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STATIONARY UTILITY AND TIME PERSPECTIVE[1]

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This paper extends an earlier study by one of the authors. A set of postulates concerning a utility function of an infinite consumption program implies the existence of a utility scale such that postponement of each of two programs by the same time delay cannot increase, and generally diminishes, the difference of their utilities. This property of "time perspective" allows previous results concerning "impatience" to be extended and generalized.

1. INTRODUCTION

IN A PREVIOUS article one of the authors [8] studied some implications of a set of postulates concerning a preference ordering of consumption programs for an infinite future. The preference ordering was assumed to be representable by a numerical utility function defined on the space of consumption programs, and the postulates were formulated as properties of that function. While these postulates themselves appeared to be concerned only with properties more immediate and elementary than any questions of timing preference, it was found that the postulates implied, at least in certain parts of the program space, a preference for advancing the timing of future satisfactions. This conclusion was expressed by the concept of *impatience*. In its simplest form this concept was defined to mean that, if in any given year the consumption of a bundle x of commodities is preferred over that of a bundle x', then the consumption in two successive years of x, x', in that order, is preferred to the consumption of x', x.

Impatience so defined is, of course, a property of the underlying preference ordering. Consequently it is also a property exhibited by every utility function representing the preference ordering. In other words, it—and for that matter all other properties of the utility function studied in the previous article—is invariant under any continuous increasing transformation of the utility scale. To emphasize this fact, the title of that article used the term "ordinal utility."

Subsequently we have found a deeper property of the preference ordering in question, to be called the property of *time perspective*. Besides being of considerable

[1] During the academic year 1959-60 the work by Koopmans on this study was supported by a grant from the National Science Foundation to the Cowles Foundation. During the summer of 1960 the work of Diamond and Koopmans, and during that of 1962 the work of Koopmans and Williamson was also carried out under Cowless Foundation auspices, mostly under Contract Nonr 3055(01) with the Office of Naval Research. During the academic year 1960-61 the work of Koopmans and Williamson was supported by Harvard University, and Diamond participated in discussions during that year.

We are indebted to Herbert Scarf for highly valuable comments that have led us to the present proof of the "weak time perspective" property, and to Herbert Simon and Menahem Yaari for valuable critical comment.

interest in itself, this property can be used to derive the previous result regarding impatience, and to generalize that result and extend it to a larger part of the program space. The time perspective property is most conveniently formulated as a (no longer ordinal) property of a certain subclass of the utility functions that represent the given preference ordering. To define it, consider two consumption programs, $(x, x', x'', \ldots)$ and $(y, y', y'' \ldots)$, of which the first is preferred to the second. Now postpone each entire program by one time unit, and insert a common consumption bundle z in the gap so created in both programs, to make $(z, x, x', \ldots)$ and $(z, y, y' \ldots)$, respectively. Then, by Postulates 3 and 4 of the previous study, the postponed first program is still preferred to the postponed second program. We shall say that a utility function chosen to represent the preference ordering has the property of time perspective if, for all programs $(x, x', x'', \ldots)$ and $(y, y', y'', \ldots)$ and for all inserts z that one may choose in the above description, the *difference* of the utilities of the postponed programs is smaller than the difference of the utilities of the original programs. Since utility differences enter into this definition, time perspective, as a property of a utility function, is not invariant for continuous increasing transformation of the utility scale. We say, however, that a preference ordering of consumption programs itself has the property of time perspective if it can be represented by at least one utility function having that property.

The term "time perspective" is derived from an analogy with perspective in space. As the timing of the differences between any two programs is made to recede into a more distant future, the utility difference between the programs diminishes, in an appropriate representation of the ordering. To be precise, we call this property *strong time perspective*, as distinct from a property to be called *weak time perspective*, in which the utility difference either remains the same or diminishes.

The proof of strong time perspective found so far takes weak time perspective as its point of departure, but requires lengthy reasoning beyond that, and also a slight strengthening of Postulate P2 below. For these reasons, the present paper is limited to weak time perspective only. However, the reasoning of the present paper will suffice to show that in any utility function exhibiting time perspective, among all pairs of programs subjected to a postponement as described, equality of utility differences before and after postponement can only be in some sense an exceptional case, whereas shrinkage of the utility difference occurs in some average sense indicated in Section 5 below.

In recent years there have been a number of studies in which postulated properties of a preference ordering were reflected in properties of a utility function representing such an ordering in simplest form. A valuable survey of the results of such investigations has been made by Aumann [1]. There is a certain similarity between the present study and a whole group of investigations by, among others, von Neumann and Morgenstern [11], Herstein and Milnor [7], L. J. Savage [9], and Debreu [4]. While these studies deal with a wide range of different choice situations, their common element has been the existence of a representing utility function of a simple additive form invariant only for linear increasing transformations of the utility scale. The term "cardinal utility" has been used to describe such utility functions. Our present results are weaker in that the "simple" property of time perspective that distinguishes an interesting class of representations of the given ordering is in general conserved by a class of scale transformations wider than just the linear ones. For this reason, we shall speak of utility functions or scales possessing the time perspective property as *quasi-cardinal.*

Results of a recent study by Diamond [5] suggest the possibility of deriving the postulates concerning a utility function, used in the present study, from postulates concerning the underlying preference ordering. We shall, however, not pursue that idea in the present study.

The notation and numbering of equations, postulates, diagrams, and theorems of the previous study by Koopmans [8] will be continued here. In particular, all equation numbers below 49 serve also as references to proofs or fuller explanations in the previous study. The reading of the present paper will be facilitated by prior reading of the previous study and by inspection of its diagrams. Nevertheless, we shall restate the postulates in Section 2, so that our statements will be complete in themselves, and also because we shall introduce a strengthening of the first postulate. This strengthening is needed to correct an error in the previous study.[2]

In Section 3 we shall summarize enough of the results of the previous study to be able to present, in Section 4, the main result of the present paper. Sections 5 and 6 discuss further implications of this result. Section 7 presents an example with a variable discount factor. Finally, in Section 8 we discuss questions of "realism" of the system of postulates as a whole, and consider some possibilities for further weakening or otherwise modifying some of the postulates.

Technical aspects of the reasoning are placed in starred sections generally bearing the same number as the section to which they refer.

