Symposium on Aggregation

The Structure of Utility Functions[1]

1. INTRODUCTION AND SUMMARY

1.1 Suppose that we are given

P1 a continuous complete preference ordering $\succcurlyeq$[2] on a product space $S = S_1 \times S_2 \ldots \times S_n$, where

P2 each S_i is topologically separable and arc connected,[3]

implying that S is so too. A well-known result of Debreu [2] then yields

Lemma 0. *There exists a continuous* utility function $U(\cdot)$ *defined on* S *such that* $U(x) \geqq U(x')$ *iff*[4] $x \succcurlyeq x'$, *each* $x, x' \in S$.

1.2. Let us write

$$x = (x_1, \ldots, x_n) = (x_i)_{i \in \Omega}, \; x_i \in S_i \text{ each } i \in \Omega, \qquad \ldots(1.1)$$

where

$$\Omega = \{1, 2, \ldots, n\} \qquad \ldots(1.2)$$

is the set of *sectors*. These may be thought of as groups of goods or periods of time, for instance. In the latter case the S_i might be function spaces, the x_i being vectors whose components are time series giving the consumption of the individual goods at each moment in the period in question.

1.3. Consider any set $A \subseteq \Omega$ of sectors and define

$$x_A = (x_i)_{i \in A}, \; S_A = \prod_{i \in A} S_i, \; \bar{A} = \Omega - A. \qquad \ldots(1.3)$$

$\succcurlyeq$ clearly defines a conditional ordering on S_A, given what happens off it, whose utility functions might be written $U_A(x_A : x_{\bar{A}}) = U(x_A, x_{\bar{A}})$. If, given some $x_{\bar{A}}$ not all elements of S_A are indifferent, say that A is *essential*, if, given any, *strictly essential*. If the conditional ordering on A is the same for all $x_{\bar{A}}$, say that A is *separable*. This is Leontief's concept of separability, somewhat generalized [13], [14]. I will use the word in this sense throughout the remainder of this paper. It has nothing to do with the topological separability of P2.

1.4. Take a *reference vector* $0 \in S$ and call its component on any S_B, $B \subseteq \Omega$, 0 also. It need not, of course represent zero consumption. If A is separable, any conditional

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[2] *Complete ordering*: $\succcurlyeq$ is transitive and reflexive, and, if $x, y \in S$, either $x \succcurlyeq y$ or $y \succcurlyeq x$, or both. Debreu, for instance, calls this a preordering, because indifference is allowed.

Continuous: $\{y \in S : x \succcurlyeq y\}$ and $\{y \in S : y \succcurlyeq x\}$ are both closed, each $x \in S$.

[3] *Topologically separable*: S_i has a countably dense subset, such as the rationals among the reals or points with rational coordinates in R^n. Many rather rich spaces are topologically separable. Note that this has nothing to do with separability in Leontief's sense, which I use below.

Arc connected: If $x_i, y_i \in S_i$ they are joined by an arc in S_i. Debreu assumes " connectivity " rather than " arc connectivity ". However I need the stronger postulate to provide the proof of (1.6), which Debreu leaves to the reader. See the Appendix.

[4] " iff " = " if and only if ".

ordering on it may be represented by the continuous *subutility function*:

$$U_A(x_A) = U(x_A, 0). \qquad \ldots(1.4)$$

Throughout this paper I will use the same reference vector $0 \in S$ for all separable sets, will write 0 for 0_A, its component in S_A, each $A \subseteq \Omega$, and will normalize so that

$$U(0) = 0. \qquad \ldots(1.5)$$

1.5. Since inessential sets affect nothing, they may be neglected.[1] I will accordingly assume throughout this paper that

P3. Each sector is essential.

For simplicity of argument I will frequently make the stronger assumption

P4. Each sector is strictly essential,

which I will however show in section 6 can be replaced by the harmless P3, except in one rather special case.

1.6. Debreu has also shown [2] [3] that the utility function may be written

$$U(x) = F(U_1(x_1), \ldots, U_n(x_n)) \qquad \ldots(1.6)$$

iff each sector is separable, and that, if at least three sectors are essential,

$$U(x) = U_1(x_1) + U_2(x_2) + \cdot + U_n(x_n) \qquad \ldots(1.7)$$

in an appropriate normalization iff each set of sectors is separable, where the U_i's are the subutility functions just defined, and $F(\cdot)$ is continuous and strictly increasing. These correspond to Leontief's theorems on separability and additive separability, but are global while Leontief's are local,[3] and depend on weaker assumptions.

1.7. There are extreme cases: What do we know about the structure of $U(\cdot)$ if we are given a general collection $\mathscr{B}$ of separable sets? Call it a *separable* collection.

1.8. Even without P4, it is clear that 0 and Ω are separable and that, if A and B are separable, so is their intersection. We will see in section 2 that, if they *overlap*, that is intersect and neither contains the other, then P4 implies that their union $A \cup B$, two differences $A-B$, $B-A$ and symmetric difference $A \Delta B = (A-B) \cup (B-A)$ are also separable. Let us say that a *collection A* of subsets of Ω is *complete* if

(i) If $A, B \in \mathscr{A}$ overlap, $A \cup B$, $A \cap B$, $A-B$, $B-A$ and $A \Delta B$ all belong to $\mathscr{A}$;

(ii) $0, \Omega \in \mathscr{A}$,

and define the *completion* $\mathscr{A}(\mathscr{B})$ *of* $\mathscr{B}$ to be the intersection of all the complete collections containing $\mathscr{B}$: that is the smallest such collection. Clearly all its elements are separable if P4 holds. We will see in section 3 that these are the only sets whose separability is implied by that of the elements of $\mathscr{B}$, even given P4, and in section 6 that their separability follows from P3 except in one rather special case.

1.9. P4 implies, then, that we can assume without loss of generality that we are given a complete separable collection $\mathscr{A}$: if the original separable collection $\mathscr{B}$ is incomplete, we merely take $\mathscr{A} = \mathscr{A}(\mathscr{B})$. Of course if $\mathscr{B}$ is complete, we have no need of P4, though P3 is needed even then.

1.10. Now $\mathscr{A}$ is partially ordered under the relation $\supseteq$, with unique maximal and minimal elements Ω and 0. Call these the *trivial elements* of $\mathscr{A}$, the others the *proper*

[1] This presumes that we know which sectors are inessential—surely not a strong requirement. P3 might be further weakened, but it would complicate the exposition, so that it hardly seems worth while.

[2] Actually he leaves the proof of (1.6) to the reader.

[3] Though Leontief's theorems can frequently be proved to hold globally. See [5].

elements, and the maximal proper elements—that is those which are not contained in any other proper elements—the *top elements of* $\mathscr{A}$.

1.11. If no two top elements intersect, there is a partition $\{\Omega_0, \Omega_1, \ldots, \Omega_m\}$ of Ω in which $\Omega_1, \ldots, \Omega_m$ are top elements of $\mathscr{A}$, and Ω_0, the *free set*, is made up of these sectors which belong to no proper element of $\mathscr{A}$, since each proper element is contained in a top element. Let $(y_0, y_1, \ldots, y_m)$ be the corresponding partition of x. Since $\Omega_1, \ldots, \Omega_m$ are separable, an immediate generalization of (1.6), proved in the appendix, shows that we can write

$$U(x) = F(y_0, U_1(y_1), \ldots, U_m(y_m)) \qquad \ldots(1.8)$$

where $F(\cdot)$ is continuous, and strictly increasing in the subutilities $U_1, \ldots, U_m$.

1.12. If two of the top elements intersect, it is shown in section 3 that there is a partition $\{\Omega_1, \Omega_2, \ldots, \Omega_m)$ of Ω, the union of every subset [1] of which belongs to $\mathscr{A}$. Let $(y_1, y_2, \ldots, y_m)$ be the corresponding partition of x. An obvious extension of (1.7), proved in the appendix, therefore implies that we can write

$$U(x) = U_1(y_1) + \ldots + U_m(y_m) \qquad \ldots(1.9)$$

in an appropriate normalization.

1.13. It is shown in section 3 that $\mathscr{A}$ may be broken down into $\mathscr{A}^*$, which is composed of Ω, 0 with $\Omega_1, \Omega_2, \ldots, \Omega_m$ in the first case discussed above, and with the union of each set of the Ω's in the second, and $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_m$ which bear the same relationship [2] to $\Omega_1, \Omega_2, \ldots, \Omega_m$ as $\mathscr{A}$ to Ω. The information in $\mathscr{A}^*$ yields (1.8) or (1.9) as the case may be; that in $\mathscr{A}_i$ permits a similar analysis of Ω_i, each i, and so down the utility tree. This permits a complete characterization of the form of the utility function in terms of $\mathscr{A}$.

1.14. Some applications are discussed in section 4, an alternative approach sketched in section 5 and P4 replaced by P3 in section 6. This weaker postulate is also used in the proof of Theorem 1 in section 2.

2. THE BASIC THEOREM

2.1. Let $\{\Omega_0, \Omega_1, \ldots, \Omega_m\}$ be a partition of Ω and $(y_0, y_1, \ldots, y_m)$ be the corresponding partition of x. It is shown in the appendix that

Lemma 1. *If* $\Omega_1, \Omega_2, \ldots, \Omega_m$ *are separable*, P1-2 *imply that we can write*

$$U(x) = F(y_0, U_1(y_1), \ldots, U_m(y_m)) \qquad \ldots(2.1)$$

where $U_1(\cdot), \ldots, U_m(\cdot)$ *are the continuous subutilities defined in* (1.4) *and* (1.5) *and* $F(\cdot)$ *is* well behaved*: that is, is continuous in all its arguments and strictly increasing in the subutilities.*

Corollary. F mirrors the subutilities: that is

$$U_i = F(0, \ldots, 0, U_i, 0, \ldots, 0), \text{ each } i. \qquad \ldots(2.2)$$

2.2. Lemma 1 then permits us to prove:

Theorem 1. *If* (0) $A, B \subseteq \Omega$ *overlap*, (i) $A, B \subseteq \Omega$ *are separable*, (ii) $B-A$ *is strictly essential*,[3] *then* P1-3 *imply that* (iii) $A \cup B$, $A \cap B$, $A-B$, $B-A$, $A \Delta B = (A-B) \cup (B-A)$ *are all separable and strictly essential.*

Remark

2.3. It will be convenient to change notation slightly and write x for x_{A-B}, y for $x_{A \cap B}$, z for x_{B-A}, θ for $x_{\bar{A} \cap \bar{B}}$, and therefore (x, y, z, θ) for x, in this section.

[1] That is, the union of each set of Ω's.
[2] That is, $\mathscr{A}_i$ satisfies the definition of completeness in 1.8 with Ω_i in the place of Ω, each i.
[3] Or, of course, $A-B$.

