Equation

$$f_k\left(\tilde{k},1\right)=\frac{1}{W_u^1\left(\tilde{c}^1,1,g^\gamma\phi^1\left(\tilde{c}^1\right)\right)}. \tag{33}$$

The existence of a unique solution to this equation is implied by the Inada conditions, (assumptions $P5$ and $P6$). We have demonstrated that our utility function is consistent with the three first order conditions (16–18). Consistency with the transversality condition, Equation (29) follows from $U5$ which bounds the discount factor and ensures that marginal utilities grow more slowly than the growth rate of the economy. ■

Theorem 3 does not assert that the balanced growth path is consistent with equilibrium for all welfare weights. For this to be true, one would require something analogous to the Inada conditions applied to preferences. It asserts instead that if discount rates vary a little bit with consumption then there exist welfare weights that are consistent with existence of a balanced growth path. The implication for a decentralized equilibrium is that there exists some initial wealth distribution that is consistent with the range of variation in discount factors permitted by Definition 4, for which an equilibrium exists.

The theorem also says nothing about convergence to the balanced growth path. If the initial capital stock happens to exactly equal the balanced growth stock k, then the solution to the social planner's problem will place the economy immediately on its balanced growth path. If the initial stock is not equal to k, the economy may or may not converge to the balanced growth path. In the case of an exchange economy Lucas and Stokey give a two-person example in which a non-inferiority condition plus the assumption of increasing marginal impatience are sufficient to guarantee local convergence to balanced growth. Intuition suggests that increasing marginal impatience is a necessary condition although we have not been able to prove this for our model, nor have we been able to find sufficient conditions that guarantee asymptotic stability of the balanced growth path.

8 Two Open Economy Examples

In this Section, we provide two examples that illustrate how our preferences may be used in open economy models. We begin with an exogenous growth economy example and then extend this example to an endogenous growth model in Subsection 8.2.

8.1 A Small Open Economy Example

Consider the example of a representative agent in a small open economy. Let world wealth be given by the expression

$$a_t = g^t a$$

and let the individual's preferences be constructed from the aggregator function

$$u_t = W(c_t, a_t, u_{t+1}) \equiv \left[c_t^{\theta} a_t^{\psi-\theta} + u_{t+1}^{\delta} a_t^{\psi-\delta} \right]^{\frac{1}{\psi}}, \tag{34}$$

where the aggregator W satisfies assumptions $U1-U6$. We also assume, $\delta < 1, \quad \theta < 1$ and $\delta > \psi$.[2] Note that this aggregator is linearly homogenous.

This specification implies that as the world gets richer, this individual feels better off. More importantly, as the world gets richer, his marginal utility of consumption increases since he values his consumption sequence in part by comparing it to world wealth. In a rich world an extra unit of consumption gives more additional utility than it would in a poor world.

The individual faces the sequence of budget constraints

$$\begin{aligned} b_t &= b_{t-1}R + w_t - c_t, \quad t = 1, ... \\ b_0 &= \bar{b}_0, \end{aligned}$$

where b_t is domestic assets at date t, w_t is exogenous wage income defined by the sequence $w_t = g^t w$, and R is the exogenous market interest factor.

[2]Since the composition of concave functiuons is concave, concavity of the aggregator $W(c, u')$ in c and u' is a sufficient condition for $u(C)$ to be concave in C. If δ and θ are strictly less than 1 then $u(C)$ is strictly concave when $\psi/\delta = 1$. It follows from continuity that $u(C)$ is strictly concave for ψ/δ in an open neighborhood of $\psi/\delta = 1$.

The utility maximizing individual will choose a consumption sequence that equates his marginal rate of substitution to the world interest factor R;

$$\left(\frac{c_t}{a_t}\right)^{\theta-1} = \left(\frac{c_{t+1}}{a_{t+1}}\right)^{\theta-1} \left(\frac{u_{t+1}}{a_{t+1}}\right)^{\delta-\psi} \frac{\delta R}{\psi} g^{\delta-\psi}. \tag{35}$$

