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Is intertemporal choice theory testable?

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Abstract

Kreps–Porteus preferences constitute a widely used alternative to time separability. We show in this paper that with these preferences utility maximization does not impose any observable restrictions on a household's savings decisions or on choices in good markets over time. The additional assumption of a weakly separable aggregator is needed to ensure that the assumption of utility maximization restricts intertemporal choices. Under this assumption, choices in spot markets are characterized by a strong axiom of revealed preferences (SSARP).

Under uncertainty Kreps–Porteus preferences impose observable restrictions on portfolio choice if one observes the last period of an individual's planning horizon. Otherwise there are no restrictions.

Keywords: Intertemporal choice; Non-parametric restrictions

1. Introduction

There is a large literature on testing individual demand data for consistency with utility maximization (see, e.g. Afriat, 1967; Varian, 1982; Chiappori and Rochet, 1987). In this literature, it is assumed that one observes how an individual's choices vary as prices and his income vary. However, data of this sort can only be obtained through experiments. If one actually records an individual's actions in markets over time, these classical tests of demand theory might be useless because they neglect the fact that an agent's choices today may be affected by his choice set tomorrow or his savings from previous periods. Tests of demand theory which use market data must be tests of intertemporal choice models. If one assumes that all agents maximize time-separable and time-invariant utility and if one only observes their choices in spot markets (i.e. saving decisions or incomes are unobservable) the analysis in Chiappori and Rochet (1987) remains valid and a strong version of the strong axiom of revealed preferences (SSARP, see Chiappori and Rochet, 1987) is necessary and sufficient

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for data to be consistent with utility maximization. However, time separability is a very strong restriction on preferences which holds only if one assumes that preference orderings on consumption streams from $t = 1, \dots, s$ are independent of an agent's expectations on his consumptions for the periods from $s + 1$ onwards.

While it seems intuitively reasonable to argue that history independence and time consistency together with some form of stationarity is enough to ensure that an agent's choice behavior is restricted by the assumption of utility maximization, we show that this intuition is wrong and that the assumption of Kreps–Porteus preferences (Kreps and Porteus, 1978) does not impose any restriction on observed choices. It follows from our analysis that such widely used concepts as time consistency or pay-off history independence are not testable if one does not use experimental data but is confined to data on individual behavior in markets.

Uncertainty adds an additional dimension to the agent's choice problem. Risk aversion will generally impose restrictions on portfolio selection when continuation utilities are identical across all possible next period states. The question then arises to what extend these restrictions are observable. If one observes the last period of an individual's planning horizon, these restrictions are reflected in the individual's portfolio holdings coming into this last period. However, if we assume that the last period is not observable, the assumption of Kreps–Porteus preferences imposes no restrictions on portfolio selection even when all off sample path choices as well as all probabilities are observable.

These negative results raise the questions under which conditions utility maximization does impose restrictions on intertemporal choices. We derive a sufficient (additional) condition on the aggregator function, which ensures that the model is testable. If the aggregator function is weakly separable then choices on spot markets must satisfy SSARP. If asset prices are unobservable, SSARP is also sufficient for the choices to be rationalizable by a time-separable utility function and the two specifications are therefore observationally equivalent.

We develop our arguments for a finite horizon choice problem.

Without stationarity assumptions, as long as the number of observed choices is finite, one cannot refute the conjecture that the agent maximizes a Kreps–Porteus style utility function over an infinite horizon consumption program. However, it seems natural to impose a Markov structure on the infinite horizon problem and to confine attention to recursive utility of the Epstein–Zin type (Epstein and Zin, 1989). An extension of these results to the infinite horizon problem is subject to future research.

The paper is organized as follows. In Section 2 we introduce the model and some notation. Section 3 proves the main result and discusses its implications for a finite horizon choice problems, both under certainty and under uncertainty.

2. The model

We consider an individual's choice problem over $\bar{T} + 1$ periods, $t = 0, \dots, \bar{T}$ with uncertainty resolving each period. We take as given an event tree Ξ with nodes $\xi \in \Xi$. Let ξ_0 be the root node, i.e. the unique node without a predecessor. For all other nodes, let ξ_- be the unique predecessor of node ξ. For all nodes $\xi \in \Xi$, let $\mathfrak{I}(\xi)$ be the set of its immediate successors. Nodes without successors, i.e. $\mathfrak{I}(\xi)$ is empty, are called terminal nodes. Finally, we collect all nodes which are possible at some period t in a set $\mathcal{N}_t$ and we denote by M the total number of nodes in the event tree. We assume that M is finite. For simplicity, we assume that there are no terminal nodes in any $\mathcal{N}_t$ for $t < T$.

