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David M. Kreps; Evan L. Porteus

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DYNAMIC CHOICE THEORY AND DYNAMIC PROGRAMMING¹

BY DAVID M. KREPS AND EVAN L. PORTEUS

Finite horizon sequential decision problems with a "temporal von Neumann-Morgenstern utility" criterion are analyzed. This criterion, as developed in [7], is a generalization of von Neumann-Morgenstern (expected) utility of the vector of rewards, wherein an individual's preferences concerning the timing of the resolution of uncertainty are taken into account. The preference theory underlying this criterion is reviewed and then extended in natural fashion to yield preferences for strategies in sequential decision problems. The main result is that value functions for sequential decision problems can be defined by a dynamic programming recursion using the functions which represent the original preferences, and these value functions represent the preferences defined on strategies. This permits citation of standard results from the dynamic programming literature, concerning the existence of (memoryless) strategies which are optimal with respect to the given preference relation.

1. INTRODUCTION AND SUMMARY

CONSIDER A FINITE HORIZON sequential decision problem, where at each time $t = 0, 1, \dots, T$, an individual must choose an action d_t . The actions available are determined by the state at time t , x_t . Some random event takes place, determining an immediate payoff z_t and the next state x_{t+1} . The probability distribution of the pair (z_t, x_{t+1}) is determined by the action d_t . The individual desires to select "best" actions contingent on states and other information that is relevant. In a conventional approach, "best" is defined through a cardinal utility function on the vector of payoffs: Each (measurable) strategy (i.e., contingent plan for selecting actions) induces a probability measure on the vector of payoffs, and strategies are ranked by the expected utility they thereby induce. An advantage of this criterion is that dynamic programming can be used to find "best" strategies, when they exist. One flaw with this sort of criterion, however, is that it ignores the temporal aspect of the resolution of uncertainty. That is, two strategies may induce the same probability distribution of payoffs, but one may cause the uncertainty to resolve at an earlier time than does the other. Insofar as there is value in earlier resolution of uncertainty, the individual may prefer the former strategy. In [7], we provided some motivation for the consideration of preference structures that account for this temporal resolution of uncertainty. (See also [2, 10, 16] for motivation in the context of consumption-investment budgeting.) Furthermore, we axiomatized, represented, and studied special cases of such preference structures. In this paper, we seek to show how dynamic programming can be used to analyze sequential decision problems when the individual has such a preference structure. We frequently cite results from [7], and we assume no further motivation for considering such preference structures is necessary. Thus, the reader may wish to refer to [7] before proceeding.

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In Section 2, we recount the relevant definitions and results from [7]. In that paper, two approaches to dynamic choice theory were given—one descriptive and the other normative. They were shown to be equivalent in a natural sense. We use only the normative approach here, as the resulting development closely parallels the conventional “expected utility” criterion development. The crucial notion introduced in Section 2 is a *temporal lottery*, which is a generalization of a lottery (or probability distribution) on the payoff vector. The generalization is that uncertainty is “dated” by the time of its resolution in a temporal lottery; in comparison, this “dating” is not encoded by a probability distribution on the payoff vector. This allows us to introduce preferences which can distinguish between temporal lotteries solely because the times at which their uncertainty resolves differ.

In Section 3, we define what we mean by a sequential decision problem and compare this with the conventional definition of such a problem and with what was termed a dynamic choice problem in [7]. After defining (measurable) strategies and policies, a crucial construction is given: Each strategy, payoff history, and state give rise to a corresponding temporal lottery, viz., the temporal lottery that the individual faces if, given the payoff history and state, he thereafter uses that strategy. (This parallels the usual construction of a probability measure on the payoff vector from a strategy, history, and state.)

This construction allows us in Section 4 to extend in a natural fashion the (assumed) preference relation on temporal lotteries to a preference relation on strategies. Given a representation of the preferences on temporal lotteries (as described in Section 2), we give a dynamic programming recursion which defines value functions for strategies, and show that these value functions represent the preferences on strategies. Optimality of strategies is defined and optimality criteria are given. Finally, we adapt our results to the general operator approach to sequential decision problems as developed in [12], which permits us to cite further results and to investigate the optimality of strategies which have special structure. As examples, we give conditions under which there exists an optimal (measurable) strategy and under which there exists an optimal “memoryless” strategy. (These results are only sketched, as the mathematics employed are quite standard. Our objective in this paper is primarily to provide the connection between the dynamic choice theory of [7] and the standard theory of sequential decision problems/dynamic programming.)