Equations (34) that defines recursive preferences and (35) that represents the first-order condition for maximization of utility can be expressed as a system of two difference equations in the transformed variables $\tilde{u}_t = u_t/a_t$ and $\tilde{c}_t = c_t/a_t$;

$$\tilde{u}_t^{\psi} = \tilde{c}_t^{\theta} + g^{\delta} \tilde{u}_{t+1}^{\delta}, \tag{36}$$

$$\tilde{c}_t^{\theta-1} = \frac{\delta R}{\psi} g^{\delta-\psi} \tilde{c}_{t+1}^{\theta-1} \tilde{u}_{t+1}^{\delta-\psi}. \tag{37}$$

Let $\tilde{b}_t$ be the ratio of net assets to world wealth, that is, $\tilde{b}_t = b_t/a_t$. The budget constraint, in transformed variables, takes the form;

$$\tilde{b}_t = \tilde{b}_{t-1}\frac{R}{g} + \tilde{w} - \tilde{c}_t, \quad t = 1, ... \tag{38}$$

$$\tilde{b}_0 = \tilde{b}_0, \tag{39}$$

where $\tilde{w}_t = w_t/g^t$. Equations (36–38) represent a system of three difference equations in three variables, $\tilde{u}_t, \tilde{c}_t, \tilde{b}_t$ with a single initial condition, $\tilde{b}_0$. The unique balanced growth path is defined by the equations

$$\bar{u} = \left(\frac{\delta R}{\psi}\right)^{\frac{1}{\psi-\delta}} \frac{1}{g}, \quad \bar{c} = \left(\bar{u}^{\psi} - g^{\delta}\bar{u}^{\delta}\right)^{\frac{1}{\theta}}, \quad \bar{b} = \frac{(\bar{c} - \tilde{w})}{R/g - 1}.$$

The Equations (36–37) form a sub-system in the two variables $\tilde{u}_t$ and $\tilde{c}_t$ that locally, (around the balanced growth path) obeys the linear equations

$$\begin{bmatrix} (1-s)\,\delta & 0 \\ \delta-\psi & \theta-1 \end{bmatrix} \begin{bmatrix} d\tilde{u}_{t+1} \\ d\tilde{c}_{t+1} \end{bmatrix} = \begin{bmatrix} \psi & -s\theta \\ 0 & \theta-1 \end{bmatrix} \begin{bmatrix} d\tilde{u}_t \\ d\tilde{c}_t \end{bmatrix}$$

where $d\tilde{u}_{t+1} = (\tilde{u}_{t+1} - \bar{u})/\bar{u}$, $d\tilde{c}_{t+1} = (\tilde{c}_{t+1} - \bar{c})/\bar{c}$ and $s = \bar{c}^{\theta}/\bar{u}^{\psi}$. This system has the representation

$$\begin{bmatrix} d\tilde{u}_{t+1} \\ d\tilde{c}_{t+1} \end{bmatrix} = J \begin{bmatrix} d\tilde{u}_{t} \\ d\tilde{c}_{t} \end{bmatrix},$$

where

$$J = \begin{bmatrix} \frac{1}{(1-s)\delta}\psi & \frac{-1}{(1-s)\delta} s\theta \\ \frac{\delta-\psi}{(1-s)\delta(1-\theta)}\psi & 1 - \frac{\delta-\psi}{(1-s)\delta(1-\theta)} s\theta \end{bmatrix}.$$

In the appendix we show that when $\delta > \psi$ this system has one root below 1 and one root above 1. We also assume that $R > g$ which is a necessary condition for the wealth of the agent to be well defined. Under these assumptions, the subsystem (36–37) defines a linear function

$$d\tilde{u}_t = \mu_1 d\tilde{c}_t$$

and a scalar difference equation

$$d\tilde{c}_{t+1} = \mu_2 d\tilde{c}_t$$

where $0 < \mu_2 < 1$. This subsystem converges back to the balanced growth path for $\tilde{c}_t$ in the neighborhood of $\bar{c}$. The initial value of this difference equation, $\tilde{c}_0$ is chosen to satisfy the transversality condition, that is, to ensure that

$$\tilde{b}_0 + \sum_{t=1}^{\infty} Q_0^t (w_t - \tilde{c}_t) = 0,$$

where

$$Q_0^t = \frac{1}{(R/g)^{t-1}}.$$

Agents in our example display an endogenous discount rate and their long-run position in the world income distribution will be a function of their preferences.