At each node ξ there are J short-lived assets with asset j paying $d_j(\zeta) \in \mathbb{R}$ at all nodes $\zeta \in \Im(\xi)$, its price being denoted by $q_j(\xi)$.

At each node $\xi \in \Xi$, the individual receives an exogenous income $I(\xi) \in \mathbb{R}_+$ (either from selling endowments or from transfers) and he is active in spot and asset markets. He faces prices $p(\xi) \in \mathbb{R}^L_{++}$ and chooses a consumption bundle $c(\xi) \in \mathbb{R}^L_+$.

The agent's consumption decisions must be supported by portfolio choices $(\theta(\xi))_{\xi\in\Xi}$, $\theta(\xi) \in \mathbb{R}^J$. All consumptions and portfolio choices $(c(\xi), \theta(\xi))_{\xi\in\Xi}$ must lie in the individual's budget set which we define as

$$\mathcal{B}((p(\xi), q(\xi), d(\xi), I(\xi))_{\xi\in\Xi}) = \{(c(\xi), \theta(\xi))_{\xi\in\Xi} : p(\xi)c(\xi) + q(\xi)\theta(\xi) \leq I(\xi) + \theta(\xi_-)d(\xi), c(\xi) \geq 0 \text{ for all } \xi \in \Xi\}$$

where we normalize $\theta(\xi_{0-}) := 0$.

The agent attaches a positive probability to each node. Given a node $\zeta \in \Xi$ and a direct successor $\xi \in \Im(\xi)$, we denote by $\mu(\xi)$ the (unconditional) probability of node ξ and by $\mu(\xi|\zeta)$ the conditional probability of ξ given ζ.

We say that an agent's utility function $u : \mathbb{R}^{LM}_+ \to \mathbb{R}$ is of the Kreps–Porteus type if $u((c(\xi))_{\xi\in\Xi}) = v_{\xi 0}$, where v_ξ, utility at node is ξ recursively defined by

$$v_\xi(c(\xi)) = W(c(\xi), \mu(\xi))$$

with

$$\mu(\xi) = \sum_{\zeta\in\Im(\xi)} \pi(\zeta|\xi)v_\zeta(c(\zeta)) \quad \text{for all non-terminal } \xi$$

We will assume throughout that the aggregator $W : \mathbb{R}^L_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is twice continuously differentiable, strictly increasing and strictly concave. We normalize $W(0, 0) = 0$ and hence impose $\mu(\xi) = 0$ for terminal nodes ξ.

We also impose two regularity conditions on the aggregator which are often needed to extend the preference specification to infinite horizon problems (see, e.g. Koopmans (1960) or Epstein and Zin (1989)).

LS1: The function $W(\cdot, \cdot)$ is bounded, i.e.

$$\sup_{x\in\mathbb{R}^L_+, y\geq 0} W(x, y) < \infty$$

LS2: The second partial derivative of $W(\cdot, \cdot)$ is bounded above by one, i.e.

$$\partial_y W(x, y) < 1 \quad \text{for all } x \in \mathbb{R}^L_+, y \geq 0$$

In a slight abuse of notation we will refer to utility functions which satisfy all of the above assumptions as 'Kreps–Porteus utility'.

2.1. Observations

In order to make our main argument, we assume that we observe choices, prices and incomes at periods $t = 0, \ldots, T \leq \bar{T}$. Under uncertainty, we assume that we observe these variables and all dividends all nodes $\xi \in \mathcal{N}_t, t = 0, \ldots, T$ as well as all relevant probabilities (which might be known when we assume objective laws of motion). In order to present our main argument as strong as possible, we assume that all last period continuation utilities $\mu(\xi)$ for all $\xi \in \mathcal{N}_T$ are known (which might justified because they are all zero and it is the last period of the individual's planning horizon, i.e. $T = \bar{T}$). When we discuss our result below, we will assess how realistic these assumptions are. We define $\Omega = \cup_{t=0}^{T} \mathcal{N}_t$ to be the set of all observable nodes in the event tree. An extended observation is then given by

$$\mathcal{O} = ((c(\xi), \theta(\xi), d(\xi), q(\xi), p(\xi), \pi(\xi))_{\xi \in \Omega}, (\mu(\xi))_{\xi \in \mathcal{N}_T})$$

The question is whether there are restrictions on this observations imposed by the assumption of Kreps–Porteus utility. It is important to note that if one does not observe an agent's choices over his entire planning horizon (i.e. if $\bar{T} > T$) one is free to choose choices as well as prices, dividends and incomes at all nodes which are not in Ω. We therefore have the following definition.

