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HABITS AND TIME PREFERENCE*

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This paper proposes a utility function incorporating both habit formation and an endogenous rate of time preference in a manner consistent with the intuition of Irving Fisher regarding the influence of past consumption on impatience. It is shown that the new specification is tractable and generates new predictions in the context of three model economies: (1) a closed economy with heterogeneous agents, (2) a small open economy with one traded good and one nontraded good, and (3) a small open economy with a traded good and domestic money.

1. INTRODUCTION

Habit formation has been increasingly incorporated into dynamic economic modeling. In the conventional specification, habits are introduced as a separate argument into the felicity function while a constant rate of discounting is maintained. (For example, in continuous time settings see Ryder and Heal 1973, Becker and Murphy 1988 and Constantinides 1990, and in discrete time see Boyer 1978 and 1983). In common with the additive utility function, this specification implies that the rate of time preference along a constant path at level c is independent of c. Since this feature of additive utility is responsible for many of its starkest implications (see Epstein and Hynes 1983, Epstein 1987, pp. 69–71), those implications are shared also by the existing models of habit formation. In particular, the above constancy of time preference leads to predictions of extreme disparities in long-run wealth holdings in a general equilibrium with heterogeneous agents (Becker 1980). In addition, tractability is a concern. For example, in models where the rate of interest is exogenous, such as in the case of a small open economy facing a given world interest rate, the well-known knife-edge property of steady states invalidates the common procedure of linearizing the dynamic system in order to conduct comparative dynamics analysis locally near the steady state.

This paper proposes a utility function that incorporates both habit formation and an endogenous rate of time preference. Table 1 contains our utility specification as well as the utility functions that are most common in dynamic economic modeling. The Uzawa (1968) function has been widely used in applications where the implications of a constant rate of time preference are particularly unappealing and/or inconvenient.[2] At one level our specification differs from the Uzawa function

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[2] They are avoided also in overlapping generations models, but these models tend to be analytically less tractable if each generation lives for $T > 2$ periods and unsuitable for short-run analysis if $T = 2$.

TABLE 1
INTERTEMPORAL UTILITY FUNCTIONS

Intertemporal utility $U(C)$	Constant RTP along constant paths	Endogenous RTP along constant paths
Preference independent of past consumption	$\int_0^\infty v(c)e^{-\beta t}\,dt$ (conventional additive utility)	$\int_0^\infty v(c)e^{-\int_0^t \beta(c)\,d\tau}\,dt$ (Uzawa 1968)
Preference dependent on past consumption	$\int_0^\infty v(c, z)e^{-\beta t}\,dt$ $\dot{z}(t) = \sigma[c(t) - z(t)]$ (Ryder and Heal 1973)	$\int_0^\infty v(c)e^{-\int_0^t \beta(z)\,d\tau}\,dt$ $\dot{z}(t) = \sigma[c(t) - z(t)]$ (this paper)

RTP–the rate of time preference; C–consumption path; $z(t)$–consumption habits.

in that the discount function β depends on an index of past consumption rather than on current consumption only. The following more meaningful distinction between the corresponding preference orders is made precise below: For the Uzawa function, the rate of time preference (along an arbitrary consumption path) depends on current and future consumption only, while given our specification it depends also on an index of past consumption. Thus, our utility function is a natural and, as we argue further below, "minimal" extension of the Uzawa function to incorporate habit formation. Note that the Uzawa function is obtained from ours in the limit as the speed of habit adjustment σ becomes arbitrarily large.

Irving Fisher (1930, p. 89) has emphasized the endogeneity of time preference and his intuitive arguments are frequently recalled in discussions and applications of the Uzawa utility function. But one of the determinants of time preference discussed by Fisher that has not been considered in the recent more formal modeling literature is "habits." Fisher (1930, pp. 337–338) clearly expresses the intuitive underpinnings for our model of time preference and intertemporal utility:

> It has been noted that a person's rate of preference for present over future income, given a certain income stream, will be high or low according to the past habits of the individual. If he has been accustomed to simple and inexpensive ways, he finds it fairly easy to save and ultimately to accumulate a little property. The habits of thrift being transmitted to the next generation, by imitation or by heredity or both, result in still further accumulation. The foundations of some of the world's greatest fortunes have been based upon thrift.
>
> Reversely, if a man has been brought up in the lap of luxury, he will have a keener desire for present enjoyment than if he had been accustomed to the simple living of the poor. The children of the rich, who have been accustomed to luxurious living and who have inherited only a fraction of their parent's means, may spend beyond their means and thus start the process of the dissipation of their family fortune. In the next generation this retrograde movement is likely to gather headway and to continue until, with the

Blanchard's (1985) continuous time version of the overlapping generations model is another possibility, though his assumption of an age-independent probability of death may seem unrealistic. Of course, the infinite horizon framework adopted in this paper does not require that agents literally live forever; under additional assumptions, the existence of appropriate intergenerational altruism justifies its use (Barro 1974).

gradual subdivision of the fortune and the reluctance of the successive generations to curtail their expenses, the third or fourth generation may come to actual poverty.

The accumulation and dissipation of wealth do sometimes occur in cycles. Thrift, ability, industry and good fortune enable a few individuals to rise to wealth from the ranks of the poor. A few thousand dollars accumulation under favorable circumstances may grow to several millions in the next generation or two. Then the unfavorable effects of luxury begin, and the cycle of poverty and wealth begins anew. The old adage, "From shirt sleeves to shirt sleeves in four generations," has some basis in fact.

While intuitive plausibility is one reason for exploring new utility functions, at least as important is that they "make a difference" in the sense of providing new predictions and theoretical insights. In particular, they should be tractable when applied to some standard dynamic models. In the second part of this paper we provide such a justification for our specification of intertemporal utility by applying it to three economies: (1) a closed economy with heterogeneous agents, (2) a small open economy with one traded good and one nontraded good, and (3) a small open economy with a traded good and domestic money. In all three economies we show that a sufficiently large degree of persistence in spending habits, through its influence on time preference, generates cyclical wealth accumulation, thus confirming Fisher's intuition expressed in the quotation above at least locally near the steady state. In the second and third economies, cyclical wealth accumulation is associated with cyclical current accounts and exchange rates and rich covariations between these two variables. We emphasize that, for the reasons outlined in our opening paragraph, the implied dynamics in all three models differ substantially from that implied by the Ryder-Heal specification of habit formation. They differ also from the dynamics implied by the Uzawa utility function, where the local stability of steady states also prevails, but convergence to the steady state is necessarily noncyclical (see Obstfeld 1981 and 1990, and Epstein 1987, for example).

The paper proceeds as follows: Section 2 formulates and analyses our model of utility. In the next three sections the three economies mentioned above are defined and analysed. Technical details and proofs are supplied in the Appendix. A more detailed set of proofs is available from the authors upon request.

2. INTERTEMPORAL UTILITY AND TIME PREFERENCE

2.1. *Utility*. A consumption path C assigns consumption $c(t) \geq 0$ to each time $t \geq 0$ over an infinite horizon. We propose the following functional form for the utility function defined on such paths:

$$U(C) = \int_0^\infty v(c) \exp\left(-\int_0^t \beta(z)\, d\tau\right) dt, \tag{2.1}$$

$$\dot{z} = \sigma(c - z), \qquad z(0) = z_0 \geq 0 \qquad \text{given,}$$

where $\sigma > 0$ and regularity conditions for v and β are described below. The dependence of $U(C)$ on z_0 is suppressed in the notation.

Integration from time $-\infty$ implies that

$$z(t) = \sigma \int_{-\infty}^{t} c(\tau) \exp(\sigma(\tau - t)) \, d\tau,$$

a weighted average of past consumption levels with weights declining exponentially into the past at the rate σ. We refer to $z(t)$ as the level of "habits" or index of past consumption and to σ as the rate of habit adjustment. Two limiting cases are noteworthy. If $\sigma = 0$, then z and $\beta(z)$ are constant which yields the conventional time-additive utility function. On the other hand, if $\sigma = \infty$ then $z \equiv c$, $\beta(z) \equiv \beta(c)$ and the Uzawa function is obtained. From the perspective of our specification (2.1), therefore, these common functional forms correspond to alternative extreme assumptions about the rate of habit adjustment.

