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Structure of Pareto Optima When Agents Have Stochastic Recursive Preferences

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A recursive method is given for constructing all Pareto optimal allocations for a dynamic economy under Markov certainty in which consumer preferences are recursive. *Journal of Economic Literature* Classification Numbers: C60, D50.

1. INTRODUCTION

In recent years, following Koopman's work (Koopmans, 1960), there have been several studies of the dynamics of Pareto allocations for agents with recursive preferences. Among these are Lucas and Stokey [9], Epstein [5], and Dana and Le Van [2]. In particular, using the fact that Pareto optima maximize a weighted sum of the utilities of the different agents, it has been shown that a Pareto optimal allocation can be viewed as a function of a trajectory of a dynamic system of capital stock and the utility "weights." This function can be determined recursively.

All the above studies considered only the case of certainty. The purpose of this work is to generalize the above characterizations of Pareto optima to a setting with Markov uncertainty, where agents have recursive preferences as introduced by Epstein and Zin [6].

Ma [10] has considered an economy similar to ours and proved the existence of equilibrium in such an economy. In this paper, we do not

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discuss the issue of existence of equilibrium; rather, we focus on the characterization of Pareto optima by means of a "representative agent." For a continuous-time analogue, see Duffie *et al.* [4].

2. Utility

We use the standard setup of a probability space $(\Omega, \mathbf{F}, Q)$ and the filtration $\mathbf{F} = \{\mathscr{F}_t : \mathscr{F}_t = \sigma(X_0, X_1, ..., X_t),\ t \in \{0, 1, ...\}\}$ of sub-sigma fields generated by a time-homogeneous Markov chain $(X_0, X_1, ...)$ which takes values in a finite set $\mathfrak{F} = \{1, ..., S\}$.

There is one commodity in our economy. A consumption process is bounded $\{\mathscr{F}_t\}$-adapted process. For a consumption process c, we denote the consumption at period t by $c(t)$. Since S is finite, we can view $c(t)$ as in element in $\mathbb{R}^{S^t}$, $t = 0, 1, ...$. Hence we can view the consumption space as the sequence space ℓ^∞ by viewing a consumption process c as an element in $\ell^\infty = \mathbb{R} \times \mathbb{R}^S \times \mathbb{R}^{S^2} \times \cdots$. Individual agents consume from the non-negative cone ℓ^∞_+. We adopt the usual norm $\|\cdot\|$ given by $\|c\|_\ell = \sup_i |c_i| < \infty$.

We define, for any consumption process $c \in \ell^\infty_+$, a new consumption process $\mathscr{T}_i c$ by $\mathscr{T}_i : \ell^\infty_+ \to \ell^\infty_+$ as follows: If $c = (c_0, c_1, c_2, ...)$, then

$$\mathscr{T}_i c = (c_i, c_{iS+1}, ..., c_{iS+S}, c_{iS^2+S+1}, ..., c_{iS^2+S^2+S}, c_{iS^3+S^2+S+1}, ..., c_{iS^3+S^3+S^2+S}, ...).$$

That is, if c is a consumption process starting today, then $\mathscr{T}_i c$ is the continuation of c starting in the next period given that the state at that period is i.

Now, we turn to consumers' preferences. Following Epstein and Zin [6] and Ma [10], we adopt a preference described by recursive utility. Recursive utility, roughly speaking, consists of two components: an aggregator $W(\cdot, \cdot)$ and a certainty equivalent $\mu(\cdot)$. Formally, we have the following definition:

Definition 2.1. A function $W: \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is an aggregator if it satisfies:

W1. W is continuous, strictly concave, and strictly increasing;

W2. There exists M such that $|W(c_0, 0)| \leqslant M$, $c_0 \in \mathbb{R}_+$;

W3. There exists $\beta > 0$ such that

$$|W(c_0, x) - W(c_0, x')| \leqslant \beta\, |x - x'|, \qquad c_0 \in \mathbb{R}_+, \quad x \in \mathbb{R}_+, \quad x' \in \mathbb{R}_+;$$

W4. $W(0, 0) = 0$.