I will show that (i) and (ii) imply, with (0) and P1-3, that

$$\text{(iv)} \quad U(x, y, z, \theta) = F(u(x)+v(y)+w(z), \theta), \text{ say,}$$

where $F(\cdot, \theta)$ is strictly increasing, which in turn clearly implies (iii): since (iii) certainly implies (i) and (ii) we have

Corollary " (i) and (ii)," " (iii) ", and " (iv) " are equivalent, given P1-3 and (0).

Proof

2.4. Since A and B are both separable, the conditional preference ordering on $S_{A\cap B}$ is independent both of (z, θ) and of (y, θ), and is therefore independent of (y, z, θ). Hence their intersection is separable, too. Lemma 1 therefore implies that we can write:

$$U(x, y, z, \theta) = G(g(x, v(y)), z, \theta) = H(x, h(v(y), z), \theta) \qquad ...(2.3)$$

where $v(y)$ is the subutility function for $A\cap B$, $g(x, v(y))$ for A, $h(v(y), z)$ for B, and all the functions are well behaved and mirror the subutility functions.[1] Since each is continuous on a connected domain, the range of each is an interval. $(x, y, z, \theta)\succ(x, y', z, \theta)$, some x, z, θ, y, y' since $A\cap B$ is essential, and hence for all x, z, θ since it is separable. Hence this interval is non-degenerate. In particular, the range of

$$v(y) = U(0, y, 0, 0) = g(0, v(y)) = h(v(y), 0) \qquad ...(2.4)$$

is a non-degenerate interval J containing 0.

2.5 Define analogously

$$u(x) = U(x, 0, 0, 0) = g(x, 0); \; w(z) = U(0, 0, z, 0) = h(0, z). \qquad ...(2.5)$$

2.6. According to 2.4,

$$U(x, y, z, \theta) \geqq U(x', y, z, \theta) \qquad ...(2.6)$$

iff

$$g(x, v(y)) \geqq g(x', v(y)), \qquad ...(2.7)$$

and also iff

$$H(x, h(v(y), z), \theta) \geqq H(x', h(v(y), z), \theta), \qquad ...(2.8)$$

so that (2.7) and (2.8) are equivalent. I will use this to show that (2.6) holds iff $u(x) \geqq u(x')$, so that $A-B$ is separable.

2.7. Say that $v\in J$ is *joined* to $v'\in J$ if there are $z, z' \in S_{B-A}$ such that

$$h(v, z) = h(v', z'). \qquad ...(2.9)$$

If so

$$g(x, v) \geqq g(x', v) \text{ iff } g(x, v') \geqq g(x', v'), \qquad ...(2.10)$$

because each is equivalent to $H(x, h, \theta) \geqq H(x', h, \theta)$, where $h = h(v, z) = h(v', z')$.

2.8. Say that v is *connected* to v' if there exists a chain $v = v_0, v_1, ..., v_t = v'$, each element of which, except the last, is joined to its successor. Clearly (2.10) holds in this case, too.

2.9 " Being connected to " is an equivalence relation, and so partitions J into equivalence classes.

2.10. These equivalence classes are intervals. To see this, let $v, v', v<v'$, belong to the same equivalence class K, and take $\bar{v}$ such that $v>\bar{v}>v'$. In the chain connecting v, v' there are a pair v_i, v_{i+1} such that $v_i \geqq \bar{v} \geqq v_{i+1}$. In an obvious notation, therefore: $h(v_i, z_i) = h(v_{i+1}, z_{i+1}) \leqq h(\bar{v}, z_{i+1}) \leqq h(v_i, z_{i+1})$ because $h(\cdot, z_{i+1})$ is increasing. Since $h(v_i, \cdot)$ is continuous on the connected space S_{B-A}, this implies that there is a $\bar{z} \in S_{B-A}$ such that $h(v_i, \bar{z}) = h(\bar{v}, z_{i+1})$. $\bar{v}$ is therefore joined to $v_i \in K$, and hence belongs to K too. K is therefore an interval.

[1] Since e.g. $h(v(y), 0) = U(0, y, 0, 0) = v(y)$. To say $h(v(v), z)$ is well behaved is to say that it is continuous, and strictly increasing in v.

2.11. I will show that there is only one equivalence class, so that each $v \in J$ is connected to 0 in particular, implying, by (2.5)-(2.7) and (2.10), that $U(x, y, z, \theta) \geqq U(x', y, z, \theta)$, iff $u(x) \geqq u(x')$ so that $A-B$ is separable with subutility function $u(x) = U(x, 0, 0, 0)$.

2.12. Suppose the contrary and let v^* be a common frontier point of two equivalence classes J_1, J_2. Since J, being an interval, is connected, such a point exists. If either J_1 or J_2 is degenerate, it may be the end point of J; otherwise it is an interior point, since both are intervals. In either case it belongs to J, and not merely to its closure. Should v^* be the upper bound of J, for instance, I_+ below is empty, but this does not affect the validity of the argument.

2.13. Since $B-A$ is strictly essential, there are $z, z' \in S_{B-A}$ such that

$$h(v^*, z) > h(v^*, z'). \qquad \ldots(2.11)$$

Since h is well behaved by 2.4 there is a $\delta > 0$ such that

$$h(v^*, z) \geqq h(v, z') \geqq h(v^*, z'),\ v \in J \cap [v^*, v^*+\delta] = I_+, \text{ say}, \qquad \ldots(2.12)$$

and since $h(v^*, \cdot)$ is continuous in z on the connected space S_{B-A}, there exists a $z^* \in S_{B-A}$ such that $h(v^*, z^*) = h(v, z')$. Hence v^* is connected to each $v \in I_+$. Similarly there exists an $\epsilon > 0$ such that v^* is connected to each $v \in J \cap [v^*-\epsilon, v^*] = I_-$, say. Since it is a common frontier point of $J_1, J_2 \subseteq J$, this implies that it belongs to both equivalence classes, which is impossible. There is therefore only one equivalence class, so that 2.11 implies that $A-B$ is separable with subutility function $u(x)$. Being separable and essential, it is strictly essential. Hence $B-A$ is separable with subutility function $w(z)$ by a similar argument.

2.14. Lemma I now implies that we can write

$$U = G(g(u, v), w, \theta) = H(u, h(v, w), \theta) \qquad \ldots(2.13)$$

in a slightly new notation where all the functions are well behaved.

2.15. I will now show that this implies that we can write [1]

$$U(x, y, z, \theta) = F(a(u(x)) + b(v(y)) + c(w(z)), \theta) \qquad \ldots(2.14)$$

where all functions are well behaved. Since this is just (iv) in a different notation, it proves both the theorem and the corollary.

2.16. According to a theorem of Aczel [2] on associative functions,

$$U = G(g(u, v), w) = H(u, h(v, w)), \qquad \ldots(2.15)$$

where all functions are well behaved, and

$$u, v, w, g, h, G, H \text{ all range over the same interval } \langle \alpha, \beta \rangle, \qquad \ldots(2.16)$$

would imply that we could write

$$g(u, v) = \phi(a(u) + b(v)),\ h(v, w) = \psi(b(v) + c(w))$$

$$G(g, w) = F(\phi^{-1}(g) + c(w)),\ H(u, h) = F(a(u) + \psi^{-1}(h)), \qquad \ldots(2.17)$$

so that

$$U = F(a(u) + b(v) + c(w)), \qquad \ldots(2.18)$$

where

$$\text{all functions are well behaved.} \qquad \ldots(2.19)$$

2.17. (2.18)-(2.19) is the sort of thing we want. There are two obstacles in the way of finding it: θ and the fact that (2.16) is not normally satisfied.

[1] If so, we can clearly normalize so that $F(\lambda, 0) = \lambda$, and $a(u(0)) = \ldots = 0$. Do so. Then $a+b+c$, b, a, c, $a+c$ are the subutility functions for $A \cup B$, $A \cap B$, $B-A$, $A-B$, $A \Delta B$ respectively. Since each of these sets is separable and essential, it is strictly essential.

[2] [1] page 321, Corollary 1 to Theorem I of page 320.

2.18. Consider the latter first. Since $u(\cdot)$, ..., are continuous on arc connected domains S_{A-B}, ..., and $A-B$, ..., are essential, the domains of u, v, w are all non-degenerate intervals. To make them all of the same type, restrict u, v, w to the interior of their domains until further notice. The domains of u, v, w, are now all open intervals. So, by their continuity on connected spaces, are those of $g(\cdot)$ and $h(\cdot)$. We can map each of these intervals onto (0, 1) by well-behaved transformations which are independent of θ. Do so, and denote the transformed concepts by the same symbols as the old, thus retaining (2.13).

2.19. 2.13 also implies that the common range of $U(\cdot, \theta)$, $G(\cdot, \theta)$ and $H(\cdot, \theta)$ is an open interval. There is a well-behaved transformation L_θ which makes (0, 1) the common range of

$$L_\theta(U) = G_\theta(g(u, v), w) = H_\theta(u, h(v, w)) \qquad \text{...(2.20)}$$

where

$$G_\theta(g, w) = L_\theta(G(g, w, \theta)),\ H_\theta(u, h) = L_\theta(H(u, h, \theta)). \qquad \text{...(2.21)}$$

All these functions are still well behaved, and (2.16) is now satisfied with G_θ, H_θ replacing G, H. By 2.18 we have (2.17)-(2.19), except that the new functions also require a subscript θ.

2.20. In particular therefore

$$g(u, v) = \phi_\theta(a_\theta(u) + b_\theta(v)), \qquad \text{...(2.22)}$$

and we can clearly normalize so that

$$a_\theta(0{\cdot}5) = b_\theta(0{\cdot}5) = c_\theta(0{\cdot}5) = 0, \text{ each } \theta. \qquad \text{...(2.23)}$$

Do so and drop the suffix 0 in the case $\theta = 0$ to get

$$\phi_\theta(a_\theta(u) - b_\theta(v)) = \phi(a(u) + b(v)). \qquad \text{...(2.24)}$$

Taking $v = 0{\cdot}5$ first and then $u = 0{\cdot}5$, we have

$$a_\theta = \phi_\theta^*(a),\ b_\theta = \phi_\theta^*(b), \qquad \text{...(2.25)}$$

so that, by (2.24),

$$\phi_\theta^*(a+b) = \phi_\theta^*(a) + \phi_\theta^*(b), \qquad \text{...(2.26)}$$

where

$$\phi_\theta^* = \phi_\theta^{-1} \qquad \text{...(2.27)}$$

is well behaved by (2.19) on its interval domain I, say, which contains the origin. The domains of a and b being non-degenerate intervals this implies

$$\phi_\theta^*(\lambda) = k_\theta \lambda, \text{ say, } \lambda \in I;\ k_\theta > 0, \text{ a constant}, \qquad \text{...(2.28)}$$

so that [1]

$$\begin{aligned} k_\theta &= a_\theta/a = b_\theta/b \\ &= c_\theta/c, \end{aligned} \qquad \text{...(2.29)}$$

by a similar argument applied to $h(\cdot)$.