Of course, there exist alternative functional forms that contain the Uzawa and additive functions as special cases. For example, inspection of Table 1 suggests the alternative

$$U(C) = \int_0^\infty v(c, z) \exp\left(-\int_0^t \beta(c) \, d\tau\right) dt. \tag{2.1'}$$

We would argue, however, that (2.1) is simpler than (2.1') in the informal but significant sense that each of its component functions β and v is a function of a single argument, while in (2.1') v depends on two arguments. On such informal grounds, (2.1) seems to us to be a "minimal" extension of the Uzawa function that incorporates habit formation in a way that captures Fisher's intuition (see (2.6) below).

Returning to our functional form (2.1), we adopt the following assumption for v and β.

ASSUMPTION 1. *v and β are twice continuously differentiable and*
(i) $\beta > 0$, $\beta' > 0$ *and* $\beta'' = 0$,
(ii) $v < 0$, $v' \geq 0$ *and* $-\log(-v)$ *concave,*
where all functions are evaluated at c and the restrictions hold for all $c > 0$.

The implications of (i) and of $\beta' > 0$ in particular will be clarified shortly by considering the rate of time preference implicit in U. Here we note merely that the additive utility model corresponds to $\beta' = 0$ and that our assumption that β is linear is adopted for the purposes of simplification and specificity. Condition (ii) guarantees, as in the case of the Uzawa function (see Epstein 1987, Lemma 1), that the intertemporal utility function U is globally increasing and strictly quasiconcave.[3]

[3] Condition (ii) can be weakened to

$$\beta v' - v\beta' > 0, \; v' \geq 0 \text{ and } \frac{v''}{v'}(\beta v' - v\beta') + \beta' v' < 0, \tag{ii'}$$

An attractive feature of Assumption 1 is that it does not depend upon σ. This allows us to examine the consequences of different degrees of habit persistence without worrying about whether changing σ destroys the basic regularity properties of U. Notice also that if Assumption 1 is satisfied only on a subset of positive consumption levels, then U is well-behaved on a suitably restricted domain and the analyses in Sections 3 through 5 are valid for a suitably restricted set of interest rates.

2.2. *Time Preference.* Let $U_T(C)$ denote the marginal utility with respect to a small increment in consumption along the path C and at times near T, in the sense made precise by the Volterra derivative. (See Wan 1970 and Ryder and Heal 1973 for a definition and applications in similar settings.) For our utility function (2.1) we have

$$U_T(C) = [v'(c(T)) + \sigma\Psi(T)] \cdot \exp\left(-\int_0^T \beta(z)\, d\tau\right), \tag{2.2}$$

where $\Psi(T)$ represents the shadow value of the stock of habits. More precisely, let ${}_TC$ denote the "tail" of C which assigns consumption $c(t + T)$ to each time $t \geq 0$. Then

$$\begin{aligned} \Psi(T) &= \frac{\partial U({}_TC)}{\partial z(T)} \\ &= -\int_T^\infty v(c(t))\left(\int_T^t \beta' e^{\sigma(T-\tau)}\, d\tau\right) \exp\left(-\int_T^t \beta(z)\, d\tau\right) dt \end{aligned} \tag{2.3}$$

and therefore $\Psi(T) > 0$ and $U_T > 0$.

In analogy with conventional practice in discrete-time models and in agreement with Epstein and Hynes (1983), we define the (local) rate of time preference ρ as the rate of decrease in marginal utility along a *locally* constant consumption path. Precisely,

$$\rho \equiv \frac{-d}{dT} \log U_T(C)|_{\dot{c}(T)=0}. \tag{2.4}$$

The restriction to locally constant paths serves to isolate the effect on the marginal utility of the time delay.[4] Performing the differentiation yields

$$\rho = \beta(z(T)) - \frac{\sigma[(\beta(z(T)) + \sigma)\Psi(T) + U({}_TC)\beta']}{v'(c(T)) + \sigma\Psi(T)}, \tag{2.5}$$

which is sufficient for U to be well-behaved locally in the neighborhood of constant paths and thus suffices for the local analyses to follow. We have chosen to adopt (ii) because of its relative simplicity.

[4] If the consumption path is such that $\dot{c}(T) \neq 0$, the expression (2.4) measures the rate at which marginal utility *would* fall with time if c were to remain momentarily fixed.

which shows the way in which ρ depends upon past consumption $z(T)$ and upon current and future consumption $_TC$ (through $c(T)$, $\Psi(T)$ and $U(_TC)$). Thus we can think of a (local) *rate of time preference function* $\rho(z, C)$, where there is no harm in setting $T = 0$.

Along a globally constant consumption path, where all past and future consumption levels equal c, we compute that $z = c$ and $\rho = \beta(z)$. Thus β describes the rate of time preference along such paths. The assumption that $\beta' > 0$ implies that time preference increases if we move from one constant consumption path to a higher one, corresponding to the assumption of increasing marginal impatience in Lucas and Stokey (1984). This assumption is necessary for dynamic stability as has been widely recognized in studies employing the Uzawa function, but there is disagreement in the literature over its intuitive appeal. For low consumption levels, decreasing impatience seems intuitive, for the reasons given by Fisher (1930, pp. 72–73), i.e., a "man must live" and thus greater poverty increases the importance attached to immediate versus future consumption. But for higher consumption levels, increasing impatience seems reasonable for the reasons given in Epstein (1987, pp. 73–74) and Obstfeld (1990), for example. Thus the reader may prefer to interpret the local dynamic analysis below as applying to a region of relative affluence.

Examination of the rate of time preference function for consumption paths that are not globally constant reveals much about the nature of our utility function. First, if we evaluate the derivative along a globally constant path then (2.5) implies that

$$\text{(2.6)} \qquad \frac{\partial}{\partial z}\rho(z, C)|_{\text{constant path}} = \frac{\beta'}{v' + \sigma\Psi}\left[v' - \frac{\sigma v \beta'}{(\beta + \sigma)(\beta + 2\sigma)}\right] > 0.$$

Hence, past consumption increases the degree of impatience, which conforms with Irving Fisher's intuition and with our attributing the time nonseparable nature of U to "habits." The dependence of ρ on past consumption distinguishes our model from Uzawa's, where ρ depends only on current and future consumption. In the Uzawa case, the rate of time preference ρ, obtained from (2.4) by letting $\sigma \to \infty$, is given by

$$\rho(z, C) = [\beta(c)v'(c) - \beta'(c)v(c)]/[v'(c) - \beta' U(C)],$$

which is independent of z.