Definition 1. An extended observation

$$\mathcal{O} = ((c(\xi), \theta(\xi), d(\xi), q(\xi), p(\xi), \pi(\xi))_{\xi \in \Omega}, (\mu(\xi))_{\xi \in \mathcal{N}_T})$$

is said to be rationalizable by Kreps–Porteus utility if there exist $c(\xi)$, $\theta(\xi)$, $p(\xi)$, $q(\xi)$, $I(\xi)$, $d(\xi)$ for all $\xi \in \Xi$, $\xi \notin \Omega$ and if there exists a Kreps–Porteus utility function $u(\cdot)$ which is consistent with the probabilities $(\pi(\xi))_{\xi \in \Omega}$ and the last period continuation utilities $\mu(\xi), \xi \in \mathcal{N}_T$ such that

$$(c(\xi), \theta(\xi))_{\xi \in \Xi} \in \arg \max_{c \in \mathbb{R}_{+}^{LM}, \theta \in \mathbb{R}^{JM}} u(c)$$

such that

$$(c(\xi), \theta(\xi))_{\xi \in \Xi} \in \mathcal{B}((p(\xi), (q(\xi), (d(\xi), (I(\xi))_{\xi \in \Xi})$$

It is well known that the absence of arbitrage is a necessary condition for the agent's choice problem to have a finite solution

Definition 2. Prices and dividends $(p(\xi), q(\xi), d(\xi))_{\xi \in \Xi}$ preclude arbitrage if there is no trading strategy $(\theta(\xi))_{\xi \in \Xi}$ with $\theta_{0-} = 0$ such that if we define

$$D^{\theta}(\xi) = \theta(\xi_{-})d(\xi) - \theta(\xi)q(\xi)$$

$$D^{\theta}(\xi) \geq 0 \quad \text{for all } \xi \in \Xi \text{ and } D^{\theta} \neq 0$$

We will assume throughout that the observed prices preclude arbitrage and that the observed choices lie in the agent's budget set. We also assume that we never observe zero consumption, i.e. that $c(\xi) \neq 0$ for all $\xi \in \Omega$ (although consumption of a given commodity might sometimes be zero, it cannot be the case that the agent chooses to consume nothing at all) and that the agent does not trade assets in the last period of his planning horizon, $\theta(\xi) = 0$ for all $\xi \in \mathcal{N}_{\bar{T}}$. Finally, we assume that $\mu(\xi) = 0$ for terminal nodes $\xi \in \mathcal{N}_{\bar{T}}$. These restrictions on observed choices are trivial restrictions and follow directly from monotonicity.

3. Observable restrictions

For our non-parametric analysis we need to derive Afriat inequalities (Afriat, 1967). These non-linear inequalities completely characterize choices which are consistent with the maximization of a Kreps–Porteus utility function.

Lemma 1. *An extended observation*

$$\mathcal{O} = ((c(\xi), \theta(\xi), d(\xi), q(\xi), p(\xi), \pi(\xi))_{\xi\in\Omega}, (\mu(\xi))_{\xi\in\mathcal{N}_T})$$

with all $c(\xi) \in \mathbb{R}^L_{++}$ is rationalizable by a Kreps–Porteus utility function if and only if there exist positive numbers $\lambda(\xi), \eta(\xi), \gamma(\xi), W(\xi))_{\xi\in\Omega}$ with $\gamma(\xi) < 1$ for all $\xi \in \Omega$, with $\eta(\xi) = \mu(\xi)$ for all $\xi \in \mathcal{N}_T$ and with

$$\eta(\xi) = \sum_{\zeta\in\Im(\xi)} \pi(\zeta|\xi)W(\zeta)$$

for all $\xi \in \mathcal{N}_t$, $t < T$, such that

- *for all $\xi \in \mathcal{N}_t$, $t < T$,*

$$\lambda(\xi)q_j(\xi) = \gamma(\xi) \sum_{\zeta\in\Im(\xi)} \pi(\zeta|\xi)\lambda(\zeta)d_j(\zeta) \quad \text{for } j = 1, \dots, J \tag{U1}$$

- *for all $\xi, \zeta \in \Omega$,*

$$W(\xi) - W(\zeta) \leq \lambda(\zeta)p(\zeta)(c(\xi) - c(\zeta)) + \gamma(\zeta)(\eta(\xi) - \eta(\zeta)) \tag{U2}$$

the inequality holds strict if $c(\xi) \neq c(\zeta)$ or if $\eta(\xi) \neq \eta(\zeta)$.

