Explaining Asset Pricing Puzzles Associated with the 1987 Market Crash *

Luca Benzoni† Pierre Collin-Dufresne‡ Robert S. Goldstein§

January 25, 2010

Abstract

The 1987 market crash was associated with minimal changes in observable macro-economic fundamentals. Yet, subsequently, the implied volatility curve for equity index options dramatically and permanently steepened. We explain this evidence in a general equilibrium framework in which expected growth in fundamentals and economic uncertainty are subject to jumps. The jumps are rare events which occur with a frequency unknown to agents. The 1987 crash and steepening in smile can be explained by such a (small) jump and the associated updating of beliefs about the probability of future jumps. Our framework also simultaneously captures salient features of individual option prices, stock returns, and interest rates.

*We thank Raj Aggarwal, Gordon Alexander, Mariacristina De Nardi, Darrell Duffie, Monika Piazzesi, Mark Rubinstein, Costis Skiadas, Stanley Zin, and seminar participants at the March 2006 NBER Asset Pricing Meeting, Chicago, the 2006 Western Finance Association conference, and the 2007 Econometric Society conference for helpful comments and suggestions. All errors remain our sole responsibility. Previous versions of this paper circulated under the title "Can Standard Preferences Explain the Prices of Out-of-the-Money S&P 500 Put Options?".

†Benzoni is at the Federal Reserve Bank of Chicago.

‡Collin-Dufresne is at the Columbia Business School and NBER.

§Goldstein is the corresponding author. He is at the Carlson School of Management, University of Minnesota, 321 19th Ave S., Minneapolis, MN 55455, phone 612-624-8581, fax 612-626-1335, golds144@umn.edu, and NBER.

Explaining Asset Pricing Puzzles Associated with the 1987 Market Crash

Previously entitled, "Can Standard Preferences Explain the Prices of Out-of-the-Money S&P 500 Put Options?"

Abstract

The 1987 market crash was associated with minimal changes in observable macro-economic fundamentals. Yet, subsequently, the implied volatility curve for equity index options dramatically and permanently steepened. We explain this evidence in a general equilibrium framework in which expected growth in fundamentals and economic uncertainty are subject to jumps. The jumps are rare events which occur with a frequency unknown to agents. The 1987 crash and steepening in smile can be explained by such a (small) jump and the associated updating of beliefs about the probability of future jumps. Our framework also simultaneously captures salient features of individual option prices, stock returns, and interest rates.

Key words. Volatility Smile; Volatility Smirk; Implied Volatility; Option Pricing; Portfolio Insurance; Market Risk.

JEL Classification Numbers. G12, G13.

1 Introduction

The 1987 stock market crash has generated many puzzles for financial economists. In spite of little change in observable macroeconomic fundamentals, market prices fell 20-25% and interest rates dropped about 1-2%. Moreover, the crash triggered a permanent shift in index option prices: Prior to the crash, implied 'volatility smiles' for index options were relatively flat. Since the crash, however, the Black-Scholes formula has been significantly underpricing short-maturity, deep out-of-the-money S&P 500 put options (Rubinstein (1994), Bates (2000)). This feature, often referred to as the 'volatility smirk,' is demonstrated in Figure 1, which shows the spread of both in-the-money (ITM) and out-of-the-money (OTM) implied volatilities relative to at-the-money (ATM) implied volatilities from 1985-2006. This figure clearly shows that on October 19, 1987 the volatility smirk spiked upward, and that this shift has remained ever since.

Not only is this volatility smirk puzzling in its own right, but it is also difficult to explain relative to the shape of implied volatility functions (IVF) for individual stock options, which are much flatter and more symmetric (see, e.g., Bollen and Whaley (2004), Bakshi, Kapadia, and Madan (2003), and Dennis and Mayhew (2002)). Indeed, Bollen and Whaley (2004) argue that the difference in the implied volatility functions for options on individual firms and on the S&P 500 index cannot be explained by the differences in their underlying asset return distributions.

In this paper, we attempt to explain these puzzles while simultaneously capturing other salient features of asset prices. In particular, we examine a representative-agent general equilibrium endowment economy that can simultaneously explain:

- The prices of deep OTM put options for both individual stocks and the S&P 500 index;
- Why the slope of the implied volatility curve changed so dramatically after the crash;
- Why the regime shift in the volatility smirk has persisted for more than twenty years;
- How the market can crash with little change in observable macroeconomic variables.

We build on the long-run risk model of Bansal and Yaron (2004, BY), who show that if agents have a preference for early resolution of uncertainty (e.g., have Kreps-Porteus/Epstein-Zin (E/Z) preferences with elasticity of intertemporal substitution EIS > 1), then persistent shocks to the expected growth rate and volatility of aggregate consumption will be associated with large risk premia in equilibrium. Their model is able to explain a high equity premium, low interest rates, and low interest rate volatility while matching important features of aggregate consumption and dividend time series. We extend their model in two dimensions. First, we add a jump component to the shocks driving expected consumption growth rate and consumption volatility. These jumps (typically downward for expected growth rates and upward for volatility) are bad news for the agent with E/Z preferences, who will seek to reduce her position in risky assets. In equilibrium, this reduction in demand leads to asset prices exhibiting a downward jump, even though aggregate consumption and dividend are smooth. That is, in our model, the *level* of consumption and dividends follows a continuous process; it is their expected growth rates and volatilities that jump. Since shocks to expected consumption growth rate and consumption volatility are associated with large risk premia, jumps in asset prices can be substantial, akin to market 'crashes.'

Our second contribution relative to BY is to allow for parameter uncertainty and learning. Specifically, we assume the jump frequency is governed by a hidden two-state continuous Markov chain, which needs to be filtered in equilibrium. This adds another source of risk to the economy, namely the posterior probability of the hidden state. We show that the risk-premium associated with revisions in posterior beliefs about the hidden state can be large, as they are a source of 'long-run risk.' In fact, we show that it can explain the dramatic shift in the shape of the implied volatility skew observed in 1987. If, prior to 1987, agents' beliefs attribute a very low probability to high jump intensities then, prior to 1987, prices mostly correspond to a no-jump Black-Scholes type economy. However, after a jump in prices occurs as a result of the jump in expected growth rates and volatility of fundamentals, agents update their beliefs about the likelihood (i.e., intensity) of future jumps occurring, which contributes to the severity of the market crash and leads to the steep skew in implied volatilities observed in the data henceforth. Because these beliefs are very persistent, the skew is long-lived after the crash.

Although the two new features, jumps and learning, dramatically impact the prices of options, we show that our model still matches salient features of the US economy in terms of quantities (e.g., dividend and consumption data) and prices (e.g., equity premium, low interest rate level and volatility). Because jumps impact the expected consumption growth rate and consumption volatility, but not the level of consumption, the consumption process remains smooth in our model, consistent with the data. Further, as noted by BY and Shephard and Harvey (1990), it is very difficult to distinguish between a purely i.i.d. process and one which incorporates a small persistent component. Indeed, we show that the dividend and consumption processes implied by the model fit the properties of the data well, in that we cannot reject the hypothesis that the observed data was generated from our model.

Nonetheless, and as in BY, the asset pricing implication of our model differ significantly from those of an economy in which dividends are i.i.d.. Specifically, the calibrated model matches the typical level of the price-dividend ratio and produces reasonable levels for the equity premium, the risk-free rate, and their standard deviation. In the same calibration we show that the pre-crash implied volatility function for short-maturity index options is nearly flat, while it becomes a steep smirk immediately after the crash. Moreover, the model predicts a downward jump in the risk free rate during the crash event, consistent with observation.

Finally, the model reproduces the stylized properties of the implied volatility functions for individual stock option prices. We specify individual firm stock dynamics by first taking our model for the S&P 500 index and then adding on idiosyncratic shocks, both of the diffusive and the jump types. We then calibrate the coefficients of the idiosyncratic components to match the distribution of returns for the 'typical' stock. In particular, we match the cross-sectional average of the high-order moments (variance, skewness, and kurtosis) for the stocks in the Bollen and Whaley (2004) sample. We simulate option prices from this model and compute Black-Scholes implied volatilities across different moneyness. Consistent with the evidence in Bollen and Whaley (2004), Bakshi, Kapadia, and Madan (2003), and Dennis and Mayhew (2002), we find an implied volatility function that is considerably flatter than that for S&P 500 options. Bakshi, Kapadia, and Madan (2003) conclude that the differential pricing of individual stock options is driven by the degree of skewness/kurtosis in the underlying return distribution in combination with the agent's high level of risk aversion. Here, we propose a plausible endowment economy that in combination with recursive utility yields predictions consistent with their empirical findings.