The nature of the intertemporal nonseparability exhibited by our utility function is further clarified by considering the dependence of ρ on future consumption. By making use of the Volterra derivative we can compute that for $t > 0$

$$\text{(2.7)} \qquad \frac{\partial \rho}{\partial c(t)}(z, C) = \beta\beta' e^{-\beta t} - (\beta + \sigma)e^{-(\beta+\sigma)t}\frac{\partial \rho}{\partial z(t)}(z(t), {}_tC),$$

where we henceforth suppress in the notation that derivatives of ρ are evaluated along constant paths. Say that current consumption and time t consumption are *complementary* if $\partial\rho(z, C)/\partial c(t) > 0$. The complementarity between future and current consumption predominates in the sense that

$$\int_0^\infty \frac{\partial\rho}{\partial c(t)}(z, C)\, dt = -2\beta'^2\sigma^2 v \frac{(v' + \sigma\Psi)^{-1}}{\beta(\beta + \sigma)(\beta + 2\sigma)} > 0. \tag{2.8}$$

There are other definitions of "complementarity" that have been used in the literature to elucidate the nature of time nonseparable utility functions. Ryder and Heal (1973) define notions of adjacent and distant complementarity in terms of the effect of a change in $c(t_3)$ at the marginal rate of substitution between $c(t_1)$ and $c(t_2)$. Adjacent complementarity prevails if for all t_3 in some neighborhood of t_1, an increase in $c(t_3)$ shifts preference towards $c(t_1)$ at the expense of $t_2 > t_1$. Roughly speaking, we are looking at the limit as $t_2 - t_1$ becomes arbitrarily small, which is natural in light of the central role of the (instantaneous) rate of time preference in determining dynamics in continuous time and also simplifying in the sense of delivering an unambiguous sign in (2.8). We further restrict attention to first order perturbations of a constant consumption path since we are interested only in local behaviour near a steady state. With these modifications of the Ryder and Heal definitions in mind, the positive signs in (2.6) and (2.8) correspond to the existence of adjacent complementarity. Our notion of complementarity should also be distinguished from that due to Edgeworth and discussed by Obstfeld (1990), where complementarity is defined as positivity of the cross partial of $U(C)$ with respect to $c(0)$ and $c(t)$. The Edgeworth notion is cardinal, while ours and the Ryder and Heal notion are based on marginal rates of substitution and thus are ordinal in nature.

3. EQUILIBRIUM WEALTH ACCUMULATION

As indicated in the introduction, in a competitive equilibrium with heterogeneous agents, each of whom has a rate of time preference that equals an exogenous constant in steady states, only the most patient individuals can have positive wealth in the long run (Becker 1980). A less extreme long run distribution of wealth is possible if the rate of time preference of each individual is suitably endogenous (Epstein 1987). Here we consider the implications for the dynamics of wealth accumulation if habits influence the rate of time preference in the way modeled by our intertemporal utility function (2.1). Of particular interest is whether the cyclical dynamics conjectured by Fisher can be confirmed.

Since our focus is on the structure of time preference (and since the paper is already quite long) we simplify the production side by assuming that the marginal product of capital is a technological constant. Then there is no market interaction between agents and the model decomposes into a number of separate problems of individual utility maximization subject to an intertemporal budget constraint with a constant rate of interest. A model of individual wealth accumulation is of independent interest and is also of use below in Section 4. Our discussion will for the most part relate to this model of individual behaviour; the implications for market equilibrium dynamics will be summarized at the end of the section.

Consider, therefore, a consumer with utility function (2.1) who faces an exogenous real interest rate $r > 0$. If x denotes wealth and if we define the discount factor

$$\alpha(t) \equiv \exp\left(-\int_0^t \beta(z)\, d\tau\right),$$

then the consumer's optimization problem can be expressed as follows:

$$\text{(P)} \qquad \max \int_0^\infty \alpha(t) v(c)\, dt$$

subject to

$$\dot{\alpha} = -\beta(z)\alpha,\ \alpha_0 = 1, \tag{3.1}$$

$$\dot{z} = \sigma(c - z),\ z_0 > 0 \text{ given}, \tag{3.2}$$

$$\dot{x} = rx - c,\ x_0 > 0 \text{ given}, \tag{3.3}$$

$$c,\ x \geq 0.$$

The associated discounted Hamiltonian is

$$H = \alpha[v(c) + \Psi\sigma(c - z) - \phi\beta(z) + \omega(rx - c)],$$

where $\Psi(t)$ is defined by (2.3) and equals the shadow price of z, and where ϕ is the utility variable

$$\phi(t) \equiv U({}_tC).$$

We have made use of the fact that $-\phi(t)$ equals the current-valued shadow price of the auxiliary state variable α. If (2.1) is used to evaluate $U({}_tC)$ then differentiation implies that

$$\dot{\phi} = \beta(z)\phi - v(c). \tag{3.4}$$

Thus the maximum principle implies the following set of differential equations in (z, ϕ, Ψ, c, x) which must be satisfied by an interior optimum for (P): equations (3.2) through (3.4) and

$$\dot{\Psi} = (\beta + \sigma)\Psi + \phi\beta', \tag{3.5}$$

$$\dot{c} = \frac{v' + \sigma\Psi}{v''}[\rho(z, c, \phi, \Psi) - r], \tag{3.6}$$

where, in a slight abuse of earlier notation, the rate of time preference function $\rho(z, c, \phi, \Psi)$ is defined in (2.5). Note the familiar relation implied by (3.6) between the sign of consumption growth and the relative magnitudes of the rate of time preference and the rate of interest. Since intertemporal utility is strictly quasiconcave, equations (3.2) through (3.6) and convergence to steady state values, described below, define the unique optimum for (P).[5]

The steady state (z^*, ϕ^*, Ψ^*, c^*, x^*) is characterized by

$$\beta(z^*) = r,\ c^* = z^*,\quad x^* = z^*/r,$$

$$\phi^* = v(c^*)/r \text{ and } \Psi^* = -v(c^*)\beta'/(r(r+\sigma)).$$

The steady state exists if $\inf \beta(\cdot) < r$, in which case it is unique.[6] Note the contrast with the Ryder and Heal specification in which β is constant; there, steady states either do not exist if $\beta \neq r$, or comprise a continuum if $\beta = r$.

Now turn to dynamics near the steady state. Such local dynamics are faithfully represented by the linearization of (3.2) through (3.6). Since there are two predetermined variables, x and z, local stability requires that the matrix J of the linearized system have exactly two eigenvalues with negative real parts (stable roots). This is also sufficient for stability under the regularity condition described in the Appendix. Convergence to the steady state is cyclical if the stable roots are complex.

The local dynamics of the optimal consumption/savings, wealth accumulation plan are summarized in the following theorem.

THEOREM 1. *Under Assumption* 1 *and* $\inf \beta(\cdot) < r$, *there exists* $\sigma_1 > 0$ *such that the dynamic system* (3.2) *through* (3.6) *is locally stable for all* $0 < \sigma \neq \sigma_1$ *and is cyclical if and only if* $\sigma < \sigma_1$. *Moreover,* σ_1 *is increasing in the steady state values of* β' *and* v''/v'.

Apart from the possible exception of σ_1, the steady state is locally stable for all positive σ. Since the rate of interest is constant, stability is due exclusively to the endogenous rate of time preference. Intuitively, two aspects of time preference affect stability. First, complementarity between future and current consumption (see (2.8)) is stabilizing. (For example, if initial wealth is smaller than the steady state level, then positive accumulation is required to reach the steady state. Along the adjustment path consumption is in some average sense below c^* and so the rate of time preference is smaller than its steady state value r. Thus consumption is induced to increase consistent with convergence to c^*.) This is the only force operating in the Uzawa model and "explains" the well-known local stability prevailing there. In the present model, ρ also varies with past consumption which dependence is also stabilizing if $\partial\rho/\partial z > 0$. (For example, if consumption has been in excess of c^* for some time, then the large value of z implies a high degree of impatience and therefore a decline in the rate of consumption as wealth is run down.) Thus stabilizing forces predominate in our model.

Turn to the results concerning the existence of cycles. First, the theorem validates Fisher's intuition only in part since the cycles are local and they dampen

[5] The proof is analogous to that of Epstein (1987, Lemma 2).

[6] Since $\rho = \beta(z^*)$ in a steady state, z^* is uniquely determined by $\beta(z^*) = r$ which is the steady state implication of (3.6). Then $c^* = z^*$, $x^* = c^*/r$ and the steady state value ϕ^* and Ψ^* are uniquely determined by (3.4) and (3.5).

towards the steady state; our model does not generate limit cycles. Second, cycles are more likely if habits adjust more slowly, or the steady state rate of time preference is more sensitive to the level of consumption, or if the desire to smooth consumption is weaker. Since cycles are impossible in the Uzawa model and since the latter is approached as $\sigma \to \infty$, it is not surprising that we obtain cycles only for sufficiently small σ. On the other hand, it is interesting to note that cyclical behaviour is to be expected if it is believed that the common additive utility model holds not exactly, but in the approximate sense of (2.1) with σ near 0.

