The Rate of Time Preference and Dynamic Economic Analysis

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Strong restrictions on the structure of preferences are a central feature in the received theory of intertemporal allocation. In fact, most of the modern literature concerned with capital-theoretic problems represents preferences by a functional in which an additive utility function is discounted by a constant rate of time preference. This specification is attractive because it is analytically tractable in dynamic models, and it clearly delineates how tastes and opportunities interact to determine an economy's (household's) paths of consumption and capital formation. However, its rigid structure (constancy of time preference) severely limits the conclusions and explanatory power of the corresponding models. This paper considers a class of utility functionals (in continuous time) which have the appealing feature that the rate of time preference depends systematically on an index of aggregate future consumption. The more flexible structure embodied in these functionals leads to important generalizations and modifications of standard conclusions. We highlight this added richness by examining five basic problems in dynamic economic analysis.

I. Introduction

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One way to understand the nature of our specification of intertemporal utility is as follows. Additivity implies that the marginal rate of substitution between consumption at t_1 and t_2 is independent of consumption at any $t \neq t_1, t_2$. We generalize preferences to allow this marginal rate of substitution to depend on consumption at any time t, $t \geq \min(t_1, t_2)$—it is required only that future consumption be weakly separable from past consumption levels. We will borrow (and abbreviate) the terminology of Blackorby, Primont, and Russell (1978) and refer to the preferences and utility functionals below as recursive.

Several important features of recursive utility should be emphasized. First, the potential nonconstancy of time preference just noted introduces a degree of generality that is contained in Irving Fisher's (1930) seminal formulation of the theory of intertemporal choice. At the same time, the specification is not so general that it precludes interesting predictions. Second, in common with additive utility, recursive utility implies the following simplification of intertemporal planning. Suppose the planner is free to revise his plans at some $t > 0$; his decisions at t will depend on the past through accumulated assets, but they will not depend directly on past consumption activities. In fact, recursive utility is the most general specification for which this is true, since such behavior is merely a restatement of the weak separability of future consumption. Such preferences are closely related to the issue of the intertemporal consistency of plans (see Blackorby et al. 1973; Deaton and Muellbauer 1980, pp. 340–43).[1] Finally, stan-

[1] Other generalizations of additive utility are conceivable and have been considered in the literature; but none share all of the noted features of recursive utility. For example, general utility functions of the sort appearing in Fisher (1930) generate few definite predictions. Wan (1970) analyzes a very general continuous-time model at the cost of weak results and complicated mathematical procedures. In models of habit formation (Pollak 1970; Heal and Ryder 1973), plans at any age t necessarily depend on past consumption in addition to accumulated wealth.

dard techniques (the maximum principle) can be applied to solve the associated planning problem.

Some of the implications of recursive preferences have been analyzed in a discrete-time framework.[2] There are far fewer instances where preference structures that are more general than the standard specifications have been introduced in continuous-time models. And there is a natural desire to formulate dynamic capital models in continuous time because of the sharp distinction between stocks and flows that is obtained. Uzawa (1968) introduced utility functionals that are closely related to those considered below, but the discussion does not illuminate the essential analytical issues. His specification has subsequently been applied to problems in international trade by Calvo and Findlay (1978), Findlay (1978), and Obstfeld (1981). Our presentation clarifies the nature of Uzawa's specification. Some, though not all, of the implications of recursive utility for dynamic economic analysis which we describe below are present—often in different models and in disguised forms—in the trade studies noted. However, due to the different objectives of these studies, the consequences of recursive utility are not emphasized and are likely to go unnoticed by economists working in other areas.

As we noted, five problems in economic dynamics are considered. First, the nature of optimal economic growth is investigated. If the production function is concave, and thus the marginal productivity of capital is declining, the standard result is confirmed—capital is adjusted monotonically to a unique stationary level. New results follow, however, if the marginal productivity of capital is constant or increasing. In either case a stable steady-state capital stock may exist, and thus the turnpike nature of optimal economic growth is relevant for more technologies than commonly analyzed.

The next two problems concern questions of comparative statics and dynamics in decentralized competitive economies. In the first, the short-run and long-run incidence of a tax on interest income is examined, and in the second we analyze the long-run real consequences of government monetary policy in a model in which real money balances enter individual utility functions (see Sidrauski 1967; Brock 1974). In both cases new conclusions emerge. In the latter case, for example, the long-run level of capital is positively related to the rate of monetary expansion, and thus a rational foundation for the Tobin effect is established.

In Section VI we model a dynamic decentralized economy with heterogeneous households and investigate the long-run distribution

[2] See Koopmans (1960) and Koopmans, Diamond, and Williamson (1964). For applications, see Beals and Koopmans (1969), Iwai (1972), and Boyer (1975).

of capital. Recently, Becker (1980) has confirmed a conjecture of Ramsey (1928). He analyzes an economy in which households have additive utility functionals but possess different constant rates of time preference. In the long-run steady state, all capital is owned by the household with the lowest rate of time preference; if several households share this lowest rate, the distribution of capital across these households is indeterminate. This extreme consequence follows because constancy of individual rates of time preference is imposed on the problem. The recursive utility functionals described below generate a more appealing model of long-run distribution.

Finally, we consider individual consumer behavior in a multicommodity framework. Consumer choice theory has traditionally dealt with short-run demand functions where income or wealth appears as an argument of the demand functions. But in the long-run steady state of an intertemporal plan, income and wealth are endogenous and consumption depends only on commodity prices and the rate of interest. There exists a steady state for the consumer if and only if the rate of discount is variable. Thus our specification of utility permits us to determine the testable implications of utility maximization for long-run consumer behavior. This we do in Section VII where the exhaustive comparative statics properties of long-run demand functions are derived. A notable result is that only the substitution effects of price changes prevail in the long run.

The paper proceeds as follows: Section II formulates and discusses the recursive utility functionals. The next five sections address the problems mentioned. Section VIII concludes, and technical details are collected in an Appendix.

II. A Class of Utility Functionals

A consumption path is represented by a function C which maps each t in $[0, \infty)$ into $c(t)$, consumption at time t. We wish to specify a utility functional U which assigns utility $U(C)$ to each consumption path C. The following specification for U is adopted:

$$U(C) = -\int_0^\infty \exp\left\{-\int_0^t u(c)d\tau\right\}dt, \tag{1}$$

where $u > 0$, $u' > 0$, and $u'' < 0$.

This specification has several desirable features. First, U is monotonic; if a consumption path C' lies everywhere above the path C, then $U(C') > U(C)$. Second, U has convex indifference curves; if $U(C) = U(C')$, then $U(\frac{1}{2}C + \frac{1}{2}C') \geqslant U(C)$. In fact, $U(\frac{1}{2}C + \frac{1}{2}C') \geqslant \frac{1}{2}U(C) + \frac{1}{2}U(C')$ for any two paths C and C'.