Related Literature: Motivated by the empirical failures of the Black-Scholes model in post-crash S&P 500 option data, prior studies have examined more general option pricing models (see, e.g., Bates (1996), Duffie et al. (2000), and Heston (1993)). A vast literature explores these extensions empirically,[1] reaching the conclusion that a model with stochastic volatility and jumps significantly reduces the pricing and hedging errors of the Black-Scholes formula.[2] These previous studies, however, focus on post-1987 S&P 500 option data. Further, they follow a partial equilibrium approach and let statistical evidence guide the *exogenous* specification of the underlying return dynamics.

Reconciling the findings of this literature in a rational expectations general equilibrium setting has proven difficult. For instance, Pan (2002) notes that the compensation demanded for the 'diffusive' return risk is very different from that for jump risk. Consistent with Pan's finding, Jackwerth (2000) shows that the risk aversion function implied by S&P 500 index options and returns post-1987 crash is partially negative and increasing in wealth (similar results are presented in Aït-Sahalia and Lo (2000) and Rosenberg and Engle (2000)). This evidence eludes the standard general equilibrium model with constant relative risk aversion utility and suggests that there may be a lack of integration between the option market and the market for the underlying stocks.

Several papers have investigated the ability of equilibrium models to explain post-1987 S&P 500 option prices. Liu, Pan, and Wang (2005, LPW) consider an economy in which the endowment is an i.i.d. process that is subject to jumps. They show that in this setting neither constant relative risk aversion nor Epstein and Zin (1989) preferences can generate a volatility smirk consistent with post-1987 evidence on S&P 500 options. They argue that in order to reconcile the prices of options and the underlying index, agents must exhibit 'uncertainty aversion' towards rare events that is different from the standard 'risk-aversion' they exhibit towards diffusive risk. This insight provides

[1] Among recent contributions, Bakshi et al. (1997, 2000), Bates (2000), and Huang and Wu (2004) focus on derivatives prices alone. Pan (2002), Broadie et al. (2007), Chernov and Ghysels (2000), Jones (2003), Eraker (2004), and Benzoni (2002) use data on both underlying and derivatives prices to fit the model.

[2] A related literature investigates the profits of option trading strategies (e.g., Coval and Shumway (2001) and Santa-Clara and Saretto (2009)) and the economic benefits of giving investors access to derivatives when they solve the portfolio choice problem (e.g., Constantinides et al. (2009), Driessen and Maenhout (2007), Liu and Pan (2003)).

a decision-theoretic basis to the idea of crash aversion advocated by Bates (2008), who considers an extension of the standard power utility that allows for a special risk-adjustment parameter for jump risk distinct from that for diffusive risk. These prior studies assume that the dividend *level* is subject to jumps, while the *expected* dividend growth rate is constant. Thus, in these models a crash like that observed in 1987 is due to a 20-25% downward jump in the dividend level.[3] Moreover, their model predicts no change in the risk free rate during the crash event. In our setting, it is the expected endowment growth rate that is subject to jumps. Thus, in our model dividends and consumption are smooth, and the market can crash with minimal change in observable macroeconomic fundamentals. Further, the risk free rate drops around crash events, consistent with empirical evidence.

Other studies explore the option pricing implications of models with state dependence in preferences and/or fundamentals; see, e.g., Bansal, Gallant, and Tauchen (2007), Bondarenko (2003), Brown and Jackwerth (2004), Buraschi and Jiltsov (2006), Chabi-Yo, Garcia, and Renault (2008), David and Veronesi (2002 and 2009), Garcia, Luger, and Renault (2001 and 2003). These papers do not study the determinants of stock market crashes, the permanent shift in the implied volatility smirk that followed the 1987 events, and the difference between implied volatility functions for individual and index stock options. To our knowledge, our paper is the first to focus on these issues.

Also related is a growing literature that investigates the effect of changes in investors' sentiment (e.g., Han (2008)), market structure, and net buying pressure (e.g., Bollen and Whaley (2004), Dennis and Mayhew (2002), and Gârleanu et al. (2009)) on the shape of the implied volatility smile. This literature argues that due to the existence of limits to arbitrage, market makers cannot always fully hedge their positions (see, e.g., Green and Figlewski (1999), Figlewski (1989), Hugonnier et al. (2005), Liu and Longstaff (2004), Longstaff (1995), and Shleifer and Vishny (1997)). As a result, they are likely to charge higher prices when asked to absorb large positions in certain option contracts. These papers, however, do not address why end users buy these options at high prices relative to the Black-Scholes value or why the 1987 crash changed the shape of the volatility smile so dramatically and permanently. Our paper offers one possible explanation.

Finally, the large impact that learning can have on asset price dynamics has been shown previously (e.g., David (1997), Veronesi (1999, 2000)). One important difference between these papers and ours is that our agent learns from jumps rather than diffusions, as in Collin-Dufresne et al. (2002), leading to different updating dynamics.

The main contribution of our paper is to explain pre- and post-1987 crash asset prices in a rational-expectation framework that is consistent with underlying fundamentals. However, to our knowledge this is also the first article to examine the effect of jumps in the Bansal and Yaron (2004) framework. This has proven to be a fruitful extension of the long-run risk framework, and has been

[3]Barro (2006) makes a similar assumption about output dynamics. His model captures the contractions associated with the Great Depression and the two World Wars, but it does not match the evidence around the 1987 crash, when the output level remained smooth.

further explored by, e.g., Drechsler and Yaron (2008), Eraker (2008), and Eraker and Shaliastovich (2008).

The rest of the paper proceeds as follows. In Section 2, we present the model and discuss our solution approach. Section 3.1 shows that the model matches the relevant asset pricing facts wile being consistent with underlying fundamentals. Section 4 concludes the paper.

2 The Model

We specify the dynamics for log-consumption ($c \equiv \log C$) and log-dividend ($\delta \equiv \log D$) as

$$
\begin{aligned}
dc &= \left(\mu_C + x - \frac{1}{2}\Omega\right) dt + \sqrt{\Omega}\, dz_C && (1)\\
d\delta &= \left(\mu_D + \phi\, x - \frac{1}{2}\sigma_D^2\Omega\right) dt + \sigma_D \sqrt{\Omega}\left(\rho_{DC} dz_C + \sqrt{1-\rho_{DC}^2} dz_D\right) && (2)\\
dx &= -\kappa_x x\, dt + \sigma_{xc}\sqrt{\Omega}\, dz_c + \sqrt{\sigma_{x0} + \sigma_{x\Omega}\,\Omega}\; dz_x + \widetilde{\nu}\, dN && (3)\\
d\Omega &= \kappa_\Omega \left(\overline{\Omega} - \Omega\right) dt + \sigma_\Omega \sqrt{\Omega}\left(\rho_{\Omega C}\, dz_C + \sqrt{1-\rho_{\Omega C}^2}\, dz_\Omega\right) + \widetilde{M}\, dN\,. && (4)
\end{aligned}
$$

These are continuous-time versions of the dividend and consumption dynamics considered in BY, except that the predictable dividend component x and the measure of economic uncertainty Ω are subject to jumps. The diffusive shocks $\{dz_C,\, dz_D,\, dz_x,\, dz_\Omega\}$ are uncorrelated Brownian motions, while jumps are governed by a Poisson process dN with $\text{Prob}(dN_t = 1|\mathcal{F}_t) = \lambda_t\, dt$. The jump intensity λ_t can take two possible values, λ^G and λ^B, and it transitions from one state to the other via the dynamics:

$$
\begin{aligned}
\pi\left(\lambda_{t+dt} = \lambda^B | \lambda_t = \lambda^G\right) &= \phi_{GB}\, dt\\
\pi\left(\lambda_{t+dt} = \lambda^G | \lambda_t = \lambda^B\right) &= \phi_{BG}\, dt. && (5)
\end{aligned}
$$

Following Eraker (2000), the jump size variables $\widetilde{\nu}$ and $\widetilde{M}$ are drawn from the distribution

$$
\pi\left(\widetilde{\nu} = \nu,\, \widetilde{M} = M\right) = \xi\, e^{-\xi M}\mathbf{1}_{\{M>0\}} \frac{1}{\sqrt{2\pi\sigma_\nu^2}}\; \exp\left(\left(-\frac{1}{2\sigma_\nu^2}\right)\left[\nu - \left(\overline{\nu} - \alpha\left(M - \frac{1}{\xi}\right)\right)\right]^2\right). \quad (6)
$$

As in BY, changes in economic fundamentals are driven by the continuous trickling down of new information, modeled through the diffusive shocks $\{dz_C,\, dz_D,\, dz_x,\, dz_\Omega\}$. Jumps add an additional source of risk to this framework and capture the notion of sudden unexpected changes in economic fundamentals. In equation (6), $\overline{\nu}$ denotes the average value of $\widetilde{\nu}$. When $\overline{\nu}$ is negative, the typical jump in the x state variable lowers the expected growth rate of the agent's endowment. Moreover, a jump increases economic uncertainty Ω by $\widetilde{M}$, whose average value is $\frac{1}{\xi}$. If the realization of $\widetilde{M}$ is higher than $\frac{1}{\xi}$, then the average jump in x is reduced to $\left[\overline{\nu} - \alpha\left(M - \frac{1}{\xi}\right)\right]$ (assuming $\alpha > 0$). Hence, α controls the level of correlation between the jump in volatility and the jump in expected consumption growth.