Throughout, we deal only with finite horizon problems (that is, problems with only a finite number of times at which actions must be taken). A treatment of countable stage decision problems must await development of a theory of preferences for countable stage temporal lotteries. Also, we do not consider “almost-optimal” criteria, although the details of a treatment of such criteria will become apparent.

Mitten [9] and Sobel [15] have dealt with general ordinal criteria for sequential decision problems. In particular, Section 5 of Sobel formally subsumes the preference structures we deal with here. But by dealing only with the special preference structures axiomatized in [7], we are able to give sharper results. A

special case of the sequential decision problems treated in this paper was considered in [12]. It was given as an example of a problem that could not be modelled using a conventional affine operator or even extrema of affine operators. However, it was not axiomatized and there was no recognition of the phenomenon of temporal resolution of uncertainty. Finally, preferences which have identical mathematical representations as those considered here, but which are motivated somewhat differently, are developed by Selden [14].

2. DYNAMIC CHOICE THEORY

Let T be a positive integer and, for each (time) $t = 0, 1, \dots, T$, let Z_t be a compact Polish (i.e., complete separable metric) space. The set Z_t is the set of possible *payoffs* at time t , with generic element z_t . Define $Y_1 = Z_0$ and, recursively, $Y_t = Y_{t-1} \times Z_{t-1}$ for $t = 2, \dots, T+1$. The set Y_t is the set of payoff *histories* up to (but not including) the time t payoff, with generic element $y_t = (z_0, \dots, z_{t-1})$. Note that Y_{T+1} is the set of complete payoff vectors. For notational convenience, we will sometimes write Y_t or y_t when $t = 0$; Y_0 can be taken to be any convenient singleton set.

The objects on which the individual is assumed to have primitive preferences are called *temporal lotteries*. There is uncertainty concerning the payoffs that will be received, and we assume that this uncertainty resolves at times $t = 0, \dots, T$ and that all uncertainty concerning z_t must resolve on or before time t . However, the uncertainty concerning z_t may partially or completely resolve before time t , and the individual may prefer earlier or later resolution of this uncertainty. The concept of a temporal lottery is a formal way of "dating" uncertainty by the time of its resolution. This is accomplished by the following definitions.

Let D_T^* be the space of Borel probability measures on Z_T , endowed with the Prohorov metric (i.e., the metric of weak convergence). Recursively, let D_t^* be the space of all Borel probability measures on $Z_t \times D_{t+1}^*$. (This construction is possible because each D_t^* , metrized with the Prohorov metric, is compact and Polish, cf. [11].) Generic elements of D_t^* are denoted by d_t . Note that each D_t^* is a mixture space: If $d_t, d'_t \in D_t^*$ and $\alpha \in [0, 1]$, let $\alpha d_t + (1 - \alpha) d'_t$ denote the element of D_t^* which, for A a Borel measurable subset of $Z_t \times D_{t+1}^*$, assigns measure $\alpha \cdot d_t(A) + (1 - \alpha) \cdot d'_t(A)$ to A . Elements of D_t^* can be depicted as probability (or event) trees with chance nodes for times $t, t+1, \dots, T$, as described in [7]. (In that paper, degenerate "choice nodes" were included in the depiction of elements of D_t^* . Here, it is convenient to delete them.)

The space D_0^* is the space of temporal lotteries. Contained within D_0^* are temporal lotteries where no uncertainty resolves until time t or after. Clearly, in these temporal lotteries the payoffs at times $0, \dots, t-1$ must be deterministic, given by some $y_t \in Y_t$ and the "beyond time t " structure is given by some $d_t \in D_t^*$. So we can write $Y_t \times D_t^*$ (with generic element (y_t, d_t)) to denote the set of these special temporal lotteries. Note that in this notation, $Y_t \times D_t^* \cong Y_{t+1} \times D_{t+1}^*$.