The recursive structure of U is evident. Denote by C_T and ${}_TC$ those portions of the consumption path C corresponding to times before and beyond T, respectively. Then $U(C)$ may be expressed in the form

$$U(C) = G[T, C_T, U(_TC)], \tag{2}$$

where

$$G(T, C_T, \phi) = -\int_0^T \exp\left\{-\int_0^t u(c)d\tau\right\}dt + \phi \cdot \exp\left\{-\int_0^T u(c)dt\right\}. \tag{3}$$

Because G is increasing in ϕ $(= U[_TC])$ the relative ranking of two consumption paths C and C', which differ only beyond some time T, is independent of their common history $C_T = C'_T$. This gives precise meaning to the statement that future consumption is weakly separable from past consumption levels. (In an additive utility functional it is also true that past consumption is weakly separable from future consumption levels. This is not true for U.)

Corresponding to the weak separability of $_TC$ is the fact that the latter may be aggregated. We will refer to $U(_TC)$ as aggregate future (beyond T) consumption or as the (sub-) utility derived from that consumption. That U itself can be used to aggregate future consumption reflects the stationarity of the preference order represented by (1), in the sense made precise in Koopmans (1960).

An essential feature of U is the rate of time preference implicit in its structure. To analyze this issue we must define marginal utilities and marginal rates of substitution between consumption at various dates. In continuous-time analysis this may be accomplished by making use of the concept of a Volterra derivative. A precise definition and alternative applications of the concept may be found in Wan (1970) and Heal and Ryder (1973). The Volterra derivative gives precise meaning to the rate at which utility changes with respect to a small increment in consumption near time T. For the present study, it suffices for the reader to accept both this fact and the intuitive nature of its calculation. The marginal utility of U with respect to a small increment in consumption near T is denoted U_T. Given the specification (1), U_T is given by

$$U_T(C) = u'[c(T)] \cdot \int_T^\infty \exp\left\{-\int_0^t u(c)d\tau\right\}dt. \tag{4}$$

The rate of time preference implicit in U can now be defined. It is helpful to proceed from a discrete-time approximation to our model. Let $\Delta > 0$ be a small interval of time and $U_{T-\Delta}(C)/U_T(C)$ the marginal rate of substitution between consumption at T and at $T - \Delta$. It is consistent with common practice to define the local rate of time preference to be $[U_{T-\Delta}(C)/U_T(C)] - 1$ on the vector where $c(T) = c(T - \Delta)$.

This definition is attractive because it leads to predictions about the local equilibrium path of consumption based on a comparison of the rate of time preference (a taste variable) with the real rate of interest (an opportunity variable). When we pass to a continuous-time framework, the definition above becomes the negative of the logarithmic rate of change of marginal utility along a locally constant path. Because U_T is given by (4) we immediately derive the following rate of time preference ρ:

$$\begin{aligned}\rho &= -\frac{d}{dT}\log U_T(C)\Big|_{\dot{c}(T)=0} \\ &= \left[\int_T^{\infty}\exp\left\{-\int_T^t u(c)d\tau\right\}dt\right]^{-1} = -[U({}_TC)]^{-1}.\end{aligned} \tag{5}$$

Thus U embodies positive discounting of future consumption, and $\rho > 0$. But the rate of time preference is not constant as in the additive model. Rather the rate is a function of aggregate future consumption $U({}_TC)$ and is summarized by the function

$$\rho(\phi) = -1/\phi, \tag{6}$$

where ϕ represents aggregate future consumption. Note that if the consumption path is globally constant, as would be the case in a stationary state, $\dot{c}(t) = 0$ for all t and $c(t) = \bar{c}$. In this situation $U({}_TC) = -1/u(\bar{c})$ and the definition of time preference becomes

$$\rho = u(\bar{c}) \tag{7}$$

given our specification of utility.

Figure 1 provides further clarification in the context of the discrete-time approximation to the model. Indifference curves between $c(T - \Delta)$ and $c(T)$ are portrayed in the figure. In the additive model such indifference curves are unaffected by consumption in periods other than $T - \Delta$ and T. Given recursive utility, consumption beyond T may shift the indifference curves via a change in $U({}_TC)$, the subutility from consumption in future periods. Thus indifference curves must be labeled according to the underlying values of future utility, and $\rho > 0$ requires that indifference curves have slopes at least equal to unity in absolute value along the ray $c(T - \Delta) = c(T) = c$. Since ρ is independent of the common consumption level c, indifference curves constructed for the same future utility value have the same slopes at A and B. Finally, that ρ increases with future utility imposes the relationship between the slopes of the two indifference curves passing through point A. Since an increment in future utility shifts preferences from $c(T)$ to $c(T - \Delta)$, at least locally around the line of equal

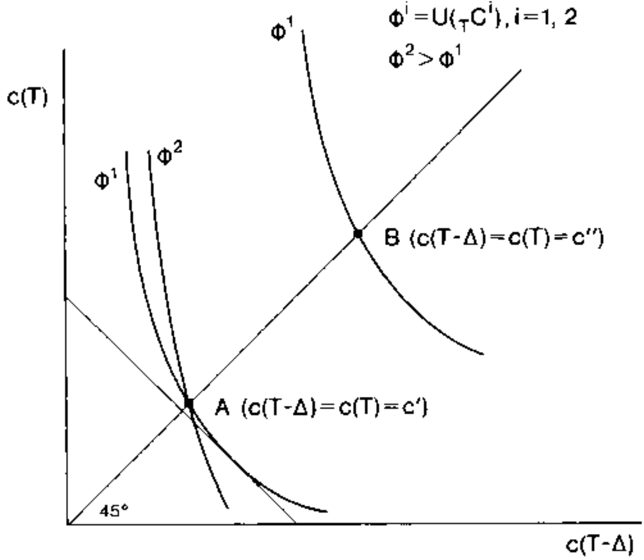

FIG. 1

consumption, one may interpret the monotonicity of ρ as reflecting a form of complementarity between $c(T - \Delta)$ and aggregate consumption in periods after T.

Some brief comments are in order regarding the links between the formulation above and both standard practice and the existing literature employing more general utility structures. It is obvious that for the conventional additive utility functional, the local and global definitions of time preference are equal. The previous contributions that introduce recursive forms (see Koopmans et al. [1964] and Uzawa [1968] and the applied studies they spawned) define an agent's time preference to equal the marginal rates of substitution minus unity on globally constant—$\dot{c}(t) = 0$ for all t—consumption paths. This focus reveals interesting properties of the steady state. However, the introduction of the precise definition of local time preference contained in (5) makes possible an elegant formulation of non-steady-state behavior in terms of a simple comparison between local time preference and the instantaneous marginal rate of transformation—the real interest rate. This feature will be illuminated in the analyses of the five problems. In addition, an important advantage of the definition in (5) is that it clearly reveals, via the monotonicity of ρ, the nature of the intertemporal complementarity implicit in U.