2. RESTATEMENT OF THE POSTULATES

In the following, ${}_1x \equiv (x_1, x_2, \ldots) \equiv (x_1, {}_2x) \equiv ({}_1x_{t-1}, {}_tx)$, $t = 2, 3, \ldots$, denotes an infinite sequence of consumption vectors $x_t \equiv (x_{t1}, \ldots, x_{tn})$ relating to successive periods $t = 1, 2, \ldots$. The postulates will be referred to as P1 (previous study), P1′ (present study), P2, etc. Interpretations of these postulates have already been given in the previous study. We shall enlarge somewhat on the interpretation of P4. Also, in Section 2*, we shall interpret the strengthening of P1 to P1′.

P1′ (Existence and Continuity): *There exists a utility function* $U({}_1x)$, *which is*

[2] This error was kindly brought to our attention by Richard Levitan of the International Business Machines Corporation.

defined for all $_1x=(x_1,x_2,\ldots)$ *such that, for all* t, x_t *is a point of a bounded convex subset* $\mathscr{X}$ *of the n-dimensional commodity space. The function* $U(_1x)$ *has the continuity property that, if U is any of the values assumed by that function, and if* U' *and* U'' *are numbers such that* $U'<U<U''$, *then there exists a positive number* δ *such that the utility* $U(_1x')$ *of every program* $_1x'$ *having a distance* $d(_1x',{}_1x)\leqq\delta$ *from some program* $_1x$ *with utility* $U(_1x)=U$ *satisfies* $U'\leqq U(_1x')\leqq U''$.

P2 (Sensitivity): *There exist first-period consumption vectors* x_1, x_1' *and a program* $_2x$ *from-the-second-period-on, such that*

$$U(x_1, {}_2x) > U(x_1', {}_2x)\,.$$

P3 (Limited Noncomplementarity): *For all* $x_1, x_1', {}_2x, {}_2x'$,

(P3a) $\quad U(x_1, {}_2x) \geqq U(x_1', {}_2x)$ *implies* $U(x_1, {}_2x') \geqq U(x_1', {}_2x')$,

(P3b) $\quad U(x_1, {}_2x) \geqq U(x_1, {}_2x')$ *implies* $U(x_1', {}_2x) \geqq U(x_1', {}_2x')$.

P4 (Stationarity): *For some* x_1 *and all* $_2x, {}_2x'$,

$$U(x_1, {}_2x) \geqq U(x_1, {}_2x') \quad \text{if and only if} \quad U(_2x) \geqq U(_2x')\,.$$

To clarify the meaning of the notation $U(_2x)$ in this postulate, we give an equivalent statement in more explicit tabular form:

P4 equivalent: *For some* x_1 *and all* $_2x, {}_2x'$, *program A below is at least as good as program B if and only if program C is at least as good as program D.*

Program	Period 1	2	3	4
A	x_1	x_2	x_3	$x_4 \ldots$
B	x_1	x_2'	x_3'	$x_4' \ldots$
C	x_2	x_3	x_4	$\ldots\ldots$
D	x_2'	x_3'	x_4'	$\ldots\ldots$

Hence Postulate P4 says that the ordering of a subset of programs that differ only from the second period on is the same as that of corresponding programs obtained by advancing the timing of every future consumption vector by one period. This does *not* imply that, after one period has elapsed, the ordering then applicable to the "then" future will necessarily be the same as that now applicable to the "present" future. All postulates are concerned with only one ordering, namely that guiding decisions to be taken in the present. Any question of change or constancy of preferences as the time of choice changes is therefore extraneous to the present study. Postulates P4 and P3b taken together express merely an invariance of the *present* ordering under postponement of entire programs, provided gaps created by such postponement are filled in the same way for all programs compared.

P5 (Extreme Programs): *There exist* ${}_1\underline{x}, {}_1\bar{x}$ *such that*

$$U({}_1\underline{x}) \leqq U({}_1x) \leqq U({}_1\bar{x}) \text{ for all } {}_1x .$$

2.* The *norm*, or concept of *distance between two programs*, used in Postulate 1, is defined by

$$(6) \qquad d({}_1x', {}_1x) \equiv \sup_t |x_t' - x_t| , \qquad |x_t' - x_t| \equiv \max_k |x_{tk}' - x_{tk}| .$$

The only difference between the previous Postulate P1 and the present Postulate P1′ is that the set $\mathscr{X}$ of all feasible one-period consumption vectors x is now required to be convex and bounded. This means (convexity) that any weighted average $\theta x + (1-\theta)x'$, $0 < \theta < 1$ of two feasible one-period consumption vectors x, x' is again a feasible consumption vector, and (boundedness) that there is a lower[3] and an upper bound to the feasible rates of consumption of any commodity.

3. SUMMARY OF PREVIOUS RESULTS

Postulates P1′, P2, P3, and P4 have been shown to imply that there exist scalar functions $u(x)$, $V(u, U)$, of a vector $x \in \mathscr{X}$ and of two scalars u, U, respectively, such that the aggregate utility function $U({}_1x)$ satisfies a recurrent relation

$$(11) \qquad U({}_1x) = V(u(x_1), U({}_2x)) .$$

Subject to supplementation on one open point discussed in Section 3* below, the *aggregator function* $V(u, U)$ has also been shown to be continuous and increasing in its two arguments u, U. For the second argument U, equation (11) specifies the aggregate utility $U({}_2x)$ of that part ${}_2x$ of the given program ${}_1x$ that starts with the second period (evaluated as if it were to start immediately). For the first argument u, (11) specifies the value assumed by an *immediate*, or *one-period*, utility function $u(x)$ for the consumption vector $x = x_1$ of the first period in the given program. The function $u(x)$ is defined and continuous on the set $\mathscr{X}$ of all feasible consumption vectors.

By using Postulate P5 also, it has been shown further that, by two independently chosen, continuous, and increasing transformations of the variables U, u, respec-

[3] While zero is a natural lower bound to all consumption proper, one may wish to treat labor of various kinds as negative consumption. In that case the absolute value of the negative lower bound for each type of labor expresses the maximal amount of that labor that can be rendered.

tively, one can make the range of each of the functions $U({}_1x)$ and $u(x)$ coincide with the closed unit interval [0, 1],

$$(12) \qquad 0 = U({}_1\underline{x}) \leqq U({}_1x) \leqq U({}_1\bar{x}) = 1 \quad \text{for all programs } {}_1x\,,$$

$$(13) \qquad 0 = u(\underline{x}) \leqq u(x) \leqq u(\bar{x}) = 1 \quad \text{for all vectors } x\,.$$