We model jumps as rare events, i.e., both λ^B and λ^G are small.[4] Yet, we specify $\lambda^B = 0.035$ to be considerably larger than $\lambda^G = 0.0005$. Thus, the agent views the economy with $\lambda_t = \lambda^B$ as the 'bad' economy and $\lambda_t = \lambda^G$ as the 'good' economy.

The agent does not observe the state of the economy directly, i.e., she does not know whether $\lambda_t = \lambda^B$ or $\lambda_t = \lambda^G$. Instead, she observes only the process $\{dN_t\}$. We define the state variable $p(t)$ as her date-t estimate that the economy is in the good state (i.e., her prior):

$$p(t) \equiv \pi(\lambda_t = \lambda^G | \, dN_t) \, . \tag{7}$$

Over each interval dt, the agent updates her prior. If a jump occurs during this interval (i.e., if $dN = 1$), the probability of being in the good state jumps downward. If no jump occurs during this interval, the probability of being in a good state drifts upward (after accounting for the transitions ϕ_{GB} and ϕ_{BG}). The solution to the filtering problem with Markov switching is studied in Liptser and Shiryaev (2001). Applying[5] their Theorem 19.6 page 332, we obtain the Bayesian dynamics for the probability of being in a good state:

$$dp = p\left(\frac{\lambda^G - \overline{\lambda}(p)}{\overline{\lambda}(p)}\right) dN + \Big[-p\left(\lambda^G - \overline{\lambda}(p)\right) + (1-p)\phi_{BG} - p\phi_{GB}\Big] \, dt \, , \tag{8}$$

where $\overline{\lambda}(p(t)) = \Big[p(t)\lambda^G + (1-p(t))\,\lambda^B\Big]$. Thus, at a time t, the agent perceives the probability of a future jump to be:

$$\text{Prob}(dN_t = 1 | \mathcal{F}_t) = \overline{\lambda}(p(t)) \, dt. \tag{9}$$

For future reference it is convenient to define the state vector as $X_t = (x_t, \Omega_t, p_t)$. Further, we write state vector dynamics as:

$$\begin{aligned} dc_t &= \mu_C(X_t)\, dt + \sigma_C(X_t)\, dz(t) \\ d\delta_t &= \mu_D(X_t)\, dt + \sigma_D(X_t)\, dz(t) + \sigma_{DD}(X_t)\, dz_D(t) \\ dX_t &= \mu_X(X_t)\, dt + \sigma_X(X_t)\, dz(t) + \widetilde{\Gamma}\, dN(t)\, , \end{aligned} \tag{10}$$

with the vector of independent Brownian motions $z = (z_c,\, z_x,\, z_\Omega)$ and the vector of jump variables $\widetilde{\Gamma}$. Note that $(c_t,\, X_t)$ and $(\delta_t,\, X_t)$ are both Markov systems.

2.1 Recursive Utility

Following Epstein and Zin (1989), we assume that the representative agent's preferences over a consumption process $\{C_t\}$ are represented by a utility index $U(t)$ that satisfies the following recursive equation:

$$U(t) = \left\{(1 - e^{-\beta dt})C_t^{1-\rho} + e^{-\beta dt}\mathrm{E}_t\left(U(t+dt)^{1-\gamma}\right)^{\frac{1-\rho}{1-\gamma}}\right\}^{\frac{1}{1-\rho}} \, . \tag{11}$$

[4] Since we focus on the 1987 crash, we do not model the smaller but more frequent jumps observed in the data. This is in contrast to other papers that considered high-frequency jumps, e.g., Drechsler and Yaron (2008) assume an average frequency of 0.8 jumps per year.

[5] See also their example 1, p. 333.

With $dt = 1$, this is the discrete time formulation of Kreps-Porteus/Epstein-Zin (KPEZ), in which $\Psi \equiv 1/\rho$ is the EIS and γ is the risk-aversion coefficient.

The properties of the stochastic differential utility in (11) and the related implications for asset pricing have been previously studied by, e.g., Duffie and Epstein (1992a,b), Duffie and Skiadas (1994), Schroder and Skiadas (1999, 2003), and Skiadas (2003). In Appendix A, we extend their results to the case in which the aggregate output has jump-diffusion dynamics. The solution to our model follows immediately from Propositions 1 and 2 in Appendix A. Specifically, let us define

$$J(t) = \left(\frac{1}{1-\gamma}\right) U(t)^{(1-\gamma)}.$$

Then, it is well-known that $J(t)$ has the following representation:

$$J(t) = \mathrm{E}_t \left[\int_t^{\infty} f(C_s, J(s))\, ds \right], \tag{12}$$

where $f(c, J)$ is the normalized aggregator defined in Duffie and Epstein (1992), which we reproduce in equation (A.7) of Appendix A.

For the cases $\rho, \gamma \neq 1$, Proposition 1 gives the agent's value function as:[6]

$$J = \frac{e^{c(1-\gamma)}}{1-\gamma}\, \beta^{\theta}\, I(X)^{\theta}, \tag{13}$$

where we have defined $\theta \equiv \frac{1-\gamma}{1-\rho}$. Further, we show in Appendix A that I is the price-consumption ratio, and satisfies the relation:

$$\begin{aligned} -\theta \;\; &= \;\; I(X)\left((1-\gamma)\mu_C(X) + (1-\gamma)^2 \frac{||\sigma_C(X)||^2}{2} - \beta\theta\right) + \frac{\mathcal{D}I(X)^{\theta}}{I(X)^{(\theta-1)}} \\ &\quad + (1-\gamma)\theta\sigma_C(X)\sigma_X(X)^{\top} I_X(X) + I(X)\lambda(X)\overline{\mathcal{J}} I(X)^{\theta}, \end{aligned} \tag{14}$$

where we define the (continuous diffusion, jump and jump-compensator) operators $\mathcal{D}, \mathcal{J}, \overline{\mathcal{J}}$ in equations (A.13)-(A.13) in Appendix A.

2.2 Pricing Kernel and Riskfree Rate

When $\rho, \gamma \neq 1$, Proposition 1 in Appendix A identifies the pricing kernel as

$$\Pi(t) \;\; = \;\; e^{\int_0^t ds\, [(\theta-1)I(s)^{-1} - \beta\theta]}\, \beta^{\theta}\, e^{-\gamma c_t}\, I(X_t)^{(\theta-1)}. \tag{15}$$

Using Itô's lemma, we obtain the dynamics of the pricing kernel which identifies both the diffusion and the jump risk-premia, as well as the risk-free rate:

$$\frac{d\Pi(t)}{\Pi(t)} = -r_t\, dt - (\gamma\sigma_C(X_t) + (\theta-1)\sigma_I(X_t))\; dz(t) + \mathcal{J} I(X_t)^{(\theta-1)}\, dN_t - \lambda(X_t)\overline{\mathcal{J}} I(X_t)^{(\theta-1)}\, dt. \tag{16}$$

[6]Other cases with either ρ or γ equal to 1 can be treated similarly, as shown in the Appendix.