We suspect that it is the general recursive structure reflected in (2), rather than the specific functional form (1), which is responsible for many of the results that follow. However, to our knowledge, there exists only one further class of functional forms for intertemporal utility consistent with (2). The following generalization of (1) also provides additional understanding of the relation of our specification to that of Uzawa (1968).[3] Consider

$$\hat{U}(C) = \int_0^\infty v(c) \exp\left\{-\int_0^t u(c)d\tau\right\}dt. \tag{8}$$

If $v(c)$ is set equal to -1, (8) is transformed into (1). If $u(c)$ is set equal to $\delta[v(c)]$ for an increasing function δ, Uzawa's utility structure is obtained. Along a globally constant consumption path the rate of time preference is a constant, $\delta[v(\bar{c})]$, and is the analogue of (7). On the other hand, the formulation of local time preference implicit in (8) with $u = \delta(v)$ is more complex than (5). And this fact results in added technical complications and ambiguities without yielding more interesting results.[4] Thus, the specification (1) is maintained. Indeed, a contribution of this section is that the essential features of weakly separable intertemporal preferences are exposed. The analysis of the five problems that follow is therefore able to illustrate most of the important implications in a clear and simple fashion and at a minimum cost in terms of technical machinery.

III. Optimal Economic Growth

The literature concerned with optimal capital accumulation is dominated by the assumption of diminishing marginal returns. This may incorporate a view about reality. But it also reflects capital theorists' interest in steady states coupled with the almost universal application of the additive utility functional. To the extent that steady states have intrinsic interest, recursive preferences greatly expand the range of technologies consistent with such positions. To examine this issue consider the following generalization of the optimal growth problem:

$$\max\,\{U(C)\colon \dot{k} = g(k) - c,\ c \geqslant 0,\ k(t) \geqslant 0\ \forall\ t,\ k(0) = k_0\}. \tag{9}$$

[3] Epstein (1983*b*) provides an axiomatic basis for this functional in the context of choice under uncertainty. Note that in finite horizon models, Epstein (1982) formulates and analyzes a much more general class of recursive functionals in continuous time. Also, a referee has noted that Nairay (1981) has studied Uzawa functionals in a continuous-time model with uncertainty.

[4] The added complexity is due to the fact that given $\hat{U}$ the rate of time preference depends also on current consumption; in fig. 1 slopes at A and B would generally differ. For a constant consumption stream at the level $\bar{c}$, it is still true that the rate of time preference equals $u(\bar{c})$. Thus steady-state analysis is not altered substantially; but analysis of behavior out of steady state is more complex. Note that Uzawa's non-steady-state analysis contains an error. The phase diagram fig. 2 and the first two complete sentences on p. 494 do not follow from his eq. (35) as claimed.

The functional $U(C)$ is as defined in (1), and the constraints are standard; k denotes capital; population growth is ignored, or all variables can be interpreted in per capita terms; and g is the production function net of depreciation. We assume that $g(0) = 0$, $g'' \geqslant (\leqslant)\ 0$ for $k < (>)\ k_1$, $g' > (<)\ 0$ for $k < (>)\ k_2$, $k_2 > k_1$. This specification allows increasing marginal productivity at an early stage of economic development and diminishing marginal productivity at later stages. If $k_1 = 0$, the common neoclassical specification of a concave production function is obtained; and if $k_2 = \infty$ and $g'' \equiv 0$, constant returns to scale are present. This latter specification plays a central role in Frank Knight's writings on capital theory. A final assumption is that $u(0) < g'(0)$. In light of (7), this requires that the rate of time preference along the path with everywhere zero consumption is less than the marginal product of capital given zero capital.

The optimization problem may be solved by standard techniques after applying the transformation in Uzawa (1968, p. 491) or that sketched in the Appendix to the present paper. We will not undertake an exhaustive analysis of the nature of solutions. Rather, we focus on behavior that can be optimal given recursive preferences but can never be optimal if the rate of time preference is constant.

With constant time preference, the solutions possess the following features: (i) if $g'' < 0$ throughout and if $g'(\infty) \leqslant 0$, there exists a unique and globally stable steady-state capital stock; (ii) if g is linear, $g(k) = rk$, there does not exist a steady state unless the rate of discount equals r, in which case every stock is a steady state; (iii) finally, if $k_1 > 0$, it is optimal either, depending possibly on the starting point, to drive the stock to zero or to converge to a steady state in the region of declining marginal productivity. It can *never* be optimal for an economy to converge to a stock $\bar{k}$ in the region $0 < \bar{k} < k_1$, where marginal productivity is increasing (Skiba 1978). Such an economy is presumably "underdeveloped."

When the rate of time preference varies as in (5) and (6) the following results are obtained: In case i the result above is confirmed. In case ii there exists a unique and globally stable steady state if $r < u(\infty)$; a steady state is possible in a world of Knightian constant returns. In case iii it may be optimal to converge to a stock in the region where marginal productivity is increasing—that is, it is conceivable that there exists a locally or even globally stable steady state $\bar{k}$, with $g''(\bar{k}) > 0$.

These assertions are proven below and in the Appendix, but the underlying intuition is conveyed by figure 2. To establish the appropriate interpretation of the latter, proceed as follows. First, recall that in the additive utility model along an optimal path, the logarithmic time derivative of the undiscounted shadow price of capital equals the

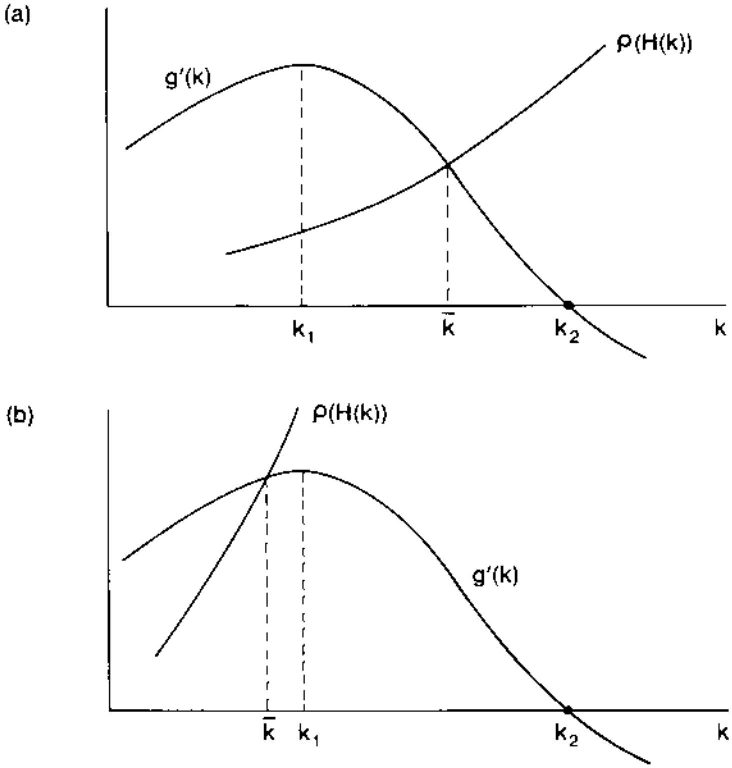

FIG. 2

rate of time preference minus the marginal product of capital. An analogous relation holds for the recursive utility specification, and this analytical convenience is made possible by our formulation of local time preference in (5). Let $H(k_0)$ be the maximum value of lifetime utility in (9). We will measure the shadow price in utility terms, but in terms of V rather than U, where V and U are ordinally equivalent and related by $U = -e^{-V}$. Let J be the corresponding lifetime utility, $H = -e^{-J}$. Then J' is the shadow price of capital in terms of utility V, and if J is twice differentiable, the logarithmic derivative of J' is

$$\frac{d}{dt} \log J'[k(t)] = J''(k)\dot{k}/J'(k) = \rho[H(k)] - g'(k). \tag{10}$$