2.21. The θ-version of (2.18) is

$$\begin{aligned} L_\theta(U) &= F_\theta(a_\theta(u) + b_\theta(v) + c_\theta(w)) \\ &= F_\theta(k_\theta(a(u) + b(v) + c(w))) \end{aligned} \qquad \text{...(2.30)}$$

by (2.29), so that

$$U = F(a(u) + b(v) + c(w), \theta), \text{ say}, \qquad \text{...(2.14)}$$

as we wished, where all the functions are well behaved.[2]

[1] a_θ, b_θ, c_θ, a, b, c all vanish together, so this creates no problems.
[2] We normalize so that $F(\lambda, 0) = \lambda$ once more, so that the argument of footnote 1, p. 371 works.

2.22. All this so far only when u, v, w are restricted to the interior of their interval domains. It holds throughout these domains by the good behaviour of all the functions in (2.14).

2.23. This completes the proof of both theorem and corollary.

2.24. Given P1, P2 and P4 we may therefore assume without loss of generality that we are given a complete collection $\mathscr{A}$ of subsets of Ω which are known to be separable, that is a complete separable collection. Of course the original separable collection $\mathscr{B}$ may already be complete. If so we will not need P4, though a certain amount of essentiality will be needed in applying the results.

2.25. The intersection of two complete collections, each containing a given collection $\mathscr{B}$, is clearly complete and contains $\mathscr{B}$. Hence there exists a smallest collection $\mathscr{A}(\mathscr{B})$ with these properties. To know that the elements of $\mathscr{B}$ are separable is to know that these of $\mathscr{A}(\mathscr{B})$ are, given P1, P2 and P4.

3. THE UTILITY TREE

3.1. In this section I will show how the partial ordering of the complete collection $\mathscr{A}$ under $\succcurlyeq$ determines the structure of $U(\cdot)$.

3.2. Consider first the partial ordering itself. For this we need no postulates at all other than the existence of the complete collection $\mathscr{A}$.

3.3. The definitions in 1.10 of a *top element* A of $\mathscr{A}$, as a maximal proper element and in 1.11 of the *free set* Ω_0 as composed of those sectors which belong to no proper element, may be rephrased as

$$A \in \mathscr{A} \text{ is a top element of } \mathscr{A} \text{ iff } A \subset B \subset \Omega \Rightarrow B \notin \mathscr{A} \qquad \text{...(3.1)}$$

$$\Omega_0 = \{i \in \Omega:\ i \in B \subset \Omega \Rightarrow B \notin \mathscr{A}\} \qquad \text{...(3.2)}$$

where " $\subset$ " denotes strict inclusion, denying the possibility of equality.

3.4. Top elements may or may not intersect. See that $\mathscr{A}$ is *top disjoint* if no pair of its top elements intersect, *top overlapping*, if at least one pair do. We will see below that there are at least three top elements in the latter case, and that each pair overlap.

3.5. **$\mathscr{A}$ top disjoint.**

3.5.1. Let $\Omega_1, \Omega_2, \ldots, \Omega_m$ be the top elements. Then

$$\{\Omega_0, \Omega_1, \ldots, \Omega_m\} \text{ is a partition of } \Omega, \qquad \text{...(3.3)}$$

where Ω_0 is the free set defined in (3.2). Let $(y_0, y_1, \ldots, y_m)$ be the corresponding partition of x. Lemma 1 implies

$$U(x) = F(y_0, U_1(y_1), \ldots, U_m(y_m)), \qquad \text{...(3.4)}$$

where the $U_i(\cdot)$ are continuous subutility functions, and $F(\cdot)$ is well behaved.

3.5.2. Suppose that a proper element C of $\mathscr{A}$ intersects one of the top elements. It is contained in it if $\mathscr{A}$ is top disjoint. This is obvious if $m = 1$, since C cannot contain free sectors. If $m \geqq 2$, let C intersect Ω_i, Ω_j. Since (3.1) implies that it contains neither, it overlaps each. Hence $B = C \cup \Omega_i$ is a proper element of $\mathscr{A}$, contradicting (3.1). Hence $\mathscr{A}$ may be listed, with trivial double counting, as the union of

$$\begin{aligned} \mathscr{A}^* &= \{\Omega, \Omega_1, \ldots, \Omega_m, 0\}, \\ \mathscr{A}_i &= \{C \in \mathscr{A}:\ C \subseteq \Omega_i\}, \quad i = 1, 2, \ldots, m. \end{aligned} \qquad \text{...(3.5)}$$

3.6. $\mathscr{A}$ **top overlapping.**

3.6.1. Call the top elements $A_1, \ldots, A_m$. At least one pair intersect. Since neither contains the other by (3.1), they overlap.

$$A_i \text{ and } A_j \text{ overlap.} \quad \ldots(3.6)$$

Then $A_i \cup A_j \in \mathscr{A}$ by Theorem 1, so that

$$A_i \cup A_j = \Omega, \quad \ldots(3.7)$$

by (3.1). Hence

$$\Omega_0 = 0, \text{ by (3.2).} \quad \ldots(3.8)$$

3.6.2. Since no other top element contains either A_i or A_j by (3.1), (3.7) implies that any other overlaps both. Hence [1]

$$(3.6) \text{ and } (3.7) \text{ hold for each } i, j, i \neq j, \quad \ldots(3.9)$$

$$A_j - A_i = \bar{A}_i = \Omega_i, \text{ say, each i}, j, i \neq j, \quad \ldots(3.10)$$

and

$$\Omega_i \cap \Omega_j = \bar{A}_i \cap \bar{A}_j = \overline{A_i \cup A_j} = \bar{\Omega} = 0 \quad \ldots(3.11)$$

while, by Theorem 1,

$$\Omega_i = A_j - A_i \in \mathscr{A}, \; \Omega_i \cup \Omega_j = A_i \Delta A_j \in \mathscr{A} \quad \ldots(3.12)$$

because of (3.6).

3.6.3. Now

$$C_I = \bigcup_{i \in I} \Omega_i \in \mathscr{A}, \text{ each } I \subseteq \{1, 2, \ldots, m\}, \quad \ldots(3.13)$$

by (3.12) and Theorem 1. This is most easily seen when $I = \{1, 2, \ldots, t\}$, say. Define in that case $D_j = \Omega_i \cup \Omega_j \in \mathscr{A}$ by (3.12). Hence $E_2 = D_2 \in \mathscr{A}$ and by repeated application of Theorem 1 so does each $E_j = E_{j-1} \cup D_j = \bigcup_{i=1}^{j} \Omega_i$. Since $C_I = E_t$ when $I = \{1, 2, \ldots, t\}$ this proves (3.13) in this particular case. The argument is clearly valid in general with trivial alterations. In particular, therefore $\bigcup_{i=1}^{m} \Omega_i \in \mathscr{A}$. It is contained in none of the top elements $A_1, A_2, \ldots, A_m$. Hence $\bigcup_{i=1}^{m} \Omega_i = \Omega$ by (3.1), so that (3.11) implies that

$$\{\Omega_1, \Omega_2, \ldots, \Omega_m\} \text{ is a partition of } \Omega, \quad \ldots(3.14)$$

and,

$$m \geqq 3, \quad \ldots(3.15)$$

since otherwise $A_1 = \Omega_2$ and $A_2 = \Omega_1$ cannot intersect, contradicting (3.6) Notice too that

$$A_i = \bar{\Omega}_i = \bigcup_{j \neq i} \Omega_j, \quad \ldots(3.16)$$

$$\bigcap_{i \in I} A_i = \bigcap_{i \in I} \bar{\Omega}_i = \overline{\bigcup_{i \in I} \Omega_i} = \bigcup_{j \notin I} \Omega_j \in \mathscr{A}. \quad \ldots(3.17)$$

3.6.4. If $(y_1, y_2, \ldots, y_m)$ is the corresponding partition of x, (3.13)-(3.15) imply that

$$U(x) = U_1(y_1) + U_2(y_2) + \ldots + U_m(y_m), \quad \ldots(3.18)$$

in an appropriate normalization, where the $U_i(\cdot)$ are continuous subutility functions, by Lemma 2.

3.6.5. Moreover no element $C \in \mathscr{A}$ overlaps an Ω_i: If it were to, it would be a proper element, and would therefore overlap $A_i = \bar{\Omega}_i$, too, by (3.1), so that $B = C \cup A_i$ would be a proper element of $\mathscr{A}$ by Theorem 1, contradicting (3.1). We can therefore once more

[1] We see that each A_k overlaps A_i if A_i overlaps any A_j. Hence each A_i overlaps A_k.

list $\mathscr{A}$, with only trivial double counting, as the union of

$$\mathscr{A}^* = \{\bigcup_{i \in I} \Omega_i \colon I \subseteq \{1, 2, \ldots, m\}\},$$
$$\mathscr{A}_i = \{C \in \mathscr{A} \colon C \subseteq \Omega_i\}, \quad i = 1, 2, \ldots, m. \qquad \ldots(3.19)$$

3.7. Let us define a *complete collection relative to* C by replacing Ω by C in the definition of a complete collection in 1.8. We may then summarize the main results in 3.5.6. in the following theorem:

Theorem 2. *Any complete collection* $\mathscr{A}$ *may be written*

(i) $\mathscr{A} = \mathscr{A}^* \cup \left(\bigcup_{i=1}^{m} \mathscr{A}_i\right)$

where

(ii) $\mathscr{A}_i = \{A \in \mathscr{A} \colon A \subseteq \Omega_i\}$

is complete relative to Ω_i, $i = 1, 2, \ldots, m$ *and where the top collection*

(iii) $\mathscr{A}^* = \{0, \Omega, \Omega_1, \ldots, \Omega_m\}$, $m \geqq 0$ *if* $\mathscr{A}$ *is top disjoint, in which case*

(iv) $\{\Omega_0, \Omega_1, \ldots, \Omega_m\}$ *is a partition of* Ω, *in which* Ω_0 *is the free set and* $\Omega_1, \ldots, \Omega_m$ *the top elements of* $\mathscr{A}$, *or*

(v) $\mathscr{A}^* = \{C_I = \bigcup_{i \in I} \Omega_i \colon I \subseteq \Phi = \{1, 2, \ldots, m\}\}$, $m \geqq 3$, *if* $\mathscr{A}$ *is top overlapping, in which case*

(vi) $\{\Omega_1, \Omega_2, \ldots, \Omega_m\}$ *is a partition of* Ω, *so that* Ω_0 *is empty and the top elements of* $\mathscr{A}$ *are* $\bar{\Omega}_1, \ldots, \bar{\Omega}_m$, $m \geqq 3$.

3.8. It is now clear how to find the structure of $U(\cdot)$, given the separable complete collection $\mathscr{A}$, and P1-3.

3.8.1. Determine the top elements of $\mathscr{A}$. Do they intersect? Determine the Ω_i, $\mathscr{A}^*$, $\mathscr{A}_i$. If $\mathscr{A}$ is top disjoint

$$U(x) = F(y_0, U_1(y_1), \ldots, U_m(y_m)), \qquad \ldots(3.4)$$

where $F(\cdot)$ is well behaved by Lemma 1. If top overlapping

$$U(x) = U_1(y_1) + U_2(y_2) + \ldots + U_m(y_m), \qquad \ldots(3.18)$$

in an appropriate normalization. This exhausts the information in $\mathscr{A}^*$.