A further piece of notation: For $d_t \in D_t^*$ and f a bounded measurable function on $Z_t \times D_{t+1}^*$, the expectation of f with respect to d_t is denoted by $E_{d_t}[f]$.

We assume, as in [7], that the individual expresses a preference ordering on elements of D_0^* , denoted by \succeq , with \sim and $>$ denoting indifference and strict preference, respectively, which satisfies the following axiom:

AXIOM N: (a) The relation \succeq is complete and transitive. (b) The relation \succeq is continuous. (c) If $d_n, d'_n \in D_n^*$ and $y_n \in Y_n$ are such that $(y_n, d_n) > (y_n, d'_n)$, then $(y_n, \alpha d_n + (1 - \alpha)d'_n) > (y_n, \alpha d'_n + (1 - \alpha)d'_n)$ for all $\alpha \in (0, 1]$ and $d'_n \in D_n$.

Parts (a) and (b) of the axiom should be clear. Part (c) is a "temporal substitution" property which allows us to apply the machinery of cardinal utility theory (cf. [4]) and obtain the following representation theorem.

THEOREM 1: A relation \succeq on D_0^* satisfies Axiom N if and only if there exist continuous functions $u_t^*: Y_t \times Z_t \times R \rightarrow R$ for $t = 0, 1, \dots, T-1$, and $u_T^*: Y_T \times Z_T \rightarrow R$ such that (a) for $t = 1, \dots, T-1$, u_t^* is strictly increasing in its third argument, and (b) if we define $U_T^*: Y_T \times D_T^* \rightarrow R$ by $U_T^*(y_T, d_T) = E_{d_T}[u_T^*(y_T, \tilde{z}_T)]$ and $U_t^*: Y_t \times D_t^* \rightarrow R$ ($t = 0, \dots, T-1$) by

$$U_t^*(y_t, d_t) = E_{d_t}[u_t^*(y_t, \tilde{z}_t, U_{t+1}^*((y_t, \tilde{z}_t), \tilde{d}_{t+1}))],$$

then $(y_t, d_t) \succeq (y_t, d'_t)$ if and only if $U_t^*(y_t, d_t) \geq U_t^*(y_t, d'_t)$.

(Here, R denotes the real line. Tildes denote random variables.) The proof of this result may be found in [7]. The functions $\{u_t^*\}$ are called *basic utilities* representing \succeq . This preference structure is *not* representable by the expectation of a single cardinal utility function on the vector of payoffs. Here, the individual may distinguish between temporal lotteries when the only difference between them is in the timing of the resolution of uncertainty. Only when the individual is indifferent to the timing of resolution of uncertainty can his preferences be represented by a single utility function. (The formal conditions for this are Axiom N and $\alpha(y_n, d_n) + (1 - \alpha)(y_n, d'_n) \sim (y_n, \alpha d_n + (1 - \alpha)d'_n)$ for all α, y_n, d_n , and d'_n .) The nature of the individual's preferences for earlier or later resolution of uncertainty are given by the shape of the u_t^* as a function of their third argument. See [7] for further details.

A special case of particular interest in the context of Markov decision problems is when the preference relation is *temporally separable*. By this we mean: If $(y_n, d_n) \succeq (y_n, d'_n)$ for some $y_n \in Y_n$, then $(y'_n, d_n) \succeq (y'_n, d'_n)$ for all $y'_n \in Y_n$. Verbally, preferences concerning what happens beyond time t are independent of previous payoffs. If this assumption holds, the representation simplifies as follows. The function u_T^* can be assumed to be a function of z_T only and, for $t < T$, u_t^* can be taken to be a function on $Z_t \times R$. (In the case where preferences can be represented by a single cardinal utility function on the vector of payoffs, this implies that the utility function is separable, a special case being the additive utility function which dominates the literature of Markov decision problems.)

