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ABSTRACT

For homothetic time and state separable preferences, the coefficient of relative risk aversion (CRRA) is equal to the reciprocal of the elasticity of intertemporal substitution (EIS). This paper shows that, when the growth rate of consumption is i.i.d., asset pricing models based upon preferences in which the CRRA and the EIS are no longer linked do not have more explanatory power. Further, in these stochastic environments, estimates of the CRRA in the standard preferences are measures of the true CRRA and not the EIS. These results are fairly accurate descriptions of economies calibrated using United States annual data.

IN THE LAST DECADE, following the work of Lucas (1978), many researchers have studied intertemporal general equilibrium asset pricing models. Typically, their analyses assume that assets can be priced using the Euler equations of an individual agent maximizing

$$(1/\gamma)E \textstyle\sum_{t=0}^{\infty} \beta^t c_t^{\gamma}. \tag{1}$$

However, in (1) the quantity $(1 - \gamma)$ is both the reciprocal of the elasticity of intertemporal substitution (EIS) and the coefficient of relative risk aversion (CRRA). Authors such as Selden (1978), Barsky (1986), and Hall (1988), among others, argue that individual attitudes toward risk and toward intertemporal variations are in fact unrelated. They claim that an econometrician using preferences in which this link does not exist would be able to distinguish these disparate attitudes and would therefore be better able to model asset pricing data.

This paper argues that this intuition is flawed. We begin by considering a complete markets exchange economy in which the growth rate of the aggregate endowment is i.i.d.[1] and require all agents in the economy to have the same preferences (although they may have different wealths); their utility function satisfies a recursive representation used by Epstein and Zin (1989a,b) and Weil

* Kellogg Graduate School of Management, Northwestern University. I had helpful conversations with George Constantinides, Larry Epstein, and Stan Zin. Lars Hansen, Kenneth Singleton, and an anonymous referee made many useful comments. They are not responsible for any errors that may remain.

[1] As in Brown and Gibbons (1985), we will use the term "independent and identically distributed" to mean independence from *all* past information, not just past growth rates.

(1988):

$$U_t = \{c_t^\sigma + \delta(E_t U_{t+1}^\alpha)^{\sigma/\alpha}\}^{1/\sigma}, \tag{2}$$

$$1 > \sigma \neq 0, \quad \alpha < 1, \quad \alpha \neq 0, \quad 1 > \delta > 0,$$

where U_t is time-t utility and c_t is time-t consumption. The class of preferences is strictly larger than (1), as can be seen by setting $\alpha = \sigma$. We prove that there exists a unique equilibrium in which asset prices are calculable from the first-order conditions of a representative agent holding the aggregate endowment; we also show that the agent's EIS and the CRRA are governed by different parameters: the former is $1/(1 - \sigma)$, while the latter is $1 - \alpha$. However, an econometrician with data on asset prices and aggregate consumption cannot separately identify both of these quantities. In fact, in this economy, (1) and (2) are *observationally equivalent*: they impose the same restrictions upon the data. In other words, even though (2) is a much larger class of preferences, it has no more explanatory power than (1).

We go on to look at the results of fitting (1) to data generated by (2). Most researchers would argue that the difficulty with implementing (1) in this way is that one cannot discern the representative agent's attitudes toward risk from estimates of γ. Indeed, economic intuition (Hall (1988)) might seem to indicate that they say more about the value of the EIS than the CRRA. We find in this framework that $(1 - \gamma)$ is unambiguously the representative consumer's CRRA. Instead, the problem with (1) is that β is playing one too many roles: the empirical investigator cannot disentangle the agents' EIS from their "true" discount factor δ.

The unrealistic assumption that endowment growth rates are i.i.d. plays a critical role in the derivation of the above two results. For that reason, we go on to look at the economies which are calibrated using United States annual data. We find that, in such a setting, it is better to interpret estimates of γ as measures of risk aversion and not of intertemporal substitution. Further, in terms of resolving the Mehra-Prescott equity premium puzzle, (2) does not have much more explanatory power than (1).

Why does the standard intuition lead us astray? Theorists do not like (1) because they are interested in the results of comparative statics exercises such as changing the CRRA or EIS. When one manipulates γ, it is unclear which effects are due to movements in the CRRA as opposed to the EIS. We can separate the implications of moving one instead of the other only by using (2). (See Epstein (1988) or Selden (1978).) Econometricians, however, are only tangentially concerned with these analyses. Their task is to learn about the representative agent's preferences from observing the first-order conditions to a single infinite dimensional maximization problem. As we shall see, the difficulties they face in this endeavor may be entirely different from those that trouble theorists.

The paper is organized as follows. In Section I, we set up a complete markets exchange economy in which agents have identical utility functions of the form (2). The homotheticity of their preferences allows us to easily characterize, and prove the existence of, a unique equilibrium in which assets can be priced from the first-order conditions of a representative consumer. Section II proves that (1) and (2) are observationally equivalent when the growth rates of the aggregate endowment are i.i.d. Section III analyzes the correct interpretation of the estimates researchers obtain when they fit (1) to data actually generated by a representative consumer maximizing (2). Sections IV and V examine to what extent these results are true in calibrated economies. In Section VI, we close with some conclusions and intuition into our results.