3.8.2. In either case each $U_i(\cdot)$ is a continuous subutility function defined on

$$S_{\Omega_i} = \prod_{j \in \Omega_i} S_j,$$

and $\mathscr{A}_i$ is complete relative to Ω_i by (ii).

3.8.3. Look at each $\mathscr{A}_i$ in turn. Discover its top elements. Do they intersect? Find the Ω_{ij}, $\mathscr{A}_i^*$, $\mathscr{A}_{ij}$, in an obvious notation. If $\mathscr{A}_i$ is top disjoint, we can write

$$U_i(y_i) = F_i(y_{i0}, U_{i1}(y_{i1}), U_{i2}(y_{i2}), \ldots) \qquad \ldots(3.4_i)$$

where $F_i(\cdot)$ is well behaved; if top overlapping

$$U_i(y_i) = F_i(\Sigma_j U_{ij}(y_{ij})), \qquad \ldots(3.18'_i)$$

in an appropriate normalization, where $F_i(\cdot)$ is continuous and strictly increasing. The presence of this latter $F_i(\cdot)$ may be surprising. It is needed because when $\mathscr{A}$ is top intersecting, the $U_i(y_i)$ will already have been normalized to secure (3.18), and so cannot now be renormalized to get rid of the $F_i(\cdot)$ in (3.18$'_i$).[1] This means, too, that the $U_{ij}(\cdot)$ in (3.18$'_i$) do not generally satisfy[2] (3.4) in 1.3. However, continuous strictly increasing transforms of them do, which is all that is required for us to be able to turn next to the $\mathscr{A}_{ij}$, legitimately, applying the same procedure, and so down the utility tree.

3.8.4. At each stage (3.4), or (3.18) or one of their analogues, is equivalent to the separability of the elements of the top collection. Since $\mathscr{A}$ is finite each of its proper elements is contained in some top collection. Hence the structure discovered by this procedure is equivalent to the separability of each element of $\mathscr{A}$. If $\mathscr{A} = \mathscr{A}(\mathscr{B})$ as defined in 1.8 and P4 holds, it is equivalent to the separability of the elements of $\mathscr{B}$.

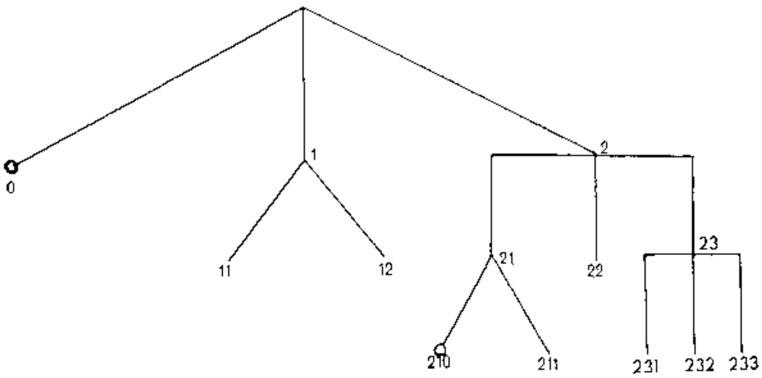
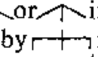
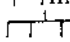


FIGURE 1

3.9. This procedure can be represented graphically. Represent a top collection by  or  in the top disjoint case, according to whether or not there are free sectors and by  in the top overlapping case, when $m = 3$. If $m = 4$, for instance, use , ,  instead. The entire collection $\mathscr{A}$ can then be represented as in Fig. 1. The sectors corresponding to each ray may be listed at its end, or separately as here:

$$\begin{array}{lll} \Omega_0 = \{1, 2\}, & \Omega_1 = \{3, 4, 5\}, & \Omega_2 = \{6, \ldots, 13\} \\ \Omega_{11} = \{3\}, & \Omega_{12} = \{4, 5\}, & \\ \Omega_{21} = \{6, 7, 8\}, & \Omega_{22} = \{9\}, & \Omega_{23} = \{10, \ldots, 13\} \\ \Omega_{210} = \{6, 7\}, & \Omega_{211} = \{8\}, & \\ \Omega_{231} = \{10\}, & \Omega_{232} = \{11\}, & \Omega_{233} = \{12, 13\}. \end{array}$$

$U(\cdot)$ can be read straight off this information. Here it is[3]

$$U(x) = F(x_1, x_2, F_1(F_{11}(x_3), F_{12}(x_4, x_5)), F_{21}(x_6, x_7, F_{211}(x_8)) + F_{22}(x_9) + F_{23}(F_{231}(x_{10}) + F_{232}(x_{11}) + F_{233}(x_{12}, x_{13}))). \quad \ldots(3.19)$$

[1] If $\mathscr{A}$ is top disjoint no such problem exists—nor does it in general if the next higher collection is.

[2] And so are not strictly subutility functions in the narrow sense in which I have used this term so far. Of course this does not matter at all since they reflect preference perfectly and are continuous.

[3] Where the F's are well behaved—that is, continuous in all their arguments, and strictly increasing in the F's among them. Having a choice between, for instance, $F_{231}(x_{10})$ and $U_{231}(x_{10})$, in the case of bottom sets, I chose the F notation for apparent consistency.

3.10. Clearly it is sufficient to list the *bottom elements* of $\mathscr{A}$ here—that is, those of its proper elements of which no proper subset belongs to $\mathscr{A}$.

3.11. I will call the partial ordering of $\mathscr{A}$ under $\supseteq$, as represented in these diagrams, the *structure* of $\mathscr{A}$.

4. SOME APPLICATIONS

4.1. In this section I will apply these results to the theory of decisions over time and to the aggregation of production relations. The latter topic is a large one; accordingly I will merely sketch its treatment. I will assume essentiality throughout but not strict essentiality: to be precise, I use P1-3, but not P4.

4.2. Consider first a person who expects to live n years and whose preferences among prospects from year i forward are independent of his plans for the interim. Formally

$$\mathscr{B} = \{B_i\colon\ B_i = \{i, i+1, \ldots, n\}, \quad i = 1, \ldots, n, B_0 = 0\}. \qquad \ldots(4.1)$$

This is a nested set of intervals, no two of which overlap, so that $\mathscr{A}(\mathscr{B}) = \mathscr{B}$. In particular, it is top disjoint at each stage. Its graph is given in Figure 2 and its utility function is

$$U(x) = F(x_1, F_{(1)}(x_2, F_{(2)}(x_3, \ldots, F_{(n-2)}(x_{n-1}, F_{(n-1)}(x_n))\ldots))) \qquad \ldots(4.2)$$

in the notation of 3.8.9 where (i) stands for i 1's in a row.

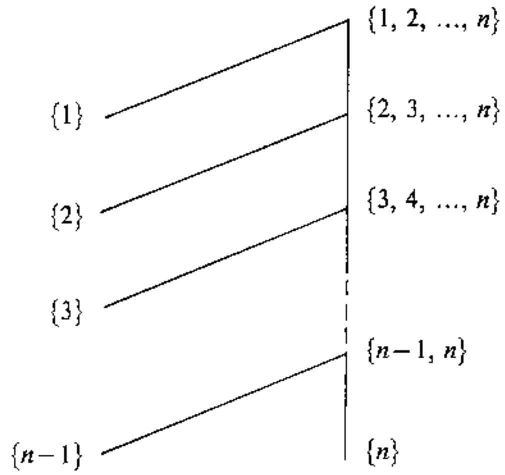


FIGURE 2

4.3. Suppose that his preferences in each of the initial periods

$$C_i = \{1, 2, \ldots, i\}, i = 1, 2, \ldots, n-1, \qquad \ldots(4.3)$$

are also independent of what he means to do later, so that

$$\mathscr{B} = \{B_0, \ldots, B_n, C_1, C_2, \ldots, C_{n-1}\}. \qquad \ldots(4.4)$$

Then

$$\{i\} \text{ and } A_i = \overline{\{i\}} \text{ are separable, } i = 1, 2, \ldots, n. \qquad \ldots(4.5)$$

$\{1\} = C_1$, $\{n\} = B_n$, $\{i\} = B_i \cap C_i$, $i = 2, 3, \ldots, n-1$, are separable, and being essential therefore strictly essential. Hence each element of $\mathscr{A}(\mathscr{B})$ is separable. Moreover $A_1 = B_2$, $A_n = C_{n-1}$ and $A_i = B_i \Delta C_i$, $i = 2, \ldots, n-1$ are the top elements of $\mathscr{A}(\mathscr{B})$ so that, if $n \geqq 3$,

$$U(x) = \Sigma U_i(x_i), \qquad \ldots(4.6)$$

in an appropriate normalization.

4.4. Consider next a person who expects to live $n \geqq 3$ years, whose preferences within each pair of adjacent years is independent of what he intends to do in the others. Formally:

$$\mathscr{B} = \{D_1, D_2, ..., D_{n-1}\}, D_i = \{i, i+1\}, i = 1, ..., n-1. \qquad ...(4.7)$$

Hence $\{i\} = D_{i-1} \cap D_i$ is separable and hence strictly essential, $i = 2, 3, ..., n-1$. Thus $\{1\} = D_1 - D_2$ and $\{n\} = D_{n-1} - D_n$ are too, so that P3 holds again, and each element of $\mathscr{A}(\mathscr{B})$ is once more separable. Hence

$$A_1 = \bigcup_{j=2}^{n-1} D_j, A_n = \bigcup_{j=1}^{n-2} D_j, A_i = \bigcup_{j=1}^{i-1]} D_j \Delta \bigcup_{k=i}^{n-1} D_k \qquad ...(4.8)$$

are the top elements of $\mathscr{A}(\mathscr{B})$. Since $\{i\} = A_i$, we have (4.6) once more.

This case was discussed in [5] and [10].

4.5. *Klein-Nataf aggregation.*[1] Suppose that we are given an economy of n firms with production frontiers

$$\phi_j(x_j) = 0, j = 1, 2, ..., n, \qquad ...(4.9)$$

where

$$x_j = (x_j^0, x_j^1, ..., x_j^m), \qquad ...(4.10)$$

gives the net output of each good and service [2] in firm j, and each x_j^i, that of those in the ith class. The classes may, for instance, be " outputs ", " labour inputs " and " capital inputs " as in [7] and [13]. If so the labour and capital inputs will appear as negative net outputs.