Accordingly, the domain of $V(u, U)$ becomes the unit square, its range the unit interval, and

$$(14) \qquad V(0,0) = 0\,, \quad V(1,1) = 1\,.$$

The key property of the function $V(u, U)$ proved and used in the previous study concerns its iterated application. We use again the notation

$$V_\tau({}_1u_\tau; U) \equiv V(u_1, V(u_2, \ldots, V(u_\tau, U) \ldots))\,,$$

where ${}_1u_\tau$ denotes the finite sequence $(u_1, u_2, \ldots, u_\tau)$. If the $u_t = u(x_t)$ are the immediate utility levels associated with the successive vectors x_t of a program ${}_1x$ then clearly for all τ,

$$U({}_1x) = V_\tau({}_1u_\tau; U({}_{\tau+1}x))\,,$$

a generalization of (11). The equation

$$(26) \qquad V_\tau({}_1u_\tau; U) = U$$

therefore expresses the condition that the postponement of a program of utility U by τ periods is just compensated for by the insertion, in the τ periods so vacated, of consumption vectors $x_1, \ldots, x_\tau$ with a sequence of one-period utility levels $u_t = u(x_t)$, $t = 1, \ldots, \tau$. Obviously the utility

$$(27) \qquad U = U({}_1x_\tau, {}_1x_\tau, {}_1x_\tau, \ldots)$$

of the program indefinitely repeating the *consumption pattern* ${}_1x_\tau \equiv (x_1, \ldots, x_\tau)$ meets this condition. It has been shown that, given the *utility pattern* ${}_1u_\tau$ associated with a consumption pattern ${}_1x_\tau$, there exists one and only one value

$$(28) \qquad U \equiv W_\tau({}_1u_\tau)$$

of U that satisfies the condition (26). The *correspondence function* $W_\tau({}_1u_\tau)$ is continuous and increasing in each of its arguments $u_1, \ldots, u_\tau$.

We are now able to state the key property (29) of $V_\tau({}_1u_\tau; U)$ derived in the previous study (and illustrated in Figure 6 of that study for the case $\tau = 2$):

$$(29) \qquad \text{If} \quad U \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} W_\tau({}_1u_\tau) \quad \text{then} \quad U \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} V_\tau({}_1u_\tau; U) \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} W_\tau({}_1u_\tau)\,.$$

This indicates that repeated application of the function $V_\tau({}_1u_\tau; U)$ to any initial value U brings about a monotonic approach to $W_\tau({}_1u_\tau)$. It has been shown in (32) that $W_\tau({}_1u_\tau)$ is also the limit for infinitely repeated application, regardless of the initial value U used.

It will be useful to compare the already proved property (29) with the yet to be proved time perspective property described in Section 1. We can now state the latter as follows: There exists a continuous increasing transformation of the utility scale, as a result of which,

$$(49)\quad \begin{cases} \text{if} \quad U' > U, \quad V_\tau({}_1u_\tau, U) = U'', \quad V_\tau({}_1u_\tau, U') = U''', \quad \tau \geqq 1, \quad \text{then} \\ \text{(49a) weak time perspective}, \quad U''' - U'' \leqq U' - U, \\ \text{(49b) strong time perspective}, \quad U''' - U'' < U' - U. \end{cases}$$

Note that the strict inequalities in (29) represent special cases of (49b) obtained by those choices of ${}_1u_\tau$ that make $U'' = U$, or $U''' = U'$, respectively. These are the only cases of (49) involving comparisons of utility levels rather than of utility differences. Thus (29) states the *ordinal* special cases contained in (49b); that is, the only cases that are invariant for continuous increasing transformations of the scale. In contrast, neither (49a) nor (49b) can be true for all equivalent ordinal scales. The main aim of the present study is to show that the ordinal comparisons in (49b) already known through (29) are sufficient, given the continuity and monotonicity of $V(u, U)$, to prove the existence of one or more scales for which the quasi-cardinal comparisons in (49a) are also valid.

3.* The real-valued function $U({}_1x)$ is defined on the Cartesian product ${}_1\mathscr{X}$ of an infinite sequence of identical sets $\mathscr{X}$, where $\mathscr{X}$ is convex and bounded. In addition U is continuous on ${}_1\mathscr{X}$ in the topology defined by (6). We now show that ${}_1\mathscr{X}$ is connected in that topology. Let ${}_1\underline{x}$ and ${}_1\underline{x}'$ be points of ${}_1\mathscr{X}$. Because $\mathscr{X}$ is convex, the segments defined by $\underline{x}_t''(\theta) = \theta\underline{x}_t + (1-\theta)\underline{x}_t'$, $0 \leqq \theta \leqq 1$, lie in $\mathscr{X}$ for each t. Because $\mathscr{X}$ is bounded the functions $\underline{x}_t''(\theta)$ from $[0,1]$ to $\mathscr{X}$ are equicontinuous. It follows that the function ${}_1\underline{x}''(\theta)$ from $[0,1]$ to ${}_1\mathscr{X}$ is continuous in the topology of definition (6), so ${}_1\mathscr{X}$ is (arcwise) connected. It follows, by the continuity of $U({}_1x)$, that the values assumed by $U({}_1x)$ for all ${}_1x$ in ${}_1\mathscr{X}$ fill an interval, which by Postulate P2 is nondegenerate. By Postulate P5, it is the closed interval $[U({}_1\underline{x}), U({}_1\bar{x})]$, which can by an appropriate continuous increasing transformation be made to be the unit interval $[0,1]$. This proves (12) and (14). In particular, one can take ${}_1x = (\underline{x}, \underline{x}, \underline{x}, \ldots)$, where $\underline{x} = \underline{x}_1$, and ${}_1\bar{x} = (\bar{x}, \bar{x}, \bar{x}, \ldots)$, where $\bar{x} = \bar{x}_1$.

By a similar argument, for any given x_1 the values assumed by the function $U(x_1, {}_2x)$ for all ${}_2x \in {}_2\mathscr{X}$ again fill an interval. By (11), (12), and since $V(u, U)$ has previously been proved to be increasing in U for all u, this interval must be

$$[U(x_1, {}_2\underline{x}),\ U(x_1, {}_2\bar{x})] = [V(u(x_1), 0),\ V(u(x_1), 1)].$$

Hence, for any given u and for $0 \leqq U \leqq 1$, $V(u, U)$ assumes all values in $[V(u,0), V(u,1)]$. Using again that $V(u, U)$ increases with U, it follows that $V(u, U)$ is continuous in U. This point, not covered by the previous study, makes available all other conclusions of that study on the basis of the strengthened Postulate P1′.