I. The Setup

We begin by constructing a parameterized theoretical economy in the following fashion. We assume that the state of the world, ω, lies in a set Ω and that (Ω, G, μ) is a probability space. The J agents do not know ω, but, at each time $t > 0$, all of them observe a state vector $s_t(\omega)$, where $s_t : \Omega \rightarrow \Re^N$ is G-measurable. Let G_t be the σ-algebra of subsets of Ω generated by the functions $\{s_1, \cdots, s_t\}$. Then, the J agents in the economy all have the same information set at time t:

$$I_t = \{x : \Omega \rightarrow \Re^*, \ x \ G_t\text{-measurable}\}, \tag{3}$$

where $\Re^*$ is the set of extended real numbers. More precisely, I_t is the set of random variables measurable with respect to the period-t information, G_t.

We assume that the agents face choices over a commodity space of nonnegative consumption processes:

$$C = \{c \in X_{s=0}^{\infty} I_s \mid c_s \geq 0 \text{ for all } s\}. \tag{4}$$

For each event $B \in G_s$, agents have a complete preference ordering over C that is representable by the appropriate component of a utility function:

$$F: C \rightarrow X_{s=0}^{\infty} I_s, \tag{5}$$

which has the recursive representation:

$$F_t(c) = \{c_t^{\sigma} + \delta(E_t(F_{t+1}(c))^{\alpha})^{\sigma/\alpha}\}^{1/\sigma}, \tag{6}$$

$$1 > \sigma \neq 0, \quad 1 > \alpha \neq 0, \quad 1 > \delta > 0.$$

(See Appendix A for a more precise description of F.) Note that F is increasing, concave, and homogeneous of degree one. If one denotes utility at time t by U_t instead of $F_t(c)$, it becomes clear that (2) and (6) are equivalent.

What is the motivation behind these preferences? Selden (1978) defined a two-period family of utility functions:

$$V(c_1, c_2) = \{c_1^{\phi} + \kappa(Ec_2^{\eta})^{\phi/\eta}\}^{1/\sigma}, \quad \phi < 1, \quad \eta < 1. \tag{7}$$

For this two-period utility function, the CRRA and the EIS are governed by different parameters: the former is $1/(1 - \eta)$, while the latter is $1/(1 - \phi)$. In (6), the individual essentially has a Selden utility function over today's consumption and tomorrow's random utility. In particular, in a two-period setting, the preferences (6) reduce to (7).

It is illustrative to consider the instance where $\alpha = \sigma$.

Example 1: When $\alpha = \sigma$, (6) becomes

$$F_t(c) = \{c_t^\sigma + \delta E_t(F_{t+1}(c))^\sigma\}^{1/\sigma}. \tag{8}$$

From this expression, it is trivial to derive an explicit formula for F_t:

$$F_t(c) = (E_t\{\textstyle\sum_{\tau=t}^{\infty} \delta^{\tau-t} c_\tau^\sigma\})^{1/\sigma}. \tag{9}$$

More generally, of course, F is state and time nonseparable.

Suppose now that agent j in the economy is endowed with a risky stream of consumption $y_j \in C$. Further, suppose that the agents in the economy trade a complete market of state- and date-contingent claims to consumption. We call $\pi : C \longrightarrow \Re^*$ a *valuation operator* if

$$\begin{aligned} \pi(x + y) &= \pi(x) + \pi(y), \\ \pi(\Theta x) &= \Theta\pi(x), \end{aligned} \tag{10}$$

for all $x, y \in C$ and $\Theta \in \Re_+$, and define an equilibrium as follows.

Definition 1: A valuation operator π^* and allocations $\{x_j : j = 1, \cdots, J\}$ form an equilibrium to the economy if

$$\begin{aligned} &\text{i) } x_j = \underset{c \in C}{\operatorname{argmax}}\ F_0(c) \\ &\qquad \text{s.t. } \pi^*(c) \le \pi^*(y_j), \\ &\text{ii) } \textstyle\sum x_j \le \sum y_j. \end{aligned} \tag{11}$$

Thus, π^* evaluates risky streams in terms of first-period consumption.

The homotheticity of the individuals' preferences, taken in conjunction with the completeness of the markets, allows us to calculate the equilibrium valuation operator by positing a "representative" consumer who holds the aggregate endowment of the economy. (See Appendix A.) Using the chain rule of differentiation, we find that this agent's marginal rate of substitution between consumption at $(t + 1)$ and at t is given by

$$MRS_t^{t+1} = \delta(E_t(F_{t+1}(y))^\alpha)^{(\sigma/\alpha)-1}(F_{t+1}(y))^{(\alpha-\sigma)}(y_{t+1}/y_t)^{\sigma-1}. \tag{12}$$

Then the following theorem holds.

THEOREM 1: *If* $|F_0(y)| < \infty$, *the economy has a unique equilibrium valuation operator*:

$$\pi^*(z) = \textstyle\sum_{s=0}^{\infty} E\, z_s(\prod_{\tau=0}^{s} MRS_\tau^{\tau+1}). \tag{13}$$

Proof: See Appendix A.

In other words, as long as the representative agent receives a finite amount of utility from the aggregate endowment, any bundle of state- and date-contingent claims to consumption can be priced using π^*.

Thus, we have developed a parameterized complete market exchange economy, indicated when it is in equilibrium, and shown what this equilibrium is.[2] Epstein

[2] What happens if we reopen markets in the future? The results of Donaldson and Johnsen (1985) tell us that, since F is "consistent" and markets are complete, the relative prices of assets remain unchanged.