4.5.1. Let us require that an aggregate production relation

$$X^0 = F(X^1, X^2, ..., X^m), \qquad ...(4.11)$$

where

$$-F(\cdot) \text{ is } \textit{well behaved} \qquad ...(4.12)$$

—that is continuous and increasing,[3] should hold for the economy as a whole, given only that the individual firms lie on their own production frontiers, whether or not the allocation between them is efficient, where

$$X^i = X^i(x^i) = X^i(x_1^i, x_2^i, ..., x_n^i), \quad i = 0, 1, ..., m, \qquad ...(4.13)$$

and

$$X^i(\cdot) \text{ is continuous and strongly increasing}^4 \text{ in each } x_j^i, \; i = 0, 1, ..., m, \; ...(4.14)$$

is an aggregate measuring the net output of the ith class of good in the economy as a whole. Notice that it depends on how the total output $\Sigma_j x_j^i$ of goods in this class is divided among the firms.[5]

4.5.2. Choose a reference vector a_j for the jth firm, which is on its production frontier. Write $x_j = \bar{x}_j + a_j$ in the equations to date, $\bar{\phi}(\bar{x}_j) = \phi_j(\bar{x}_j + a_j)$, $\bar{X}^i(\bar{x}^i) = X^i(\bar{x}^i + a^i) - X^i(a^i)$, $\bar{F}(\bar{X}^1, ..., \bar{X}^m) = F(\bar{X}^1 + X^1(a^1), ..., \bar{X}^m(a^m)) - F(X^1(a^1), ..., X^m(a^m))$, and drop the bars to get (4.9)-(4.14) with

$$\phi_j(0) = x^i(0) = F(0) = 0, \text{ each } i, j. \qquad ...(4.15)$$

[1] A full discussion of this topic would require a separate paper. Hence the rather sketchy analysis and *ad hoc* assumptions below.

[2] From now on I will use " goods " to cover services too.

[3] This terminology is sufficiently consistent with that of Sections 2-3 not to be confusing, I hope.

[4] *Strongly increasing in each* x_j^i:

$X^i(x^i) \geq X^i(y^i)$ if $x^i \geqq y^i$, and $X^i(x^i) > X^i(y^i)$ if $x^i \geqq y^i$, $x_j^i > y_j^i$, some j.

[5] There is nothing, for instance, to ensure that different firms make equally efficient use of the same inputs = negative net outputs.

4.5.3. Define subaggregates

$$X^i_j(x^i_j) = X^i(0, \ldots, x^i_j, 0, \ldots), \text{ so that } X^i_j(0) = 0 \qquad \ldots(4.16)$$

setting $x_k = 0$, $k \neq j$, in (4.11) we have

$$X^0_j = F(X^1_j, X^2_j, \ldots, X^m_j), \; X^i_j = X^i_j(x^i_j) \qquad \ldots(4.17)$$

whenever the ith firm is on its production frontier. Let us require that the aggregate production relation (4.11) reflect the technologies of the individual firms in the sense that, if all but one of them are on their production frontiers, it holds only if the remaining one is too. Then (4.17) is the equation of this frontier.

4.5.4. Choose now a vector x_j on the jth firm production frontier, each j. Vary x^0_j, holding $X^0_j(x^0_j)$ and each x^i_j, $i \neq 0$, constant. Each firm remains on its production frontier (4.17) and X^i remains unchanged, $i>0$. (4.11) therefore implies that $X^0(x^0)$ remains unchanged also. Hence

$$X^0(x^0) = G^0(X_1(x^0_1), \ldots, X^0_n(x^0_n)), \text{ say,} \qquad \ldots(4.18)$$

at least when x_0 is efficiently producible—that is when each x^0_j lies on its firm's production frontier for some $(x^i_j), i \neq 0$. Let us call the set of such x^0_j, S^0_j, and restrict each x^0_j to S^0_j. Then (4.18) holds everywhere. Solving (4.11) for each X^i in turn, as we can by (4.12) restricting each x^i_j to S^i_j defined in a similar manner, and applying a similar argument we get

$$X^i = G^i(X^i_1, X^i_2, \ldots, X^i_n), \quad i = 0, 1, 2, \ldots, m, \qquad \ldots(4.19)$$

where each

$$G^i(\cdot) \text{ is well behaved, } G^i(0) = 0, \; G^i(0, \ldots, X^i_j, \ldots, 0) = X^i_j, \qquad \ldots(4.20)$$

if each

$$S^i_j \text{ is tube connected,}^1 \qquad \ldots(4.21)$$

a reasonable assumption which I will make.

4.5.5. Substituting from (4.17) and (4.19) into (4.11), we get

$$\begin{aligned} X^0 &= G^0(X^0_1, \ldots, X^0_n) = G^0(F(Y_1), \ldots, F(Y_n)) \\ &= F(X^1, \ldots, X^m) \\ &= F(G^1(Y^1), G^2(Y^2), \ldots, G^m(Y^m)) \\ &= -U(X), \text{ say} \end{aligned} \qquad \ldots(4.22)$$

where

$$X = (X^1_1, \ldots, X^m_n), \; Y^i = (X^i_1, \ldots, X^i_n), \; Y_j = (X^1_j, \ldots, X^m_j) \qquad \ldots(4.23)$$

and

$$U(\cdot), \; -F(\cdot), \text{ each } G^i(\cdot), \text{ well behaved.} \qquad \ldots(4.24)$$

4.5.6. This would seem to be just the sort of problem to which our results should be applied, $U(\cdot)$ ranking as an honorary utility function. Nevertheless two difficulties remain—we need to have (i, j) essential, each $i, j>0$, and the space S of X must be a product space. The former can, as it happens, be considerably relaxed, but the latter is really needed.

[1] S^i_j *is tube connected*: if $x^i_j, y^i_j \in S^i_j$ they are connected by an arc all of whose other points belong to the interior of S^i_j. Continuity follows as in Lemma 1, the strictly increasing property as in footnote 10 to Section IV of [6].

4.5.7. Now it might be thought that we could just take

$$S = \prod_{i=1}^{m} \prod_{j=1}^{n} S_j^i, \qquad ...(4.25)$$

which is certainly a product space. Unfortunately this is not so. Suppose for instance that firm j neither produces nor uses any good in class 0. Then the equation (17) of its production frontier is

$$F(X_j^1, X_j^2, ..., X_j^m) = X_j^0 = 0. \qquad ...(4.26)$$

In other words it is a constraint on $X_j^1, ..., X_j^m$. Since the aggregate production relation (4.22) is only required to hold when each firm is on its production frontier, it only holds for X constrained as in (4.26), and therefore not over the entire product space S. I will arbitrarily assume that this does not happen, and that (4.22) holds for all $X \in S$.

4.5.8. For essentiality it is sufficient to assume that each S_j^i contains a pair x_j^i, y_j^i with $x_j^i > y_j^i$. I do so.[1] Since (i, j) in the obvious notation is certainly separable, it is strictly essential.

4.5.9. I am now in a position to apply the analysis of section 3 with

$$\Omega = \{(1, 1), ..., (m, n)\},$$

each (i, j) strictly essential and

$$\mathscr{B} = \{B^1, ..., B^m, C_1, ..., C_n\} \qquad ...(4.27)$$

where

$$B^i = \{(i, 1), ..., (i, n)\},\ C_j = \{(i, j), ..., (m, j)\}, \text{ each } i, j > 0. \qquad ...(4.28)$$

Now

$$\overline{\{(i, j)\}} = A_j^i = (B^i \Delta C_j) U(\underset{k \neq j}{U} C_k) \in \mathscr{A}(\mathscr{B}) \qquad ...(4.29)$$

and is clearly a top element, each $i, j > 0$. Hence

$$U(X) = \Sigma U_j^i(X_j^i) \qquad ...(4.30)$$

in an appropriate normalization, which merely comes to the application of a well-behaved transformation to X^0, which is certainly legitimate, and leaves (4.9)-(4.30) unchanged after trivial change in notation. Apply it. Of course

$$U(\cdot), \text{ and each } U_j^i(\cdot), \text{ are well behaved,} \qquad ...(4.31)$$

$$U_j^i(0) = U(0) = 0, \text{ each } i, j > 0. \qquad ...(4.32)$$

4.5.10. Define now

$$f^i(X^i) = -F(0, ..., 0, X^i, 0 ...). \qquad ...(4.33)$$

It is well behaved and $f^i(0) = 0$, so that we can apply this transformation to X^i without affecting any of (4.9)-(4.33) or the interpretation of the aggregates. Do so. Setting $X_j^k = 0, k \neq i$ in (4.22) yields

$$X^i = \Sigma_j U_j^i(X_j^i) \qquad ...(4.34)$$

because of (4.20) and (4.32). Setting $X_k^i = 0, k \neq j$ in this yields

$$X_j^i = U_j^i(X_j^i), \qquad ...(4.35)$$

so that

$$X^i = \Sigma_j X_j^i. \qquad ...(4.36)$$

[1] This assumption can be greatly weakened. Set $a_{ij} = 1$ if S_j^i contains such a pair and 0 if it does not, $i, j > 0$. Then (4.22) holds if A is *connected*: that is, if its rows and columns cannot be independently permuted to put A in block diagonal form—the blocks in general being rectangular.

(4.22) and (4.30) now yield

$$\sum_{i=0}^{m} X^i(x^i) = 0, \qquad \text{...(4.37)}$$

and, when we put $x_k^i = 0$, $k \neq j$,

$$\sum_{i=0}^{m} X_j^i(x_j^i) = 0. \qquad \text{...(4.38)}$$

4.5.11. Since (4.38) clearly implies (4.37) which is of the form (4.11), with the definitions (4.36), such aggregation is possible iff well behaved subaggregates $X_j^i(x_j^i)$ exist for each firm j and class of goods i, such that the equations of the production frontiers for the individual firms can be written in the form (4.38), and then the aggregates for the economy are the sum (4.36) of the corresponding subaggregates from the individual firms —as Nataf discovered in 1948.[1]

4.5.12. There is one apparent paradox. (4.38) implies that

$$\Sigma \bar{X}^i = 0 \qquad \text{...(4.39)}$$

too, where

$$\bar{X}^i(x^i) = \Sigma_j a_j X_j^i(x_j^i), \qquad \text{...(4.40)}$$

which is well behaved if each $a_j > 0$. Yet these $\bar{X}^i$ are not mere transforms of the X^i, and (4.39) represents quite a different locus in the space of the x_j^i than does (4.37). This is certainly true and an obvious source of worry for statisticians fitting aggregate production functions, but it does not upset our analysis. We looked for a particular relation and found it—it was never required that there should not be others.[2]

5. AN ALTERNATIVE APPROACH

5.1. In this section I will discuss briefly an alternative approach which is sometimes easier to apply than that in section 3, at least in the case of complete additive separability.[3]

5.2. Say that $C \subseteq \Omega$ is a *component* of $\mathscr{B}$ if no element of $\mathscr{B}$ overlaps it. 0, Ω are the *trivial components*, the others *proper components*. The *component collection* $\mathscr{C}(\mathscr{B})$ *of* $\mathscr{B}$ is made up of the components of $\mathscr{B}$. It satisfies the following lemma.

Lemma 3.