A function $f: \ell_+^\infty \to \mathbb{R}_+^S$ is defined to be in B if $f(\cdot)$ is bounded and continuous. We define a norm on B by

$$\|f\|_B = \sup_{c \in \ell_+^\infty} \|f(c)\|_S,$$

where $\|x\|_S = \sup_i |x_i|$, $x \in \mathbb{R}^S$. Under this norm, B is a complete metric space. Under assumptions given below, a consumer's utility function u is in B, treating the ith component of u, u_i, as the utility when the initial state is i.

DEFINITION 2.2. A function μ: $\mathbb{R}_+^S \to \mathbb{R}_+^S$ is a certainty equivalent if:

C1. μ is continuous, strictly increasing, and strictly concave.

C2. There exists $\beta_\mu \in (0, 1/\beta)$ such that $\|\mu(u) - \mu(v)\|_S \leqslant \beta_\mu \|u - v\|_S$ for all $u, v \in B$.

C3. $\mu(0) = 0$.

C4. If $u = (\bar{u}, \bar{u}, ..., \bar{u})$, then $\mu(u) = u$.

Condition C4 is nowhere used in this paper, but since it is commonly assumed in the literature, we keep it in our definition. It merely says that if there is no uncertainty for tomorrow's utility, its certainty equivalent is equal to itself.

The ith component of $\mu(u)$, $\mu_i(u)$, is the certainty equivalent of the next period utility u, given that the state in the current period is i.

For additional notation, given $c_0 \in \mathbb{R}_+$ and $x \in \mathbb{R}^S$, we let

$$\mathscr{W}(c_0, x) = (W(c_0, x_1), W(c_0, x_2), ..., W(c_0, x_S)),$$

and for $u \in B$ we let $u(\mathscr{T}c) = (u_1(\mathscr{T}_1 c), u_2(\mathscr{T}_2 c), ..., u_S(\mathscr{T}_S c))$.

We have the following lemma, which gives the definition of the recursive utility function u corresponding to (W, μ).

LEMMA 2.1. *Every aggregator W and certainty equivalent μ together define an operator $T_{\mathscr{W}}$ on B into B by $T_{\mathscr{W}}\, u(c) = \mathscr{W}[c_0, \mu(u(\mathscr{T}c))]$. There exists a unique $u \in B$ such that $u = T_{\mathscr{W}}\, u$. Moreover, u is strictly increasing and strictly concave, with $u(0) = 0$.*

Since the proof is similar to that of Lucas and Stokey [9], we omit it here. Please also see Ma [10].

3. DYNAMIC PROGRAMMING FOR PARETO OPTIMAL ALLOCATIONS

The objective of this section is to introduce a recursive method that generates all Pareto optimal allocations. The result given in this section are in parallel to those in Lucas and Stokey [9]. Since the proofs are a straightforward generalization of theirs, we omit them here and only state the result.

We begin by defining our economy and the economy's feasible utility set and its support function. Lemma 3.1 establishes their properties. Proposition 3.1 give the major result of this section: A recursive method that generates all Pareto optimal allocations.

In our economy, there are n infinitely lived consumers. Consumer l has recursive utility u^l given above, corresponding to (W^l, μ^l). We use supercripts, as in c^l, $l \in \{1, 2, ..., n\}$, to denote the consumptions of different consumers.

An endowment process is an element of ℓ^{∞}_{++}, the set of pointwise strictly positive elements of ℓ^{∞}.

First, for any endowment $e \in \ell^{\infty}_{++}$, the feasible consumption set is

$$C(e) = \left\{ c \in (\ell^{\infty}_{+})^n : \sum_{l=1}^{n} c^l \leqslant e \right\}.$$

For any $e \in \ell^{\infty}_{++}$ and any $i \in \mathfrak{F}$, the feasible utility set is

$$U_i(e) = \{ u_i(c) \equiv (u_i^1(c^1), u_i^2(c^2), ..., u_i^n(c^n)) : c \in C(e) \}.$$

That is, $U_i(e)$ is the set of all the utilities that can be attained given that the state at $t = 0$ is i. We record two properties: (i) If $0 \leqslant u' \leqslant u$ and $u \in U_i(e)$, then $u' \in U_i(e)$. (ii) $U_i(e)$ is compact and convex.