If for some node $\xi \in \Omega$, the observed consumption, $c(\xi)$, lies on the boundary of $\mathbb{R}^L_+$ the conditions remain sufficient but are no longer necessary.

Proof. For the necessity part, consider the agent's first-order condition (which are necessary and sufficient for optimality of interior choices):

At any node $\xi \in \Xi$,

$$\partial_c W(c(\xi), \mu(\xi)) - \lambda(\xi)p(\xi) = 0$$

and

$$\eta(\xi)q_j(\xi) = \sum_{\zeta\in\Im(\xi)} \eta(\zeta)d_j(\zeta) \quad \text{for } j = 1, \dots, J \text{ and for all non-terminal } \xi \in \Xi$$

where $\lambda(\xi_0) = \eta(\xi_0)$ and where for all $\xi \neq \xi_0 \in \Xi$,

$$\lambda(\xi) = \lambda(\xi-)\frac{\eta(\xi)}{\pi(\xi|\xi-)\eta(\xi-)\partial_\mu W(c(\xi-), \mu(\xi-))}$$

Defining $\gamma(\xi) = \partial_\mu W(c(\xi), \mu(\xi))$, these first-order condition, together with the assumption that $W(\cdot, \cdot)$ is concave and the usual characterization of concave functions proves necessity: (U1) stems from the second set of first-order conditions and inequality (U2) characterizes strict concavity of $W(\cdot)$, where the first optimality condition is used to substitute for $\partial_c W$. The assumption that $\gamma(\xi) < 1$ for all $\xi \in \Omega$ follows from condition LS2.

For the sufficiency part, assume that the unknown numbers exist and satisfy the inequalities. We can then construct a piecewise linear aggregator function following Varian (1982):

Define

$$W(c, \mu) = \min_{\xi \in \Omega} \left\{ U(\xi) + \begin{pmatrix} \lambda(\xi) \\ \gamma(\xi) \end{pmatrix} \left[\begin{pmatrix} p(\xi)c \\ \mu \end{pmatrix} - \begin{pmatrix} p(\xi)c(\xi) \\ \eta(\xi) \end{pmatrix} \right] \right\}$$

The resulting function is clearly concave and strictly increasing and the function rationalizes the observation $\mathcal{O}$. Furthermore, the approach in Chiappori and Rochet (1987) can be used to construct a strictly concave and smooth aggregator function. Their argument goes through without any modification.

Since $\gamma(\xi) < 1$ for all ξ it follows immediately that LS2 must hold. LS1 follows from the fact that all constructed numbers are finite.

When $T < \bar{T}$, we can construct future dividends, prices and consumptions such that they are consistent with period T portfolio holdings and period T continuation utilities. The key is to observe that for all possible observed continuation utilities $\mu(\xi), \xi \in \mathcal{N}_T$ and for all last period portfolios $\theta(\xi), \xi \in \mathcal{N}_T$ there will exist unobserved next period dividends to rationalize them. □

We now use this characterization to show that the assumption of Kreps–Porteus utility is in general not testable using only market data since it imposes very few restrictions on observed choices.

Theorem 1. *Any possible extended observation $\mathcal{O}$ for which $\mu(\xi) \neq \mu(\zeta)$ for all nodes $\xi \neq \zeta \in \mathcal{N}_T$ can be rationalized by a Kreps–Porteus utility function.*

The following lemma is crucial for the proof of the theorem. While the lemma appears simple its proof turns out to be quite tedious.

Lemma 2. *For any finite event tree Ω, probabilities $(\pi(\xi))_{\xi \in \Omega}$ and positive numbers $\eta(\xi)$ for all terminal $\xi \in \Omega$ with $\eta(\xi) \neq \eta(\zeta)$ for all $\xi \neq \zeta$ there exist a $\bar{\gamma} > 0$, $(W(\xi), \gamma(\xi))_{\xi \in \Omega}$, $1 > \gamma(\xi) \geq \bar{\gamma}$ and $W(\xi) > 0$ for all $\xi \in \Omega$ as well as a number $\delta > 0$ such that*

$$W(\zeta) - W(\xi) + \gamma(\zeta)(\eta(\xi) - \eta(\zeta)) > \delta \quad \text{for all } \zeta, \xi \in \Omega \tag{1}$$

with

$$\eta(\xi) = \sum_{\zeta\in\Im(\xi)} \pi(\zeta|\xi) W(\xi) \quad \textit{for all non-terminal } \xi \in \Omega$$

Proof. We construct these number recursively. Let T denote the number of periods in Ω, let m denote the number of nodes in Ω and fix some $\varepsilon < 1/(m+2)$. Fix δ to ensure that $0 < \delta < \varepsilon$.