Note that the rate of time preference in (10) is defined along an optimal path, $\rho[H(k)] = -[H(k)]^{-1} = e^{J(k)}$, and is indirectly a function of the stock size k.[5]

[5] Let C be the optimal consumption path and let $k(t)$ be the optimal stock at time t in

Equation (10) describes the dynamics of the capital stock and may be applied to the analysis of the stability of steady states. Suppose that $J'' < 0$—the shadow price of capital is a decreasing function of the stock size.[6] (This is necessarily the case if $g'' \leqslant 0$ but may be violated in some regions if g is convex for small stocks.) Then $\dot{k}$ has the sign of $g'(k) - \rho[H(k)]$. In particular $\bar{k}$ is a steady state if and only if $g'(\bar{k}) = \rho[H(\bar{k})]$, in which case it is necessarily globally stable as long as $J'' < 0$, for example, if $g'' \leqslant 0$.

Turn to figure 2 and assume $J'' < 0$. The marginal product schedule can be viewed as a demand schedule for productive capital, while the time preference schedule can be viewed as a supply schedule of capital on the part of households. Long-run equilibrium in the capital market occurs at steady-state levels of capital and the rate of interest. Moreover, the stability of the steady state in the planning problem (9) corresponds precisely to the stability of market equilibrium in the Marshallian sense. Thus stability of $\bar{k}$ requires that the curve $\rho[H(k)]$ cut the marginal product curve from below. As figure 2b shows, this is possible even if $g'' \geqslant 0$.[7] In the standard additive utility model, similar diagrams apply with the rate of time preference curve perfectly horizontal. It is evident that stability requires $g'' < 0$.

Though figure 2b is suggestive, it does not prove that the behavior described is indeed optimal for some specification of u and g. We therefore present a concrete example that is consistent with figure 2.[8]

Take

$$u(c) = \frac{B(1 + \beta)(c + a)^{\beta/(1+\beta)}}{(B\beta)^{\beta/(1+\beta)}},$$

$$g(k) \equiv (\Delta + k)^2 - B(\Delta + k)^{\beta+1} - a,$$

where $B > 0$, $\beta > 1$, $B > [(\beta + 1)2^{\beta-1}]^{-1}$, $[1/B(\beta + 1)]^{1/(\beta-1)} < \Delta < [2/B(\beta + 1)]^{1/\beta}$, and $a \equiv \Delta^2 - B\Delta^{\beta+1} > 0$. Then $g(0) = 0$ and the

(9). Then $H[k(t)] = U({}_tC)$ because of the recursive structure of preferences or the intertemporal consistency of plans. Then apply (5).

[6] The assumption that J'' exists and is negative is especially restrictive when $g'' > 0$ in some regions, as it rules out some forms of behavior described by Skiba (1978). We adopt these assumptions because it is our purpose to focus on behavior that is conceivably optimal given recursive but nonadditive utility. An exhaustive analysis of optimal growth with recursive utility and a convex-concave technology would require a separate paper.

[7] Multiple steady states are possible where both the marginal product and rate of time preference curves are upward sloping. Note that both figures are consistent with our assumption that $\rho[H(0)] = u(0) < g'(0)$.

[8] The example is an application of the general procedure described in the Appendix for constructing examples. It is readily adapted to the case of a constant rate of time preference. Because even then concrete examples of solutions with convex-concave production functions are not available in the literature, our procedure may be of independent interest.

marginal product curve for g has the appearance depicted in figure 2 with $k_1 = [2/\beta(\beta + 1)B]^{1/(\beta - 1)} - \Delta$ and $k_2 = [2/B(\beta + 1)]^{1/\beta} - \Delta$. For such utility and production functions, the problem (9) possesses a solution for any $k_0 > 0$ and yields lifetime utility $H(k_0) = -(k_0 + \Delta)^{-1}$. Given optimizing behavior, the rate of time preference varies with stock size according to the linear rule $\rho[H(k)] = k + \Delta$. Thus from (10) capital evolves according to the differential equation

$$\dot{k} = 1 - B(\beta + 1)(\Delta + k)^{\beta - 1}. \tag{11}$$

If $k_0 > 0$ capital converges to the steady-state stock $\bar{k} = [1/B(\beta + 1)]^{1/(\beta - 1)} - \Delta > 0$. If $\beta > 2$, then $\bar{k} > k_1$, $g''(\bar{k}) < 0$, and we have the situation depicted in figure 2*a*. On the other hand, if $\beta < 2$, then $k_1 > \bar{k}$ and figure 2*b* applies.

IV. Short-Run and Long-Run Tax Incidence

Recent contributions to taxation theory have focused on the difference between the short-run and long-run incidence of a factor tax. The long-run shifting of factor taxes has been analyzed in a growth-theoretic framework, and the intertemporal welfare costs of such taxes have been computed.[9] In this section we investigate the impact of a tax on capital income in an intertemporal general equilibrium model in which a representative household with infinite life maximizes an intertemporal utility function and possesses perfect foresight with respect to future factor prices and government transfer payments. The latter correspond to the government's rebate of tax revenues. Thus we follow common practice in analyzing the differential impact of an interest tax versus a lump sum tax.

Suppose the household has a constant rate of time preference δ (as in Chamley 1980). Then while capital bears the entire burden of the tax in the short run, it shifts the burden completely in the long run because the long-run after-tax rate of return to capital must equal the constant δ. Steady-state capital, consumption, and hence welfare undergo "large" reductions.

These extreme results are imposed a priori by the maintained hypothesis of a perfectly elastic supply curve in the market for capital (fig. 2*a*). If the rate of time preference is not constant and the supply curve is upward sloping, the model is more flexible and is capable of generating a broader range of quantitative outcomes depending on

[9] See Diamond (1970), Feldstein (1974), and Kotlikoff and Summers (1979) for analyses of shifting and Chamley (1980) for the computation of welfare costs. The model in the latter study resembles ours except for the specification of the utility function.

the particular nature of preferences and the technology.[10] We now show this in a more formal analysis.