Here, we have defined the diffusion-volatility of the price-consumption ratio as $\sigma_I(X) = \frac{1}{I(X)} I_X(X)^\top \sigma_X(X)$, and the risk-free rate via:[7]

$$
\begin{aligned}
r(X) \quad &= \quad \beta + \rho\left(\mu_C(X) + \frac{||\sigma_C(X)||^2}{2}\right) - \gamma(1+\rho)\frac{||\sigma_C(X)||^2}{2} - (1-\theta)\sigma_I(X)^\top\left(\sigma_C(X) + \frac{1}{2}\sigma_I(X)\right) \\
&\quad + \lambda(X)\left(\frac{\theta-1}{\theta}\overline{\mathcal{J}}I^\theta - \overline{\mathcal{J}}I^{(\theta-1)}\right). \qquad (17)
\end{aligned}
$$

Note that this result reduces to the standard result for the CRRA exchange economy if $\rho = \gamma$ (i.e., $\theta = 1$). Instead, if agents have a preference for resolution of uncertainty, then risk-aversion affects the interest rate via the precautionary savings motive if consumption has positive volatility. Further, if agents display preference for early resolution ($\gamma > \rho$) and if the EIS is greater than 1 (i.e, $\rho < 1$) then the greater the volatility of the price-dividend ratio, the lower the equilibrium interest rate, as agents want to divest from the risky asset because of long-run risk (this follows since under these conditions $1 - \theta = \frac{\gamma-\rho}{1-\rho} > 0$).

2.3 Price Dividend Ratio and Equity Premium

The stock market portfolio is a claim to aggregate dividends $D(t)$. Thus, its value is obtained by the standard discounted cash-flow formula:

$$
S(t) = \mathrm{E}_t\left[\int_t^\infty \frac{\Pi(s)}{\Pi(t)} D(s)\,ds\right]. \qquad (18)
$$

This equation implies that the excess return on the stock is given by:[8]

$$
\frac{1}{dt}\mathrm{E}_t\left[\frac{dS(t)}{S(t)}\right] + \frac{D(t)}{S(t)} - r_t = -\frac{1}{dt}\mathrm{E}_t\left[\frac{d\Pi(t)}{\Pi(t)}\frac{dS(t)}{S(t)}\right]. \qquad (19)
$$

Defining the price-dividend ratio via $S_t = D_t\, L(X_t)$ and substituting into equation (19), we obtain a PDE for $L(X)$ similar to that obtained for the price consumption ratio. To save on space, we relegate this equation to the Appendix (see equation (B.7) in Appendix B). Using the definition of the pricing kernel, we can compute the right-hand side of equation (19) more explicitly to obtain the following expression for the risk-premium on the dividend claim:

$$
\begin{aligned}
\mu_S(X) + \frac{1}{L(X)} - r(X) \quad &= \quad \left(\gamma\sigma_C(X) + (1-\theta)\sigma_I(X)\right)^\top\left(\sigma_D(X) + \sigma_L(X)\right) \\
&\quad + \lambda(X)\left(\overline{\mathcal{J}}\{I(X)^{\theta-1}L(X)\} - \overline{\mathcal{J}}I(X)^{\theta-1} - \overline{\mathcal{J}}L(X)\right), \qquad (20)
\end{aligned}
$$

where we have defined $\sigma_L(X) = \frac{1}{L(X)} L_X(X)^\top \sigma_X(X)$ as the diffusion of the price-dividend ratio. Note that when $(1-\theta) = \frac{\gamma-\rho}{1-\rho} > 0$ (which holds, in particular, when agents have preference for early resolution of uncertainty, that is $\gamma > \rho$, and the EIS is greater than 1, that is $\rho < 1$) then, the higher the volatility of the price-consumption ratio, the greater the equity premium.

[7] Again we only present results for the case where ρ, $\gamma \neq 1$. The other cases are treated in the Appendix.

[8] This follows from Itô's lemma, the dynamics of $\Pi(t)$, and the fact that $\Pi(t)S(t) + \int_0^t \Pi(s)D(s)\,ds$ is a P-martingale.

2.4 Pricing Options on the Market Portfolio and Individual Stocks

The date-t value of an European call option on the stock market portfolio $S(t)$, with maturity T and strike price K, is given by

$$C(S(t), X(t), K, T) = \mathrm{E}_t^Q \left[e^{-\int_t^T r(X(s))\, ds} \; (S(T) - K)^+ \right], \tag{21}$$

where the expectation is computed under the risk-neutral measure Q. The risk-neutral dynamics of the stock price, $S_t = D_t\, L(X_t)$ are:

$$\begin{aligned} \frac{dS}{S} &= \left(r_t - \frac{1}{L(X_t)} \right) dt + \left(\sigma_D(X_t) + \sigma_L(X_t)\right) dz^Q(t) + \sigma_{DD}(X_t)\, dz_D^Q(t) \\ &\quad + \mathcal{J} L(X_t)\, dN_t - \lambda^Q(X_t) \overline{\mathcal{J}} L(X_t)\, dt\,. \end{aligned} \tag{22}$$

The risk-neutral dynamics for D, x, Ω, and p are given in Appendix B.4.

As in Bakshi, Kapadia, and Madan (2003), we specify return dynamics on an individual stock, $\frac{dS_i}{S_i}$, as a sum of a systematic component and an idiosyncratic component. In particular, we assume that individual firm dynamics follow

$$\frac{dS_i}{S_i} = \frac{dS}{S} + \sigma_i\, dz_i + \left[\left(e^{\widetilde{\nu}_i} - 1 \right) dN_i - \mathrm{E}\left[e^{\widetilde{\nu}_i} - 1 \right] \lambda_i\, dt \right], \tag{23}$$

where the market return dynamics $\frac{dS}{S}$ are given in equation (22). Here, σ_i captures the volatility of the idiosyncratic diffusive shock, while the diversifiable jump component has Poisson arrival rate N_i with constant intensity λ_i and normally-distributed jump size $\widetilde{\nu}_i \rightsquigarrow N(\mu_{\nu_i}, \sigma_{\nu_i})$. The free parameters ($\sigma_i$, λ_i, μ_{ν_i}, σ_{ν_i}) are chosen to match historical moments of the return distribution on individual firms. By definition, the diversifiable shocks do not command a risk premium, while the risk adjustments on the systematic component are identical to those that we have applied to price the options on the S&P 500 index. Thus, the price of an option on an individual stock is given by a formula similar to equation (21).[9]

3 Data and Model Implications

Here, we calibrate the model coefficients to match economic fundamentals, we solve the model numerically, and study its asset pricing implications.

[9] There might be a *potential* concern that the dynamics (23) for the individual firms and the dynamics (22) for the aggregate index are not self-consistent. That is, the terminal value of a strategy that invests an amount $S(0) = \sum_{i=1}^N S_i(0)$ in the index does not necessarily have the same terminal value of a strategy that invests an amount $S_i(0)$ in each of the individual stocks, $i = 1, \ldots, N$. However, we find in unreported simulations that the discrepancy is negligible, i.e., $S(T) \approx \sum_{i=1}^N S_i(T)$. Intuitively, the idiosyncratic shocks that we specify are in fact diversifiable when the portfolio is composed of a sufficiently large number of firms.

3.1 Baseline Model Coefficients

Table 1 reports the coefficients for our baseline model calibration, expressed with a yearly decimal scaling. They are organized in seven groups, which we briefly discuss below.

1. *Preferences:*

   We use a time discount factor coefficient $\beta = 0.0176$, and fix the coefficient of relative risk aversion γ at 10, a value that is generally considered to be reasonable (e.g., Mehra and Prescott (1985)). The magnitude of the EIS coefficient Ψ is more controversial. Hall (1988) argues that the EIS is below 1. However, Guvenen (2001) and Hansen and Singleton (1982), among others, estimate the EIS to be in excess of 1, and Attanasio and Weber (1989), Bansal, Gallant, and Tauchen (2007), and Bansal, Tallarini, and Yaron (2006) find it to be close to 2. Accordingly, we fix $\Psi = 2$ in our baseline case.

2. *Aggregate Consumption and Dividends:*

   We fix $\mu_C = 0.018$ and $\mu_D = 0.025$, consistent with the evidence that, historically, dividend growth has exceeded consumption growth (Section 3.2 below). As in, e.g., BY, we set $\phi > 1$ to allow the sensitivity of dividend growth to shocks in x to exceed that of consumption growth. Setting $\rho_{DC} > 0$ guarantees a positive correlation between consumption and dividends.

3. *Predictable Mean Component, x:*

   Similar to BY, in the dynamics (3) we use $\kappa_x = 0.2785$, which makes x a highly persistent process (if we adjust for differences in scaling and we map the BY AR(1) ρ coefficient into the κ_x of our continuous-time specification, we find $\kappa_x = 0.2547$). We decompose the shocks to the x process into two terms that are orthogonal and parallel to consumption shocks, with $\sigma_{xc} > 0$, $\sigma_{x\Omega} > 0$, and $\sigma_{x0} = 0$.