The nature of the consumption function $c(x, z)$ generated by the utility maximization problem (P) is of interest. In the Appendix we show that consumption is normal and also that it is *addictive* in the sense that c is increasing in z. (Becker and Murphy 1988 define addiction in this way.) This adds to the justification provided in Section 2.2 and inequality (2.6) in particular for our use of the term "habit formation" to describe the time nonseparable feature of our utility function.

Finally, consider briefly the implied equilibrium dynamics in an economy populated by many such consumers and with a constant real interest rate. Theorem 1 applies to describe the wealth accumulation path followed by each consumer. In particular, unlike the case where steady state rates of time preference are constant as in the Ryder and Heal specification, all consumers can own positive wealth in the long run even if rates of time preference, in the sense of the functions $\beta(\cdot)$, differ across consumers. And, unlike the case where everyone has a Uzawa utility function, convergence to the steady state can be cyclical for some consumers, e.g., for those with large degrees of habit persistence.

4. THE REAL EXCHANGE RATE AND CURRENT ACCOUNT

We modify and reinterpret the model of individual wealth accumulation just analyzed so that it represents the behaviour of a representative agent in a small open economy in which there exists a traded and a nontraded good. The model is used to investigate the effect of habits on the dynamics of the real exchange rate and current account.

4.1. *The Model.* At each instant the representative agent consumes a traded good and nontraded good at rates denoted f (for foreign) and d (for domestic). Real expenditure (an index of aggregate consumption) at t, $c(t)$, is given by the relation

$$c(t) = u(f(t),\ d(t)), \tag{4.1}$$

where u is a positive, increasing, concave and linearly homogeneous "aggregator" function.

Intertemporal utility U is defined via (2.1) where C is the intertemporal program of real expenditure. We specialize Assumption 1 by requiring that

$$v \equiv -1. \tag{4.2}$$

Epstein and Hynes (1983) emphasize the analytical advantages of the comparable assumption in an Uzawa-type model where time preference is unaffected by habits.

Note that the rate of habit formation $\dot{z}$ depends on the consumption levels f and d only via c.

Adopt the traded good as numeraire and denote by q the (relative) price of the nontraded good or the real exchange rate and by r the constant real world interest rate. Then the optimizaton problem solved by the representative agent is

$$\text{(4.3)}\quad \begin{array}{ll} \max U & \\ \text{subject to (4.1)} & \text{and } \dot{x} = rx + y - f - qd, \\ & x(0) = x_0 \text{ given and } \lim_{t\to\infty} x(t)e^{-rt} \geq 0. \end{array}$$

Here x denotes the claims in foreign assets or external balance, $y = y_f + qy_d$ is aggregate domestic output with y_f and y_d denoting outputs of traded and nontraded goods respectively. Both outputs are assumed to remain constant over time.

Because of the fact that f and d enter U only via real expenditure c, a two-stage budgeting procedure is valid for (4.3). For any chosen $c(t)$, it is optimal to choose consumption at t to solve

$$\text{(4.4)}\qquad c(t)e(q(t)) = \min_{f,d} \{f + q(t)d\colon\ u(f, d) \geq c(t)\},$$

where the homogeneity of u has been used to express the value of this atemporal budget allocation problem in the form shown on the left. The function $e(\cdot)$ is the unit cost function dual to u, with the price of f having been normalized to unity. In particular, $e(\cdot)$ is increasing and concave (see Diewert 1982). Then

$$\text{(4.5)}\qquad p \equiv e(q)$$

serves as the price index for real expenditure and the intertemporal problem can be transformed into problem (P) of Section 2 with the single exception that the wealth accumulation equation is

$$\text{(4.6)}\qquad \dot{x} = rx + y - pc.$$

Denote this modified problem by (P1).

An equilibrium is a tuple $\{f(t), d(t), c(t), q(t), x(t)\}_0^\infty$ such that intertemporal utility is maximized and the market for the nontraded good clears, i.e., $d(t) = y_d$ for all t. (For the traded good, the excess domestic demand $f(t) - y_f$ can be met by a perfectly elastic world supply at price $q(t)$.) Shephard's Lemma (or the envelope theorem) applied to (4.4) implies that it is optimal to satisfy $d = ce'(q)$ at any instant. In conjunction with market clearing this implies

$$\text{(4.7)}\qquad c(t)e'(q(t)) = y_d, \quad \text{or} \quad q(t) = e'^{-1}(y_d/c(t)), \quad \text{for all } t.$$

By straightforward modifications of the arguments in Section 3, we can derive the dynamic system that describes an equilibrium. In particular, the maximization problem gives

$$\frac{e'}{e}\dot{q} = \frac{\dot{p}}{p} = -(\rho - r). \tag{4.8}$$

From (4.5), (4.7) and (4.8), real expenditure follows a dynamic rule similar to (3.6):

$$\frac{\dot{c}}{c} = -\frac{e''}{e'}\dot{q} = -\frac{1}{\kappa}(\rho - r), \quad \kappa \equiv -\frac{e'^2}{ee''}. \tag{4.9}$$

The wealth accumulation equation can be rewritten (using (4.5) and (4.7)) as

$$\dot{x} = rx + y - e[e'^{-1}(y_d/c)]c. \tag{4.10}$$

Equilibrium is defined by (4.9), (4.10), the appropriate forms of (3.2), (3.4) and (3.5) and by convergence to steady state values.

4.2. *Dynamics*. The natural counterpart of Theorem 1 is readily proven for the present dynamic system and much of the discussion of Theorem 1 may be translated. In particular, the steady state is unique if inf $\beta(\cdot) < r$ and is locally stable for all $\sigma \neq \sigma_1$ for some σ_1, convergence is cyclical if and only if $\sigma < \sigma_1$ and σ_1 depends in an intuitive manner on $\beta(\cdot)$. Here we concentrate on implications and interpretation that are specific to the present open economy model.

Since a cyclical adjustment implies a cyclical current account, a large degree of habit persistence (small σ) leads to cyclical behaviour in the current account. This is in contrast to the informal literature (see Lawrence 1987) where the view is expressed that persistence in spending is responsible for a persistent current account.

Another factor influencing the qualitative dynamics is the parameter κ defined in (4.9). The cut-off value σ_1 is decreasing in κ. To interpret this relation, specialize the aggregator U from (4.2) to the CES form

$$u(f, d) = [\delta f^{\nu} + (1 - \delta)d^{\nu}]^{1/\nu}, \quad \nu < 1, \ \delta \in (0, 1), \tag{4.11}$$

in which case we compute

$$\kappa = (1 - \nu)(\delta^{-1} - 1)^{1/1-\nu} q^{\nu/\nu - 1}, \tag{4.12}$$

which is decreasing in δ. Thus our model confirms the intuition that a large budget share for nontraded goods (δ smaller) contributes to persistence in the current account in the sense of making oscillatory convergence less likely.[7] The intuition is as follows: Capital accumulation increases demand for both traded and nontraded goods, and hence exerts an upward pressure on the real exchange rate q (since the supply of the nontraded good is fixed) and in turn on the price index. The larger the budget share for nontraded goods, the more sensitive is the price index to the real exchange rate and hence to capital accumulation. Large adjustments in the price index imply that real expenditure need not adjust rapidly and hence cycling is less likely.

To conclude, consider the effects of a once-and-for-all increase (from time 0) in

[7] We can also show that a larger δ increases the speed of convergence to the steady state and, when convergence is cyclical, reduces the period of cycles.

government expenditure on the traded good, modeled in the usual way as a lump sum tax with no offsetting benefits to utility. The long run effects are similar to those in models with Uzawa utility (see Penati 1987). Namely, the steady state levels of consumption of both traded and nontraded goods and of the real exchange rate do not change, but the external balance of the country is improved because consumers accumulate foreign assets to finance expenditure.