(1988) proves the existence of a unique equilibrium for a representative consumer economy in which the single agent's preferences satisfy (2) and the state of the world follows an i.i.d. process. However, our development of the preferences is different from his, and Theorem 1 generalizes his result by allowing for a possibly nontrivial distribution of wealth and for arbitrary stochastic processes of aggregate consumption.

II. Observational Equivalence

At this point, we will find it convenient to make an assumption.

Assumption 1: $\lambda_t = y_t/y_{t-1}$ is i.i.d. over time; more precisely, λ_t and λ_{t+1} have the same distribution for all t, and λ_{t+1} is independent of G_t for all t.

In this setting, the representative consumer's utility depends in a time- and state-independent fashion upon the aggregate endowment; in other words, there exists a time-invariant constant K such that the representative consumer's utility at time t:

$$F_t(y) = Ky_t \tag{14}$$

for all t. Plugging this expression into (6), we can see that the consumer's utility is finite if $1 > \delta(E\lambda^{\alpha})^{\sigma/\alpha}$. Similarly, if we substitute (14) into (12), we find that the consumer's marginal rates of substitution have a correspondingly simple form:

$$MRS_t^{t+1} = \delta(E\lambda^{\alpha})^{(\sigma/\alpha)-1}\lambda_{t+1}^{\alpha-1}. \tag{15}$$

Theorem 1 tells us that, when $F_t(y)$ is finite, the equilibrium valuation operator π^* is completely characterized by these marginal rates of substitution.

Brown and Gibbons (1985) follow Rubinstein (1976) and show that, when agents maximize the state-separable preferences $(1/\gamma)E\sum\beta^t c_t^{\gamma}$ and Assumption 1 is true, one can write the marginal rates of substitution for the representative consumer in terms of the return to the market portfolio. It is simple to extend their logic into our more general setting using the gross return to the market portfolio:

$$R_t^m = \lambda_t/\{\delta(E\lambda^{\alpha})^{\sigma/\alpha}\}. \tag{16}$$

Substituting, we find that

$$MRS_t^{t+1} = \delta^{\alpha}(E\lambda^{\alpha})^{\sigma-1}(R_{t+1}^m)^{\alpha-1}. \tag{17}$$

The whole purpose of enlarging (1), though, was to introduce a wedge between the CRRA and the reciprocal of the EIS. Equation (16) illustrates that, in this economy where the growth rate of the aggregate endowment is i.i.d., the return to the market portfolio is also i.i.d. Hence, the representative individual derives utility in period t from two sources, today's consumption and tomorrow's wealth, according to the following formula:

$$U_t = (1/\sigma)c_t^{\sigma} + \beta A\{(E_t W_{t+1}^{\alpha})\}^{\sigma/\alpha}, \tag{18}$$

where A is a time-invariant constant. Consider an individual with an expected utility of wealth function EW^{α}. Such an agent will be indifferent between the random wealth W_{t+1} and the deterministic wealth $(EW_{t+1}^{\alpha})^{1/\alpha}$. Following Selden (1978), we can term the latter the *certainty equivalent* of W_{t+1}. In other words, the individual has the CES utility function, with elasticity of substitution $1/(1-\sigma)$, over today's consumption and a certainty equivalent of tomorrow's wealth. It is natural to think of *risk* preferences determining the certainty equivalent and of *time* preferences shaping choices over bundles of today's consumption and deterministic levels of wealth in the next period. In this sense, the individual has a CRRA of $1-\alpha$ and an EIS of $1/(1-\sigma)$; they are governed by different parameters.

It is also worth noting that an individual with the preferences (2) is not indifferent to the timing of the resolution of uncertainty. To understand this concept of Kreps and Porteus (1978), it is best to consider the two consumption trees in Figure 1 (taken from Epstein and Zin (1989a)). Agents with utility function (1) will be indifferent between these profiles. In contrast, individuals with preferences (2) care about when the uncertainty surrounding their final-period consumption is resolved. In particular, if their CRRA $(1-\alpha)$ is larger than the reciprocal of their EIS $(1-\sigma)$, they prefer early resolution (the first tree). Conversely, if $\alpha > \sigma$, they prefer late resolution (the second tree).[3]

Now, let $\beta = \delta(E\lambda^{\alpha})^{(\sigma/\alpha)-1}$ and $\gamma = \alpha$, and consider a different economy, one in which all the agents seek to maximize (1):

$$(1/\gamma)E\sum \beta^t c_t^{\gamma}.$$

The equilibrium valuation operator in this economy is also uniquely determined by the marginal rates of substitution of a representative agent holding the aggregate endowment:

$$MRS_t^{t+1} = \beta\lambda_{t+1}^{\gamma-1}. \tag{19}$$

However, (19) is the same as (15)! We can compress this logic into the following theorem.

THEOREM 2: *Suppose there exist two economies with the same endowment process that satisfies Assumption* 1. *In the first economy, all agents have preferences* (2):

$$U_t = (c_t^{\sigma} + \delta(E_t U_{t+1}^{\alpha})^{\sigma/\alpha}\}^{1/\sigma},$$

$$1 > \sigma \neq 0, \quad 1 > \alpha \neq 0, \quad 1 > \delta > 0,$$

while, in the latter, all agents have period-t utility $E_t\sum_{s=t}^{\infty}\beta^{t-s}c_s^{\gamma}$, *where*

$$\gamma = \alpha \text{ and } \beta = \delta(E\lambda^{\alpha})^{(\sigma/\alpha)-1}. \tag{20}$$

The equilibrium valuation operators in the two economies are the same.