The representative household has the utility function U of (1). It accumulates capital k at the rate

$$\dot{k}(t) = r(t)k(t) + w(t) + R(t) - c(t), \quad k(0) = k_0 > 0. \tag{12}$$

Here $r(t)$ and $w(t)$ represent expectations of future returns to capital and labor, respectively. (We assume that one unit of labor is supplied inelastically.) The expected lump sum rebate from the government is $R(t)$.

The household maximizes U subject to (12) and its expectations, and it accumulates capital according to the profile $\{k^*(t)\}_0^\infty$. A perfect foresight intertemporal equilibrium occurs when expectations are consistent with equilibrium in all future factor markets, that is,

$$r(t) = (1 - \sigma)g'[k^*(t)], \quad w(t) = g[k^*(t)] - k^*(t)g'[k^*(t)], \tag{13}$$

and

$$R(t) = \sigma g'[k^*(t)]k^*(t), \quad \text{for all } t.$$

Here σ is the rate of taxation of capital income and g is a neoclassical production function, $g' > 0$, $g'' < 0$.

We focus on steady-state equilibria. In a steady state the household equates its rate of time preference to the after-tax return on saving. Because of (7) this amounts to $u(\bar{c}) = \bar{r}$. (A bar over a variable indicates a steady-state value.) Perfect foresight requires that $\bar{c} = \bar{r}\bar{k} + \bar{w} + \bar{R} = g(\bar{k})$ and $\bar{r} = (1 - \sigma)g'(\bar{k})$. Thus the steady-state $\bar{k}$ is the unique solution to

$$u[g(\bar{k})] = (1 - \sigma)g'(\bar{k}). \tag{14}$$

A solution always exists if g satisfies the Inada conditions $g'(0) = \infty$, $g'(\infty) = 0$. The solution is unique since $u[g(k)]$ is increasing and $g'(k)$ is decreasing in k.

Comparative statics analysis of (14) yields

$$\frac{d\bar{k}}{d\sigma} = \frac{-g'}{[u'g' - (1 - \sigma)g'']} < 0 \tag{15}$$

and

$$\frac{d\bar{r}}{d\sigma} = u'g' \frac{d\bar{k}}{d\sigma} < 0.$$

[10] Strictly speaking, the demand-supply analogy is not valid because a tax change not only shifts the net marginal product curve but also causes the rate of time preference curve to shift downward because lifetime utility falls. But the shift in the ρ curve only serves to reinforce our argument that capital will bear part of the tax burden in the long run. Thus we continue to use the demand-supply analogy to convey the intuition for our results.

Thus the steady-state capital stock falls if the rate of capital taxation is increased but not as much as it would if the rate of time preference were constant. (In that case [15] would apply with $u' = 0$.) Moreover, the net return to capital also falls, so capital bears some of the burden even in the long run.

The elasticity of $\bar{k}$ with respect to σ can be expressed in the form

$$\frac{\sigma}{\bar{k}}\frac{d\bar{k}}{d\sigma} = -\frac{1}{[\gamma \cdot (\bar{k}g'/g) + (1 - \sigma)(\bar{k}g''/g)]}, \tag{16}$$

where $\gamma \equiv u'[g(\bar{k})]g(\bar{k})/u[g(\bar{k})]$ is the elasticity of the rate of time preference along a constant consumption path. The elasticity is taken with respect to the constant level of consumption and is evaluated at $g(\bar{k})$. As this elasticity approaches infinity, the elasticity of $\bar{k}$ with respect to σ approaches zero, a result that is polar to that generated by a constant rate of time preference. More generally a range of results is possible depending on the empirically relevant value of γ.

A more general perspective on the analysis in this section may be derived from figure 2*a* and the market for capital. The effect of an interest tax is to shift down the demand curve. The short-run supply curve of capital is perfectly inelastic at the existing stock. If the rate of discount is constant, the long-run supply curve is perfectly elastic so that short-run and long-run impacts of the tax increase differ substantially. The variability of ρ allows this difference between short-run and long-run impacts to be determined empirically rather than to be imposed a priori. The relationship between the short-run and long-run consequences of parameter changes is of interest in a variety of contexts.[11] In all such instances, the utility functional (1) seems to be appropriate.

V. The Long-Run Neutrality of Money

A central question in monetary theory is how the path of the money supply affects the steady-state stock of capital in an economy populated by rational agents. The seminal investigations of this problem are those of Sidrauski (1967) and Brock (1974). Each author proves a strong invariance theorem when the services of money are modeled by entering real balances in the utility function and all other interactions are excluded. In both analyses it is the assumption of constant time preference that permits a partition of the monetary and real sectors in the long run. When intertemporal preferences are recursive, this separation is not possible. A formal proof is outlined below.

[11] Nagatani (1981) emphasizes the importance of the distinction for macroeconomic modeling.

The utility functional of the representative individual is adapted from (1):

$$U(C, B) = -\int_0^\infty \exp\left\{-\int_0^t u(c, b)d\tau\right\}dt, \tag{17}$$

where $u_c, u_b > 0$, $u > 0$, and u is concave; B is a path of real balances with value $b(t)$ at time t; and U so defined still has a recursive structure. One can define a rate of time preference for each of consumption and real balances. In both cases restrict the paths so that $\dot{c}(T) = \dot{b}(T) = 0$ locally. Then both rates of time preference are equal and are given by $\rho = -[U(_TC, {}_TB)]^{-1}$.

The representative consumer, who possesses perfect foresight, solves the following optimization problem:

$$\max -\int_0^\infty \exp\left\{-\int_0^t u(c, b)d\tau\right\}dt \tag{18}$$

subject to $c + \dot{k} + \dot{b} + (\dot{p}/p)b = w + rk + v$, $k(0) = k_0$, $p(0)b(0) = M_0$. Here k_0 and M_0 denote initial physical and nominal monetary assets, respectively; p is the money price of consumption and capital goods; w is the real wage rate; r is the real return to capital; and v is real government transfers. One unit of labor is supplied inelastically.

Production is carried out by means of the neoclassical production function g, $g' > 0$, $g'' < 0$. Perfect competition in factor markets and perfect foresight imply that, along the path that is optimal in (18), we have

$$w = g - kg'(k), \quad r = g'(k), \quad v = \theta b, \tag{19}$$

where θ is the constant rate of expansion of the nominal stock of money.

It is shown in the Appendix that any perfect foresight equilibrium path satisfies

$$\frac{\dot{u}_c(c, b)}{u_c(c, b)} = \rho[H(k + b)] - g'(k), \tag{20a}$$

$$\dot{k} = g(k) - c, \tag{20b}$$

and

$$\frac{\dot{b}}{b} = \theta + g'(k) - \frac{u_b(c, b)}{u_c(c, b)}. \tag{20c}$$

The maximum lifetime utility in (17) is given by H, and $\rho[H(k + b)]$ is the "derived" rate of time preference as a function of total assets $k + b$. (If ρ is set equal to a constant, one obtains the differential equation system corresponding to the standard additive utility functional [Calvo 1979, p. 93].)