4. PROOF OF THE WEAK TIME PERSPECTIVE PROPERTY

It will be useful to shift the discussion from points U on the utility scale to (closed nondegenerate) intervals, for which we shall use the interchangeable notations

(50) $\mathscr{U} \equiv [\underline{U}, \overline{U}] \equiv \{U | \underline{U} \leq U \leq \overline{U}\}$, where $\underline{U} < \overline{U}$.

In particular, the unit interval will be denoted

(51) $\mathscr{I} \equiv [0, 1]$.

The shift to intervals has the advantage that the set inclusion symbol $\supset$ can be used to represent sets of inequalities occurring frequently in the reasoning:

(52) $\mathscr{U} \supset \mathscr{U}'$ stands for $\underline{U} \leq \underline{U}' < \overline{U}' \leq \overline{U}$.

Because $V(u, U)$ is continuous and increasing in U, insertion in V of all the points U of an interval $\mathscr{U}$ gives another interval, which we denote by

(53) $V(u, \mathscr{U}) \equiv [V(u, \underline{U}), V(u, \overline{U})]$.

This operation can be iterated for a finite sequence ${}_1u_\tau$ of values of u, expressing the effect of postponement of all programs with utilities in the interval $\mathscr{U}$ by τ periods, with insertion of a common consumption sequence ${}_1x_\tau$ with an associated one-period utility sequence ${}_1u_\tau$ in the gap created. For further simplification of notation, we shall use V as an operator symbol to denote any operation of this kind:

(54) $\mathscr{U}' = V\mathscr{U}$ stands for $\mathscr{U}' = V_\tau({}_1u_\tau; \mathscr{U})$ for some $\tau \geq 1$ and some ${}_1u_\tau$.

We shall now list those properties of the class $\mathscr{V}$ of all these "postponement" operations that enter into the proof of weak time perspective.

(a) Successive application of two operations V, V' of $\mathscr{V}$ yields another operation $V'' = V'V$ of $\mathscr{V}$ (i.e., $\mathscr{V}$ is a *semi-group*).

This property follows directly from the definition (54) of the generic operation V. For, if $\mathscr{U}' = V\mathscr{U}$, and $\mathscr{U}'' = V'\mathscr{U}'$, then obviously

(55) $\mathscr{U}'' = V_{\tau'}({}_1u'_{\tau'}; V_\tau({}_1u_\tau; \mathscr{U})) = V_{\tau'+\tau 1}(u'_{\tau'}, {}_1u_\tau; \mathscr{U})$.

We have a semi-group rather than a group (in which each operation can be undone by an inverse operation) because the future has a beginning but no end. Hence the postponement of a program only creates a gap to be filled, whereas a program cannot be advanced without suppressing one or more consumption vectors.

(b) As applied to points, each V in $\mathscr{V}$ is a continuous increasing transformation from the unit interval $\mathscr{I}$ onto a subinterval $V\mathscr{I}$ thereof.

This property follows from the continuity and increasing character of $V(u, U)$ with respect to U.

(c) If U, U' are any given points with $U' \neq 0$ or 1, then there exists an operation V in $\mathscr{V}$ such that $VU = U'$.

(d) As applied to intervals, no V transforms any interval $\mathscr{U}$ into an interval $\mathscr{U}'$ containing $\mathscr{U}$:

$$\text{(56)} \qquad \text{If } \mathscr{U}' = V\mathscr{U} \text{ then } \mathscr{U}' \not\supseteq \mathscr{U}\,.$$

Properties (c) and (d) will be proved in Section 4* below.

It may be emphasized again that all the properties (a), (b), (c), and (d) are ordinal. In particular, the translation (56) of the key property (29) into "interval language" uses only the ordinal concept of one interval not being contained in another.

It will be clear that, if (56) were violated by any operation V and interval $\mathscr{U}$, then at least the strong time perspective condition (49b) could not be satisfied. For, if any $V\mathscr{U}$ were to contain $\mathscr{U}$, then there could be no scale in which $V\mathscr{U}$ is shorter than $\mathscr{U}$. It is somewhat less obvious that an almost converse statement is also true: that if (56) holds throughout, then at least a scale with the weak time perspective property (49a) can be constructed. According to a mathematical theorem, to be published elsewhere by two of the present authors [**12**], the conditions (a), (b), (c), and (d) above suffice for the existence of at least one, and possibly infinitely many, such scales. A few further remarks on the nature of the proof are given in Section 4* below.

We record the result in a theorem involving only the aggregate utility function $U({}_1x)$ to which the postulates refer.

THEOREM 2 (Weak Time Perspective): *If* P1′, P2, P3, P4, *and* P5 *are satisfied, there exists a continuous increasing transformation* $U^* = \Phi(U)$ *such that, if* $U^*({}_1x) > U^*({}_1x')$, *then, for all* $\tau \geqq 1$ *and all* ${}_1x_\tau$,

$$0 < U^*({}_1x_\tau, {}_1x) - U^*({}_1x_\tau, {}_1x') \leqq U^*({}_1x) - U^*({}_1x')\,.$$

So far, we have not been able to make sure that scales with the time perspective property exist that have a finite range. That is, in the new scale the utility levels associated with the worst and best programs ${}_1\underline{x}$ and ${}_1\bar{x}$, respectively, may have to be assigned the values

$$\text{(57)} \qquad U^*({}_1\underline{x}) = -\infty\,, \qquad U^*({}_1\bar{x}) = +\infty\,.$$

4.* Thus far we have allowed independent transformations of the arguments u,

U of $V(u, U)$. It will now be convenient, rather than necessary, to apply the transformation (23) of the previous study to the one-period utility scale, so as to make, in accordance with (14),

$$(58) \qquad W(u) \equiv W_1(u) = u \;, \quad \text{so} \quad V(U, U) = U \;, \qquad \text{for all } u, U \in \mathscr{I} \;.$$

These relations will be conserved if from here on we apply any required transformations simultaneously to u and U.

To prove Property (c) we note that, if $U < U' < 1$, the sequence defined by

$$U_1 = U \;, \qquad U_{t+1} = V(1, U_t) \;, \qquad (t = 1, 2, \ldots)$$

is, by (29) and (58), an increasing sequence of which the limit is 1 by (58), (32). Hence there is a τ such that $U_{\tau-1} \leqq U' < U_\tau$, and a u such that $U' \leqq u < 1$ and

$$V_\tau(u, 1, \ldots, 1;\, U) = V(u, U_{\tau-1}) = U' \;,$$

because $V(u, U)$ is continuous and increasing in u. The proof is similar for $U > U' > 0$. In case $U = U'$, clearly $U' = V(U, U)$ by (58).