Proof: To complete the proof, note that the condition guaranteeing the existence of equilibrium in both economies is the same: $1 > \beta E\lambda^{\gamma}$. Q.E.D.

Thus, for any economy of individuals who seek to maximize (2), there exists

[3] One intuition (or at least a mnemonic) for the directions of these inequalities is as follows. Utility of today's wealth is AW^{σ}, while utility of next period's wealth is AEW^{α}. In some sense, agents have a CRRA of $(1-\sigma)$ for bets that resolve sooner and a CRRA of $(1-\alpha)$ for bets that resolve later; thus, if $(1-\sigma) < (1-\alpha)$, bets that resolve earlier appear less risky than those that resolve later.

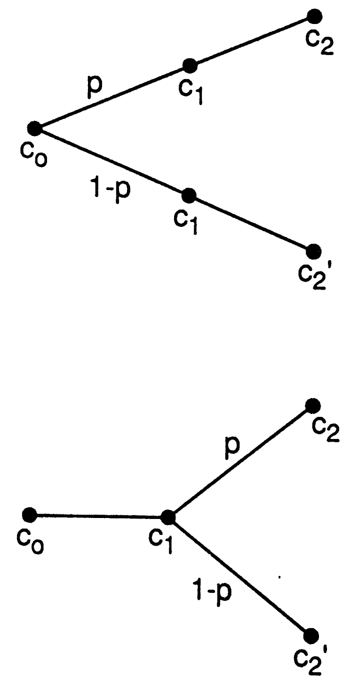

Figure 1. Preferences over the resolution of uncertainty. If $\sigma < \alpha$, the individual prefers later resolution of uncertainty and prefers the bottom consumption tree to the top one. If $\sigma > \alpha$, the individual prefers early resolution of uncertainty and likes the top tree better than the bottom one.

an economy of agents who seek to maximize (1) with the same endowment process and asset prices. To understand the implications of this result, consider econometricians who observe the joint distribution of all asset prices and aggregate quantities in an economy in which all agents have preferences of the form (2). They cannot tell from these data whether agents have state-separable preferences (1). In this environment, attempts to identify both the individual's time and risk preferences fail because the econometrician only sees the individual's first-order conditions from a *single* (infinite dimensional) maximization. In other words, when Assumption 1 is true, (2) has no more explanatory power than does (1).

III. Interpreting Point Estimates

Suppose now that econometricians who know the joint distribution of all asset prices and aggregate consumption (or, equivalently, use the distribution-free Generalized Method of Moments technique of Hansen and Singleton (1982)) were to try to fit and test a model based on (1) to the above economy where individuals actually have preferences (2). The above analysis shows that they could not reject (1). However, another issue remains: how should they best interpret their point estimates of (1)'s preference parameters β and γ?

The typical response to this question is that, in a world where the representative agent's CRRA is not equal to the reciprocal of the EIS, it is impossible to tell whether estimates of $(1 - \gamma)$ are assessments of the former or the latter. (Hall (1988) argues that economic intuition would dictate that they are evaluations of 1/EIS but is not precise about why.) From Theorem 2, though, we can see that estimates of γ correspond to ones of α. Hence, γ is unambiguously a measure of the individual's degree of risk aversion.

What if econometricians were to use the approach of Brown and Gibbons (1985) in this economy, as opposed to that of Hansen and Singleton (1982)? They would attempt to fit

$$MRS_t^{t+1} = \beta^\gamma (E\lambda^\gamma)^{\gamma-1}(R_{t+1}^m)^{\gamma-1} \tag{17'}$$

to their observations and would be unable to reject it as a model of the data. Moreover, we can see from (17) that the estimate of $(1 - \gamma)$ would again be an assessment of the individual's true CRRA.

On the other hand, estimates of β are not ones of the agent's "true" discount factor δ. Rather, they correspond to the discount factor of an agent who is holding the stochastic process y of aggregate consumption. Thus, if an econometrician fits (1) to data generated by preferences (2), the resultant estimate of β is an assessment of $\delta(E\lambda^\alpha)^{(\sigma/\alpha)-1}$. The empirical problem with (1) relative to (2) is not that it is impossible to distinguish attitudes toward temporal and state-contingent variations in consumption. Instead, the difficulty with (1) is that one cannot disentangle the EIS and the discount factor.

In the empirical literature, point estimates of β often exceed one. (See Hansen and Singleton (1982), among others.) Fortunately, the above analysis suggests a possible explanation for these findings. If $(E\lambda^\alpha)^{(\sigma/\alpha)-1}$ is sufficiently large, estimates of β may exceed one even though the true discount factor, δ, is less than one. (See Appendix B for a numerical example.)