8.2 An Endogenous Growth Example

We now discuss an example of endogenous growth model in which agents have the same preferences as those described in Subsection 8.1. We will show that this example gives a novel interpretation of why two countries with different GDP per capita might converge over time. In our example a country's growth rate will increase or decrease because the representative agent cares about his relative position in the world income distribution.

Let preferences be given by the same aggregator function as Subsection 8.1 but let the technology be represented by the function

$$k_{t+1} = Rk_t - c_t, \tag{40}$$

$$k_0 = \bar{k}_0 \tag{41}$$

As before, $a_t = g^t a$ is world wealth. Since we assume that the same technology is available to all countries in the world there is no incentive to borrow and lend internationally; the domestic interest factor R is equal to the world interest factor and is pinned down by technology. The world growth rate g is a function of the preferences of the other countries in the world.

Rewriting this system in transformed variables gives the system

$$\tilde{u}_t^{\psi} = \tilde{c}_t^{\theta} + g^{\delta}\tilde{u}_{t+1}^{\delta}, \tag{42}$$

$$\tilde{c}_t^{\theta-1} = \frac{\delta R}{\psi} g^{\delta-\psi}\, \tilde{c}_{t+1}^{\theta-1}\, \tilde{u}_{t+1}^{\delta-\psi}, \tag{43}$$

$$\tilde{k}_{t+1} = \frac{R}{g}\tilde{k}_t - \frac{\tilde{c}_t}{g}, \tag{44}$$

$$\tilde{k}_0 = k(0), \tag{45}$$

which has a balanced growth path

$$\bar{u} = \left(\frac{\delta R}{\psi}\right)^{\frac{1}{\psi-\delta}} \frac{1}{g}, \quad \bar{c} = \left(\bar{u}^{\psi} - g^{\delta}\bar{u}^{\delta}\right)^{\frac{1}{\theta}}, \quad \bar{k} = \frac{\bar{c}}{R-g}.$$

This economy has the same stability properties as the previous example, but a different interpretation. Since $R > g$, Equation (44) is an unstable difference equation. Since the subsystem (42–43) is a saddle, there is a unique value of $\tilde{u}_0$ and a unique $\tilde{c}_0$ such that the system converges to the balanced growth path. During the transition, the economy may grow faster than the world growth rate g, or it may grow slower.

If the economy starts out relatively poor, the domestic agent will devote a relatively large share of capital to investment and, temporarily, the domestic economy will grow faster than the world growth factor, g. As the economy becomes richer, the agent becomes more impatient and domestic growth slows down until it reaches the world growth rate.

If the economy starts out relatively rich, the reverse happens. The agent consumes a large fraction of his wealth and the economy grows slower than g. As the agent becomes poorer he becomes more patient and the growth rate of the domestic economy increases. All economies grow, asymptotically, with growth factor g because agents care about their relative position in the world income distribution.

9 Conclusion

In this paper we have studied the circumstances under which one can model an endogenous income distribution in a growing economy. We have shown that the homogeneity of the welfare aggregator (a property that is required for balanced growth) has strong implications in multiple agent environments. In general, balanced growth equilibria do not exist in a multi-agent economy except for the special case where all agents have the same constant rate of time preference. This case is uninteresting since it eliminates meaningful preference heterogeneity which is one of the key motivations for studying recursive preferences to begin with.

Given that balanced growth provides a fairly good description of the industrialized economies, these findings highlight a problematic feature of recursive preferences. We have suggested a generalization of the recursive preference structure that permits the coexistence of balanced growth equilibria with multiple agent economies and recursive preferences. Our extension requires preferences to be explicitly time dependent and the aggregator to be homogeneous in current consumption, future utility and an exogenous (time dependent) growth factor. Our specification is consistent with models where the externality in preferences comes through average per capita income or through per-capita world wealth.

Appendix

The Roots of J. The product of the roots (the determinant of determinant of J) is given by

$$D = \frac{1}{(1-s)}\frac{\psi}{\delta}$$

and the sum of the roots, (the Trace of J) is:

$$T = \frac{1}{(1-s)}\frac{\psi}{\delta} + \frac{1-\psi/\delta}{(1-s)(\theta-1)}s\theta + 1.$$

When $\psi/\delta = 1$ the roots are given by 1 and $1/(1-s)$.