4. *Economic Uncertainty, Ω:*

   In the Ω-dynamics (4), we fix $\overline{\Omega}$ and σ_Ω at values similar to those used in BY. However, in our calibration $\kappa_\Omega = 1.0484$, which makes the Ω process much less persistent than the x process (the half life of a shock is around 7-8 months). This is in contrast to the high persistence of the volatility shocks in the BY calibration, and more akin to the calibration in Drechsler and Yaron (2008). This feature is important in the presence of jumps to volatility, since with a highly persistent Ω process, as in BY, volatility would remain high for years after a jump. Finally, we allow for negative correlation between shocks to consumption and volatility, $\rho_{\Omega C} < 0$.

5. *Jumps:*

   We set $\lambda^G = 0.0005$ and $\lambda^B = 0.035$. Thus, if the jump intensity λ_t equals λ^G, a jump occurs about once every 2000 years. In contrast, if $\lambda_t = \lambda^B$, the average jump time is approximately

30 years. Jumps in x have negative mean, $\overline{\nu} < 0$, i.e., the typical jump carries bad news for the growth prospects of the economy.[10] Similarly, jumps to Ω are positive and increase the level of economic uncertainty. Finally, we set the transition probabilities $\phi_{BG} = 0.0025$ and $\phi_{GB} = 0.025$. With these values, in steady state the economy is in the 'good' state λ^G with probability $\phi_{BG}/(\phi_{BG} + \phi_{GB}) = 0.91$.

6. *Individual Stock Returns:*

   For each of the 20 stocks in the Bollen and Whaley (2004) study, we compute standard deviation, skewness, and kurtosis by using daily return series for the sample period from January 1995 to December 2000 (the same period considered by Bollen and Whaley). For each of these statistics, we evaluate cross sectional averages. We find an average standard deviation of 37.6% per year, and average skewness and kurtosis of 0.12 and 7.12, respectively. Four coefficients characterize the distribution of the idiosyncratic shocks in equation (23): the standard deviation of the diffusive firm-specific shock, σ_i, the intensity of the diversifiable jump component, λ_i, and the mean and standard deviation of the jump size, μ_{ν_i} and σ_{ν_i}. After some experimentation, we fix the jump intensity to $\lambda_i = 5$, which corresponds to an expected arrival rate of 5 jumps per year. We choose the remaining coefficients to match the average standard deviation, skewness, and kurtosis reported above. This approach yields $\sigma_i = 0.3137$, $\mu_{\nu_i} = 0.0036$, and $\sigma_{\nu_i} = 0.0632$. To confirm that the results are robust to this approach, we solve for σ_i, μ_{ν_i}, and σ_{ν_i} when λ_i takes different values in the 1-10 range. The results are similar to those discussed below.

7. *Initial Conditions:*

   Before the crash, the agent is nearly sure that the economy is in the good state $\lambda_t = \lambda^G$. Specifically, we set $p^{Pre} = 99.85\%$. When the agent perceives a jump in fundamentals, she updates her prior according to equation (8). In our calibration, this yields $p^{Post} = 90\%$. We fix the remaining state variables at their steady-state values, $x_0 = \overline{\nu}\,\overline{\lambda}/\kappa_x$ and $\Omega_0 = \overline{\Omega} + \overline{\lambda}/(\xi\kappa_\Omega)$.

3.2 Aggregate Consumption and Dividends

Here we demonstrate that the calibration discussed in the previous section matches the historical data well. In Table 2, we report summary statistics for the series of yearly growth rates on aggregate consumption (Panel A) and dividends (Panel B). We focus not only on low-order moments, like mean, standard deviation, auto- and cross-correlations, but also on higher-order moments like skewness and kurtosis. We report empirical results for two sample periods. The first spans 80 years of data, from 1929 to 2008, the second spans the post World War II period, 1946-2008. In addition to point estimates for these moments, we report standard errors robust with respect

[10] In our calibration jumps are rare extreme events. This sets the paper apart from other studies that considered higher-frequency jumps. For instance, Drechsler and Yaron (2008) assume that jumps have mean zero and occur on average 0.8 times per year.

to both auto-correlation and heteroskedasticity. The measure for aggregate consumption is the real (chained 2000) yearly series of per-capita consumption expenditures in nondurable goods and services from the National Income and Product Accounts (NIPA) tables published by the Bureau of Economic Analysis. Following Fama and French (1988), we obtain a monthly dividend proxy by subtracting returns without dividends from returns with dividends on the value-weighted market index as reported by the Center for Research in Security Prices (CRSP). We sum the monthly dividends to obtain the yearly dividend, and we deflate the yearly dividend series using Consumer Price Index (CPI) data.

To examine the model implications, we simulate 10,000 samples of monthly consumption and dividend data, each spanning a period of 80 years (same as the length of the 1929-2008 sample period). We aggregate the monthly series to obtain yearly dividends and consumption, and we compute the series of yearly growth rates $\frac{\Delta C}{C}$ and $\frac{\Delta D}{D}$. For each of the 10,000 samples, we compute summary statistics for these series. In the table, we report the mean value of these statistics, as well as the 5th, 50th, and 95th percentiles. In the simulations, we use two different initial conditions. First, we initialize the Markov chain for the λ process at $\lambda(t=0)=\lambda^G$. That is, we assume that at the beginning of time the economy is in the 'good' state. Second, we initialize $\lambda(t=0)=\lambda^B$, i.e., we assume that at $t=0$ the economy is in the 'bad' state. In the table we report the results for each of these two cases.

In both sets of simulations, the moments of $\frac{\Delta C}{C}$ and $\frac{\Delta D}{D}$ are very close to the sample moments. For most moments, the mean and the median computed in model simulations are essentially identical to those computed with the data. See, for instance, $AC(1)$ for $\frac{\Delta C}{C}$: it is 0.42 in the data (1929-2008 sample), compared to a median simulated value of 0.42 when $\lambda(t=0)=\lambda^G$, and 0.44 when $\lambda(t=0)=\lambda^B$.

In a few cases, the median values in model simulations do not perfectly match the estimates in the data. However, the sample estimates are in the 90% confidence interval computed from model simulations. See, e.g., the skewness of $\frac{\Delta D}{D}$. It is 0.38 in the 1929-2008 sample, a value that falls well within the model confidence interval of $[-0.20, 0.71]$ when $\lambda(t=0)=\lambda^G$, and $[-0.24, 0.71]$ when $\lambda(t=0)=\lambda^B$. These results reflect the fact that some statistics are imprecisely estimated. For instance, the skewness of $\frac{\Delta D}{D}$ is much higher in the 1946-2008 sample, and the standard error associated to this estimate is very high. Similarly, the model cannot match the extreme $\frac{\Delta C}{C}$ skewness estimated over 1929-2008 (due to the drop in consumption during the Great Depression), but it gets close to the -0.53 estimate for the 1946-2008 period.

There is one fact that the model does not capture well. The kurtosis of $\frac{\Delta D}{D}$ is 5.30 in the 1946-2008 sample, and 9.06 in the 1929-2008 sample. Both values exceed the 4.16 upper bound in model simulations. We feel, however, that this fact is not a shortcoming of the model for two reasons. First, kurtosis is very imprecisely estimated in the data (note the difference in point estimates across the two sample periods and the huge standard errors). Second, it is arguably a good thing that our results are not driven by an excess of skewness/kurtosis built in the model.

3.3 Stock Market Return and Riskfree Rate

Before turning to option prices, we further validate our calibration by showing that the model is also consistent with a wide range of asset pricing facts. Table 3 reports key asset pricing moments computed with data spanning the 1929-2008 and 1946-2008 sample periods. The real annualized total market return, $(\frac{\Delta S}{S} + \frac{D}{S})$, is the yearly return, inclusive of all distributions, on the CRSP value-weighted market index, adjusted for inflation using the CPI. The real risk-free rate r_f is the inflation-adjusted three-month rate from the 'Fama Risk Free Rates' database in CRSP. In computing the logarithmic price-dividend ratio, $log(S/D)$, we consider two measures of dividends. The first is the real dividend on the CRSP value-weighted index, which we have already discussed in Section 3.2 above. The second is the real dividend on the CRSP value-weighted market index, adjusted for share repurchases (Boudoukh et al. (2007)).