One point concerning this dynamic choice theory should be carefully noted. We are implicitly assuming that the individual's preferences are temporally consistent in the following sense. His preferences for "time t " temporal lotteries (elements of D_t^*) at time t depend on the payoff history y_t and are given by \succeq restricted to $\{y_t\} \times D_t^*$. (This is *not* an assumption of "stationarity" of preferences or what Sobel [15] calls "temporal persistence of preferences." Such an assumption would only make sense in our context in an infinite horizon model (ours has a finite horizon) and would entail requiring that preferences be stationary over time rather than that they be consistent.) This assumption does rule out consideration of "changing preferences" in the sense of Hammond [5] (among others).

3. SEQUENTIAL DECISION PROBLEMS, POLICIES, AND STRATEGIES

DEFINITION: A *sequential decision problem* over payoff spaces $\{Z_t; t = 0, \dots, T\}$ is a collection $\{X_t, D_t, A_t(x_t); t = 0, \dots, T\}$ of (a) *state spaces* X_t with generic element x_t which are Polish spaces, (b) *action spaces* D_t with generic d_t , where D_t is the space of Borel probability measures on $Z_t \times X_{t+1}$, metrized throughout by the Prohorov metric, and (c) for each $x_t \in X_t$, sets $A_t(x_t)$ of *feasible actions* from state x_t , which are subsets of D_t .

As in Section 2, we write $E_{d_t}[f]$ for the expectation of a bounded Borel measurable function f on $Z_t \times X_{t+1}$ taken with respect to d_t . Also, X_{T+1} is taken for notational convenience to be some singleton set. Note that D_t , as metrized is Polish and is compact if X_{t+1} is.

This differences between our definition and the standard definitions of a sequential decision problem found in the literature (see Blackwell [1], Hinderer [6], Strauch [17], etc.) are as follows. The notion of a state is just as in the literature—the state is a statistic which tells the individual which actions are feasible. An action here, however, is *identified* as the probability distribution on pairs of immediate payoff and next state. In comparison, the literature typically has transition probabilities as functions of states and actions. This difference is purely semantic. In [7], we used the terminology dynamic choice problem rather than sequential decision problem. There, x_t was used to denote the (necessarily closed) set of feasible actions itself, rather than a statistic indexing that set. That is, $A_t(x_t)$ would simply be x_t . The definition used here allows for greater generality— $A_t(\cdot)$ needn't be a continuous correspondence in the X_t topology. Note that we do not (yet) wish to assume that $A_t(\cdot)$ has any structural properties, such as being a measurable correspondence.

DEFINITIONS: Given payoff spaces $\{Z_t\}$ and a sequential decision problem $\{X_t, D_t, A_t(x_t)\}$, an admissible *policy* for time t is any Borel measurable function $\delta_t: Y_t \times X_t \rightarrow D_t$ such that $\delta_t(y_t, x_t) \in A_t(x_t)$ for all $y_t \in Y_t$ and $x_t \in X_t$. The set of (admissible) policies for time t is denoted by Δ_t . An (admissible) *strategy* is a vector of policies $\pi = (\delta_0, \delta_1, \dots, \delta_T) \in \Delta_0 \times \dots \times \Delta_T = \Pi$.

A policy for time t specifies a feasible action for each pair of payoff history and state. Note that we do not formally allow the selection of an action to depend on previous states and/or actions but only on previous payoffs. (Of course, one can technically incorporate the state history into the payoff history in applications (after compactifying X_t) and then use the machinery of Section 4 to show that the state history can be considered to be irrelevant if it is irrelevant to the individual's preferences.) Since nothing has been assumed about the measurability of $A_t(\cdot)$, the requirement that policies are measurable makes it possible that some A_t (and thus I) is empty.

For each strategy π , time t payoff history y_t and state x_t , there exists a corresponding temporal lottery—the temporal lottery which arises if at time t with history y_t and state x_t , the individual uses strategy π . The formal definition of this temporal lottery follows.