The determinant is clearly increasing in ψ/δ. The trace is given by the expression

$$f(\psi/\delta) = \frac{1}{(1-s)}\frac{\psi}{\delta} + \frac{1-\psi/\delta}{(1-s)(\theta-1)}s\theta + 1,$$

and since $0 < s < 1$, the slope

$$\frac{\partial f(\psi/\delta)}{\partial\psi/\delta} = \frac{1-\theta(1-s)}{(1-s)(1-\theta)},$$

is positive for $0 < \theta < 1$. Hence the trace is also increasing in ψ/δ.

Since the determinant and the trace are both increasing in ψ/δ it follows that as we reduce ψ below δ, one root will remain above 1 and the other will decrease below 1. A proof of this statement is providing by examining Figure 1.

The curve in Figure 1 plots the function

$$f(x) = x^2 - TRACE\,x + DET.$$

The zeros of this function are the roots of the characteristic polynomial of J. The figure shows $f(x)$ for $\delta = \psi$ (the dashed curve) and $f(x)$ for $\delta > \psi$ (the solid curve).

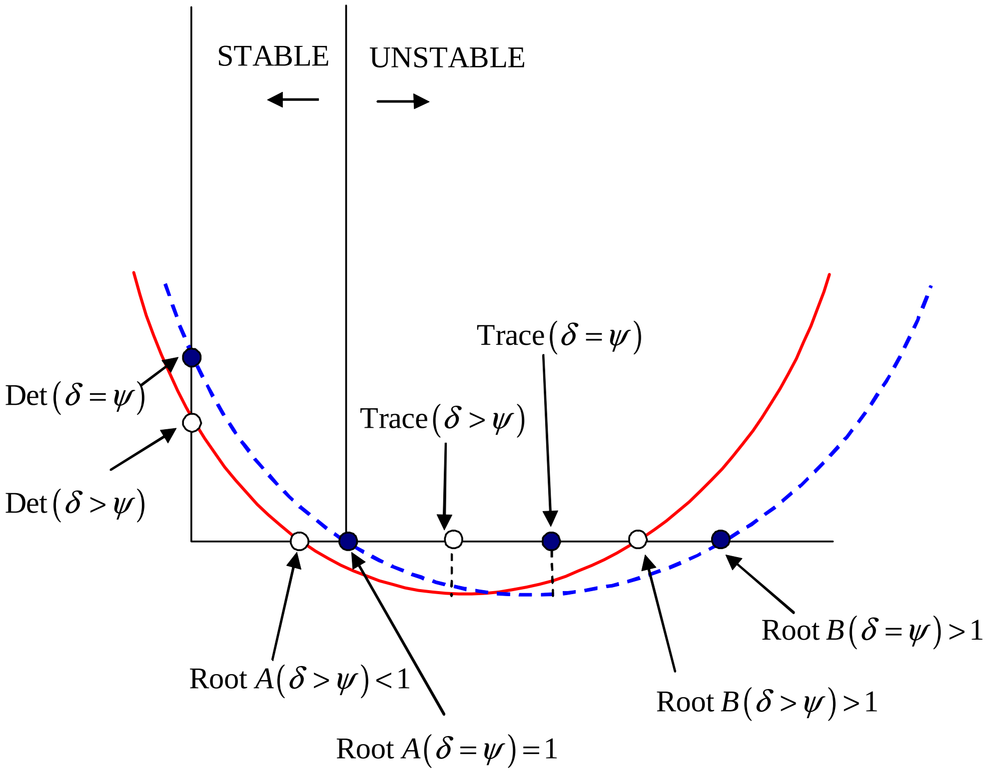

Figure 1

As ψ is lowered below δ, the trace (this is given by $2 \arg\min f(x)$) and the determinant (this is represented by the intercept) both decrease. If the two roots of the system before and after the change are denoted $\{x_{A1}, x_{B1}\}$ and $\{x_{A2}, x_{B2}\}$, the figure illustrates that these two roots move from $\{x_{A1} = 1, x_{B1} > 1\}$ to $\{x_{A2} < 1, x_{B2} > 1\}$. Hence, for values of ψ a little below δ the subsystem defined by equations (36) and (37) is a saddle. ∎

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