IV. Interpreting Point Estimates in the Presence of Serial Dependence

The above results—the observational equivalence and interpretation of estimated parameters—depend critically on Assumption 1. Unfortunately, this restriction may be counterfactual.[4] In the next two sections, we will see to what extent our findings are true in models specifically calibrated to accord with features of United States annual data. To this end, let $U_t = F_t(y)$ denote the period-t utility of a representative consumer who owns the aggregate endowment. Recall that the marginal rates of substitution of the representative consumer have the form:

$$MRS_t^{t+1} = \delta(E_t U_{t+1}^\alpha)^{(\sigma/\alpha)-1} U_{t+1}^{\alpha-\sigma} \lambda_{t+1}^{\sigma-1}. \tag{21}$$

Define $z_{t+1} = (E_t U_{t+1}^\alpha)^{(\sigma/\alpha)-1} U_{t+1}^{\alpha-\sigma}$. Then,

$$MRS_t^{t+1} = \delta z_{t+1} \lambda_{t+1}^{\sigma-1}. \tag{22}$$

Suppose that, as in the previous section, econometricians attempt to fit the state-separable preferences (1) to data on consumption and asset prices generated

[4] Breeden, Gibbons, and Litzenberger (1986) find that monthly consumption growth rates are significantly negatively correlated. Based on this finding, it would seem plausible that more aggregated growth rates will depend on past information in some fashion.

by some element of the more general class of preferences (2). In other words, they try to find β and γ such that $\beta\lambda_{t+1}^{\gamma-1}$ is a good approximation to the above marginal rates of substitution. The expression (22) provides some insight into how they should interpret their estimates of γ. If z_t is close to being a time-invariant constant, as it is when $U_{t+1} \in I_t$ for all t, their estimates of γ are, as Hall (1988) surmises, assessments of σ. On the other hand, if $z_t\lambda_t^{\sigma-\alpha}$ is close to being a time-invariant constant, as it is when Assumption 1 is satisfied, their estimates of γ are measures of the risk aversion parameter α. Thus, to determine whether to interpret estimates of $(1 - \gamma)$ as assessing risk aversion or intertemporal substitution, we need to know whether z_t or $z_t\lambda_t^{\sigma-\alpha}$ is closer to being a time-invariant constant.

To analyze this question, we use Tauchen's (1986a) methodology to specify an information structure and an endowment growth process designed to accord in some respects with United States data. More precisely, we assume that the state of the United States economy at time t, s_t, is fully summarized by the vector (ln λ_t, y_t, ln η_t, d_t), where d_t is the dividend of the S&P 500, while η_t is its growth rate. We then require in our artificial economy that $s_t' =$ (ln λ_t, ln η_t) follow a 16-state Markov chain with transition matrix P; we select P so that an econometrician who regresses s_t' against s_{t-1}' in the artificial economy obtains the same coefficient estimates as one who performs this autoregression using actual annual data from the period 1890–1982. The results of following this procedure are described in Appendix C.

Having explicitly delineated the information structure and technology in the economy, it is possible to calculate the stationary distribution of the variable z_t in (22) for various specifications of the preference parameters. In particular, I set $\beta = 0.99$, and let σ and α both range over the set $\{0.5, -0.5, -1.5, \cdots, -10.5\}$; I then calculate the coefficient of variation of z_t ($CV(z_t)$) and of $z_t\lambda_t^{\alpha-\sigma}$ whenever $\sigma \neq \alpha$ and an equilibrium exists. (When the two parameters are equal—the expected utility case—both z_t and $z_t\lambda_t^{\sigma-\alpha}$ are time-invariant constants.) For each parameter specification, I calculate the ratio $CV(z_t)/CV(z_t\lambda_t^{\sigma-\alpha})$. The maximum value this ratio assumes is

$$\text{Max} = 645.5,$$

while its minimum value is

$$\text{Min} = 52.017.$$

What can we conclude from these results? Over the range of parameters considered, the coefficient of variation of $z_t\lambda_t^{\sigma-\alpha}$ is always much smaller than that of z_t. Hence, for this model economy in which the information structure and aggregate endowment process have been specified so as to accord with actual data, an econometrician would best interpret estimates of γ as measures of the degree of the agents' aversion to risk and not of the degree of their aversion to intertemporal variations.[5]

[5] We can write down a version of (2) which corresponds to α being equal to zero:

$$(*) \qquad U_t = \{c_t^\sigma + \delta \exp(\sigma E_t(\ln U_{t+1}))\}^{1/\sigma}.$$

V. Explanatory Power in the Presence of Serial Dependence

Earlier, we saw that, in the situation where endowment growth rates are i.i.d., (2) has no more explanatory power than (1). By looking at the general form of the marginal rates of substitution given by (22),

$$MRS_t^{t+1} = \delta z_{t+1} \lambda_{t+1}^{\sigma-1},$$

we can see that this conclusion is actually true more generally. In particular, if, for all specifications of the preference parameters (σ, α, δ), z_t is a scalar multiple of λ_t raised to some power, (2) has no more explanatory power than (1). It remains to be seen, though, to what extent this property is true of United States asset pricing data.

To answer this question (in an admittedly limited fashion), we examine whether freeing the link between the CRRA and the reciprocal of the EIS allows us to resolve the equity premium puzzle of Mehra and Prescott (1985).[6] As they do, we now require λ to follow a two-state Markov chain with state space

$$(1.054, 0.982)$$

and transition matrix

$$\begin{bmatrix} 0.47 & 0.53 \\ 0.53 & 0.47 \end{bmatrix}$$

and let $s_t = (\lambda_t, y_t)$; hence, the current state of the world is described wholly by

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Epstein and Zin (1989b) find in fact that these preferences fit monthly asset pricing data best. For this utility function, one can show using Euler's law for homogeneous functions that

$$(**) \qquad E_t \ln R_{t+1}^m = (1 - \sigma) E_t \ln \lambda_{t+1} - \ln \delta,$$

where R_t^m is the return to the market portfolio.