However, the transition of the economy, in particular the adjustment of the current account, can be very different in our model. Although the country experiences a current account surplus at the beginning of the transition, a deficit can emerge later. When the adjustment path is cyclical, in particular, the current account changes positions many times between a surplus and a deficit. As the cycling continues, the real exchange rate undergoes various periods of appreciation and depreciation, and such appreciation or depreciation accompanies a current account surplus at one time and a deficit at another. (One can verify the behaviour of the current account and the real exchange rate by solving them as functions of (x, z), as in Part 4 of Section 2 of the Appendix.) In contrast, in models employing the Uzawa utility function, an increase in government expenditure on the traded good generates a current account surplus and appreciation in the real exchange rate during the transition.

5. MONEY GROWTH AND CYCLICAL DYNAMICS

In this section, we investigate whether cyclical current accounts can be optimal responses to international investment opportunities. The conventional model (e.g., Obstfeld 1981) implies monotonic current accounts and also, in apparent contradiction to empirical evidence (Sachs 1981), that the exchange rate's depreciation falls short of the rate of monetary growth if and only if the current account is in surplus. Our model is consistent with cycles in the current account and exchange rate and thus also with nonstandard correlations between these variables.

5.1. *The Model.* The model is a variation of the real model of the last section in which the nontraded good d is replaced by domestic real money balances m. The traded good f is numeraire and the domestic nominal price is ξ, which can also be interpreted as the reciprocal nominal exchange rate. Let $\pi \equiv \dot{\xi}/\xi$ be the inflation rate. Then the opportunity cost of holding real balances, or the relative price of m, is $q \equiv r + \pi$. Otherwise, the notation and utility specification are as above. In particular, we assume that at any instant consumption of the traded good and real money balances may be aggregated into "real expenditure" c which has price p satisfying (4.5).

A simplifying assumption is adopted. We assume the Cobb-Douglas form for the aggregator u, i.e., take $\nu = 0$ in (4.11) to obtain

$$c = u(f, m) = [\delta^{\delta}(1-\delta)^{1-\delta}]^{-1} f^{\delta} m^{1-\delta} \quad \text{and} \tag{5.1}$$

$$p = e(q) = q^{1-\delta}. \tag{5.2}$$

The representative agent's optimization problem is

$$\max U$$
$$\text{subject to (5.2)}, \quad \dot{w} = rw + L - pc,$$
$$w(0) = w_0 > 0 \quad \text{given}, \quad \lim_{t \to \infty} e^{-rt} w(t) \geq 0,$$

where L is a lump-sum transfer and w denotes total real wealth. An optimal program that also clears the money market and corresponds to government budget balance defines an *equilibrium*. Since foreigners do not hold domestic money, market clearing requires that

$$\dot{m}/m = \dot{M}/M - \pi = \gamma - \pi, \tag{5.3}$$

where γ is the constant rate of growth of money supply and M is the nominal stock of money. Secondly, lump-sum transfers are financed by money growth,

$$L = m\dot{M}/M. \tag{5.4}$$

5.2. *Dynamics*. The dynamic system that characterizes equilibrium is described in the Appendix. The existence of a unique steady state is easily proven as in earlier models if $\gamma + r > 0$. However, the dynamic system is larger (it is 6-dimensional) than those studied above because the inflation rate is determined endogenously. Thus the analysis of local dynamics is more complicated. Nevertheless, adequate mathematical machinery exists in the form of Routh's criterion (Gantmacher 1964) for local stability and Marden (1966, Theorem 41.2) p. 191, for determining whether the path to the steady state involves cycling. Roughly speaking, Marden describes a procedure for counting the number of zeros of a (characteristic) polynomial in any given sector of the complex plane.

The dynamic system has two predetermined variables, habits z and wealth. It is convenient to work with net claims on foreign assets x, rather than total wealth, where w and x satisfy

$$w = y/r + x + m;$$

y denotes the constant domestic output of the traded good. Making use of (5.1) through (5.4) we can derive the law of motion

$$\dot{x} = rx + y - f, \quad x(0) = x_0 \quad \text{given}, \tag{5.5}$$

and we can work with the state variables z and x.

The existence and uniqueness of the steady state can be established readily as before. Local stability of the dynamic system is shown in the Appendix. For local cycling, the following theorem is established.

THEOREM 2. *There exists a pair of values for* σ, $\sigma^{(1)}$ *and* $\sigma^{(2)}$ $(0 < \sigma^{(1)} \leq \sigma^{(2)})$, *such that the equilibrium dynamic system is cyclical for* $\sigma < \sigma^{(1)}$ *and noncyclical for* $\sigma > \sigma^{(2)}$. *Moreover, for each* $\sigma \in (\sigma^{(3)}, \infty)$, *where*

$$\sigma^{(3)} \equiv (r + \sqrt{r^2 + 16r\delta\beta^{-1}(r)\beta'/(1-\delta)})/2,$$

the system is cyclical for γ sufficiently large and noncyclical for γ sufficiently near $-r$.

As in previous models, sufficiently persistent habits (small σ) cause cycling and sufficiently low persistence (large σ) leads to monotonic adjustment.[8] A feature that is specific to the model in this section and that differentiates it from models based on Uzawa utility (Obstfeld 1981), is that the money growth rate can affect the qualitative dynamics of the system, a strong form of nonneutrality of monetary policy. For instance, when habit persistence is sufficiently low, high (low) money growth rates are associated with cyclical (monotonic) adjustment to the steady state.

Another feature we emphasize here is the equilibrium relation between the current account, $\dot{x}$, and the rate of growth of real money balances. In a model with Uzawa utility, Obstfeld (1981) shows that when the current account is in a surplus ($\dot{x} > 0$), real balances must be rising and the nominal exchange rate must be depreciating at a rate lower than the money growth rate ($\pi < \gamma$), and vice versa. This fundamental relationship generally fails in the current model with $\sigma < \infty$. To see this, note that the equilibrium level of real money balances m depends on both the external balance x and habits z, i.e., $m = m(x, z)$. Thus

$$\dot{m} = m_x \dot{x} + m_z \dot{z}.$$

Clearly, therefore if $m_z \neq 0$, then $\dot{m}$ and $\dot{x}$ have opposite signs for suitable magnitudes of x and z relative to their steady state values. Finally, we show in the Appendix that m_z cannot vanish if the adjustment towards the steady state is cyclical.

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APPENDIX

Section 1. Marden's Method. Consider a linear differential equation

$$\dot{X} = JX \tag{A1.1}$$

where J is an $n \times n$ matrix with a real characteristic polynomial

$$f(\theta) \equiv \det(\theta I - J) = \sum_{j=0}^{n} a_j \theta^j, \ a_n \neq 0. \tag{A1.2}$$

Without loss of generality, let $a_n = 1$. To examine stability of (A1.1), we have to know the number of stable and unstable roots of $f(\theta)$; to examine cycling of (A1.1), we have to know the number of stable complex roots of $f(\theta)$. Routh's criterion helps us in the first task and we refer to Gantmacher (1964, pp. 173–250) for a

[8] The Appendix also shows that the dynamic system is cyclical when $\beta \to \infty$ and noncyclical when $\beta \to 0$, in common with the model of Section 3.

description. Marden's method helps us in the second and here we outline a specialization for determining the number of roots on the nonnegative real line.