We solve the model numerically (Appendix C discusses the numerical approach) and simulate 10,000 samples of monthly stock market returns, riskfree rates, and price-dividend ratios, each spanning a period of 80 years. We aggregate the monthly series at the yearly frequency. For each of the 10,000 simulated samples, we compute summary statistics for these series. We report the mean value of these statistics, as well as the 5th, 50th, and 95th percentiles. We repeat the analysis with two simulation schemes. In the first set of simulations, we initialize the probability process p at $p(t = 0) = p^{Pre}$, which corresponds to the pre-crash economy. In the second set, we initialize $p(t = 0) = p^{Post}$, which corresponds to the post-crash economy.

Panel A in Table 3 shows the moments for the real stock market return. The sample mean estimate is very close to the mean in model simulations. The sample standard deviation for the yearly return is a bit high relative to the model predictions when estimated over the 1929-2008 sample. However, the estimate computed with post World War II data falls well within the model confidence bands, and the model matches the (annualized) monthly return standard deviation estimate well. Moreover, the model fits the market return kurtosis accurately, both in monthly and annual data. The sample skewness is more imprecisely estimated, but remains reasonably close to the model predictions, especially in monthly data.

Panel B shows the mean and standard deviation of the riskfree rate. While the model matches the mean accurately, the standard deviation is somewhat higher in the data than in the model. This is a well known feature of the long-run risk setup. It is arguably a desirable property of the model, rather than a weakness. For instance, Beeler and Campbell (2009, p. 8) note that "the data record the ex post real return on a short-term nominally riskless asset, not the ex ante (equal to ex post) real return on a real riskless asset. Volatile inflation surprises increase the volatility of the series in the data, but not in the model."

The properties of the stock market return and the riskfree rate extend to the equity premium, which the model matches quite well (Panel C). Finally, the model seems to underestimate the mean level of the logarithmic price-dividend ratio (Panel D). However, when we account for share repurchases in the measure of dividends, as in, e.g., Boudoukh et al. (2007), the sample estimate of the price-dividend ratio is revised downward and is perfectly in line with the model predictions.

In sum, these results support two main conclusions. First, the model matches several important asset pricing moments quite well. Second, the asset pricing moments predicted by the model pre- and post-1987 crash are similar. Yet, we show in the next section that option prices differ quite dramatically before and after the crash.

3.4 Option Prices

Figure 1 shows the spread of in-the-money and out-of-the-money implied volatilities relative to at-the-money implied volatilities from 1985-2006. (Appendix D explains how we constructed the implied volatility series.) Prior to the crash, 10% OTM puts with one month to maturity had an average implied volatility spread of 1.83%. Similarly, the spread for 2.5% ITM put options averaged -0.13% prior to the crash. On some dates the implied volatility function had the shape of a mild 'smile' and on others it was shaped like a mild 'smirk'. Overall, the Black Scholes formula priced all options relatively well prior to the crash, underpricing deep OTM options only slightly. This all changed on October 19, 1987, when the spread for OTM puts spiked up to a level above 10%. Since then, implied volatilities for deep OTM puts have averaged 8.21% higher than ATM implied volatilities. Moreover, since the crash, implied volatilities for ITM options have been systematically lower than ATM implied volatilities, with an average spread of -1.34%.

We simulate the price of options on the S&P 500 index, with one month to maturity, across different strike prices. Figure 2 illustrates the results for the pre-crash economy, i.e., $p = p^{Pre}$, and for the post-crash economy, i.e., $p = p^{Post}$. The two plots capture the stark regime shift in index option prices. Prior to the crash, the implied volatility function is nearly flat, with a very mild upward tilt. In the model, the pre-crash spread between OTM and ATM implied volatilities is 1.68%, while the spread between ITM and ATM implied volatilities is -0.06%. After the crash, the implied volatility function tilts into a steep smirk: in the model, the spread between OTM and ATM implied volatilities is 8.39%, while the spread between ITM and ATM implied volatilities is -0.38%. These values closely match the numbers we find in the data.

Another important property of S&P 500 option prices, which is evident from Figure 1, is that the regime shift in S&P 500 options has persisted since the 1987 crash. The model captures this evidence, too. Figure 3 shows the implied volatility function for S&P 500 options, computed across different strike prices, as a function of p, i.e., the probability the agent assigns to being in the 'good' economy. (Figure 3 adds a p-dimension to Figure 2, since Figure 2 depicts two sections of the plot in Figure 3, those for which $p = p^{Pre}$ and $p = p^{Post}$.) In our calibration, prior to the crash the agent is nearly sure the economy is in the 'good' state, i.e., p^{Pre} is very close to unity. In 1987, the agent perceives a jump in fundamentals and updates her prior, which leads to $p^{Post} \approx 0.9$. There may be additional jumps in fundamentals after that. If that were to happen, the agent would revise her posterior probability p further down. However, Figure 3 shows that lowering p below p^{Post} would have a minimal effect on the shape of the smirk. That is, the model predicts the magnitude of the shift to be persistent.

3.4.1 Sensitivity Analysis

We illustrate the sensitivity of the pre- and post-crash volatility functions to some key underlying parameters:

Preferences Coefficients Figure 4 shows that when the coefficient of risk aversion is lowered to 7.5, most of the post-crash volatility smirk remains intact. Increasing the value of γ to 12.5 steepens the post-crash smirk considerably. Most importantly, even when $\gamma = 12.5$, a value that exceeds the range that most economists find to be 'reasonable' (e.g., Mehra and Prescott (1985)), the pre-crash smirk remains relatively flat.

As noted previously, researchers have obtained a wide array of estimates for the EIS parameter Ψ. In our baseline case we use $\Psi = 2$. Figure 5 shows that even lower estimates for Ψ, such as 1.5, still produce steep post-crash volatility smirks.

Jump Coefficients Figure 6 illustrates the effect of a one-standard-deviation perturbation of the average jump size coefficient. Not surprisingly, the steepness of the post-crash smirk is quite sensitive to the level of this coefficient. Lowering $\overline{\nu}$ increases the steepness of the smirk, especially in the post-crash economy.

In contrast, a change in the expected size of the jump in volatility, $1/\xi$, has a limited effect on the steepness of the smirk. This is evident from Figure 7, which shows results for $\xi = \xi_{high}$, a value that corresponds to an average jumps size $1/\xi_{high} \approx 0$, and $\xi = \xi_{low}$, a value that corresponds to an average jumps size that is double the baseline case. This result is due to the low persistence of the Ω processes in our calibration: Unlike shocks to x, shocks to Ω are short lived. Moreover, jumps are rare events in our calibration. Thus, in sum, changing the expected volatility jump size has little impact on the volatility smirk.

3.4.2 Options on Individual Stocks

We now turn to the pricing of individual stock options. We simulate option prices for a typical stock, as discussed in Section 2.4, and extract Black-Scholes implied volatilities for different option strike prices. Figure 8 compares this implied volatility function to the volatility smirk for S&P 500 options. Consistent with the empirical evidence, our model predicts that the volatility smile for individual stock options is considerably flatter than that for S&P 500 options.

Bakshi, Kapadia, and Madan (2003) conclude that the differential pricing of individual stock options is driven by the degree of skewness/kurtosis in the underlying return distribution in combination with the agent's high level of risk aversion. Here, we propose a plausible endowment economy that, in combination with recursive utility, yields predictions consistent with their empirical findings. Combined with our results discussed above, this evidence suggests that the market of S&P 500 and individual stock options, as well as the market for the underlying stocks, are well integrated.

3.5 The Change in Stock and Bond Prices around the 1987 Crash

In our model, the 1987 crash is caused by a downward jump $\widetilde{\nu}$ in expected consumption growth x and a simultaneous upward jump $\widetilde{M}$ in consumption volatility Ω. Here we quantify the magnitude of these jumps implied by our model. Prior to the crash, the stock market price-dividend ratio is $L^{Pre} = L(x^{Pre}, \Omega^{Pre}, p^{Pre})$, where p^{Pre} is the agent's prior on the probability that $\lambda_t = \lambda^G$. In our calibration, $p^{Pre} \approx 1$. Immediately after the crash, the price-dividend ratio drops to $L^{Post} = L(x^{Pre}+\widetilde{\nu}, \Omega^{Pre}+\widetilde{M}, p^{Post})$, where p^{Post} is the agent's posterior probability that $\lambda_t = \lambda^G$. Similarly, the change in the riskfree rate at the time of the crash is

$$\Delta r = r^{Post} - r^{Pre} = r(x^{Pre} + \widetilde{\nu}, \Omega^{Pre} + \widetilde{M}, p^{Post}) - r(x^{Pre}, \Omega^{Pre}, p^{Pre}) . \tag{24}$$

Using S&P 500 data, we find $L^{Pre} = 32.64$ and $L^{Post} = 25.96$, while the change in the three-month Treasury bill yield measured over the two weeks before and after the crash is -1.39%. The model matches these numbers when $p^{Pre} = 99.85\%$, $x^{Pre} = 0.0317$, $\widetilde{\nu} = -0.0034$, $\Omega^{Pre} = 0.000533$, and $\widetilde{M} = 0.0001689$.