Focus on steady-state equilibria $(\bar{c}, \bar{b}, \bar{k})$, which are defined by the equations

$$g'(\bar{k}) = \rho[H(\bar{k} + \bar{b})] = u(\bar{c}, \bar{b}), \tag{21a}$$

$$\bar{c} = g(\bar{k}), \tag{21b}$$

and

$$\frac{u_b(\bar{c}, \bar{b})}{u_c(\bar{c}, \bar{b})} = \theta + g'(\bar{k}). \tag{21c}$$

Assume that $u(c, b)$ is such that both consumption and real balances are normal in an atemporal problem, where u is the utility index. Then it is straightforward to show that the steady state is unique for any given θ. A steady state exists if $g'(0) = \infty$, $g'(0) = \infty$, and, for any $c > 0$, $\lim_{b\to 0} [u_b(c, b)/u_c(c, b)] = \infty$, $\lim_{b\to\infty} [u_b(c, b)/u_c(c, b)] = 0$.[12] Finally, a local analysis of the dynamic system (20) in a neighborhood of the steady state shows that the steady state is locally stable.[13]

Comparative statics analysis of (21) yields the following results:

$$\frac{d\bar{b}}{d\theta} < 0, \tag{22a}$$

$$\frac{d\bar{k}}{d\theta} > 0, \tag{22b}$$

and

$$\frac{d\bar{c}}{d\theta} > 0. \tag{22c}$$

A higher rate of monetary expansion increases the rate of price inflation and thus raises the opportunity cost of holding money. The steady-state quantity of real balances is therefore reduced, but the steady-state levels of consumption and the capital stock are increased. The source of these results is clear from figure 2*a*. The rate-of-time-preference schedule is a function of total assets: $\rho[H(k + b)]$ rather than $\rho[H(k)]$ is the correct specification. When real balances decline in

[12] For constant paths $U = -[u(c, b)]^{-1}$, so U and u are ordinally equivalent. Thus assumptions on u are readily translated into assumptions on the nature of the preference order defined by U on constant (consumption and real balances) paths. For example, the normalcy assumption is equivalent to the requirement that both consumption and real balances are normal goods in intertemporal optimization problems in which U is the utility index and only constant paths C and B are allowed.

[13] The analysis is similar to Calvo (1979, pp. 96–99) and Obstfeld (1981, pp. 1149–51) and is omitted.

response to the rise in opportunity cost, the time preference curve drawn against k shifts down. Thus the intersection with the marginal productivity schedule occurs at a larger level of capital. When the rate of time preference is maintained to be constant, the equation $g'(\bar{k}) = \rho$ uniquely defines the steady-state capital stock, and the neutrality of money is imposed. Recursive preferences therefore produce the Tobin effect as a feature of rational choice, and with a minimum specification of interactions.[14]

This nonneutrality theorem has implications for two central and related issues in monetary economics. A basic feature of the business cycle is that real output is serially correlated, and equilibrium models of output fluctuations introduce a Tobin effect to generate this behavior (see Lucas 1975; Fischer 1979). The result above provides a basis for this effect that is consistent with rational intertemporal choice. An associated problem concerns interest rate behavior in the United States. Before World War II, changes in nominal interest rates are less than changes in expected inflation. After World War II, changes in nominal interest rates with respect to changes in expected inflation are less than predicted when tax distortions are introduced (see Summers 1982).[15] These results are consistent with the specification of preferences in (17). The equilibrium real interest rate and the equilibrium rate of monetary expansion are negatively related. The change in the real rate will thus moderate the response of the nominal interest rate when equilibrium inflation is changed.

Brock (1974, pp. 773–74) also obtains a nonneutrality result in a model with endogenous labor supply in which the marginal rate of substitution between consumption and leisure at any time T depends on real balances at that time.[16] Such interdependence is *not* underlying our nonneutrality result.

Expand our model to include a labor-leisure choice. Let L be a leisure path with value $l(t)$ at time t. Suppose

$$U(C, B, L) = -\int_0^\infty \exp\left\{-\int_0^t u(c, b, l)d\tau\right\}dt \tag{23}$$

[14] A referee informs us that Michener (1981) has studied the Tobin effect in the discrete-time framework of Koopmans et al. (1964).

[15] Assume that the only distortion is a tax on interest income at the rate σ. If the after-tax real rate is constant—as would be the case with additive preferences—the change in the nominal interest rate with respect to a change in expected inflation is $1/(1-\sigma)$. This latter expression exceeds unity. The Summers study (1982) indicates that, given U.S. tax parameters, nominal rates change by less than is consistent with a constant real rate. It should also be noted that Summers's results also indicate that effects other than simple rational nonneutralities are present.

[16] That is, v_c/v_l is not independent of b where the utility function is $\int_0^\infty e^{-\delta t}v(c, b, l)dt$.

and

$$u(c, b, l) = u^1(c, l) + u^2(b).$$

Then the marginal rate of substitution (according to U) between $c(T)$ and $l(T)$, defined using Volterra derivatives, is $u_c^1[c(T), l(T)]/u_l^1[c(T), l(T)]$, which *is* independent of $b(T)$. In spite of this fact, a nonneutrality result can be obtained. The explanation is clear: Brock imposes intertemporal independence (additivity) so that real balances can influence consumption-leisure choices only via within-period interdependence. In our specification the latter is not required because of the intertemporal complementarity embodied in U. From (5), appropriately extended, the form taken by this intertemporal interdependence is evident: the rate of time preference falls if future real balances are reduced.

VI. The Long-Run Distribution of Wealth and Consumption

Ramsey (1928) conjectured that in the long run the most patient household will acquire all the capital stock. Becker (1980) has verified this result in a dynamic general equilibrium model with heterogeneous households in which each household discounts future consumption at a constant rate. This extreme and unappealing long-run distribution result is a direct consequence of the assumed constancy of households' rates of time preference. We show, in the context of a simplified model, that a more appealing long-run distribution emerges if household preferences have the form (1).

There are H households, $h = 1, 2, \ldots, H$. Household h has the utility functional U^h,

$$U^h(C) = -\int_0^\infty \exp\left\{-\int_0^t u^h(c)d\tau\right\}dt, \tag{24}$$

where u^h satisfies the properties specified in Section II.

We need to make precise the statement that household h is more patient than household j. Because rates of time preference are not constant, the following definition seems appropriate. Let ρ^h and ρ^j be the rate of time preference functions for households h and j, respectively. Household h is no less patient than j if $\rho^h \leq \rho^j$ whenever both households face identical future consumption streams. In light of (5), this is equivalent to the requirement that

$$\left[\int_0^\infty \exp\left\{-\int_0^t u^h(c)d\tau\right\}dt\right]^{-1} \leq \left[\int_0^\infty \exp\left\{-\int_0^t u^j(c)d\tau\right\}dt\right]^{-1}$$

for all consumption paths C. But this is equivalent to $u^h(c) \leq u^j(c)$ for all c.