If $n_t = \sharp\mathcal{N}_t$ denotes the number of nodes at period t, we can define a function $\xi_t(i)$ by $\eta(\xi_t(1)) < \eta(\xi_t(2)) < \cdots < \eta(\xi_t(n_t))$. Since by assumption $\eta(\xi) \neq \eta(\zeta)$ for all $\xi, \zeta \in \mathcal{N}_T$, this function exists for $t = T$.

For T we can choose the associated $\gamma(\xi)$ such that

$$1 - \epsilon = \gamma(\xi_T(1)) = \gamma(\xi_T(2)) + \epsilon = \cdots = \gamma(\xi_T(n_T)) + (n_T - 1)\epsilon$$

Now choose $W(\xi_T(l)) > \eta(\xi_T(n_T))$ and define for $i = 2, \ldots, n_T$

$$W(\xi_T(i)) = W(\xi_T(i-1)) + \gamma(\xi_T(i-1))(\eta(\xi_T(i)) - \eta(\xi_T(i-1))) - \delta$$

Given $(W(\xi), \gamma(\xi))_{\xi\in\mathcal{N}_t}$ we can construct $(\eta(\xi), W(\xi), \gamma(\xi))$ for $\xi \in \mathcal{N}_{t-1}$ as follows: For all $\xi \in \mathcal{N}_{t-1}$, compute the new $\eta(\xi) = \sum_{\zeta\in\Im(\xi)} \pi(\zeta|\xi) W(\zeta)$. One can choose δ to ensure that $\eta(\xi) \neq \eta(\zeta)$ for all $\xi, \zeta \in \mathcal{N}_{t-1}$ and that the function ξ_{t-1} is well defined. Then define

$$\gamma(\xi_{t-1}(1)) = \gamma(\xi_t(n_t)) - \epsilon$$

and

$$\gamma(\xi_{t-1}(i)) = \gamma(\xi_{t-1}(i-1)) - \epsilon \quad \text{for } i = 2, \ldots, n_{t-1}$$

Also define

$$W(\xi_{t-1}(1)) = W(\xi_t(n_t)) + \gamma(\xi_t(n_t))(\eta(\xi_{t-1}(1)) - \eta(\xi_t(n_t))) - \delta$$

and

$$W(\xi_{t-1}(i)) = W(\xi_{t-1}(i-1)) + \gamma(\xi_{t-1}(i-1)) \times (\eta(\xi_{t-1}(i)) - \eta(\xi_{t-1}(i-1))) - \delta \quad \text{for } i = 2, \ldots, n_{t-1}$$

We can repeat the construction up to $W(\xi_0)$, $\gamma(\xi_0)$. Since there are finitely many nodes sufficiently small δ, ϵ can be found to ensure that for all t and all $\xi, \zeta \in \mathcal{N}_t, \eta_\xi \neq \eta_\zeta$. Furthermore, since $W(\cdot)$ is constructed as a piecewise linear increasing and concave function inequalities (1) must hold. □

With this lemma, the proof of the theorem is very short.

Proof of Theorem 1. For any $\varepsilon>0$ and $\bar{\gamma} > 0$, if for all $\xi \in \Omega, \gamma(\xi) \geq \bar{\gamma}$, one can find $(\lambda(\xi))_{\xi\in\Omega}$ which solve (U1) and which satisfy $0 < \lambda(\xi) < \varepsilon$. This follows from the absence of arbitrage and the fact that we can choose $\lambda(\xi_0)$ without any restrictions. Therefore, for any $\delta > 0$ and any observation on spot prices and consumptions one can find $\lambda(\xi)$ which satisfy (U1) and for which

$$\sup_{\xi,\zeta\in\Omega} |\lambda(\xi)p(\xi)(c(\xi)-c(\zeta))| < \delta$$

But now, Lemma 2 implies that inequalities (U2) must hold as well since inequalities (1) hold. □

3.1. Interpretation of the main theorem

We want to argue that Theorem 1 implies that the assumption of Kreps–Porteus utility imposes no restriction on individual choice behavior.

The point is easiest to illustrate in a model with no uncertainty. In this case, we can assume that one observes the behavior of an individual throughout his lifetime and that there is a unique terminal node. There is a unique $\zeta \in \mathfrak{I}(\xi)$ for all non-terminal $\xi \in \Xi$, $\pi(\xi) = 1$ and $\mu(\xi) = v(\zeta)$. The assumption that $\mu(\xi)$ is observable is justified if we assume that this terminal node denotes the last period of the individual's planning horizon. In this case, we know that $\mu(\xi) = 0$ and Theorem 1 immediately implies that the assumption of Kreps–Porteus utility imposes no restrictions on individual choices in markets.