Let $\theta = \nu \exp(\hat{\imath}\Phi)$, where $\hat{\imath} \equiv (-1)^{1/2}$. Write

$$f(\nu e^{\hat{\imath}\Phi})/(a_n e^{\hat{\imath}n\Phi}) \equiv F_0(\nu, \Phi) + \hat{\imath}\Phi F_1(\nu, \Phi).$$

When $\Phi \to 0$, F_0 and F_1 approach the following:

$$F_0(\nu) = a_0 + a_1\nu + \cdots + a_{n-1}\nu^{n-1} + \nu^n \tag{A1.3}$$

$$F_1(\nu) = -(na_0 + (n-1)a_1\nu + \cdots + a_{n-1}\nu^{n-1}).$$

The Sturm sequence of polynomials generated from (F_0, F_1), denoted $\{F_0(\nu), \ldots, F_K(\nu)\}$, is given by the following division algorithm:

$$F_{k-1}/F_k = G_k - F_{k+1}/F_k, \; k = 1, \ldots, K-1, \tag{A1.4}$$

where F_K is the polynomial with the lowest order which can be generated by (A1.4). We shall assume that F_0 and F_1 have no common divisor so that F_K is a constant polynomial. (If F_0 and F_1 have a common divisor, first factorize $f(\theta)$ and then apply Marden's method to the factors separately.) Without loss of generality, let $K = n$. Denote by $V(\nu)$ the sequential sign variations (given ν) of the Sturm sequence generated from (F_0, F_1). The following results are immediate implications of Marden (1966, Theorem (41.2), p. 191).

THEOREM A1.1. *Let the polynomial $f(\theta)$ have N ($\leq n$) zeros on the nonnegative real line. Then*

$$N = V(0) - V(\infty). \tag{A1.5}$$

THEOREM A1.2. *Let $f(\theta)$ and N be as in Theorem A1.1, let $V(Q)$ be the number of sequential sign variations of a sequence Q and write*

$$F_k(\nu) = \sum_{j=0}^{n-k} a_{n-k,j}\,\nu^j, \qquad k = 0, 1, \ldots, n.$$

Then the a's are generated sequentially by the following formula: for $k = 1, \ldots, n-1$ and $j = 1, \ldots, n-k$,

$$a_{n-k-1,n-k-j} = c_k a_{n-k,n-k-1-j} + d_k a_{n-k,n-k-j} - a_{n-k+1,n-k-j}, \tag{A1.6}$$

where

$$c_k = a_{n-k+1,n-k+1}/a_{n-k,n-k} \quad \textit{and}$$

$$d_k = -\det\begin{bmatrix} a_{n-k+1,n-k+1} & a_{n-k+1,n-k} \\ a_{n-k,n-k} & a_{n-k,n-k-1} \end{bmatrix} \Big/ a^2_{n-k,n-k}.$$

Furthermore,

(A1.7) $N = V(a_{n,0}, a_{n-1,0}, \cdots, a_{0,0}) - V(1, a_{n-1,n-1}, \cdots, a_{0,0})$.

We can summarize the process of calculation as follows. Beginning with F_0 and F_1 in (A1.3), calculate c_1 and d_1, then $a_{n-2,j}$, using (A1.6). This creates polynomial $F_2(\nu)$. Continue the process until $a_{0,0}$ is obtained. This being done, (A1.7) gives the number of roots on the nonnegative real line.

Section 2. Proofs for Section 3. We divide the proof into six parts: 1. Describe the linearized dynamic system corresponding to (3.2) through (3.6) in Section 3 and find the eigenvalues of the matrix J in the linearized system. 2. Show that for all $\sigma > 0$ matrix J has exactly two stable roots (those with negative real parts) and three unstable roots (those with positive real parts). 3. Prove that the stable roots are complex if and only if $\sigma \in (0, \sigma_1)$. 4. Establish the properties of σ_1 stated in Theorem 1. 5. Specify a regularity condition (sufficient for local stability given 2) and show that it is satisfied if and only if $\sigma \neq \sigma_1$. 6. Derive the consumption function and show that it has the properties stated in Section 3. Parts 1 through 5 form a proof of Theorem 1.

Part 1. The linearized system and the eigenvalues. The linearized system is

(A2.1) $$(\dot{z}, \dot{\phi}, \dot{\psi}, \dot{c}, \dot{x})^T = J(z - z^*, \phi - \phi^*, \psi - \psi^*, c - c^*, x - x^*)^T,$$

where

$$J = \begin{bmatrix} -\sigma & 0 & 0 & \sigma & 0 \\ \nu\beta'/r & r & 0 & -\nu' & 0 \\ \psi\beta' & \beta' & r+\sigma & 0 & 0 \\ \nu'\beta'/\nu'' & -\sigma\beta'/\nu'' & -\sigma(r+\sigma)/\nu'' & 0 & 0 \\ 0 & 0 & 0 & -1 & r \end{bmatrix}.$$

All variables in this matrix are evaluated at the steady state and asterisks are suppressed.

Obviously, r is an eigenvalue of J with multiplicity one. Other eigenvalues of J, denoted by θ, are those of the upper-left four by four sub-matrix in J. The characteristic polynomial of this sub-matrix is

$$f(\theta = [\theta(\theta - r)]^2 - \Gamma\theta(\theta - r) + A,$$

where

$$\Gamma = \sigma(\sigma + r + 2v'\beta'/v'') \quad \text{and} \quad A = -\sigma(\beta'/v'')[(rv' - v\beta')\sigma + r^2 v'].$$

Thus the eigenvalues θ ($\neq r$) of J are

(A2.2) $$\theta_{ij} = \frac{r \pm (r^2 + 4k_i)^{1/2}}{2}, \qquad i = 1, 2; j = 1, 2$$

where

$$k_{1,2} = \frac{\Gamma \pm (\Gamma^2 - 4A)^{1/2}}{2}. \tag{A2.3}$$

Part 2. The numbers of stable and unstable roots. We can directly examine the eigenvalues in (A2.2). However, for larger dynamic systems, such as the one in Section 5, it may be difficult to solve for the eigenvalues. To unify arguments in the Appendix, we use Routh's criterion to identify the numbers of stable and unstable roots.

LEMMA A2.1. *For all $\sigma > 0$, matrix J in Part 1 has exactly two stable and three unstable roots.*

PROOF. Since r (> 0) is a root of J, we only have to show that polynomial $f(\theta)$ in Part 1 has exactly two stable and two unstable roots. Applying Routh's criterion reveals that the number of unstable roots equals the sequential sign variations of the following sequence:

$$1,\ -2r,\ r^2 - \Gamma/2,\ \frac{2rA}{r^2 - \Gamma/2} + r\Gamma,\ A. \tag{A2.4}$$

It can be verified that (A2.4) has two sequential sign variations. Q.E.D.

Part 3. The existence of σ_1. The dynamic system exhibits cycling if the two stable roots of J are complex conjugates. Again one can determine the conditions for complex roots from (A2.2), but we choose to use Marden's method for the same reason as stated in Part 2.

LEMMA A2.2. *The two stable roots of J are complex conjugates if and only if $\Gamma^2 - 4A < 0$.*

PROOF. Equivalently, we show that the two unstable roots of $f(-\theta)$ are complex if and only if $\Gamma^2 - 4A < 0$. Applying Theorem A1.2 to $f(-\theta)$ gives us the number N of unstable real roots of $f(-\theta)$. Thus the current lemma can be restated as $N = 0 \Leftrightarrow \Gamma^2 - 4A < 0$.

We compute F_0 and F_1 in (A1.3) and form the first two rows of the algorithm in Theorem A1.2, yielding

$$F_0 = A - r\Gamma\nu + (r^2 - \Gamma)\nu^2 + 2r\nu^3 + \nu^4$$

$$F_1 = -4A + 3r\Gamma\nu - 2(r^2 - \Gamma)\nu^2 - 2r\nu^3, \text{ and}$$

$$1,\ 2r,\ r^2 - \Gamma,\ -r\Gamma,\ A$$

$$-2r,\ -2(r^2 - \Gamma),\ 3r\Gamma,\ -4A.$$

That is, $a_{4,4} = 1 > 0$, $a_{4,0} = A > 0$, $a_{3,3} = -2r < 0$ and $a_{3,0} = -4A < 0$. Further calculation by (A1.6) shows that $a_{0,0} > 0$ and

$$a_{2,2} = -\Gamma(1 + 2\Gamma/r^2)/2,\ a_{2,0} = A(1 + 2\Gamma/r^2),$$

$$a_{1,1} = \frac{\Gamma^2 - 4A}{\Gamma^2(1 + 2\Gamma/r^2)^2}[r\Gamma + 4(\Gamma^2 - 2A)/r],\ a_{1,0} = \frac{-4A(\Gamma^2 - 4A)}{\Gamma^2(1 + 2\Gamma/r^2)}.$$

$\Gamma^2 - 4A \geq 0 \Rightarrow N = 2$: If $\Gamma^2 - 4A > 0$, the remark following Lemma A2.3 (below) implies $\Gamma > 0$, and hence implies $a_{2,2} < 0$, $a_{2,0} > 0$, $a_{1,1} > 0$ and $a_{1,0} < 0$. Thus $N = 2$ by (A1.7). If $\Gamma^2 - 4A = 0$, $f(-\theta)$ has a root with multiplicity two and again one can show that $N = 2$.