To prove (d), we shall show that the assumption that $\mathscr{U}' = V\mathscr{U} \supset \mathscr{U}$, and hence $\underline{U}' \leqq \underline{U} < \overline{U} \leqq \overline{U}'$, contradicts (29). If we should have $\underline{U}' = \underline{U}$, a contradiction with (29) would already have occurred. But if $\underline{U}' = V_\tau({}_1u_\tau; \underline{U}) < \underline{U}$, then by (58) and the fact that $V(u, U)$ increases with u, at least one of the elements u_t, $t = 1, \ldots, \tau$ in ${}_1u_\tau$ must satisfy

$$(59) \qquad u_t < \underline{U} \,.$$

We arrange the u_t for which (59) holds in order of increasing t, and increase each of these in succession continuously from the given value up to $\underline{U}$ until, by the continuity and increasing property of $V(u, U)$ with regard to both of its variables, we have reached a sequence ${}_1u'_\tau$ such that

$$\underline{U}'' \equiv V_\tau({}_1u'_\tau;\, \underline{U}) = \underline{U}, \qquad \overline{U}'' \equiv V_\tau({}_1u'_\tau;\, \overline{U}) > V_\tau({}_1u_\tau;\, \overline{U}) = \overline{U}' \geqq \overline{U} \;,$$

again contradicting (29). Such a sequence ${}_1u'_\tau$ is bound to be reached because, if we continue the increases in the u_t satisfying (59) until all of them have been raised to $\underline{U}$, we shall obtain a sequence ${}_1u''_\tau$ such that, using (58),

$$u''_t \geqq \underline{U}, \qquad t = 1, \ldots, \tau; \qquad \text{hence} \quad V_\tau({}_1u''_\tau;\, \underline{U}) \geqq \underline{U} \,.$$

The construction of a scale with the weak time perspective property, given in [**12**], is analogous to, but not identical with, the construction of Haar measure [**3**] or [**6**, Ch. XI]. It starts from a "counting function" $\mathscr{U} : \mathscr{T}$ of two intervals, $\mathscr{U}$, $\mathscr{T}$, which, while using only ordinal concepts, roughly measures $\mathscr{U}$ using $\mathscr{T}$ as a measuring rod. If we call any interval $V\mathscr{T}$ for $V \in \mathscr{V}$ an *image* of $\mathscr{T}$, the counting function is defined as the minimum number of images of $\mathscr{T}$ required to cover $\mathscr{U}$.

To derive a continuous measure from this function, one needs to form the ratio

(60) $(\mathscr{U}:\mathscr{T})/(\mathscr{S}:\mathscr{T})$

of the count of $\mathscr{U}$ to that of a fixed *standard interval* $\mathscr{S}$, before one can shrink the interval $\mathscr{T}$ down to an arbitrarily chosen point T_0. One wishes to make that limit transition in such a way as to obtain an additive interval function; that is, a function $\lambda(\mathscr{U})$ satisfying

(61) if $\overline{U}=\underline{U}'$ then $\lambda(\mathscr{U}\cup\mathscr{U}')=\lambda(\mathscr{U})+\lambda(\mathscr{U}')$.

This can be achieved by using a generalized limit[4] [2, II, §3]

(62) $$\lambda(\mathscr{U})\equiv \operatorname*{Lim}_{\substack{\overline{T}-\underline{T}\to 0\\ \overline{T}\geqq T_0\geqq \underline{T}}} (\mathscr{U}:\mathscr{T})/(\mathscr{S}:\mathscr{T})\,.$$

The resulting function is found to be positive and finite if $\mathscr{S}$ and $\mathscr{U}$ are nondegenerate intervals in the interior $\mathscr{I}^\circ$ of $\mathscr{I}$, and if $T_0\neq 0$ or 1. One also has, for all $U_0\neq 0$ or 1,

(63) $$\operatorname*{Lim}_{\substack{\overline{U}-\underline{U}\to 0\\ \overline{U}\geqq U_0\geqq \underline{U}}} \lambda(\mathscr{U})=0\,.$$

Finally, due to the properties of the counting function used in the construction of λ, one obtains

(64) $\lambda(V\mathscr{U})\leqq\lambda(\mathscr{U})$

for all $V\in\mathscr{V}$ and all $\mathscr{U}\subset\mathscr{I}^\circ$. It follows that the continuous increasing transformation

(65) $$U^*=\Phi(U)\equiv\begin{cases} C+\lambda([\tfrac12, U]) & \text{if } U>\tfrac12\,,\\ C & \text{if } U=\tfrac12\,,\\ C-\lambda([U,\tfrac12]) & \text{if } U<\tfrac12\end{cases}$$

defines a utility scale satisfying (49a), provided the aggregator function $V(u, U)$ is likewise transformed by

(66) $$V^*(u^*, U^*)\equiv\Phi(V(\Phi^{-1}(u^*),\,\Phi^{-1}(U^*)))\,.$$

The construction is not necessarily unique. From simple examples such as $V(u, U)=\frac12(u+U)$ it is easily seen that in general no *unique* scale with either the weak or the strong time perspective property exists, even if one were to prescribe the utilities $U^*({}_1x)$, $U^*({}_1x')$ of two nonequivalent programs ${}_1x, {}_1x'$ in the scale in question.

[4] In the reference, instead of a construction using Banach's generalized limit, an existence proof along the lines of Halmos' discussion of Haar measure is given.

5. WEAK VERSUS STRONG TIME PERSPECTIVE

The following elementary consideration suffices to show that any scale with the weak time perspective property must in some average sense exhibit strong time perspective. Consider the effect of postponement of the best and worst programs by one period. In terms of the original scale where (12), (13), and (14) hold, this effect is subject to the inequalities

$$\text{(67)} \qquad \text{if } 0 \begin{Bmatrix} = \\ < \\ \end{Bmatrix} u \begin{Bmatrix} \\ < \\ = \end{Bmatrix} 1 \quad \text{then} \quad 0 \begin{Bmatrix} = \\ < \\ < \end{Bmatrix} V(u,0) < V(u,1) \begin{Bmatrix} < \\ < \\ = \end{Bmatrix} 1 ,$$

because of (14) and the monotonicity of $V(u, U)$. Being ordinal, (67) goes over into any new scale $u^* = \Phi(u)$, $U^* = \Phi(U)$, constructed to have the weak time perspective property, provided 0 and 1 are replaced by $0^* = \Phi(0)$, $1^* = \Phi(1)$, respectively. If, contrary to (57), 0^* and 1^* are finite, then obviously for all u^*

$$\text{(68)} \qquad \frac{V^*(u^*, 1^*) - V(u^*, 0^*)}{1^* - 0^*} < 1 .$$

Since for any partition $0^* = U_0^* < U_1^* < \ldots < U_N^* = 1^*$ of $[0^*, 1^*]$ the left hand member in (68) is a suitably weighted average of the corresponding ratios

$$\frac{V^*(u^*, U_{n+1}^*) - V^*(u^*, U_n^*)}{U_{n+1}^* - U_n^*}$$

for all intervals of the partition, the latter ratios average out at less than 1, whereas none exceeds 1. If, on the other hand, one or both of 0^*, 1^* are infinite, one can for any finite u^* construct a similar argument in which 0^*, 1^* are replaced by any $\underline{u}^*$, $\bar{u}^*$ such that $\underline{u}^* \leqq u^* \leqq \bar{u}^*$ and $\underline{u}^* < \bar{u}^*$.