(i) *If* $\mathscr{B}^* \supseteq \mathscr{B}$, $\mathscr{C}(\mathscr{B}^*) \subseteq \mathscr{C}(\mathscr{B})$,

(ii) *Component collections are complete,*

(iii) $\mathscr{C}(\mathscr{B}) = \mathscr{C}(\mathscr{A}(\mathscr{B}))$.

Proof of (i). The components of $\mathscr{B}^*$ satisfy the conditions required of those of $\mathscr{B}$.

Proof of (ii). Take a pair A, B of overlapping elements of $\mathscr{C}(\mathscr{B})$ and a $D \in \mathscr{B}$. I will show that

$$D \text{ does not overlap any of } A \cup B,\ A \cap B,\ A - B,\ B - A \text{ or } A \Delta B. \qquad \text{...(5.1)}$$

Since $(0, \Omega) \in \mathscr{C}(\mathscr{B})$, this will prove (ii).

A, for instance, does not overlap D. Hence

$$\text{at least one of } D \supseteq A,\ A \supseteq D,\ D \cap A = 0, \text{ holds.} \qquad \text{...(5.2)}$$

[1] Though his results were local while these are global, and depend on weaker assumptions. In particular he needed $n \geqq 3$, while we only need $m, n \geqq 2$ here.

[2] This point is discussed in [4] and touched on in [6].

[3] *Complete additive separability*: The case in which we can write $U(x) = \Sigma U_i(x_i)$ in an appropriate normalisation.

Considering B, too, we have the following possibilities.

$D \supseteq B$, say: If $A \supseteq D$, $A \supseteq B$; if $D \cap A = 0$, $D \cap B = 0$. Neither is consistent with A and B overlapping. Hence $D \supseteq A$, implying that $D \supseteq A \cup B$.

$D \subseteq A, B$: implying that $D \subseteq A \cap B$,

$D \subseteq A$, $B \cap D = 0$, say: implying that $D \subseteq A - B$,

$A \cap D = B \cap D = 0$: implying that $(A \cup B) \cap D = 0$.

In none of these cases can D overlap any of $A \cup B$, $A \cap B$, $A - B$, $B - A$, $A \Delta B$. Hence (i) and therefore (ii) holds.

Proof of (iii). Since $\mathscr{A}(\mathscr{B}) \supseteq \mathscr{B}$, $\mathscr{C}(\mathscr{A}(\mathscr{B})) \subseteq \mathscr{C}(\mathscr{B})$ by (i), so that we need only show that

$$\mathscr{C}(\mathscr{A}(\mathscr{B})) \supseteq \mathscr{C}(\mathscr{B}). \qquad \text{...(5.3)}$$

Take $\mathscr{C} \in \mathscr{C}(\mathscr{B})$ and define

$$\mathscr{B}_1 = \{B \in \mathscr{B}:\ B \subseteq C\},\ \mathscr{B}_2 = \{B \in \mathscr{B}:\ B \supseteq C \text{ or } B \cap C = 0\} = \mathscr{B} - \mathscr{B}_1 \qquad \text{...(5.4)}$$

since C does not overlap any element of $\mathscr{B}$. All the proper elements of $\mathscr{A}_1 = \mathscr{A}(\mathscr{B}_1)$ are contained in C, and each element of $\mathscr{A}_2 = \mathscr{A}(\mathscr{B}_2)$ either contains C or does not intersect it—as is obvious when one remembers that we could treat C as a single sector in constructing $\mathscr{A}(\mathscr{B}_2)$. Hence C is a component of $\mathscr{A} = \mathscr{A}_1 \cup \mathscr{A}_2$. Moreover $\mathscr{A}_1$ and $\mathscr{A}_2$ are each complete and no element of either overlaps the other. Hence $\mathscr{A}$ is complete. Since it certainly contains $\mathscr{B}$, $\mathscr{A} \supseteq \mathscr{A}(\mathscr{B})$. Hence C is a component of $\mathscr{A}(\mathscr{B})$; which immediately yields (5.3) and therefore (iii).

5.3 (ii) implies that we can define the top elements of $\mathscr{C}(\mathscr{B})$ in the same manner as we did those of $\mathscr{A}(\mathscr{B})$ in section 3. Call them *top components* of $\mathscr{B}$— or of $\mathscr{A}(\mathscr{B})$. In particular, therefore, $\mathscr{C}(\mathscr{B})$ is either top disjoint or top overlapping. Indeed, it is top disjoint if $\mathscr{A}(\mathscr{B})$ is top overlapping and, except in one highly special case, top overlapping if $\mathscr{A}(\mathscr{B})$ is top disjoint.

5.4. If $\mathscr{A}(\mathscr{B})$ is top disjoint, 3.5.2. implies that the union of any collection of the Ω_i, $i > 0$ and any of the free sectors, is a component of $\mathscr{B}$. The top components are therefore found by dropping from Ω any one free sector or any one top element Ω_i, $i > 0$, of $\mathscr{A}(\mathscr{B})$. If $m+p > 2$, where m is the number of top elements, and p of free sectors, $\mathscr{C}(\mathscr{B})$ is therefore top overlapping. If $m+p \leqq 2$ it is clearly top disjoint.

5.5. If, on the other hand, $\mathscr{A}(\mathscr{B})$ is top overlapping, 3.6.4. implies that $\Omega_1, \Omega_2, \ldots, \Omega_m$ as defined in (3.10) are the top components so that $\mathscr{C}(\mathscr{B})$ is top disjoint. Hence we have:

Theorem 2. *If* $\{\Omega_1, \Omega_2, \ldots, \Omega_m\}$, $\{y_1, y_2, \ldots, y_m\}$ *are corresponding partitions of* Ω *and* x, *and* $m \geqq 3$, *P*1, 2, 4, *imply that we can write*

$$U(x) = \Sigma U_i(y_i) \qquad \text{...(5.3)}$$

in an appropriate normalization, iff

$$\textit{each } \Omega_i \textit{ is a top component.} \qquad \text{...(5.4)}$$

Each $U_i(\cdot)$ *is continuous in this case.*

Corollary. P1, P2 and P4 imply that we can write

$$U(x) = \Sigma U_i(x_i) \qquad \text{...(5.5)}$$

in an appropriate normalization, iff

$$\{1\}, \{2\}, \ldots, \{n\} \text{ are the only proper components of } \mathscr{B}, \text{ when } n \geqq 3. \qquad \text{...(5.6)}$$

5.6. This is much the most useful result to be derived from this approach. Given P1, P2 and P4 it would have yielded $U(x) = \Sigma U_i(x_i)$ immediately in 4.3 and 4.4. Combined with the results of section 6, it would have yielded the same results immediately, given

P1-3. It is possible to give a complete analysis of the structure of $U(\cdot)$ using these methods, paralleling that in section 3, but the existence of free sectors in the top overlapping case, and the possibility that $m+p \leqq 2$, makes it relatively clumsy. I will therefore forbear to give it here.

5.7. However, one relation between the two approaches is worth bringing out. Let $r = ij \ldots k$, $k>0$, in $n+1$—ary notation, or a blank, and say $r \in R$ if $ij \ldots k$ is the label of a node or terminus in a diagram such as Fig. 1 or 2. Then

$$\tilde{\mathscr{A}} = \{\Omega_r : r \in R\} \cup \{0\} \qquad \ldots(5.7)$$

is complete and is composed of just those elements of $\mathscr{A}$ which are also components of it.[1]

6. STRICT ESSENTIALITY RELAXED

6.1. In none of the examples discussed in section 4 did I have to use strict essentiality. The reason was that in each case at least one pair of overlapping separable sets A, B had $A-B$ strictly essential because it was, or contained, the intersection of a number of separable sets, which, itself separable and essential, was strictly essential. Theorem 1 therefore implied that $B-A$, too, was strictly essential, which in turn implied that various other sets were, until I had enough strict essentiality to prove all I needed, without explicitly assuming any at all. Since ordinary essentiality is rather nugatory—inessential sets affect nothing and so may be neglected if known—it would be interesting to know whether this were commonly the case. In this section I will show that it is. Since a full and rigorous analysis would take up rather a lot of space, and this article is already too long, I will merely sketch the analysis at some points.

6.2. $\mathscr{B} = \{A, B\}$, where A and B overlap, is the simplest non-trivial, or *proper*, example of a *connected collection*—that is one each pair of distinct elements A, B of which are *linked in* $\mathscr{B}$ by a *chain*,

$$A = B_1, B_2, \ldots, B_n = B, \qquad \ldots(6.1)$$

of its elements, each of which, except the last, overlaps its successor.[2] In this case the proper elements of $\mathscr{A}(\mathscr{B})$ are clearly $\{A \cap B, A-B, B-A\}$ and their various combinations—in fact just those sets whose separability is implied by P1-3 iff $\mathscr{B}$ is separable and $A-B$ strictly essential. In fact this result is true in general as we will see in Theorem 3 below.

6.3. Call the elements $P_1, \ldots, P_s$ of the partition $\mathscr{P}(\mathscr{B})$ of the *overset*

$$\Omega(\mathscr{B}) = \bigcup_{B \in \mathscr{B}} B \qquad \ldots(6.2)$$

of a proper connected collection $\mathscr{B}$, found by superimposing the partitions $\{B, \Omega(\mathscr{B})-B\}$, $B \in \mathscr{B}$, the *parts* of $\mathscr{B}$, and

$$\mathscr{R}(\mathscr{B}) = \{C : C = \bigcup_{p \in S} P_p, \text{ some } S \subseteq \{1, \ldots, s\}\}, \qquad \ldots(6.3)$$

the *power collection* of $\mathscr{B}$.[3] We then have

Theorem 3. *If a separable connected collection* $\mathscr{B}$ *contains an overlapping pair of elements* A, B *whose difference* $A-B$ *is strictly essential,* P1-3 *imply that* $\mathscr{A}(\mathscr{B})$ *is separable.*

Proof of theorem. I will proceed by proving a sequence of Lemmas.

Lemma 4. *If* $\mathscr{B}$ *is connected and* $0 \subset \mathscr{C} \subset \mathscr{B}$, *there are an overlapping pair* $A \in \mathscr{C}$, $B \in \mathscr{B}-\mathscr{C}$.

[1] $0, \Omega \in \tilde{\mathscr{A}}$ clearly. Since each Ω_r is a component of $\mathscr{A}$ and an element of it, no two overlap.

[2] Let us ban empty connected collections. The *trivial* connected collections are then those containing only one element.

[3] In normal terminology, $\mathscr{R}(\mathscr{B})$ is the power collection of $\mathscr{P}(\mathscr{B})$. If $\mathscr{B}$ is trivial, $\mathscr{R}(\mathscr{B}) = \mathscr{B} \cup \{0\}$.

Proof. A chain linking $C \in \mathscr{C}$ with $D \in \mathscr{B} - \mathscr{C}$ has a first element in $B - \mathscr{C}$. Call it B and its predecessor A.