Hansen and Singleton (1983) assume that $(\ln \lambda, \ln R^m)$ is conditionally Gaussian and homoscedastic and estimate γ in preferences (1) using the expression:

$$(+) \qquad E_t \ln R_{t+1}^m = (1 - \gamma) E_t \ln \lambda_{t+1} + k.$$

From (**) we see that, when α is near zero, Hansen and Singleton's (1983) estimate of $1/(1 - \gamma)$ is a measure of the individual's EIS, not his or her CRRA.

However, this analysis does not contradict the argument in the paper. It is critical to remember that the econometrician in the paper either knows the joint distribution of all asset returns and consumption or, equivalently, is using a distribution-free method to obtain his or her parameter estimates. Suppose the representative consumer is maximizing a utility function of the form (*). It is then impossible for $(\ln \lambda, \ln R^m)$ to be conditionally Gaussian and homoscedastic. (Actually, there are two exceptions to this statement: σ could equal zero—the expected utility case—or consumption growth rates could be i.i.d. ln normal. In the latter case, (+) is a nonsensical regression.) In this situation, a highly well informed econometrician such as the one in the paper would not test (+) as a way of implementing (1).

In other words, if the representative agent is maximizing preferences of the form (2) in which $\alpha = 0$, Hansen and Singleton's (1983) maximum-likelihood estimate of γ is a measure of intertemporal preferences. However, from the argument in this paper, their (1982) GMM estimate of γ is a better measure of risk preferences than of time preferences. Thus, the two estimates will be the same only if $\sigma = \alpha$.

[6] After this section was completed, I became aware of similar simulations performed contemporaneously by Philippe Weil (1988); his results corroborate mine.

the current level and growth rate of the aggregate endowment. (The first and second moments of this endowment growth process are the same as that of United States annual per capita consumption growth from 1889 to 1978.) In this economy, using Euler's Law of homogeneous functions, it can be shown that the return to the market portfolio is

$$R_t^m = U_t^\sigma \lambda_t^{1-\sigma}/\{\beta(E_{t-1}U_t^\alpha)^{\sigma/\alpha}\}, \tag{23}$$

while the risk-free rate is

$$R_t^f = 1/E_t MRS_t^{t+1}, \tag{24}$$

where MRS_t^{t+1} is defined as in (21).

Mehra and Prescott report that, in the United States from 1889 to 1978, the average annual risk-free rate was 0.8%, while the average annual risk premium to the S&P 500 was 6.2%. In the above model economy, they calculate the expected risk-free rate and expected risk premium for the preference parameters:

$$\delta \in (0, 1),$$
$$\sigma \in [-9, 1],$$
$$\sigma = \alpha.$$

(Recall that δ is the individual's discount factor, $(1 - \sigma)$ is the reciprocal of the EIS, and $(1 - \alpha)$ is the CRRA.) When an equilibrium exists, they find that it is impossible for the expected risk premium to be larger than 0.35% unless the expected risk-free rate is larger than 4%.

What happens if σ and α are not equal? I calculate the risk premium to a share of the market portfolio for the following set of preference parameters:

$$\delta = 0.995,$$
$$\sigma \in \{0.5, -0.5, -1, \cdots, -10.5\},$$
$$\alpha \in \{0.5, -0.5, -1 \cdots, -10.5\}.$$

(The results are not that sensitive to δ as long as it is restricted to be less than one.) When an equilibrium exists, it is possible to generate an expected risk premium larger than 1.6% only if the expected risk-free rate is larger than 4%. Thus, freeing up the link between CRRA and the EIS does not help greatly in resolving the equity premium puzzle.

VI. Conclusions and Extensions

The contributions of this paper are threefold. The first is technical. We develop in a state preference setting a set of utility functions that satisfy the recursive representation (2). When agents all have identical utility functions in this class and markets are complete, we prove that there exists a unique equilibrium in which asset prices are characterized by the Euler equations of the representative consumer. In a more substantive vein, we show that, although different parameters describe the EIS and the CRRA of (2), when Assumption 1 is satisfied, (2) has no more explanatory power than (1) and econometricians who know the joint distribution of consumption and all asset returns (or, equivalently, use a distribution-free estimation method such as GMM) should interpret their estimates of γ in the standard preferences (1) as measuring the CRRA and not the EIS. Finally, we show that, even though Assumption 1 is counterfactual, (a) and (b) are fairly accurate descriptions of artificial economies that are calibrated to accord with United States annual data.

What is the intuition behind these results? Equations (1) and (2) are observationally equivalent to an econometrician with data on asset prices and aggregate consumption because such data only provide information on first-order conditions. When Assumption 1 is satisfied, we are enlarging (1) by relaxing state separability (while preserving state independence). State separability is a second-order restriction; thus, to the first order, (1) and (2) look the same.