A Pareto optimal allocation given e and current state i is a consumption process $c \in C(e)$ such that there is no $\tilde{u} \in U_i(e)$ for which $\tilde{u} \geqslant u_i(c)$ and $\tilde{u} \neq u_i(c)$.

We now define the support function of U_i, $v_i : \ell^{\infty}_{++} \times \Delta^{n-1} \to \mathbb{R}$, by

$$v_i(e, \theta_i) = \sup_{u \in U_i(e)} \sum_{l=1}^{n} \theta_i^l u^l, \tag{3.1}$$

where $\theta_i \in \Delta^{n-1} \equiv \{\theta_i \in \mathbb{R}^n_+ : \sum_l \theta_i^l = 1\}$. Since $U_i(e)$ is compact, the supremum is attained.

LEMMA 3.1. (i) *For each* $i \in \mathfrak{F}$, v_i *is bounded and continuous.* (ii) $u \in U_i(e)$ *if and only if* $u \geqslant 0$ *and* $v_i(e, \theta_i) - \theta_i \cdot u \geqslant 0$ *for all* $\theta_i \in \Delta^{n-1}$.

Now, we turn to our major result: The support function v_i and the corresponding allocations satisfy the equations

$$v_i(e, \theta_i) = \max_{c_0 \in \mathbb{R}^n_+,\, u \in (\mathbb{R}^S_+)^n} \sum_{l=1}^{n} \theta_i^l W^l(c_0^l, \mu_i^l(u^l)) \tag{3.2}$$

subject to

$$\sum_{l=1}^{n} c_0^l \leqslant e_0 \tag{3.3}$$

and subject to

$$\min_{\bar{\theta}_j \in \Delta^{n-1}} v_j(\mathcal{T}_j e, \bar{\theta}_j) - \sum_{l=1}^{n} \bar{\theta}_j^l u_j^l \geqslant 0, \qquad \text{for all} \quad j \in \mathfrak{F}, \tag{3.4}$$

where e_0 is the first element of $e \in \ell_{++}^{\infty}$, that is, e_0 is the endowment at time 0.

The idea is that given the endowment process and the weight vector θ_i in the current period, we can choose the Pareto-optimal allocation by choosing optimally a feasible current allocation c_0 of consumptions and a vector u of utilities from the next period on, subject to the constraint that these utilities are feasible. The weights $\bar{\theta}$ that attain the minimum in (3.4) will be the new weights used to select the next period's allocations, and so on, ad infinitum. This determines a path of allocations $c(t)$, weight $\theta(t)$, and utilities $u(t)$. We refer to this path of allocations as being recursively generated from (e, θ_i). Formally, we have the following central result.

PROPOSITION 3.1. (i) *There exists a unique bounded and continuous solution to* (3.2)–(3.4); *moreover, the solution is the value function v_i defined in* (3.1). (ii) *An allocation $c \in C(e)$ is Pareto optimal if and only if it is generated recursively from (e, θ_i).*

Remark 1. It can be shown that for all $i \in \mathfrak{F}$, $v_i(e, \theta_i)$ is strictly concave in e and strictly convex in θ_i. Therefore, for each (i, e, θ_i), there is a unique $(c_0, \bar{\theta}, u)$ solving problem (3.2)–(3.4).

Remark 2. The functions v_i are representative agent utilities in the sense of Duffie [3, pp. 9–10]. Under additional regularity conditions, they also define state price processes. To see why, note first that under suitable differentiability assumptions for agregators and certainty equivalents, each agent's recursive utility has a gradient at any consumption path that is bounded away from zero. (See Kan [7] for the proof.) One can then show, by adapting the arguments in Benveniste and Scheinkman[1] [1], that $v_i(\cdot, \theta_i)$, $\theta_i >> 0$, has a gradient at any endowment process e for which each agent's consumption is bounded away from zero in the associated Pareto optimal allocation. This gradient defines a state price process as in Duffie [3]. A general expression for this gradient is given in Kan [7].

[1] The author thanks an anonymous referee of JET for pointing out this.

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