3.1.1. Time consistency

Following Strotz (1956), there have been various attempts to formalize 'dynamic inconsistency of preferences', the human tendency to prefer immediate rewards to later rewards in a way that our 'long-run selves' do not appreciate (see e.g. Gul and Pesendorfer (2001) and the references therein).

Many papers studying time-inconsistent preferences have also searched for empirical proof that people have such preferences. It follows from Theorem 1 that it is impossible to find such empirical proof from observing individuals' choices in markets.[1] Since Kreps–Porteus utility is time consistent by construction, this immediately implies that the assumption of time consistency imposes no restriction on choices in markets. For any present-biased preference specification and any resulting observation of choices there exists a Kreps–Porteus utility function which yields exactly the same choices.

3.1.2. Uncertainty

In a model with uncertainty, Lemma 1 imposes a non-trivial restriction on an extended observation. Theorem 1 is not applicable to all situation since it requires that at different terminal nodes the continuation utilities are different. If the last period of the model is interpreted as the end of an agent's planning horizon, it makes sense to assume that $\mu(\xi) = \mu(\zeta) = 0$ for all terminal nodes ξ and ζ. An example now shows that under this assumption portfolio choices are restricted by the assumption of Kreps–Porteus utility.

Example 1. Consider a two-period model with two possible states in the second period. The states are numbered 0 (today), 1, 2 and the probabilities are $\pi_1 = \pi_2 = 1/2$. Assume

[1] The existence of external commitment devices and experimental evidence might offer a different perspective.

for simplicity that there is only one good and that the price of this good is one at each node. Assume that there are two arrow securities, one paying one unit in state 1, the other paying one unit in state 2 and that $q_1 > q_2$. Suppose that $c_1 > c_2$ and that the portfolio choice satisfies $\theta_l > \theta_2$.

The observed portfolio choice is inconsistent with Lemma 1. Since $c_1 > c_2$, by (U2), $\lambda_2 > \lambda_1$. However, by (U1) this implies that $q_1 < q_2$—a contradiction.

While the example only shows that there are restrictions on portfolio choices at time $T - 1$, there might also exist restrictions at other nodes. Consider for example an economy with identical consumptions at all last period nodes. This implies that μ_ξ has to be identical for all $\xi \in \mathcal{N}_{T-1}$, i.e. in the second to last period and Example 1 can be extended to this case.

However, in general, observed consumption will be different at all terminal nodes, leading to different continuation utilities at different nodes at $T - 1$. Theorem 1 then implies that Kreps–Porteus utility only imposes restrictions on consumptions at T and portfolio choices at $T - 1$ but on no other variables.

Moreover, it is clear that when period T is not the last period in the individual's planning horizon and it is impossible to observe $\mu(\xi)$ for $\xi \in \mathcal{N}_T$ there are no restrictions whatsoever on behavior.

Apart from the special case where last period's choices are restricted, the assumption of Kreps–Porteus utility therefore imposes no restrictions on intertemporal choice under uncertainty.

3.1.3. Observability

If one observes a household's choices throughout time it is unlikely that the weak restrictions on last period choices are actually observable. While under certainty it is conceivable that choices and prices are observable at every period, under uncertainty, one can only observe one sample path of an underlying stochastic process. One has to make stationarity assumptions on the underlying stochastic processes for prices and incomes to imbue the model with empirical content. Under a stationarity assumption, one can estimate the processes and one therefore knows prices, dividends and incomes at all nodes of the event tree. However, while prices, dividends and incomes might be stationary, the life cycle aspect of the agent's finite horizon maximization problem implies that choices are in general not stationary. Although given a finite data set, it is always possible to construct an event tree and a stationary process for prices, dividends and endowments such that the observed variables form a sample path and the assumption of stationarity of the exogenous variables cannot be refuted, it is implausible that all variables jointly follow a first-order Markov chain. Kreps–Porteus utility only imposes restrictions on last period choices under these additional stationarity assumptions.

We also assume throughout that the agent evaluates uncertain income streams according to the true (known) probabilities. While this might seem like a very strong assumption, it is standard in the applied literature and it is clear that without any assumption an agent's beliefs, Theorem 1 will become trivial. In this case the agent could put zero probability on all but one sample path. If this happens to be the observed sample path, the model is the same as under certainty.