$\Gamma^2 - 4A < 0 \Rightarrow N = 0$: If $\Gamma^2 - 4A < 0$, then $a_{2,0}$ and $a_{1,0}$ have the same sign. Therefore, $V(a_{4,0}, a_{3,0}, a_{2,0}, a_{1,0}, a_{0,0}) = 2$. Since $V(a_{4,4}, a_{3,3}, a_{2,2}, a_{1,1}, a_{0,0}) \geq 2$, then $N \leq 0$ by (A1.7). However it is impossible that $N < 0$. Hence $N = 0$. This completes the proof of Lemma A2.2. Q.E.D.

Next show that there exists σ_1 such that $\Gamma^2 - 4A < 0 \Leftrightarrow \sigma \in (0, \sigma_1)$. Calculate

$$\Gamma^2 - 4A = \sigma[\sigma h(\sigma) + 4r^2 v'\beta'/v'']$$

where

$$h(\sigma) \equiv (\sigma + r + 2v'\beta'/v'')^2 + 4\beta'(rv' - v\beta')/v''.$$

It can be verified that h assumes a global minimum at σ^*, where

$$\sigma^* \equiv -r - 2v'\beta'/v'' \text{ and } h(\sigma^*) = 4\beta'(rv' - v\beta')/v'' < 0.$$

LEMMA A2.3. *There exists* $\sigma_1 > \max(0, \sigma^*)$ *such that*

$$\Gamma^2 - 4A < 0 \Leftrightarrow \sigma \in (0, \sigma_1).$$

PROOF. For $\sigma \in (0, \infty)$, $\Gamma^2 - 4A < 0 \Leftrightarrow$

$$h(\sigma) < -4r^2 v'\beta'/(\sigma v'').$$

We briefly call the right-hand side of this inequality RHS (σ) and point out the following properties:

(1) RHS $(\sigma) > 0\ \forall \sigma \in (0, \infty)$, $\lim_{\sigma \to 0}$ RHS $(\sigma) = \infty$, $\lim_{\sigma \to \infty}$ RHS $(\sigma) = 0$;

(2) RHS (σ) is strictly decreasing, strictly convex in $(0, \infty)$.

Since $h(\sigma)$ is quadratic, there exists at least one positive solution denoted σ_1, to

$$h(\sigma_1) = -4r^2 v'\beta'/(\sigma_1 v'').$$

We show that σ_1 is unique. Uniqueness is clear when $\sigma^* \leq 0$. When $\sigma^* > 0$, it suffices to show that $h(0) \leq 0$. However, one can verify that $h(0)$ is decreasing in r so that by Assumption 1

$$h(0) < h(0)|_{r=0} = 4\beta'^2(v'^2 - vv'')/v''^2 \leq 0.$$

It is clear that $\sigma_1 > \sigma^*$. This completes the proof of Lemma A2.3. Q.E.D.

REMARK. The above proof has shown that $\Gamma > 0 \Leftrightarrow \sigma > \sigma^*$. Thus

$$\Gamma^2 - 4A \geq 0 \Rightarrow \Gamma > 0.$$

Part 4. The properties of σ_1. One can directly verify the properties of σ_1 stated in Theorem 1.

Part 5. The regularity condition. The regularity condition for local stability requires that the projection of the stable manifold of the linearized system onto the (x, z) plane coincide with that plane (see Scheinkman 1976, p. 20, for example). This condition allows one to deduce the local stability of both the system (3.2) through (3.6) and its linearization. Since the eigenvectors corresponding to the negative eigenvalues of J span the stable manifold, the regularity condition can be specified via these eigenvectors. Denote the negative eigenvalues of J by θ_{12} and θ_{22}, and the corresponding eigenvectors by $Y^i = (y_1(\theta_{i2}), \dots, y_s(\theta_{i2}))^T$. The regularity condition is the following.

REGULARITY. The following 2×2 matrix is nonsingular:

$$\begin{bmatrix} y_1(\theta_{12}) & y_1(\theta_{22}) \\ y_5(\theta_{12}) & y_5(\theta_{22}) \end{bmatrix}.$$

With matrix J, one can verify that this is equivalent to

$$\theta_{12} \neq \theta_{22} \Leftrightarrow k_1 \neq k_2 (\text{by (A2.2)}) \Leftrightarrow \Gamma^2 - 4A \neq 0 (\text{ by (A2.3)})$$

$$\Leftrightarrow \sigma \neq \sigma_1 \text{ (by Lemma A2.3)}.$$

Therefore the regularity condition is satisfied if and only if $\sigma \neq \sigma_1$. This completes part 5 and the proof of Theorem 1.

Part 6. The consumption function. Let $\theta_{i2} (i = 1, 2)$ be the two stable roots of the system and $Y^i \equiv (y_1(\theta_{i2}), \dots, y_5(\theta_{i2}))^T$ the corresponding eigenvectors of J. The stable manifold is characterized by

$$[z(t) - z^*, \dots, c(t) - c^*, x(t) - x^*]^T = (Y^1, Y^2) \begin{bmatrix} s_1 \exp(\theta_{12} t) \\ s_2 \exp(\theta_{22} t) \end{bmatrix}$$

where s_1 and s_2 are constants which, under the regularity condition, are uniquely determined by the initial condition $(z(0), x(0))$. Thus optimal consumption can be expressed as a function of $(z(0), x(0))$. This permits derivation of

$$\partial c/\partial x = (r - \theta_{12})(r - \theta_{22})/(r + \sigma) \text{ and } \partial c/\partial z = (\theta_{12} + \sigma)(\theta_{22} + \sigma)/[\sigma(r + \sigma)].$$

Obviously $\partial c/\partial x > 0$ and hence consumption is normal. Also, we can show that $\partial c/\partial z > 0$ using (A2.2):

$$\partial c/\partial z > 0 \Leftrightarrow (\theta_{12} + \sigma)(\theta_{22} + \sigma) > 0$$

$$\Leftrightarrow \sigma^2(r + \sigma)^2 + k_1 k_2 - \sigma(r + \sigma)(k_1 + k_2) > 0$$

$$\Leftrightarrow \sigma^2(r + \sigma)^2 + A - \sigma(r + \sigma)\Gamma > 0$$

$$\Leftrightarrow (r + \sigma)(r + 2\sigma)v'\beta' - \sigma v \beta'^2 > 0,$$

which is true by Assumption 1.

Section 3. Proofs for Section 5. We proceed in three stages: 1. Describe the dynamic system in Section 5 and show that on the equilibrium path, $m_z \neq 0$ if the path is cyclical (see the discussion at the end of Section 5). 2. Prove local stability of the system. 3. Prove Theorem 2.

Part 1. The dynamic system. As in the analysis of the consumer's maximization problem (P) from Section 3, q obeys the law of motion (4.8). Since $m = (1 - \delta)pc/q = (1 - \delta)c/q^{-\delta}$, (4.8) and (5.3) imply

$$\dot{c}/c = r + \gamma - q - \frac{1}{\kappa}(\rho - r). \tag{A3.1}$$

Also $\kappa = 1/\delta - 1$ (let $\nu \rightarrow 0$ in (4.12)). The dynamic system consists of (4.8), (A3.1), (5.5), the appropriate forms of (3.2), (3.4) and (3.5) with initial conditions and convergence to the steady state.