Now, $\widetilde{\nu} = -0.0034$ is a very small jump in x. The expected growth rate of a persistent process is difficult to measure. Thus, it will be difficult for the econometrician to detect this jump in ex-post dividend data. The jump in economic uncertainty, $\widetilde{M} = 0.0001689$, is bigger relative to the long-run mean of Ω, but in the calibration the persistence of the economic uncertainty process Ω is much lower than the persistence of the growth process x. After a jump, the process Ω is quickly pulled back towards its steady-state value by its drift (the half-life of a shock to Ω is around 7-8 months). Moreover, prior to the crash the value of economic uncertainty, $\Omega^{Pre} = 0.000533$, is not far from the steady state value of Ω, 0.0006 (the standard deviation of a diffusive shock is 0.000068). Thus, this jump will be hardly detectable in low-frequency dividend and consumption data as well.

These computations show that the model explains the change in stock and bond prices around the 1987 market crash with minimal change in fundamentals. In fact, the crash is mainly driven by the updating of the agent's beliefs. If we omit the effect of the jumps $\widetilde{\nu}$ and $\widetilde{M}$ and focus only on the effect of the change in the agent's prior from p^{Pre} to p^{Post}, we find $L(x^{Pre}, \Omega^{Pre}, p^{Post}) = 26.70$. That is, Bayesian updating over p alone drives an 18.20% drop in prices ($26.70/32.64-1 = -0.1820$). Similarly, we find $r(x^{Pre}, \Omega^{Pre}, p^{Post}) - r(x^{Pre}, \Omega^{Pre}, p^{Pre}) = -0.95\%$, which accounts for most of the drop in the riskfree rate.

3.6 Final Thoughts

We conclude this section with two observations. First, Figure 1 conveys two main points. One, as mentioned previously, there has been a permanent shift in the shape of the implied volatility function due to the crash. Two, there are daily fluctuations in the shape of the smirk. This second feature has been studied extensively in the literature. Prior contributions have shown that these fluctuations can be understood in both a general equilibrium framework (e.g., David and Veronesi (2002 and 2008)) and a partial equilibrium setting (e.g., Bakshi et al. (1997 and 2000), Bates (2003), Pan (2002), and Eraker (2004)). Such high-frequency fluctuations can be captured within the context of our model by introducing additional state variables that drive high-frequency changes in expected dividend growth and/or volatility. However, since these daily fluctuations have already been explained, we do not investigate such variables in order to maintain parsimony.

Second, another aspect of S&P 500 options is that expected return volatility computed under the risk-adjusted probability measure is typically higher than expected return volatility computed under the actual probability measure. The difference between these two expected volatility measures is often termed the 'variance risk premium,' or VRP. Moreover, previous studies have shown that the VRP fluctuates over time and predicts future stock market returns at the short/medium horizon (Bollerslev, Tauchen, and Zhou (2009) and Drechsler and Yaron (2008)). Our model does not capture this evidence, but it can be extended to include higher-frequency jumps (similar to Drechsler and Yaron (2008)) or time-varying volatility-of-volatility (as in Bollerslev, Tauchen, and Zhou (2009)). Since this is not our contribution, we point the interested reader to those studies for more details.

4 Conclusions

The 1987 stock market crash is associated with many asset pricing puzzles. Examples include: i) Stocks fell 20-25%, interest rates fell approximately 1-2%, yet there was minimal impact on observable economic variables (e.g., consumption), ii) the slope of the implied volatility curve on index options changed dramatically after the crash, and this change has persisted for more than 20 years, iii) the magnitude of this post-crash slope is difficult to explain, especially in relation to the implied volatility slope on individual firms. We propose a general equilibrium model that can explain these puzzles while capturing many other salient features of the US economy. We accomplish this by extending the model of Bansal and Yaron (2004) to account for jumps and learning. In particular, we specify the representative agent to be endowed with Epstein-Zin preferences and assume that the aggregate dividend and consumption processes are driven by a persistent stochastic growth variable that can jump. Economic uncertainty fluctuates and is also subject to jumps. Jumps are rare and driven by a hidden state the agent filters from past data. In such an economy there are three sources of long-run risk: expected consumption growth, volatility of consumption growth, and posterior probability of the jump intensity in expected growth rates and volatility.

Jumps in fundamentals, even small, can lead to substantial jumps in prices of long-lived assets because of the updating of beliefs about the likelihood of future such jumps. In that sense, learning acts as an amplifier of long-run risk premia associated with small persistent jumps in growth rates and their volatility. Indeed, we identify a realistic calibration of the model that matches the prices of short-maturity at-the-money and deep out-of-the-money S&P 500 put options, as well as the prices of individual stock options. Further, the model, calibrated to the stock market crash of 1987, generates the steep shift in the implied volatility 'smirk' for S&P 500 options observed around the 1987 crash. This 'regime shift' occurs in spite of a minimal change in observable macroeconomic fundamentals.

In sum, our model points to a simple mechanism, based on learning about the riskiness of the economy, that explains why market prices suddenly crashed with little change in fundamentals, and why buyers of OTM put options were willing to pay a much higher price for these securities after the crash. Of course, we acknowledge that other mechanisms probably also contributed to the crash. For example, portfolio insurance and its implementation via dynamic hedging strategies is often cited as a major culprit. Let us just point out that, while not directly a 'shock to fundamentals,' the failure of portfolio insurance could well have contributed to deteriorating prospects about economic fundamentals through a 'financial accelerator' mechanism. It is common belief that the growth rate of consumption, and consumption volatility, are tied to the strength of the financial system. Thus, if the crash revealed that risk-sharing was not as effective as previously thought, then this could have negatively affected investors' expectations about the future prospects of the economy. In this respect, further learning about economic fundamentals occurs through the experience of a crash in prices, and might result in a further drop in prices via the mechanism we describe. Explicitly modeling this feedback mechanism between prices and economic fundamentals is outside the scope of the present paper, but seems an interesting avenue for future research.

A Equilibrium Prices in a Jump-Diffusion Exchange Economy with Recursive Utility

There are several formal treatments of stochastic differential utility and its implications for asset pricing (see, e.g., Duffie and Epstein (1992a,b), Duffie and Skiadas (1994), Schroder and Skiadas (1999, 2003), and Skiadas (2003)). For completeness, in this Appendix we offer a simple formal derivation of the pricing kernel that obtains in an exchange economy where the representative agent has a KPEZ recursive utility. Our contribution is to characterize equilibrium prices in an exchange economy where aggregate output has particular jump-diffusion dynamics (Propositions 1 and 2).

A.1 Representation of Preferences and Pricing Kernel

We assume the existence of a standard filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ on which there exists a vector $z(t)$ of d independent Brownian motions and one counting process $N(t) = \sum_i \mathbf{1}_{\{\tau_i \leq t\}}$ for a sequence of inaccessible stopping times τ_i, $i = 1, 2, \ldots$.[11]

Aggregate consumption in the economy is assumed to follow a continuous process, with stochastic growth rate and volatility, which both may experience jumps:

$$d \log C_t = \mu_C(X_t)\, dt + \sigma_C(X_t)\, dz(t) \tag{A.1}$$

$$dX_t = \mu_X(X_t)\, dt + \sigma_X(X_t)\, dz(t) + \widetilde{\Gamma}\, dN(t)\,, \tag{A.2}$$

where X_t is a n-dimensional Markov process (we assume sufficient regularity on the coefficient of the stochastic differential equation (SDE) for it to be well-defined, e.g., Duffie (2001) Appendix B). In particular μ_X is an $(n,1)$ vector, σ_X is an (n,d) matrix and $\widetilde{\Gamma}$ is a $(n,1)$ vector of i.i.d. random variable with joint density (conditional on a jump $dN(t) = 1$) of $\jmath(\nu)$. We further assume that the counting process has a (positive integrable) intensity $\lambda(X_t)$ in the sense that $\left(N(t) - \int_0^t \lambda(X_s)\, ds\right)$ is a $(P, \mathcal{F}_t)$ martingale.