We intend to return in a later paper to the problem of constructing a scale exhibiting strong time perspective throughout.

6. TIME PERSPECTIVE AND IMPATIENCE

The time perspective property (49a) or (49b), whichever is applicable, directly implies two extensions of the results of the previous study with regard to impatience. Omitting asterisks, assume that the aggregator function $V(u, U)$ satisfies (49a), and that the scales of u and U have been made comparable by the transformations (23) leading to (58). Let there be two consumption vectors x', x'' with immediate utilities u', u'' such that

$$\text{(69)} \qquad u'' = u(x'') < u(x') = u' .$$

Consider two programs ${}_1x'=(x',x'',{}_3x)$ and ${}_1x''=(x'',x',{}_3x)$ of which the common continuation ${}_3x$ from period 3 on is such that

(70) $$u''<U\equiv U({}_3x)<u' .$$

Then, by (58) and the monotonicity of $V(u,U)$,

(71) $$U''\equiv V(u'',U)<U<V(u',U)\equiv U' ,$$

and, by (49a) with $\tau=1$,

(72) $$\begin{cases} V(u',U)-V(u',U'')\leqq U-U'' , \\ V(u'',U')-V(u'',U)\leqq U'-U . \end{cases}$$

By adding the inequalities (72) and using the definitional equalities in (71), we obtain

(73) $$V(u',V(u'',U))\geqq V(u'',V(u',U)) ,$$

the inequality defining *weak impatience* for the program ${}_1x'$. If (49b) had been available, we would have proved the presence of *strong impatience*, with the > sign in (73), defined simply as "impatience" in Definition 1 of the previous study.

In the previous study strong impatience was established for U in the "central" interval

(74) $$''U'\equiv W_2(u'',u')\leqq U\leqq W_2(u',u'')\equiv 'U''$$

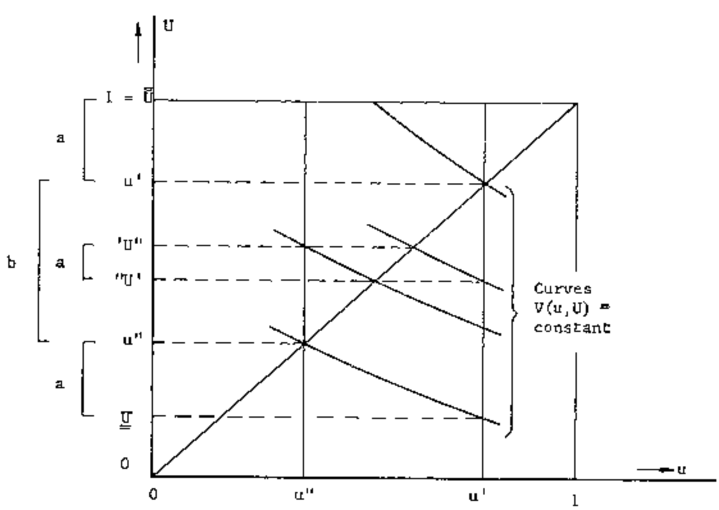

Figure 1. — Zones (a) of strong impatience previously found and zone (b) of weak impatience added in the present study.

or in either of the "lateral" intervals

(75) $\underline{U} \leqq U \leqq u''$, $u' \leqq U \leqq \overline{U}$,

where $\underline{U}$ and $\overline{U}$ are defined by

(76) $V(u', \underline{U}) = u''$, $V(u'', \overline{U}) = u'$,

if solutions to these equations exist, and by $\underline{U}=0$ and/or $\overline{U}=1$ otherwise. The presently established interval of weak impatience contains the "central" interval and is adjacent to both "lateral" intervals, thus closing the gaps as indicated in Figure 1.

Nothing conclusive can be said about impatience in the "outlying" intervals $0 \leqq U < \underline{U}$ and $\overline{U} < U \leqq 1$ for any u'', u' for which these intervals are nonempty. While $V(u, U) = \frac{1}{2}(u + U)$ is an obvious example where strong impatience holds for all programs, it is not difficult to construct other examples where the function $V(u, U)$, while satisfying the strong time perspective condition (49b), is such that for some (u'', u', U) in an outlying zone the $\geqq$ sign in (73) changes to $<$, a case which might be called *strong patience*. In Section 6* below we give a lemma that facilitates the determination of subintervals of impatience and of patience in the lateral intervals. This lemma is based entirely on the monotonicity properties of $V(u, U)$.

The second extension of previously announced results arises from the observation that in all previous and present proofs of impatience relations the symbols u'', u' can without any change in the proof be reinterpreted as finite sequences, ${}_1u''_{\tau''}$, ${}_1u'_{\tau'}$ of one-period utility levels. In that case the symbols u'', u' where occurring as scalars rather than as arguments of V must be replaced by $W_{\tau''}({}_1u''_{\tau''})$ and $W_{\tau'}({}_1u'_{\tau'})$, respectively, and expressions such as $V(u', U)$ must be read as iterated functions $V_{\tau'}({}_1u'_{\tau'}; U)$. The proof of (73) thus comes to rest on (49a) for arbitrary values of τ. Careful reading of Section 13* of the previous study will show that its results are subject to the same generalizing reinterpretations.

We state these results in the form of a theorem which again is formulated entirely in terms of the aggregate utility function $U({}_1x)$.