Fix $\pi = (\delta_0, \delta_1, \dots, \delta_T)$ and define $d_T^\pi: Y_T \times X_T \rightarrow D_T^*$ by $d_T^\pi(y_T, x_T) = \delta_T(y_T, x_T)$. For $t = T - 1, \dots, 0$, recursively define $d_t^\pi: Y_t \times X_t \rightarrow D_t^*$ by

$$(1) \quad d_t^\pi(y_t, x_t) = \delta_t(y_t, x_t) \circ (1, d_{t+1}^\pi((y_t, \cdot), \cdot))^{-1},$$

where 1 here denotes the identity map on Z_t . Equation (1) can be interpreted as follows. Suppose (inductively) that $d_{t+1}^\pi(y_{t+1}, x_{t+1})$ is the D_{t+1}^* "part" of the temporal lottery associated with π, y_{t+1} and x_{t+1} . That is, $d_{t+1}^\pi(y_{t+1}, x_{t+1})$ is a measure on $Z_{t+1} \times D_{t+2}^*$. Fix y_t, x_t , policy δ_t (from strategy π), and a measurable subset A of $Z_t \times D_{t+1}^*$. From (y_t, x_t) and using δ_t , the decision maker will be "in" A if the ensuing (z_t, x_{t+1}) are such that $(z_t, d_{t+1}^\pi((y_t, z_t), x_{t+1})) \in A$. That is, he will be "in" A if $(z_t, x_{t+1}) \in (1, d_{t+1}^\pi((y_t, \cdot), \cdot))^{-1} A$. The probability of this is of course $\delta_t(y_t, x_t) \circ (1, d_{t+1}^\pi((y_t, \cdot), \cdot))^{-1} A$. (An implication of this construction is that if $f: Y_{t+1} \times D_{t+1}^* \rightarrow R$ is bounded and measurable and $g: Y_{t+1} \times X_{t+1} \rightarrow R$ is defined by $g(y_{t+1}, x_{t+1}) = f(y_{t+1}, d_{t+1}^\pi(y_{t+1}, x_{t+1}))$, then $E_{d_t^\pi(y_t, x_t)}[f] = E_{\delta_t(y_t, x_t)}[g]$.) Of course, the temporal lottery corresponding to π, y_t and x_t is $(y_t, d_t^\pi(y_t, x_t))$.

4. DYNAMIC PROGRAMMING

Taken as primitive is a binary relation \succeq on the set of temporal lotteries which satisfies Axiom N. Let $\{u_i^*\}$ be basic utilities which represent \succeq in the sense of Theorem 1.

We begin by using \succeq to induce preference relations on strategies. For each pair (y_t, x_t) , it is natural to say that strategy π is (y_t, x_t) -preferred to π' if the temporal lottery which arises from π, y_t and x_t is preferred to that arising from π', y_t and x_t . Formally,

$$\pi \succeq_{(y_t, x_t)} \pi' \quad \text{if} \quad (y_t, d_t^\pi(y_t, x_t)) \succeq (y_t, d_t^{\pi'}(y_t, x_t)).$$

Obviously, each $\succeq_{(y_t, x_t)}$ is complete and transitive. Furthermore, π is said to be optimal if $\pi \succeq_{(y_t, x_t)} \pi'$ for all y_t, x_t , and π' .

The connection with dynamic programming is that we shall use the basic utilities which represent \succeq to represent the relations $\succeq_{(y_t, x_t)}$. For $\pi = (\delta_0, \delta_1, \dots, \delta_T)$, define value functions for strategy π by the following recursion. Let

$v_T(\pi; y_T, x_T) = E_{\delta_T(y_T, x_T)}[u_T^*(y_T, \tilde{z}_T)]$ and, for $t = T - 1, \dots, 0$, define $v_t(\pi; \cdot, \cdot): Y_t \times X_t \rightarrow R$ by

$$(2) \quad v_t(\pi; y_t, x_t) = E_{\delta_t(y_t, x_t)}[u_t^*(y_t, \tilde{z}_t, v_{t+1}(\pi; (y_t, \tilde{z}_t), \tilde{x}_{t+1}))].$$

Also, define *optimal value functions* by

$$f_t(y_t, x_t) = \sup_{\pi \in \Pi} v_t(\pi; y_t, x_t),$$

where the suprema are taken pointwise. (The terminology used will be justified shortly.)

LEMMA: For $\pi \in \Pi$, $y_t \in Y_t$ and $x_t \in X_t$, $v_t(\pi; y_t, x_t) = U_t^*(y_t, d_t^\pi(y_t, x_t))$.