The more perplexing result, perhaps, is that estimates of $(1 - \gamma)$ are unambiguously measures of the representative agent's CRRA. To better understand the intuition behind this conclusion, consider the interest rate in a world with constant growth ν:

$$1 + r = 1/\beta(1 + \nu)^{\gamma-1}, \tag{25}$$

where a representative consumer maximizes (1). Observations of r are only enough to reveal the single agent's rate of time preference, but not the discount factor or EIS. Similarly, in the i.i.d. growth rate case, data on asset prices are not sufficient to distinguish the EIS and the discount factor. On the other hand, in a complete market economy, asset prices fully reveal the individual's interstate marginal rates of substitution, which is equivalent to pinpointing the CRRA.

Epstein and Zin (1989b) attempt to fit a model based upon (2) to *monthly* asset pricing data using the GMM techniques of Hansen and Singleton (1982). If the monthly growth rate of consumption in the United States were i.i.d., implementations of (2) would meet the same fate as those of (1). In actuality, monthly growth rates of consumption exhibit some serial correlation; whether there is enough serial dependence to allow (2) to fare significantly better than (1) as a model of the data remains to be seen. Epstein and Zin's preliminary results are intriguing. However, they test a set of first-order conditions that involve measuring the return to the market portfolio, which is difficult to do accurately. Thus, their work is best viewed as an interesting first step.[7]

The major point of this paper is that, in trying to resolve the many econometric "puzzles" that have been observed in recent years in connection with intertemporal asset pricing models, it is important to remember that they are first-order problems and that changing second-order restrictions will not alleviate them. Thus, the link between the CRRA and the EIS in the standard preferences is not the cause of their empirical failure.

[7] Giovannini and Weil (1988) also perform some preliminary tests of (2) as opposed to (1); their empirical work is based upon the assumption that asset returns and growth rates are jointly lognormal. As footnote 4 points out, this restriction is inconsistent with an individual maximizing (2).

Appendix A

Our first task in this appendix is to define a utility function F which satisfies the recursive form (6). Define a truncated commodity space:

$$C^t = \{c \in X_{s=0}^t \; I_s \mid c_s \geq 0\}. \tag{A1}$$

In each event $B \in G_s$, $s \leq t$, agents are assumed to have a complete preference ordering over C^t that is representable by the appropriate component of a utility function:

$$F^t\colon C^t \rightarrow X_{s=0}^t \; I_s \tag{A2}$$

defined recursively by

$$F_t^t(c) = c_t,$$

$$F_s^t(c) = \{c_s^\sigma + \delta(E_s(F_{s+1}^t(c))^\alpha)^{\sigma/\alpha}\}^{1/\sigma}, \quad 0 \leq s \leq t-1, \tag{A3}$$

where $1 > \sigma \neq 0$, $0 < \delta < 1$, and $\alpha \leq 1$, $\alpha \neq 0$. We can now take limits to specify the agents' utility functions over C:

$$F\colon C \rightarrow X_{s=0}^\infty \; I_s,$$

$$F_s(c) = \lim_{t\rightarrow\infty} F_s^t(c_0, \cdots, c_t). \tag{A4}$$

This pointwise limit is well defined as an element of I_s since

$$F_s^{t+1}(c_0, \cdots, c_{t+1}) \geq F_s^t(c_0, \cdots, c_t) \quad \text{for all } t \geq s.$$

Our first theorem follows Eichenbaum, Hansen, and Richard (1987) to prove that the fact that the agents' preferences are homothetic, concave, and identical allows us to aggregate them into those of a representative consumer.

THEOREM A1: *Suppose* $|F_0(y)| < \infty$. π^* *and* $\{x_j : j = 1, \cdots, J\}$ *form an equilibrium such that* $\pi^*(y_j) > 0$ *for all* j *if and only if* π^* *is an equilibrium valuation operator to the representative consumer economy in which the single agent holds the aggregate endowment and has preferences defined by* F.

Proof: Define $\theta_j = \pi^*(y_m)/\pi^*(y)$ for all j. Suppose that π^* and $\{x_j : j = 1, \cdots, J\}$ form an equilibrium where $\pi^*(y_j) > 0$ for all j and that $\pi^*(y) \geq \pi^*(z)$ for some $z \in C$. Then,

$$\pi^*(x_j) \geq \pi^*(\theta_j z), \tag{A5a}$$

$$F_0(x_j) \geq F_0(\theta_j z), \tag{A5b}$$

$$F_0(x_j/\theta_j) \geq F_0(z). \tag{A5c}$$

Since F is quasi-concave,

$$F_0(y) = F_0(\textstyle\sum \theta_j (x_j/\theta_j)) \tag{A6a}$$

$$> \min_j F_0(x_j/\theta_j) \tag{A6b}$$

$$\geq F_0(z). \tag{A6c}$$

Now suppose π^* is an equilibrium to a representative consumer economy such that $\pi^*(y_j) > 0$ for all j and that $\pi^*(\theta_j y) \geq \pi^*(z)$ for some $z \in C$. Then,

$$\pi^*(y) \geq \pi^*(z/\theta_j), \tag{A7a}$$

$$F_0(y) \geq F_0(z/\theta_j), \tag{A7b}$$

$$F_0(\theta_j y) \geq F_0(z). \tag{A7c}$$

Thus, π^* and $\}\theta_j y$: $j = 1, \cdots, J\}$ form an equilibrium to the J agent economy. Q.E.D.