THEOREM 3 (Weak Impatience): *If* P1′, P2, P3, P4, *and* P5 *are satisfied, and if* ${}_1x_\tau$, ${}_{\tau+1}x_\sigma$, $1 \leqq \tau \leqq \sigma - 1$, *are program segments such that*

$$U^{(1)} \equiv U({}_1x_\tau, {}_1x_\tau, {}_1x_\tau, \ldots) > U({}_{\tau+1}x_\sigma, {}_{\tau+1}x_\sigma, {}_{\tau+1}x_\sigma, \ldots) \equiv U^{(2)} ,$$

then the weak impatience inequality

$$U({}_1x_\tau, {}_{\tau+1}x_\sigma, {}_{\sigma+1}x) \geqq U({}_{\tau+1}x_\sigma, {}_1x_\tau, {}_{\sigma+1}x)$$

is satisfied for any continuation ${}_{\sigma+1}x$ *of the programs such that*

$$U^{(2)} \leqq U({}_1x_\tau, {}_{\sigma+1}x), \qquad U({}_{\tau+1}x_\sigma, {}_{\sigma+1}x) \leqq U^{(1)} .$$

Subzones in the space of ${}_{\sigma+1}x$ where strong impatience has been proved can be derived from Theorem 1 of the previous study by replacing x_1, x_2, ${}_3x$ by ${}_1x_\tau$, ${}_{\tau+1}x_\sigma$, ${}_{\sigma+1}x$, respectively.

It should be noted that the conclusions of Theorems 1 and 3 are entirely ordinal. Hence, although the existence of a transformation giving $U({}_1x)$ the (quasi-cardinal) weak time perspective property was used in the proof of Theorem 3, its conclusions are independent of the scale in which $U({}_1x)$ is expressed.

Finally, one readily shows that Theorems 1 and 3 also cover the case where the two program segments being interchanged are not contiguous in time, by taking ${}_1x_\tau=({}_1x_\rho, {}_{\rho+1}x_\tau)$ and ${}_{\tau+1}x_\sigma=({}_1x_\rho, {}_{\tau+\rho+1}x_\sigma)$ and by dropping or modifying the first ρ vectors of the programs so obtained.

6.* We return to the case of interchange of two consecutive elements x', x'' to indicate a lemma that permits us to conclude from the presence of either impatience or patience with regard to a given continuation ${}_3x$ to the same with regard to other continuations. As before, only $u'=u(x')$, $u''=u(x'')$, and $U=U({}_3x)$ matter, and for simplicity the lemma is formulated in terms of the function $V_2(u',u''; U)$.

LEMMA 3: *If* $u'>u''$ and $V(u',u''; U')=V(u'',u'; U'')$, *then, according as there is* (a) *strong impatience,* (b) *neutrality, or* (c) *strong patience for either* $U=U'$ *or* $U=U''$, *we have* (a) $U''>U'$, (b) $U''=U'$, *or* (c) $U''<U'$, *and* (a) *strong impatience for* $U'\leqq U\leqq U''$, *or* (c) *strong patience for* $U''\leqq U\leqq U'$.

PROOF: In case (a) of strong impatience at $U=U'$ we have $V_2(u'',u'; U')<V_2(u',u''; U')=V_2(u'',u'; U'')$, hence $U'<U''$ because $V_2(u'',u'; U)$ increases with U. Now let $U'<U\leqq U''$. Then, for the same reason,

$$V_2(u'',u'; U)\leqq V_2(u'',u'; U'')=V_2(u',u''; U')<V_2(u',u''; U) .$$

The proof for the other cases is similar.

Clearly, Lemma 3 can again be extended to the case where u',u'' represent finite sequences of utility levels.

7. AN EXAMPLE WITH A VARIABLE DISCOUNT FACTOR

The question arises whether one can exhibit an example of a utility function $U({}_1x)$ showing that the postulates of this study are not in contradiction with each other. In the previous study a somewhat special class of examples was already obtained,

$$U({}_1x)=\sum_{t=1}^{\infty}\alpha^{t-1}u(x_t) , \qquad 0<\alpha<1 , \tag{41)(77}$$

where $u(x)$ satisfies (13) and has on $\mathscr{X}$ a strong continuity property analogous to that of P1′. This class is special in that the discount factor α is a constant, independent of the utility level attained.

The previous study also showed that, if the aggregator function $V(u, U)$ is differentiable, one can more generally define a discount factor[5]

$$(48)(78) \qquad \alpha(U) \equiv \left(\frac{\partial V(u, U)}{\partial U}\right)_{u=W^{-1}(U)}$$

which is invariant for changes in the utility scale. For a fuller motivation of this definition consider a constant program $x_t = x$, $t = 1, 2, \ldots$, and compare the effects on aggregate utility $U({}_1x)$ of a given small change $x' = x + h\xi$ of x in a specified direction ξ in the commodity space $\mathscr{X}$, applied successively to first-period consumption x_1 only, and to second-period consumption x_2 only. It is readily seen that the ratio of the effects, as measured by the derivatives of aggregate utility with respect to h in each case, is then precisely

$$(79) \qquad \frac{\left(\dfrac{\partial U(x, x+h\xi, x, x, \ldots)}{\partial h}\right)_{h=0}}{\left(\dfrac{\partial U(x+h\xi, x, x, \ldots)}{\partial h}\right)_{h=0}} = \alpha(U(x, x, x, \ldots)),$$

the discount factor (78) associated with the utility level $U(x, x, x, \ldots)$ of the constant program considered. The scale-invariance of (79) is due to the fact that, for $h=0$, the arguments of $\partial U/\partial h$ in numerator and denominator are identical.

The question now arises whether the present postulates permit utility functions with a variable discount factor $\alpha(U)$. We shall give an example of such a case, in which the commodity space $\mathscr{X}$ is one-dimensional. Assuming that, within the interval $0 \leqq x \leqq 1$ more of this commodity (bread, say) is always welcome, we can then use the amount x of bread as the simplest one-period utility indicator

$$(80) \qquad u(x) = x.$$

As a result, the recursive relation (11) simplifies to

$$(81) \qquad U({}_1x) = V(x, U({}_2x)).$$

However, we shall *not* make the transformation (23) that would result in (58) and hence in $U(x, x, x, \ldots) = x$ as well.

Consider the aggregator function

$$(82) \qquad V(x, U) = \frac{1}{\theta} \log(1 + \beta x^\delta + \gamma U), \qquad 0 \leqq x \leqq 1, \quad 0 \leqq U \leqq 1,$$

where β, γ, δ, θ satisfy the following compatible conditions,

$$(83) \qquad \theta = \log(1 + \beta + \gamma), \quad \beta, \gamma, \delta > 0, \quad \gamma < \theta, \quad \delta < 1.$$

[5] $u = W^{-1}(U)$ is the inverse of the correspondence function $U = W_1(u)$, defined by (28).