Lemma 5. *If $\mathscr{B}$ is connected with $A, B \in \mathscr{B}$ overlap, we can write $\mathscr{B} = \{B_1, ..., B_k\}$, say, where $B_1 = A$, $B_2 = B$ and each $\mathscr{B}_j = \{B_1, B_2, B_{j-1}\}$ is connected.*

Proof. $\mathscr{B}_1$ and $\mathscr{B}_2$ are clearly connected. Suppose $\mathscr{B}_j$ is, some $2 \leqq j < k$. According to Lemma 4, $\mathscr{B} - \mathscr{B}_j$ contains an element which overlaps an element of $\mathscr{B}_j$, and is therefore linked to all of them. Call it B_{j+1}. $\mathscr{B}_{j+1}$ is connected.

Lemma 6. *Theorem 3 holds if $\mathscr{A}(\mathscr{B})$ is replaced by $\mathscr{R}(\mathscr{B})$.*

Proof. I will proceed by an induction on the number n of elements of $\mathscr{B}$.

Lemma 6 follows immediately from Theorem 1 when $n = 2$. I will assume it for $n = k-1$ and prove it for $n = k$.

Define $\mathscr{C} = \mathscr{B}_{k-1}$ in the notation of Lemma 5. The inductive hypothesis implies that $\mathscr{C}$ satisfies Lemma 6, and we wish to prove that $\mathscr{B}$ does too.

Set

$$\mathscr{P}(\mathscr{C}) = \{Q_1, ..., Q_t\} \qquad ...(6.4)$$

so that the elements of $\mathscr{P}(\mathscr{B})$ are the non-empty ones among

$$(a)\ \Omega(\mathscr{B}) - \Omega(\mathscr{C}) = B_k - \Omega(\mathscr{C}),$$

$$(b)\ Q_p - B_k, \text{ each } p, \qquad ...(6.5)$$

$$(c)\ Q_p \cap B_k, \text{ each } p.$$

I will first show that each of these and

$$\Omega(\mathscr{B}) \qquad ...(6.6)$$

are separable and so strictly essential.

(6.6) implies, of course that we can talk of components relative to $\Omega(\mathscr{B})$, while their definition implies that the parts $P_1, ..., P_s$ of $\mathscr{B}$ may be taken as sectors and that they are the only components of $\mathscr{B}$ relative to $\Omega(\mathscr{B})$. Because of Lemma 2 the separability of the elements in (6.5)(*a*) and (*b*) and (6.6) is therefore all that is required to prove the lemma.

I will now prove that these elements are all separable.

Note first that $Q_p - B_k$ is certainly separable unless Q_p and B_k overlap, and $B_k - \Omega(\mathscr{C})$ if $B_k \subseteq \Omega(\mathscr{C})$. Since

$$\Omega(\mathscr{C}) \nsubseteq B_k \qquad ...(6.7)$$

because B_k overlaps some $B_j \in \mathscr{C}$ by Lemma 4, it remains to show that

$$Q_q - B_k \text{ is separable when it is not empty,} \qquad ...(6.8)$$

$$B_k - \Omega(\mathscr{C}) \text{ is separable when it is not empty.} \qquad ...(6.9)$$

This I will now do.

Suppose first that B_k intersects each of the parts Q_p of $\mathscr{C}$. Since it overlaps some $B_j \in \mathscr{C}$, it overlaps at least one Q_p. Suppose it overlaps Q_q. Then $B_k - Q_q$ contains each other $B_k \cap Q_q$ which, being the intersection of separable sets by the inductive hypothesis, is separable and hence strictly essential. Hence $B_k - Q_q$ is strictly essential and so, by Theorem 1, $Q_q - B_k$ is separable, proving (6.8) in this case. Moreover $\Omega(\mathscr{C}) - B_k \supseteq Q_q - B_k$, which, being separable is strictly essential. Since $\Omega(\mathscr{C})$ is separable by the inductive hypothesis this immediately implies (6.9) too. This completes the proof of Lemma 6 in this case.

Suppose next that B_k does not intercept each Q_p, so that some

$$Q_r \subseteq \Omega(\mathscr{C}) - B_k. \qquad ...(6.10)$$

$\Omega(\mathscr{C}) - B_k$ is therefore strictly essential, implying (9) by Theorem 1. When B_k overlaps some Q_q, two cases arise. If $B_k - \Omega(\mathscr{C}) \neq 0$ it is strictly essential by (6.9), so that

$B_k - Q_q \supseteq B_k - \Omega(\mathscr{C})$ is strictly essential too, and $Q_q - B_k$ therefore separable. If

$$B_k - \Omega(\mathscr{C}) = 0,$$

B_k must intercept another part Q_r of $\mathscr{C}$, since it overlaps a $B_j \in \mathscr{C}$. Hence $B_k - Q_q \supseteq B_k \cap Q_r$ which is separable and therefore strictly essential. Hence Theorem 1 implies once more that $Q_q - B_k$ is separable, completing the proof of the lemma.

Lemma 7. *$\mathscr{A}(\mathscr{B}) = \mathscr{R}(\mathscr{B}) \cup \{\Omega\} = \mathscr{D}(\mathscr{B})$, say.*

Proof. Lemma 6 immediately implies

$$\mathscr{A}(\mathscr{B}) \supseteq \mathscr{D}(\mathscr{B}). \qquad ...(6.11)$$

If now C, $D \in \mathscr{D}(\mathscr{B})$ and overlap, they are the unions of parts of $\mathscr{B}$. So therefore are their differences, union . . . which therefore belong to $\mathscr{D}(\mathscr{B})$ also. Since $0, \Omega \in \mathscr{D}(\mathscr{B})$, it is complete. It clearly contains $\mathscr{B}$. Hence

$$\mathscr{A}(\mathscr{B}) \subseteq \mathscr{D}(\mathscr{B}). \qquad ...(6.12)$$

(6.11) and (6.12) prove Lemma 7. Substituting into Lemma 6, we prove Theorem 3.

6.4. Define now the *nucleus* and *periphery*

$$N(\mathscr{B}) = \bigcap_{B \in \mathscr{B}} B,\ P(\mathscr{B}) = \Omega(\mathscr{B}) - N(\mathscr{B}) \qquad ...(6.13)$$

of a collection $\mathscr{B}$, and say that a proper connected collection is a *tangle*, or *tangled collection*, if its nucleus is empty, and a *star*, or *star collection*, if it is not. There are then three distinct types of connected collection: trivial—with one or no elements—star and tangled. We have

Corollary to Theorem 3. If $\mathscr{B}$ is a tangled collection, P1-3 imply that $\mathscr{A}(\mathscr{B})$ is separable.

Proof. Clearly $N(\mathscr{C}) = \bigcap_{B \in \mathscr{C}} B$ is separable, each $0 \subset \mathscr{C} \subseteq \mathscr{B}$. Since $N(\mathscr{B}) = 0$ there is at least one collection $\mathscr{C}$ for which $N(\mathscr{C}) \neq 0$, $N(\mathscr{D}) = 0$, each $\mathscr{D} \supset \mathscr{C}$. According to Lemma 4 there is an overlapping pair $A \in \mathscr{C}$, $B \in \mathscr{B} - \mathscr{C}$. $A - B \supseteq N(\mathscr{C})$ which being separable, is strictly essential. Hence $A - B$ is strictly essential; proving the corollary.

6.5. To apply this analysis is to a general separable collection $\mathscr{B}$, one breaks it down into connected pieces.

Say of two elements A, B of $\mathscr{B}$, that they are *joined in $\mathscr{B}$* if $A = B$, or they are linked in $\mathscr{B}$. " Being joined in $\mathscr{B}$ " is an equivalence relation and so partitions $\mathscr{B}$ into equivalence classes or *pieces*.

$$\mathscr{B}^t,\ t \in T, \text{ say}, \qquad ...(6.14)$$

with the property that elements in the same piece are joined while those in different pieces are not. Pieces are connected collections, indeed maximal connected collections in the obvious sense of the word. They may, of course be trivial connected collections, containing only one element, but as elements of a partition, they cannot be empty.

6.6. *Theorem* 4. If $\mathscr{B}^t$, $t \in T$, are the pieces of $\mathscr{B}$,

$$\mathscr{A}(\mathscr{B}) = \bigcup_{t \in T} \mathscr{A}(\mathscr{B}^t). \qquad ...(6.15)$$

Proof. Since $\mathscr{A}(\mathscr{B}) \supseteq \mathscr{A}(\mathscr{B}^t)$,

$$\mathscr{A}(\mathscr{B}) \supseteq \bigcup_{t \in T} \mathscr{A}(\mathscr{B}^t) = \mathscr{D}, \text{ say}. \qquad ...(6.16)$$

I will now show that

$$\mathscr{D} \supseteq \mathscr{A}(\mathscr{B}), \qquad ...(6.17)$$

also, thus proving (6.15). To do so, note that $\mathscr{D} \supseteq$ each $\mathscr{B}^t$. Hence $\mathscr{D} \supseteq \mathscr{B}$. It is also complete: it certainly contains 0, Ω, and if C, $D \in \mathscr{D}$ overlap, they both belong to the same $\mathscr{A}(\mathscr{B}^t)$, so that their union, differences, etc., do so too. Since $\mathscr{A}(\mathscr{B})$ is the intersection of the complete collections containing $\mathscr{B}$, this implies (6.17). With (6.16) this proves the theorem.

Corollary.

$$\mathscr{A}(\mathscr{B}) = \{\Omega\} \cup \bigcup_{t \in T} \mathscr{R}(\mathscr{B}^t). \qquad \ldots(6.18)$$

6.7. (6.18) is a simple recipe for calculating $\mathscr{A}(\mathscr{B})$.

First one finds the pieces $\mathscr{B}^t$, $t \in T$ of $\mathscr{B}$. Since it is easy to see whether sets overlap, this is easily done. Then one finds the parts of the proper pieces. This is again easily done.

Consider now the collection $\mathscr{F}(\mathscr{B})$ composed of the oversets $\Omega(\mathscr{B}^t)$, $t \in T$, and the parts of the proper pieces, together with Ω and 0. Since none of its elements overlap, it is complete, and its structure, as defined in section 3, is easily discovered.

So far there are no overlapping subcollections. However this is easily corrected. If an overset corresponds to a proper piece $\mathscr{B}^t$, $\mathscr{R}(\mathscr{B}^t)$ is overlapping. Mark $\Omega(\mathscr{B}^t)$ with a +. All other nodes are disjoint.[1]

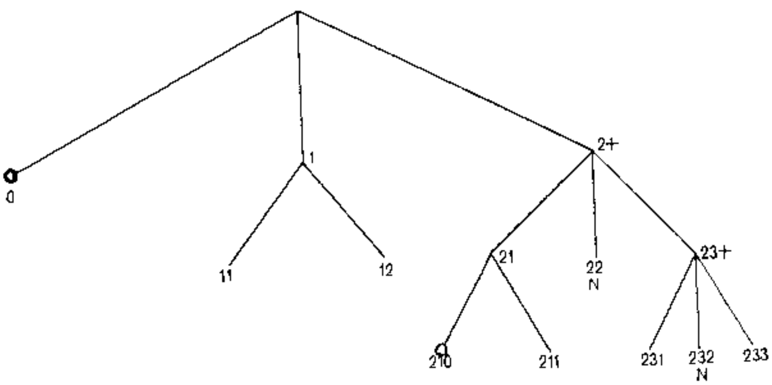


FIGURE 3

Clearly such a diagram exhausts all the information in $\mathscr{A}(\mathscr{B})$. Figure 3 is the analogue of Figure 1 in section 3.[2] How to transform one into the other is obvious.