Again, we follow the lead of Eichenbaum, Hansen, and Richard (1987) in the following proof.

Proof of Theorem 1: Since F is concave over C, it satisfies a subgradient inequality:

$$F_0(x) - F_0(y) \leq \{\pi^*(x) - \pi^*(y)\}\{F_0(y)\}^{1-\sigma} y_0^{\sigma-1} \tag{A8}$$

for all $x \in C$. Suppose $F_0(x) - F_0(y) > 0$. Then, $\pi^*(x) - \pi^*(y) > 0$, and π^* is an equilibrium valuation operator. We can use the representative consumer's first-order conditions to realize that π^* is unique. Q.E.D.

Appendix B

Example of $\hat{\beta} > 1$: Assume that $\ln \lambda$ has a distribution $N(0.02, 0.001)$. Set $\sigma = 0.6$, $\alpha = 0.5$, $\delta = 0.999$. Then the econometrician's estimate of β is

$$\begin{aligned} \hat{\beta} &= \delta (E\lambda^{\alpha})^{(\sigma/\alpha)-1} \\ &= \delta \exp(\{(\sigma/\alpha) - 1\}\{0.02\alpha + 0.5(0.001)\alpha^2\}) \\ &= (0.999)\exp\{(0.2)(0.01 + 0.0005(0.25))\} \\ &= 1.001. \end{aligned}$$

Appendix C

Tauchen (1986b) fits the following vector autoregression to United States annual data from the period 1890–1982:

$$\begin{aligned} \ln \eta_t &= 0.003 + 0.073 \ln \eta_{t-1} + 0.62 \ln \lambda_{t-1} + \varepsilon_t^1, \\ \ln \lambda_t &= 0.022 + 0.015 \ln \eta_{t-1} - 0.122 \ln \lambda_{t-1} + \varepsilon_t^2, \end{aligned} \tag{C1}$$

where

$$\operatorname{var}(\varepsilon) = \begin{bmatrix} 0.0130 & 0.00224 \\ 0.00224 & 0.0016 \end{bmatrix}. \tag{C2}$$

Using Tauchen's (1986a) quadrature method, we generate a transition matrix P (on the following page), with the state space:

$(\lambda,$	$\eta)$
0.929	0.707
0.990	0.773
1.051	0.841
1.120	0.919
0.929	0.829
0.990	0.906
1.051	0.984
1.120	1.076
0.929	0.960
0.990	1.050
1.051	1.141
1.120	1.247
0.929	1.125
0.990	1.230
1.051	1.336
1.120	1.461

If we regress ($\ln \eta$, $\ln \lambda$) upon its lagged value using data generated from this Markov chain, we obtain the same coefficients as above. In the transition matrix P which follows, P_{ij} is the probability of going to state j from state i.

0.008	0.105	0.132	0.017	0.019	0.236	0.294	0.038	0.005	0.059	0.074	0.009	0.000	0.001	0.002	0.000
0.005	0.047	0.047	0.005	0.026	0.257	0.252	0.025	0.015	0.146	0.143	0.014	0.001	0.008	0.008	0.001
0.002	0.017	0.013	0.001	0.027	0.205	0.162	0.013	0.033	0.252	0.200	0.015	0.004	0.032	0.025	0.002
0.001	0.004	0.002	0.000	0.020	0.117	0.073	0.004	0.055	0.327	0.203	0.012	0.017	0.099	0.061	0.004
0.006	0.089	0.121	0.017	0.016	0.226	0.308	0.043	0.005	0.064	0.087	0.012	0.000	0.002	0.002	0.000
0.004	0.038	0.041	0.004	0.022	0.237	0.254	0.027	0.014	0.153	0.164	0.018	0.001	0.010	0.010	0.001
0.002	0.013	0.011	0.001	0.022	0.184	0.158	0.014	0.030	0.254	0.218	0.019	0.004	0.036	0.031	0.003
0.000	0.003	0.002	0.000	0.016	0.101	0.068	0.005	0.049	0.318	0.216	0.014	0.017	0.110	0.075	0.005
0.005	0.075	0.110	0.017	0.014	0.216	0.319	0.049	0.005	0.068	0.101	0.016	0.000	0.002	0.003	0.000
0.003	0.031	0.036	0.004	0.019	0.218	0.254	0.030	0.014	0.158	0.184	0.022	0.001	0.011	0.013	0.002
0.001	0.010	0.010	0.001	0.018	0.164	0.153	0.014	0.027	0.254	0.237	0.022	0.004	0.041	0.038	0.004
0.000	0.002	0.002	0.000	0.012	0.087	0.064	0.005	0.043	0.308	0.226	0.016	0.017	0.121	0.089	0.006
0.004	0.061	0.099	0.017	0.012	0.203	0.327	0.055	0.004	0.073	0.117	0.020	0.000	0.002	0.004	0.001
0.002	0.025	0.032	0.004	0.015	0.200	0.252	0.033	0.013	0.162	0.205	0.027	0.001	0.013	0.017	0.002
0.001	0.008	0.008	0.001	0.014	0.144	0.146	0.015	0.025	0.252	0.257	0.026	0.005	0.047	0.047	0.005
0.000	0.002	0.001	0.000	0.009	0.074	0.059	0.005	0.038	0.295	0.236	0.019	0.017	0.132	0.105	0.008

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