PROOF: We proceed by backward induction in t . For $t = T$, $v_T(\pi; y_T, x_T) = E_{\delta_T(y_T, x_T)}[u_T^*] = U_T^*(y_T, \delta_T(y_T, x_T)) = U_T^*(y_T, d_T^\pi(y_T, x_T))$. Assume the result is true for $t + 1$. Then

$$\begin{aligned} U_t^*(y_t, d_t^\pi(y_t, x_t)) &= E_{d_t^\pi(y_t, x_t)}[u_t^*(y_t, \tilde{z}_t, U_{t+1}^*((y_t, \tilde{z}_t), \tilde{d}_{t+1}))] \\ &= E_{\delta_t(y_t, x_t)}[u_t^*(y_t, \tilde{z}_t, U_{t+1}^*((y_t, \tilde{z}_t), d_{t+1}^\pi((y_t, \tilde{z}_t), \tilde{x}_{t+1})))] \\ &\quad \text{(by (1))} \\ &= E_{\delta_t(y_t, x_t)}[u_t^*(y_t, \tilde{z}_t, v_{t+1}(\pi; (y_t, \tilde{z}_t), \tilde{x}_{t+1}))] \text{ (by induction)} \\ &= v_t(\pi; y_t, x_t) \quad \text{(by definition).} \end{aligned} \quad Q.E.D.$$

Immediate from the lemma and the definitions is the following.

THEOREM 2: For all π, π', y_t and x_t , $\pi \succeq_{(y_t, x_t)} \pi'$ if and only if $v_t(\pi; y_t, x_t) \geq v_t(\pi'; y_t, x_t)$. Strategy π is optimal if and only if $v_t(\pi; y_t, x_t) = f_t(y_t, x_t)$ for all y_t and x_t .

With this result and the recursive definition of $\{v_t\}$ given by equation (2), we can call upon the extant theories of finite horizon dynamic programming to investigate issues such as the existence and possible structure of optimal strategies. The general operator approach of [12] can be utilized as follows.

Let V_t denote the set of bounded Borel measurable real valued functions defined on $Y_t \times X_t$ for $t = 0, \dots, T + 1$. For $\delta \in \Delta_t$, $v \in V_{t+1}$, and $t < T$, define the operator $H_{t\delta}$ on V_{t+1} by $[H_{t\delta}v](y_t, x_t) = E_{\delta(y_t, x_t)}[u_t^*(y_t, \tilde{z}_t, v((y_t, \tilde{z}_t), \tilde{x}_{t+1}))]$. For $t = T$, define $H_{T\delta}$ on V_{T+1} by $[H_{T\delta}v](y_T, x_T) = E_{\delta(y_T, x_T)}[u_T^*(y_T, \tilde{z}_T)]$. The continuity of u_t^* insures that $H_{t\delta}: V_{t+1} \rightarrow V_t$. Also, for $v \in V_{t+1}$, define $A_t v = \sup_{\delta \in \Delta} H_{t\delta}v$, where the supremum is taken pointwise. Standard examples (see, for example [17]) show that $A_t v$ need not be in V_t .

Suppose one can define a distinguished subset of V_t , denoted V_t^* and called the set of *structured value functions*, and distinguished subsets of each Δ_t , denoted Δ_t^* and called the sets of *structured policies*, such that the following condition is met.

CONDITION PA (Preservation and Attainment): If $v \in V_{t+1}^*$ (for $t \leq T$), then $A_t v \in V_t^*$ and there exists $\delta \in \Delta_t^*$ such that $H_{t\delta} v = A_t v$.

Then the standard procedure of dynamic programming can be utilized.

THEOREM 3: If condition PA holds, then (a) $f_t \in V_t^*$ for $t = 0, \dots, T$, (b) $f_t = A_t f_{t+1}$ for $t = 0, \dots, T$, (c) $\pi = (\delta_0, \dots, \delta_T)$ is optimal if and only if $H_{t\delta_t} f_{t+1} = A_t f_{t+1}$ for each t , and (d) there exists a strategy $\pi \in \Delta_0^* \times \Delta_1^* \times \dots \times \Delta_T^*$ which is optimal.