6.8. There remains the difficulty that we have not assumed strict essentiality. As we have already seen, it is implied except possibly in the case of star pieces [3] of $\mathscr{B}$. For such, mark the nucleus with an N as in Fig. 3.[4] We then have merely to see whether any peripheral part of the star piece in question is strictly essential by virtue of having a separable subset, as has Ω^{21} in Fig. 3. Clearly we need only worry about *bottom star pieces*—that is those such as $\mathscr{B}^{23}$ in Fig. 3, none of whose peripheral elements contains an element of $\mathscr{B}$, or, equivalently of $\mathscr{A}(\mathscr{B})$.

What happens in such cases is discussed somewhat cursorily in [7]. To discuss it fully would require several new concepts, quite a lot of space, and more of the reader's time than I feel justified in asking for at this stage.

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[1] The only element of a trivial complete collection intersects no other, nor does a part of a proper complete collection.

[2] Except for the N's which I will explain below.

[3] A star piece is a piece which is a star.

[4] I am assuming that $\mathscr{B}^2$ and $\mathscr{B}^{23}$ in the obvious notation are both stars rather than tangles.

APPENDIX

Debreu's results quoted in (1.6) and (1.7) of section 1 underlie all my analysis in the slightly more general forms given below. He states but does not prove (1.6) in [3], and his proof of (1.7), which follows almost immediately from Theorem 1, is difficult. Hence this appendix.

Lemma 1.[1] *If* $\{\Omega_0, \Omega_1, \ldots, \Omega_m\}$, $(y_0, y_1, \ldots, y_m)$ *are corresponding partitions of* Ω, x, P1-2 *imply that we can write*

$$U(x) = F(y_0, U_1(y_1), \ldots, U_m(y_m)) \qquad \ldots(\text{A.1})$$

where

$$F(y_0, \cdot) \text{ is strictly increasing,}^2 \qquad \ldots(\text{A.2})$$

iff

$$\Omega_i \text{ is separable, } i = 1, 2, \ldots, m. \qquad \ldots(\text{A.3})$$

If so, we can take

$$U_i(y_i) = U(0, \ldots, 0, y_i, 0, \ldots); \; U_i(0) = U(0) = 0, i = 1, \ldots, m, \qquad \ldots(\text{A.4})$$

without loss of generality, in which case

$$F(\cdot) \text{ is continuous.} \qquad \ldots(\text{A.5})$$

Proof.

(A.1)-(A.2)⇒(A. 3): obvious.

(A.3)⇒(A.1)-(A.2). Take a continuous utility function $U(\cdot)$, normalize trivially to get $U(0) = 0$, and define the subutility functions as in (A.4).[3] Take $x', x'' \in S$. Move each y_i in turn from y_i' to y_i'', holding y_j, $j \neq i$ constant while doing so. The definition of separability in 1.3 implies that

$$y_0' = y_0'', \; U_i(y_i') = U_i(y_i''), \; i = 1, \ldots, m, \Rightarrow U(x') = U(x'') \qquad \ldots(\text{A.6})$$

$$y_0' = y_0'', \; U_i(y_i') \geqq U_i(y_i''), \; i = 1, \ldots, m, \text{ with at least one strict inequality} \Rightarrow U(x') > U(x''). \qquad \ldots(\text{A.7})$$

(A.6) is equivalent to (A.1) and hence (A.7) to (A.2).

(1)-(4)⇒(5). Define

$$z = (y_0, U_1, \ldots, U_m) = (y_0, u), \; Z = T_0 x J = T_0 x \prod_{i=1}^{m} J_i \qquad \ldots(\text{A.8})$$

where

$$T_i = S_{\Omega_i}{}^4, \; i = 0, \ldots, m; \quad J_i = \text{range of } U_i(y_i), \; i = 1, \ldots, m. \qquad \ldots(\text{A.9})$$

I will prove $F(\cdot)$ continuous at $z \in \text{Int}\, z$. Only trivial modifications are required at the frontier.[5]

Let then

$$z^t \to z \in \text{Int}\, Z, \text{ each } z^t \in Z \qquad \ldots(\text{A.10})$$

and choose

$$y_i', y_i'' \in T_i, \text{ such that } U_i(y_i') < U_i < U_i(y_i''), \; i = 1, \ldots, m, \qquad \ldots(\text{A.11})$$

$$C_i \text{ an arc in } T_i \text{ connecting } y_i' \text{ to } y_i'', \; i = 1, \ldots, m. \qquad \ldots(\text{A.12})$$

[1] I am grateful to Kenneth Arrow for showing me how to streamline my original clumsy proof.

[2] $F(y_0, \cdot)$ *strictly increasing*: $F(y_0, U_1, \ldots, U_m)$ strictly increasing in $(U_1, U_2, \ldots, U_m)$.

[3] As in (1.4) and (1.5) of 1.4.

[4] That is $T_i = \bigcup_{j \in \Omega_i} S_j$. See (1.3) in 1.3.

[5] If U_i is at the upper limit of J_i, for instance, take $y_i'' \in T_i$ such that $U_i(y_i'') = U_i$. If J_i is degenerate—that is Ω_i is inessential—take $y_i' = y_i''$ such that $U_i(y_i')$, in which case each $y_i^t = y_i'$.

Ultimately

$$U_i(y_i') < U_i^t < U_i(y_i''), \quad i = 1, \ldots, m, \ t \geqq T, \text{ say.} \qquad \ldots(A.13)$$

Since the $U_i(\cdot)$ are continuous, we can find

$$y_i^t \in C_i \text{ such that } U_i(y_i^t) = U_i^t, \quad i = 1, \ldots, m, \ t \geqq T, \qquad \ldots(A.14)$$

so that

$$(y_1^t, \ldots, y_m^t) \in \prod_{i=1}^{m} C_i = C, \text{ say,} \qquad \ldots(A.15)$$

which is compact since each C_i is a homeomorphism of $[0, 1]$. Since $y_0^t \to y_0$ this implies that $\{x^t\}$ [1] has a convergent subsequence. On any convergent subsequence of $\{x^t\}$,

$$F(z^t) = U(x^t) \to U(x) = F(z), \qquad \ldots(A.16)$$

by the continuity of $U(\cdot)$, $U_1(\cdot), \ldots, U_m(\cdot)$, x being the limit point of the particular subsequence. Suppose now that $F(z^t) \not\to F(z)$ in the full sequence $\{z^t\}$. Then there is an $\varepsilon > 0$ and a subsequence such that

$$| F(z^t) - F(z) | > \varepsilon, \qquad \ldots(A.17)$$

on the subsequence in question. The corresponding subsequence of $\{x^t\}$ itself has a convergent subsequence. Hence $F(z^t) \to F(z)$ for the subsequence of $\{z^t\}$ corresponding to this last, contradicting (A.17). Hence

$$F(z^t) \to F(z) \text{ on the full sequence.} \qquad \ldots(A.18)$$

Lemma 2. *If* $\{\Omega_1, \ldots, \Omega_m\}$, $(y_1, \ldots, y_m)$ *are corresponding partitions of* Ω, x, *and* $m \geqq 3$, P1, 2, 4 *imply that we can write*

$$U(x) = \Sigma U_i(y_i) \qquad \ldots(A.19)$$

in an approriate normalization, iff

$$C_I = \bigcup_{i \in I} \Omega_i \text{ is separable, each } I \subseteq \Phi = \{1, 2, \ldots, m\}. \qquad \ldots(A.20)$$

Proof.

(A. 19)$\Rightarrow$(A. 20): obvious.

(A. 20)$\Rightarrow$(A. 19): Let

$$\{I, J\};\ \{I', I''\};\ \{J', J''\} \text{ by partitions of } \Phi;\ I;\ J. \qquad \ldots(A.21)$$

Then Theorem 1 and (4) imply that we can represent the preference ordering by either of the continuous utility functions [2]

$$U(x) = U_{I'}(y_{I'}) + U_{I''}(y_{I''}) + U_J(y_J) = U_I(y_I) + U_J(y_J) \qquad \ldots(A.22)$$

$$V(x) = V_I(y_I) + V_{J'}(y_{J'}) + V_{J''}(y_{J''}) = V_I(y_I) + V_J(y_J) \qquad \ldots(A.23)$$

in possibly different normalizations,[3] each consistent with (A.4). Now

$$V = F(U);\ F(\cdot) \text{ continuous and strictly increasing.}^4 \qquad \ldots(A.24)$$

Hence, by (A.4),

$$F(U_I + U_J) = V_I + V_J = F(U_I) + F(U_J). \qquad \ldots(A.25)$$

[1] $x^t = (y_0^t, y_1^t, \ldots, y_m^t)$, of course.

[2] Take $A = C_I$, $B = C_{I'' \cup J}$ as defined in (A.20) for (A.22), $A = C_{I \cup J'}$, $B = C_J$ for (A.23).

[3] Theorem 1 implies that the utility function can be put in the form $F_{I', I'', J}(u(y_{I'}) + v(y_{I''}) + w(y_J))$ for instance, where $F(\cdot)$ is continuous and strictly increasing.

[4] The proof that (A.1)-(A.4)$\Rightarrow$(A.5) in Lemma 1 implies that $F(\cdot)$ is continuous and strictly increasing here.

Since U_I, U_J range over non-degenerate intervals containing the origin (A.24)-(A.25) imply

$$F(U) = \lambda U, \lambda > 0 \qquad \text{...(A.26)}$$

on the entire range of U. It is trivial [1] to normalize V so that $\lambda = 1$ and

$$U_J = U_{J'} + U_{J''}. \qquad \text{...(A.27)}$$

This really completes the proof, since it obviously permits us to split Φ into smaller and smaller pieces, until, ultimately, we get down to the individual Ω_i. For formal completeness however, let us make the inductive hypothesis

$$U = U_1 + U_2 + \ldots + U_k + U_J, \; J = \{k+1, \ldots, m\} \qquad \text{...(A.28)}$$

which holds for $k = 2$ by (A.22) with $I' = \{1\}$, $I'' = \{2\}$ and implies

$$U = U_1 + U_2 + \ldots + U_{k+1} + U_{J''}, \; J'' = \{k+2, \ldots, m\} \qquad \text{...(A.29)}$$

by (A.27) with $I' = \{1, 2, \ldots, k-1\}$, $I'' = \{k\}$, $J' = \{k+1\}$. Hence (A.28) holds for all $k \geqq 2$, and hence for $k = m$, which is (A.19).

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[1] And unnecessary as it happens.

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