3.2. Assumptions on the aggregator

In order to obtain restrictions one has to make additional assumptions on the aggregator function $W(\cdot, \cdot)$. One possibility is to require that the agents' indifference curves over current consumption are identical at all nodes. For this, we assume that $W(x, z)$ can be written as $F(w(x), z)$, where $F : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is assumed to be increasing and concave and where $w : \mathbb{R}^L_+ \to \mathbb{R}_+$ is the concave and increasing utility function for spot consumption. We call this aggregator function weakly separable. The assumption of weak separability ensures that marginal rates of substitution between different spot commodities are not affected by different future utilities. If there is only one good, i.e. $L = 1$ this assumption does not guarantee refutability. The assumption imbues the model with empirical content for $L > 1$ because it restricts possible choices on spot markets. There are many utility functions satisfying this assumption—for example, any nesting of concave CES-utility functions will give rise to a weakly separable aggregator.

The model is now testable. In fact, choices on spot markets together with prices for commodities $(p(\xi), c(\xi))_{\xi\in\Xi}$ must satisfy the strong version of the strong axiom of revealed preferences.

Definition 3 (Chiappori and Rochet, 1987). $(p(\xi), c(\xi))_{\xi\in\Omega}$ satisfies SSARP if for all sequences $\{i_1, \dots, i_n\} \subset \Omega$

$$p_{i_1}c_{i_1} \geq p_{i_1}c_{i_2},\ p_{i_2}c_{i_2} \geq p_{i_2}c_{i_3}, \dots, p_{i_{n-1}}c_{i_{n-1}} \geq p_{i_{n-1}}c_{i_n}$$

implies

$$c_{i_n} = c_{i_1}, \quad \text{or} \quad p_{i_n}(c_{i_1} - c_{i_n}) > 0$$

and if for all $\xi, \zeta \in \Omega p(\xi) \neq p(\zeta)$ implies $c(\xi) \neq c(\zeta)$.

Chiappori and Rochet (1987) show that in the context of static choice SSARP is necessary and sufficient for the data to be rationalizable by a smooth, strictly concave and strictly increasing utility function. In the intertemporal context, SSARP implies that choices are rationalizable by a separable (time invariant) expected utility function if asset prices or portfolio choices are unobservable.

We say that a utility function $u(\cdot)$ is time separable if it is Kreps–Porteus and if there exists a $\beta \in [0, 1]$ such that the aggregator can be written as

$$W(x, y) = w(x) + \beta y$$

The following theorem is the main result of this section.

Theorem 2. *The following statements are equivalent.*

(a) *An extended observation*

$$\mathcal{O} = ((c(\xi), \theta(\xi), d(\xi), q(\xi), p(\xi), \pi(\xi))_{\xi\in\Omega}, (\mu(\xi))_{\xi\in\mathcal{N}_T})$$

which satisfies $\mu(\xi) \neq \mu(\zeta)$ for all nodes $\xi \neq \zeta \in \mathcal{N}_T$ and $c(\xi) \in \mathbb{R}^L_{++}$ for all $\xi \in \Omega$ is rationalizable by a Kreps–Porteus utility function with weakly separable aggregator.

(b) *There are* $V(\xi), U(\xi) \in \mathbb{R}_+$, $\lambda(\xi) \in \mathbb{R}_{++}$ *and* $\gamma(\xi) \in \mathbb{R}^2_{++}$ *for all* $\xi \in \Omega$ *such that,*
(1) *For all* $\xi \in \Omega, \xi \notin \mathcal{N}_T$,

$$q(\xi)\lambda(\xi) = \gamma_2(\xi) \sum_{\zeta \in \Im(\xi)} d(\zeta)\pi(\zeta|\xi)\lambda(\zeta) \tag{2}$$

$$\mu(\xi) = \sum_{\zeta \in \Im(\xi)} \pi(\zeta|\xi)U(\zeta)$$

(2) *For all* $\xi \neq \zeta \in \Omega$,

$$U(\xi) \leq U(\zeta) + \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \left[\begin{pmatrix} V(\xi) \\ \mu(\xi) \end{pmatrix} - \begin{pmatrix} V(\zeta) \\ \mu(\zeta) \end{pmatrix} \right] \tag{3}$$

as well as

$$V(\xi) \leq V(\zeta) + \frac{\lambda(\zeta)}{\gamma_1(\zeta)} p(\zeta) \cdot (c(\xi) - c(\zeta)) \tag{4}$$

The inequality holds strict whenever $c(\xi) \neq c(\zeta)$.

(c) *The prices and spot market choices* $(p(\xi), c(\xi))_{\xi \in \Omega}$ *satisfy SSARP*.