The proof of Theorem 1 in [12] can easily be modified to give this result. (Note that $H_{t\delta}$ is isotone because u_t' is nondecreasing in its third argument. Also, the proof in [12] nominally requires that the $H_{t\delta}$ be Lipschitzian, which may not be the case here, but is only used in [12] for ε -optimality. In fact, the Lipschitzian condition would not be necessary even if our objective was ε -optimality—the compactness of the Z_t allows us to replace the Lipschitzian condition with one of uniform continuity.)

We conclude with a pair of lemmas and corollaries to Theorem 3. The lemmas give conditions under which Condition PA holds (i) for Δ_t^* being the set of all Borel measurable policies and (ii) for Δ_t^* being the set of “memoryless” policies. The corollaries then harvest the results of Theorem 3 for these cases. We make use of the selection theorem of Schäl [13, Corollary 4] which slightly generalizes that of Dubins and Savage [3].

LEMMA: If $A_t(x_t)$ is compact in D_t and $A_t(\cdot)$ is an upper semi-continuous correspondence for all t , then condition PA holds for $\Delta_t^* = \Delta_t$ and V_t^* the set of bounded u.s.c. functions on $Y_t \times X_{t+1}$.

A special case of the lemma is the case in which all the Z_t and X_t are discrete and each $A_t(x_t)$ is compact. Then A_t is automatically u.s.c.

PROOF: If v is a bounded u.s.c. function on $Y_{t+1} \times X_{t+1}$, then $u_t^*(y_t, z_t, v((y_t, z_t), x_{t+1}))$ is as well, because u_t^* is continuous and nondecreasing in its third argument. Then (cf. Maitra [8, Lemma 4]) $w((y_t, x_t), d_t) = E_{d_t}[u_t^*(y_t, \tilde{z}_t, v((y_t, \tilde{z}_t), \tilde{x}_{t+1}))]$ is bounded and u.s.c. on $Y_t \times X_t \times D_t$. That w and $A_t(\cdot)$ are u.s.c. imply that $\sup_{d_t \in A_t(x_t)} w((y_t, x_t), d_t)$ is bounded and u.s.c. Thus $A_t v \in V_t^*$. Finally, [13, Theorem 2] yields the existence of $\delta \in \Delta_t^*$ such that $A_t v = H_{t\delta} v$. Q.E.D.

COROLLARY: If $A_t(x_t)$ is compact in D_t and $A_t(\cdot)$ is an u.s.c. correspondence, then there exists an optimal (measurable) strategy.

DEFINITIONS: A policy $\delta \in \Delta_t$ is *memoryless* (it could be called Markov) if δ_t is x_t -measurable; that is, if $\delta(y_t, x_t) = \delta(y_t', x_t)$ for all x_t, y_t , and y_t' . A strategy π is memoryless if it is composed of memoryless policies.

LEMMA: If $A_t(x_t)$ is compact in D_t , $A_t(\cdot)$ is an u.s.c. correspondence, and preferences are temporally separable (cf. Section 2), then condition PA holds with V_t^* the set of bounded u.s.c. x_t -measurable functions and Δ_t^* the set of memoryless (time t) policies.

The proof is apparent, following the sketch of the proof of the previous lemma.

COROLLARY: If $A_t(x_t)$ is compact in D_t , $A_t(\cdot)$ is an u.s.c. correspondence, and preferences are temporally separable, then there exists an optimal memoryless strategy.

Note that in this case, the recursions analogous to "Bellman's equations" are

$$f_t(x_t) = \sup_{d \in A_t(x_t)} E_d[u_t^*(\tilde{z}_t, f_{t+1}(\tilde{x}_{t+1}))],$$

instead of the standard equations for separable von Neumann-Morgenstern utility:

$$f_t(x_t) = \sup_{d \in A_t(x_t)} E_d[\alpha_t(\tilde{z}_t) + \beta_t(\tilde{z}_t) \cdot f_{t+1}(\tilde{x}_{t+1})]$$

(for $\beta_t > 0$).

Stanford University

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