Individual households solve the following problem:

$$\max U^h(C^h) \text{ subject to } \dot{k}^h = rk^h - c^h,\ k^h(0) = k_0^h > 0, \tag{25}$$

where $r > 0$ is the constant rate of return. For simplicity we assume a linear technology, $g(k) = rk$, $k \equiv \Sigma k^h$, and thus static expectations are rational. We assume that for each h, $u^h(0) < r < u^h(\infty)$.

The problem (25) was solved in Section III. Household stocks are adjusted monotonically to their steady-state values $\bar{k}^h$, $h = 1, \ldots, H$, where

$$u^h(r\bar{k}^h) = r. \tag{26}$$

Thus the economy converges to a steady-state configuration of positive consumption $(\bar{c}^1, \ldots, \bar{c}^h, \ldots, \bar{c}^H)$ such that

$$u^h(\bar{c}^h) = r, \quad h = 1, \ldots, H. \tag{27}$$

All households own positive stocks and enjoy positive consumption in the long run. Patience is rewarded by larger steady-state consumption; that is, $u^h(c) \leqslant u^j(c)$ for all c implies that $\bar{c}^h \geqslant \bar{c}^j$.[17]

VII. Long-Run Consumer Demand Functions

Traditional demand theory summarizes behavior by consumption functions in which income or wealth appears as an argument. But in the long run both are choice variables for the consumer. In fact, long-run or steady-state consumption depends only on prices and the rate of interest. We now show this precisely in a model where preferences are described by (1), and some striking results emerge. (A stable steady state does not exist if the rate of discount is constant, as was noted in Sec. III.) Moreover, we thoroughly analyze the comparative statics properties of the long-run consumption functions and thus derive the implications of utility maximization for long-run consumer behavior.[18]

The following intertemporal optimization problem is solved:

$$\max -\int_0^\infty \exp\left\{-\int_0^t u(c_1, \ldots, c_n)d\tau\right\}dt, \tag{28}$$

subject to $\dot{w} = rw - \Sigma p_i c_i$ and $w(0) = w_0 > 0$; w denotes wealth, r denotes the rate of return to saving, and $p_1, \ldots, p_n$ are the prices of the n consumption goods. Since we wish to focus on steady states, r

[17] After completing this paper, we saw a paper by Lucas and Stokey (1982) that undertakes an extensive analysis of growth with many households in the discrete-time framework of Koopmans et al. (1964).

[18] Such implications in the context of a habit formation model are examined by Pollak (1970, 1976).

and the prices are assumed to be constant. The function u is increasing, concave, and positive, and $u(0, \ldots, 0) < r < u(\infty, \ldots, \infty)$.

Steady-state consumption $(\bar{c}_1, \ldots, \bar{c}_n)$ is a function of r and $p_1, \ldots, p_n$. Arguments similar to those used in previous sections show that the steady-state consumption functions, $\bar{c}_1(r, p_1, \ldots, p_n), \ldots, \bar{c}_n(r, p_1, \ldots, p_n)$, are the unique solutions of the following equations:[19]

$$\frac{u_i}{u_j} = \frac{p_i}{p_j}, \quad i, j = 1, \ldots, n, \tag{29}$$

and

$$u(\bar{c}_1, \ldots, \bar{c}_n) = r.$$

But these are precisely the conditions which characterize solutions to the following problem:

$$\min_{c_1, \ldots, c_n} \Sigma p_i c_i \tag{30}$$

subject to $u(c_1, \ldots, c_n) = r$. Remarkably, it follows that long-run consumption functions can be viewed as solving a problem of expenditure minimization subject to an "output" constraint where "output" equals the rate of interest r.

The well-known comparative statics properties associated with the problem (30) imply the following testable restrictions on long-run consumption functions: (*a*) each $\bar{c}_i$ is homogeneous of degree zero in commodity prices; (*b*) $\Sigma p_i \bar{c}_i$ is increasing in r; and (*c*) the matrix of price derivatives $(\partial \bar{c}_i / \partial p_j)$ is symmetric and negative semidefinite. Moreover, these restrictions are exhaustive—any set of functions satisfying *a–c* may be viewed as long-run consumption functions for some u and U.[20] The extensive and exhaustive testable long-run implications of the model (28) constitute an appealing feature of that model.

The relevance of problem (30) and the negative semidefiniteness of the matrix of (uncompensated) price derivatives give precise meaning to the statement that "income effects vanish in the long run." Intuitively one suspects that this corresponds to the fact that income is an endogenous variable in the long run. That is not quite accurate, however, because the special comparative statics properties of the $\bar{c}_i$'s do not extend to the utility specification based on (8).

[19] If the $u = r$ indifference curve is asymptotic to all axes in n-dimensional commodity space, then positive $\bar{c}_i$'s that solve (29) always exist. Otherwise, for some prices the analysis would have to be modified to allow for zero consumption of some goods.

[20] See Deaton and Muellbauer (1980) for a brief discussion of the problem of integrability. Actually *a–c* permit integration to a function u that is quasi concave. It may be shown that u is concave if and only if the functions $\bar{c}_i$ also satisfy *d*. The expression $\Sigma p_i \bar{c}_i$ is a convex function of r.

The model (28) also has strong implications for long-run welfare analysis. Along steady-state consumption paths $U = -u^{-1} = -r^{-1}$. Thus steady-state utility increases with the rate of interest and is independent of commodity prices. The burden of higher prices is borne entirely in the transition to a new steady state. Steady-state wealth $\bar{w} = (\Sigma p_i \bar{c}_i)/r$ is adjusted precisely to compensate for price changes in the long run.

An interesting additional illustration of these welfare implications is provided by the analysis of the different consequences of inflation in closed and open economies. The steady-state value of (17) is

$$\bar{U} = -\frac{1}{u(\bar{c}, \bar{b})} = -\frac{1}{\bar{r}}. \tag{31}$$

In a closed economy, an increase in the equilibrium rate of inflation results in capital deepening and a decline in $\bar{r}$. Inflation is therefore unambiguously welfare reducing. However, for a small open economy embedded in a world of perfect capital mobility, $\bar{r}$ is parametric. It therefore follows from (31) that inflation is welfare neutral in the long run. Only a pure substitution effect between money and real capital in the portfolios of home residents is present.

VIII. Concluding Comments

We have formulated a class of utility functionals for which rates of time preference depend on future consumption. These functionals were applied to a broad range of problems in economic dynamics to illustrate the critical role played by the structure of the rate of time preference. (Further applications may be found in the trade studies cited.) The models we have investigated are highly stylized. But we hope the new results which emerge are of sufficient interest to justify consideration of (1) as an alternative to the standard specification with a constant rate of discount.

Appendix

In this Appendix we prove many of the assertions made in the text. The following lemma is of use in Sections II and III:

LEMMA 1: Let the functional V be defined by $U(C) = -e^{-V(C)}$, where U is defined in (1). Then V is concave.