(d) *There exist asset prices* $(\bar{q}(\xi))_{\xi \in \Omega}$, *incomes* $(\bar{I}(\xi))_{\xi \in \Omega}$ *and portfolio holdings* $(\bar{\theta}(\xi))_{\xi \in \Omega}$ *such that the observation* $(c(\xi), \bar{\theta}(\xi), d(\xi), \bar{I}(\xi), \bar{q}(\xi), p(\xi), \pi(\xi))_{\xi \in \Omega}$ *can be rationalized by a time-separable utility function*.

Proof. The proof of Lemma 1 above implies that (a) is equivalent to (b). The additional requirement of weak separability gives rise to the set of inequalities (3) and (4).

The crucial part of the proof is to show that (b) is equivalent to (c): According to Afriat's Theorem (see Chiappori and Rochet (1987)) SSARP is necessary and sufficient for the existence of numbers $(V(\xi)), \alpha(\xi)_{\xi \in \Omega}, \alpha(\xi) > 0$ which satisfy

$$V(\xi) - V(\zeta) \leq \alpha(\zeta)p(\zeta)(c(\xi) - c(\zeta)) \tag{5}$$

for all $\xi \neq \zeta \in \Omega$, with the inequality holding strict for $c(\xi) \neq c(\zeta)$.

To show that (b) implies (c), we define $\alpha(\xi) = \lambda(\xi)/\gamma_1(\xi)$ inequality (4) then implies inequality (5).

For sufficiency, assume that there exist numbers $(V(\xi)), \alpha(\xi)_{\xi \in \Omega}, \alpha(\xi) > 0$ which satisfy (5).

We can then choose $(\gamma_1(\xi))_{\xi \in \Omega}$ small enough to ensure that inequality (3) has a solution—this follows from the same argument as in the proof of Theorem 1: We take the $V(\xi)$ as given and construct $\gamma_2(\xi)$ analogous to the number $\gamma(\xi)$ in the previous proof. Since we do not impose restrictions on $\gamma_1(\xi)$ except bounding them from above, it is easy to ensure that $(\lambda(\xi))/(\gamma_1(\xi)) = \alpha(\xi)$ by choosing $\lambda(\xi)$ sufficiently small. Equality (2) can be satisfied because all these inequalities are homogeneous in $(\lambda(\xi))_{\xi \in \Omega}$ and impose no lower bound on $\inf_{\xi \in \Omega} \lambda(\xi)$.

Finally, we have to show that (c) is equivalent to (d): The Afriat inequalities for time-separable utility are particularly easy. Inequalities (4) must hold with $\gamma_1(\xi) = 1$. Eq. (2) must hold with $\gamma_2(\xi) = \beta$. Therefore, the observation can be rationalized by a time-separable utility function if and only if in addition to inequality (5) we also have

$$\alpha(\xi)q(\xi) = \beta \sum_{\zeta \in \Im(\xi)} \alpha(\zeta)\pi(\zeta|\xi)d(\zeta)$$

Since we are free to choose the $(q(\xi))$, this can always be satisfied as long as there is no arbitrage. Portfolio choices $\theta(\xi)$ and incomes $I(\xi)$ must then be chosen to ensure that the budget constraints are satisfied. □

It is important to point out that weakly separable Kreps–Porteus utility is not observationally equivalent with time-separable utility if portfolio choices are observable. In this case, time separability puts restrictions on portfolio holdings—weakly separable Kreps–Porteus utility does not.

4. Conclusion

Assuming the existence of utility functions to explain the behavior of consumers is standard in economics. In order to imbue models which use utility functions with empirical content one would hope that by watching the behavior of individuals throughout their life, one can test the hypothesis that these individuals maximize utility. However, we show in this paper that this is only possible under additional assumptions on the utility function. Kreps–Porteus utility with a weakly separable aggregator is one class of utility functions which imposes restrictions on individual behavior. These restrictions can be formulated in a tractable way one can test a large data set for consistency with utility maximization (see Varian (1982) for such tests). Without this additional assumption there are no restrictions and the theory cannot be tested by observing the choices of a single individual. In this case, one needs to use panel data and assume that similar individuals have identical preferences.

The situation is more complicated when there is no data on individual choices and when one has examine restrictions on aggregate data. Brown and Matzkin (1996) show that there exist observable restrictions for the case where one can observe how aggregate consumption varies as prices and the income distribution vary. The criticism in this paper against traditional tests of utility maximization which use individual data applies to the analysis in Brown and Matzkin (which uses aggregate data) as well. In Kubler (2003), we extend their analysis to a multi-period model where the observations consist of a time series on aggregate data.

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