Let $\omega = c - c^* - (c^*/q^{*\delta})(q - q^*)$ and $\omega^* = 0$. The linearized system is

(A3.2) $$(\dot{z}, \dot{\phi}, \dot{\omega}, \dot{\psi}, \dot{q}, \dot{x})^T = J(z - z^*, \phi - \phi^*, \omega - \omega^*, \psi - \psi^*, q - q^*, x - x^*)^T,$$

where J is a 6×6 matrix:

$$J = \begin{bmatrix} -\sigma & 0 & \sigma & 0 & \sigma\delta z/q & 0 \\ \phi\beta' & r & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -z & 0 \\ \psi\beta' & \beta' & 0 & r + \sigma & 0 & 0 \\ 0 & \dfrac{q\beta'}{(1-\delta)\psi} & 0 & \dfrac{q(r+\sigma)}{(1-\delta)\psi} & 0 & 0 \\ 0 & 0 & -\delta q^{1-\delta} & 0 & -\delta z/q^{\delta} & r \end{bmatrix}.$$

All variables in the matrix take their steady state values and asterisks are suppressed.

Before examining local stability, we show that $m_z \neq 0$ if the equilibrium path is cyclical. Since x and z are the only exogenous state variables, the equilibrium values of q and f can be expressed by means of functions $q(x, z)$ and $f(x, z)$. Since a one-dimensional differential equation cannot produce cycles, we deduce from (5.5) that $f_z \neq 0$.

Intratemporal optimality and Cobb-Douglas atemporal utility imply that $f = \delta c q^{1-\delta}$. Combining this relation with the analogue of (4.8) and equation (A3.1) yields

$$\dot{f}/f = r + \gamma - q + \delta^{-1}\dot{q}/q.$$

It follows from $f_z \neq 0$ that $q_z \neq 0$. Finally, $\dot{m}/m = \gamma - q + r$ implies $m_z \neq 0$.

Part 2. Local stability. We show that J has the proper number of stable eigenvalues, and hence the dynamic system described above is locally stable under a regularity condition (see Section 2 of the Appendix for a description of the regularity condition).

LEMMA A3.1. *Matrix J has two stable roots and four unstable roots.*

PROOF. Apparently, r (> 0) is an eigenvalue of J. Other eigenvalues of J are the roots of polynomial $f(\theta)$:

$$f(\theta) = \theta^5 - 2r\theta^4 + a_2\theta^3 + a_3\theta^2 + a_4\theta + a_5 \tag{A3.3}$$

where $a_2 = r^2 - s$, $a_3 = rs > 0$, $a_4 = \delta D > 0$, $a_5 = -qD < 0$, $s \equiv \sigma(r + \sigma)$ and $D \equiv rsz\beta'/(1 - \delta)$. (To economize on notation, we use the same notation as in Section 2 of this Appendix.)

Routh's criterion can be verified to show that $f(\theta)$ has two stable and three unstable roots. Q.E.D.

Part 3. The occurrence of cycling: Proof of Theorem 2. We apply Theorem A1.2 to $-f(-\theta)$ to find N, the number of roots on the nonnegative real line ($f(\theta)$ is given in (A3.3)). The dynamic system is cyclical if $N = 0$ and noncyclical if $N = 2$. The first two rows of the algorithm in Theorem A1.2 are

$$1,\ 2r,\ a_2,\ -a_3,\ a_4,\ -a_5$$

$$-2r,\ -2a_2,\ 3a_3,\ -4a_4,\ 5a_5,\ 0.$$

That is, $a_{5,5} = 1 > 0$, $a_{5,0} = -a_5 > 0$; $a_{4,4} = -2r < 0$ and $a_{4,0} = 5a_5 < 0$. Further calculation by (A1.6) shows that $a_{3,3} < 0$, and $a_{3,0} > 0$. Thus (A1.7) implies

$$N = 1 + V(a_{3,0},\ a_{2,0},\ a_{1,0},\ a_{0,0}) - V(a_{3,3},\ a_{2,2},\ a_{1,1},\ a_{0,0}). \tag{A3.4}$$

The signs of $a_{j,0}$ and $a_{j,j}$ ($j = 0, 1, 2$) are in general ambiguous, so we proceed under additional assumptions on the parameters σ and γ.

(1) The dependence of N on σ: Let $\sigma \to \infty$, one can show that $a_{2,2} > 0$, $a_{2,0} < 0$, $a_{1,1} > 0$ and $a_{1,0} > 0$. Thus $N = 2$ by (A3.4). Next we show that $N = 0$ when $\sigma \to 0$. If $\sigma \to 0$, (A1.6) implies $a_{0,0} > 0$ and

$$\operatorname{sgn}(a_{2,2}) = \operatorname{sgn}\,(5qD/r + 16(\delta D/r)^2/s - 2\delta D),$$

$$\operatorname{sgn}(a_{1,1}) = \operatorname{sgn}\,\{4(\delta D)^3 + qD[\delta D(3rs + 10\delta D/r)$$
$$+ qD(35\delta D/r^2 - 3s - (5/r)^3 qD/2)]\},$$

$$\operatorname{sgn}\,(a_{1,0}) = \operatorname{sgn}\,\{qD[6\delta D - 25qD/(2r)] + 2r(\delta D)^2\}.$$

If $q \geq r\delta$, then $a_{2,2} > 0$, $a_{1,1} < 0$, $a_{1,0} < 0$ and hence $N = 0$ by (A3.4). Let $q < r\delta$. One can show that $a_{1,0} \geq 0 \Rightarrow a_{1,1} > 0$. If $a_{1,0} > 0$ then $N = 0$ by (A3.4); if $a_{1,0} = 0$, then (A1.6) implies $a_{0,0} = -a_{2,0} < 0$ and again $N = 0$. The only case left is $a_{1,0} < 0$. However, $a_{1,0} < 0 \Leftrightarrow q > 2r\delta/5 \Rightarrow a_{2,2} > 0$. As a result, $d_4 > 0$ (see Theorem A1.2 for the calculation of d_4) and $a_{0,0} = d_4\ a_{1,0} - a_{2,0} < 0$. Again $N = 0$ by (A3.4). That is, the system is cyclical when $\sigma \to 0$.

(2) The dependence on γ: The parameter γ enters the a's in (A3.4) only through the

steady state value of q $(= r + \gamma)$. If $\gamma \to \infty$ $(q \to \infty)$, then $a_{2,2} > 0$, $a_{1,1} < 0$ and $a_{1,0} < 0$ hence $N = 0$. If $\gamma \to -r$ $(q \to 0)$, then $a_{1,1} > 0$, $a_{0,0} < 0$ and

$$\operatorname{sgn}(a_{2,0}) = \operatorname{sgn}[8\delta D(3 + 5s/r^2) - s^2(7 + 12s/r^2)],$$

$$\operatorname{sgn}(a_{2,2}) = \operatorname{sgn}[(4\delta D - s^2)(2\delta D/r^2 - s(1 + 4s/r^2)/4)],$$

$$\operatorname{sgn}(a_{1,0}) = \operatorname{sgn}\{(4\delta D - s^2)[40(\delta D/r)^2 - r^2\delta D(12s^2/r^4 - 13s/r^2 - 3) - r^2s^2(1 + 4s/r^2)]\}.$$

If $4\delta D - s^2 < 0$, i.e., $\sigma > \sigma^{(3)}$ (see Theorem 2 for the definition of $\sigma^{(3)}$), then $a_{2,0} < 0$, $a_{0,0} > 0$ and hence $N = 2$. Therefore, in the region $\sigma \in (\sigma^{(3)}, \infty)$, the system is cyclical if $\gamma \to \infty$ and noncyclical if $\gamma \to -r$.

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