This function satisfies (14) and increases monotonically in x and in U. It satisfies an inequality

$$\text{(84)} \qquad \text{if} \quad U > U' \quad \text{then} \quad V(x, U) - V(x, U') \leqq \frac{\gamma}{\theta}(U - U'),$$

slightly stronger (because $\gamma < \theta$) than strong time perspective (49b). Because $V(x, U)$ satisfies the "Lipschitz condition" (84) with $\gamma/\theta < 1$, (81) and (82) define $U({}_1x)$ uniquely by

$$\text{(85)} \qquad U(x) = \lim_{\tau \to \infty} V_\tau({}_1x_\tau, U),$$

regardless of which U is used. It can readily be verified that $U({}_1x)$ satisfies all the postulates of the present study.

The relation $V(x, U) = U$ implicitly defining the correspondence function is equivalent to

$$\text{(86)} \qquad 1 + \beta x^\delta + \gamma U = e^{\theta U}.$$

From this and the definition (78) one obtains the discount factor

$$\text{(87)} \qquad \alpha(U) = \left(\frac{\gamma}{\theta(1 + \beta x^\delta + \gamma U)}\right)_{x = W^{-1}(U)} = \frac{\gamma}{\theta} e^{-\theta U}.$$

This function decreases monotonically from a value $\gamma/\theta < 1$ for $U = 0$ to a value $e^{-\theta}(\gamma/\theta) > 0$ for $U = 1$. It represents a case in which greater wealth leads the decision maker to discount future satisfactions more, in comparison with present ones. Examples with a discount factor that increases with U also exist.

7.* The present example has two further properties that an economist would like to see present, but that have no further rôle in the discussions of the present study. In the first place, $U({}_1x)$ is strictly concave in the entire domain $0 \leqq x_t \leqq 1$, $t = 1, 2, \ldots,$ of its definition. Secondly, there are no inferior periods anywhere in the future. That is, if for any τ we freeze ${}_{\tau+1}x$ and maximize $U({}_1x)$ by purchase from a given budget c at given positive prices π_t, $t = 1, \ldots, \tau$,

$$\max_{{}_1x_\tau} U({}_1x_\tau, {}_{\tau+1}x) \text{ subject to } \sum_{t=1}^{\tau} \pi_t x_t \leqq c, \quad \text{where} \quad \pi_t > 0,$$

then the maximizing purchases $x_t(c)$ are strictly increasing functions of c in all periods for which these purchases are positive.

8. COMMENTS ON THE POSTULATES

We conclude with some comments on the "realism" of the system of postulates used for this study and on the possibility of further weakening of some of the postulates, at a cost of greater complexity.

In several ways the system of postulates implies an expectation of a constant or stationary world. This comes out in the implied assumption that the utility function orders programs assumed available with certainty, in the finite list of commodities, in the boundedness of the set of feasible consumption vectors, and in the stationarity postulate P4. These are obviously simplifications adopted in order at first to study the significance of timing in relation to preference in isolation from other equally significant aspects of economic choice.

The assumption of convexity of $\mathscr{X}$ made in P1′ rules out indivisible consumption goods. A weaker assumption allowing some goods to be indivisible is considered in Section 8* below.

The limited noncomplementarity assumption P3 is rather restrictive. We conjecture that weak forms of complementarity across a finite time span would permit similar results to be obtained. However, this would require a corresponding reformulation of the stationarity postulate P4 and of the theorems to be proved.

We are less concerned about possible lack of realism of P5. In situations in which that postulate is deemed unrealistic, one can by P2 choose suitable vectors $\underline{x}$, $\bar{x} \in \mathscr{X}$ such that $u(\underline{x}) < u(\bar{x})$, and thereafter curtail the set $\mathscr{X}$ to

$$\mathscr{X}' \equiv \{x \in \mathscr{X} \mid u(\underline{x}) \leqq u(x) \leqq u(\bar{x})\} .$$

P5 then holds if $\mathscr{X}'$ is substituted for $\mathscr{X}$.

8*. The question may be raised whether, now that P1′ has been strengthened from P1 by asserting the boundedness of $\mathscr{X}$, a mild further assumption of closedness of $\mathscr{X}$ in P1′ might make P5 into an implication of the other postulates. We do not think that this is so, because compactness of $\mathscr{X}$ does not, in the topology defined by (6) in Section 2*, imply compactness of the Cartesian product ${}_1\mathscr{X}$ of a denumerable sequence of identical spaces $\mathscr{X}$.

The assumption of convexity of the feasible consumption set $\mathscr{X}$ is used only in Section 3* in proving the continuity of $V(u, U)$ with regard to U. This proof goes through also [6] if $\mathscr{X}$ is arcwise connected with a finite *interior diameter*, defined as the least upper bound over all pairs of points x, $x' \in \mathscr{X}$ of the greatest lower bound of the lengths of all arcs in $\mathscr{X}$ connecting x and x'. This assumption implies boundedness of $\mathscr{X}$.

The existence of indivisible consumption goods cannot be recognized as long as $\mathscr{X}$ is connected. However, one can also permit $\mathscr{X}$ to be the union of a finite collection $\mathscr{C}$ of arcwise connected finite-diameter sets $\mathscr{X}^{(n)}$, $n = 1, \ldots, N$, provided $\mathscr{C}$ is *connected in utility*. Two connected sets $\mathscr{X}^{(1)}$, $\mathscr{X}^{(2)}$ are called connected in utility if there exist $x^{(1)} \in \mathscr{X}^{(1)}$, $x^{(2)} \in \mathscr{X}^{(2)}$ such that $u(x^{(1)}) = u(x^{(2)})$. A collection $\mathscr{C}$ of sets

[6] A still further weakening of P1′ which is of mathematical interest is made possible by the following result of R. Strichartz [10]: If $\mathscr{X}$ is a Peano space (i.e., is compact, metric, connected, and locally connected), then the Cartesian product ${}_1\mathscr{X}$ of a denumerable sequence of identical sets $\mathscr{X}$ is arcwise connected.

$\mathscr{X}^{(n)}$ is called connected in utility if for each pair $(\mathscr{X}^{(n)}, \mathscr{X}^{(l)})$ of C there is a finite sequence $(\mathscr{X}^{(n)}, \mathscr{X}^{(m)}, \ldots, \mathscr{X}^{(p)}, \mathscr{X}^{(l)})$ in C, of which each pair of successive sets is connected in utility. This weakening of P1′ enables one to recognize all indivisible goods, the loss of which can in suitable situations be compensated for by an increased allotment of perfectly divisible goods.

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