PROOF: By induction on T we can show that for each $T > 0$,

$$-\log\left[\sum_0 e^{-\sum_{\tau=0}^{t} u(c_\tau)}\right]$$

is concave in $(c_0, \ldots, c_T)$. By standard though lengthy limiting arguments this result may be extended to continuous time and an infinite horizon. ||

Consider the growth problem (9) of Section III. Introduce the state variable $z = \int_0^t u(c)d\tau$ and rewrite the problem in the form

$$\max - \int_0^\infty e^{-z}dt, \tag{32}$$

subject to $\dot{z} = u(c)$, $\dot{k} = g(k) - c$, $z(0) = 0$, $k(0) = k_0$. Let $\hat{H}(k_0, z_0)$ be the value of the corresponding problem where $z(0) = z_0$, $H(k_0) = \hat{H}(k_0, 0)$. It is easy to see that $\partial\hat{H}(k_0, z_0)/\partial z_0 = -\hat{H}(k_0, z_0)$. Thus the Hamilton-Jacobi equation for (32) implies $0 = \max_c\{-e^{-z} - \hat{H}(k, z)u(c) + \hat{H}_k(k, z)[g(k) - c]\}$. Take $(k, z) = (k, 0)$ and derive

$$\rho[H(k)] = e^{J(k)} = \max_c \{u(c) + J'(k)[g(k) - c]\}. \tag{33}$$

(Recall that $H = -e^{-J}$.) To prove equation (10), differentiate (33) with respect to k and apply the envelope theorem.

Next it is asserted in the text that $J'' < 0$ if $g'' \leqslant 0$. To show this, argue as follows: $J(k_0) = \max V(C)$ subject to the constraints of problem (9). By lemma 1 V is a concave functional. Now standard arguments (e.g., Long 1979) show that J is concave. Therefore, if J'' exists, as we assume, then $J'' \leqslant 0$. From (10), $J''(k) < 0$ except possibly at $k = \bar{k}$, where $\rho[H(\bar{k})] = g'(\bar{k})$. From (31), $u'[c^*(k)] = J'(k)$, where $c^*(k)$ is optimal consumption given stock k. Therefore $J''(\bar{k}) = 0$ if and only if $c_k^*(\bar{k}) = 0$. But that can be ruled out by a local analysis around $[c^*(\bar{k}), \bar{k}]$ paralleling the analysis in Calvo (1979, pp. 97–98) or Obstfeld (1981, pp. 1157–58). Thus $J'' < 0$ as desired.

The asserted nature of optimal accumulation paths when $g'' \leqslant 0$ follows directly from $J'' < 0$, (7), (10), and the assumptions made in the text regarding the relative magnitudes of u and g' at endpoints; for example, $u(0) < r < u(\infty)$ if $g(k) = rk$. To confirm the possibility of solutions depicted in figure 2, *b* in particular, we construct a concrete example. In fact, the next lemma describes a general procedure for constructing examples of solutions to problems which have the form (9), even given a convex-concave technology.

LEMMA 2: Let J be defined on the nonnegative real line, $J(0) > -\infty$, $J' > 0$, and $J'' < 0$. Let ψ be defined on $D \equiv [J'(\infty), J'(0)]$ such that on D $\inf\{\psi(z): z \in D\} > 0$, $\psi' < 0$, and ψ is convex. Define $g(k)$ by

$$g(k) \equiv \{e^{J(k)} - \psi[J'(k)]\}/J'(k), \quad k > 0. \tag{34}$$

Suppose that g defined in this way satisfies our assumptions for production functions. Suppose also that the differential equation

$$\dot{k} = [-J'(k)/J''(k)][g'(k) - e^{J(k)}], \quad k(0) = k_0, \tag{35}$$

generates solutions $k^*(t; k_0)$ which converge to $\bar{k} > 0$, $g(\bar{k}) > 0$, for all $k_0 > 0$. Finally, define $u(c) \equiv \inf\{\psi(z) + zc: z \in D\}$, $c \geqslant 0$. Then $u > 0$ and u is nondecreasing and concave for $c > 0$. Moreover, for all $k_0 > 0$, $k^*(t; k_0)$ is the optimal capital stock in the appropriate version of (9) and $H(k_0) = -e^{-J(k_0)}$ is the maximum value of intertemporal utility.

An important step in the proof is to establish (33) and the "inverse" proposition that $u(c) = \inf_{k>0}\{\exp[J(k)] - J'(k)[g(k) - c]\}$. Similar arguments are applied in proofs in Epstein (1981, theorems 1 and 2) and Epstein (1983*a*, theorem 2).[21] The example in the text is a special case of the lemma with $\psi(z) = Bz^{-\beta} + az$ and $J(k) = \log(k + \Delta)$.

[21] The procedure may be extended to allow for $J'' > 0$ in some regions and to construct examples in which it may be optimal to drive the capital stock to zero.

Turn to the optimization problem (18) of Section V. Rewrite the problem in the form

$$\max -\int_0^\infty e^{-z}dt \tag{36}$$

subject to $\dot{z} = u(c, a - k)$, $\dot{a} = (w + rk + v) - (\dot{p}/p)(a - k) - c$, $z(0) = 0$, and $a(0) = k_0 + [M_0/p(0)]$; $a \equiv k + b$ denotes total real wealth. The Hamiltonian L is given by

$$L \equiv -e^{-z} + \lambda\left[w + rk + v - \frac{\dot{p}}{p}(a - k) - c\right] + \mu u(c, a - k). \tag{37}$$

Maximization with respect to c and k yields the first-order condition

$$\frac{u_b}{u_c} = r + \frac{\dot{p}}{p}. \tag{38}$$

The (initial) utility shadow price of the stock z is $\mu(0)$. It is evident from (36)—or it can be derived from the maximum principle—that $\mu(0) = \int_0^\infty e^{-z}dt$, where z follows an optimal path. Similarly, $\mu(t) = \int_t^\infty e^{-z}d\tau$ for any t. Thus $\dot{\mu}/\mu = -[\int_t^\infty e^{-(z-t)}d\tau]^{-1} = [U({}_tC, {}_tB)]^{-1}$, where C and B are optimal profiles. We conclude that

$$\frac{\dot{\mu}(t)}{\mu(t)} = -\rho[U({}_tC, {}_tB)]. \tag{39}$$

The maximum principle implies that λ evolves according to $\dot{\lambda} = -r\lambda$ and that $\lambda = \mu u_c$. Logarithmic differentiation of the latter and substitution of (39) yield

$$\frac{\dot{u}_c}{u_c} = \rho - r. \tag{40}$$

Finally, $b = M/p \Rightarrow \dot{b}/b = \theta - (\dot{p}/p) = \theta + r - (u_b/u_c)$ by (38). This proves (20c). Equation (20b) is obvious, and (20a) follows from (19) and (40).

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