Following Epstein and Zin (1989), we assume that the representative agent's preferences over a consumption process $\{C_t\}$ are represented by a utility index $U(t)$ that satisfies the following recursive equation:

$$U(t) = \left\{ (1 - e^{-\beta dt}) C_t^{1-\rho} + e^{-\beta dt} \mathrm{E}_t \left(U(t+dt)^{1-\gamma} \right)^{\frac{1-\rho}{1-\gamma}} \right\}^{\frac{1}{1-\rho}}\,. \tag{A.3}$$

With $dt = 1$, this is the discrete time formulation of KPEZ, in which $\Psi \equiv 1/\rho$ is the EIS and γ is the risk-aversion coefficient.

To simplify the derivation let us define the function

$$u_\alpha(x) = \begin{cases} \frac{x^{1-\alpha}}{(1-\alpha)} & 0 < \alpha \neq 1 \\ \log(x) & \alpha = 1\,. \end{cases}$$

[11] $N(t)$ is a pure jump process and hence is independent of $z(t)$ by construction (in the sense that their quadratic co-variation is zero).

Further, let us define

$$g(x) = u_\rho(u_\gamma^{-1}(x)) \equiv \begin{cases} \frac{((1-\gamma)x)^{1/\theta}}{(1-\rho)} & \gamma, \rho \neq 1 \\ u_\rho(e^x) & \gamma = 1, \rho \neq 1 \\ \frac{\log((1-\gamma)x)}{(1-\gamma)} & \rho = 1, \gamma \neq 1\,, \end{cases}$$

where

$$\theta = \frac{1-\gamma}{1-\rho}\,.$$

Then, defining the 'normalized' utility index J as the increasing transformation of the initial utility index $J(t) = u_\gamma(U(t))$, equation (A.3) becomes:

$$g(J(t)) = (1 - e^{-\beta dt})u_\rho(C_t) + e^{-\beta dt}\, g\left(\mathrm{E}_t\left[J(t+dt)\right]\right)\,. \tag{A.4}$$

Using the identity $J(t + dt) = J(t) + dJ(t)$ and performing a simple Taylor expansion we obtain:

$$0 = \beta u_\rho(C_t)dt - \beta g(J(t)) + g'(J(t))\,\mathrm{E}_t\left[dJ(t)\right]\,. \tag{A.5}$$

Slightly rearranging the above equation, we obtain a backward recursive stochastic differential equation which could be the basis for a formal definition of stochastic differential utility (see Duffie Epstein (1992), Skiadas (2003)):

$$\mathrm{E}_t[dJ(t)] = -\frac{\beta u_\rho(C_t) - \beta g(J(t))}{g'(J(t))}dt\,. \tag{A.6}$$

Indeed, let us define the so-called 'normalized' aggregator function:

$$f(C, J) = \frac{\beta u_\rho(C) - \beta g(J)}{g'(J)} \equiv \begin{cases} \frac{\beta u_\rho(C)}{((1-\gamma)J)^{1/\theta - 1}} - \beta\theta J & \gamma, \rho \neq 1 \\ (1-\gamma)\beta J \log(C) - \beta J \log((1-\gamma)J) & \gamma \neq 1, \rho = 1 \\ \frac{\beta u_\rho(C)}{e^{(1-\rho)J}} - \frac{\beta}{1-\rho} & \gamma = 1, \rho \neq 1\,. \end{cases} \tag{A.7}$$

We obtain the following representation for the normalized utility index:

$$J(t) = \mathrm{E}_t\left(\int_t^T f(C_s, J(s)) + J(T)\right)\,. \tag{A.8}$$

Further, if the transversality condition $\lim_{T\to\infty} \mathrm{E}_t(J(T)) = 0$ holds, letting T tend to infinity, we obtain the simple representation:

$$J(t) = \mathrm{E}_t\left(\int_t^\infty f(C_s, J(s))ds\right)\,. \tag{A.9}$$

Fisher and Gilles (1999) discuss many alternative representations and choices of the utility index and associated aggregator as well as their interpretations. Here we note only the well-known fact that when $\rho = \gamma$ (i.e., $\theta = 1$) then $f(C, J) = \beta u_\rho(C) - \beta J$ and a simple application of Itô's lemma shows that

$$J(t) = \mathrm{E}_t\left(\int_t^\infty e^{-\beta(s-t)}\beta u_\rho(C_s)ds\right)\,.$$

To obtain an expression for the pricing kernel note that under the assumption (which we maintain throughout) that an 'interior' solution to the optimal consumption-portfolio choice of the agent exists, a necessary condition for optimality is that the gradient of the Utility index is zero for any small deviation of the optimal consumption process in a direction that is budget feasible. More precisely, let us define the utility index corresponding to such a small deviation by:

$$J^\delta(t) = \mathrm{E}_t\left(\int_t^\infty f\left(C_s^* + \delta\tilde{C}(s), J^\delta(s)\right)\, ds\right).$$

Then we may define the gradient of the utility index evaluated at the optimal consumption process $C^*(t)$ in the direction $\tilde{C}(t)$:

$$\begin{aligned}
\nabla J(C_t^*; \tilde{C}_t) &= \lim_{\delta\to 0} \frac{J^\delta(t) - J(t)}{\delta} \\
&= \lim_{\delta\to 0} \mathrm{E}_t\left[\int_t^\infty \frac{f(C_s^* + \delta\tilde{C}(s), J^\delta(s)) - f(C_s, J^\delta(s))}{\delta} ds\right] \\
&= \mathrm{E}_t\left[\int_t^\infty f_C(C_s^*, J(s))\tilde{C}_s + f_J(C_s, J(s))\nabla J(C_s^*; \tilde{C}_s) ds\right]. \qquad \text{(A.10)}
\end{aligned}$$

Assuming sufficient regularity (essentially the gradient has to be a semi-martingale and the transversality condition has to hold: $\lim_{T\to\infty} \mathrm{E}_t[e^{\int_t^T f_J(C_s,J_s)ds}\nabla J(C_T^*; \tilde{C}_T) = 0)$, a simple application of the generalized Itô-Doeblin formula gives the following representation:

$$\nabla^\delta J(C_t^*; \tilde{C}_t) = \mathrm{E}_t\left(\int_t^\infty e^{\int_t^s f_J(C_u,J_u)du} f_C(C_s, J_s)\tilde{C}_s ds\right). \qquad \text{(A.11)}$$

This shows that

$$\Pi(t) = e^{\int_0^t f_J(C_s,J_s)ds} f_C(C_t, J_t) \qquad \text{(A.12)}$$

is the Riesz representation of the gradient of the normalized utility index at the optimal consumption. Since a necessary condition for optimality is that $\nabla J(C_t^*; \tilde{C}_t) = 0$ for any feasible deviation $\tilde{C}_t$ from the optimal consumption stream C_t^*, we conclude that $\Pi(t)$ is a pricing kernel for this economy. (See Chapter 10 of Duffie (2001) for further discussion.)

A.2 Equilibrium Prices

Assuming the equilibrium consumption process given in equations (A.1)-(A.2) above, we obtain an explicit characterization of the felicity index J and the corresponding pricing kernel Π.

For this we define, respectively, the continuous diffusion, jump, and jump-compensator operators for any $h(\cdot) : \mathbb{R}^n - \mathbb{R}$:

$$\begin{aligned}
\mathcal{D}h(x) &= h_x(x)\mu_X(x) + \frac{1}{2}\mathrm{trace}(h_{xx}\sigma_X(x)\sigma_X(x)^\top) \\
\mathcal{J}h(x) &= \frac{h(x+\tilde{\nu})}{h(x)} - 1 \\
\overline{\mathcal{J}}h(x) &= \mathrm{E}[\mathcal{J}h(x)] = \int \ldots \int \frac{h(x+\nu)}{h(x)} \jmath(\nu) d\nu_1 \ldots d\nu_n \; - \; 1, \qquad \text{(A.13)}
\end{aligned}$$

where h_x is the $(n,1)$ Jacobian vector of first derivatives and h_{xx} denotes the (n,n) Hessian matrix of second derivatives. with these notations, we find:

Proposition 1 *Suppose $I(x) : \mathbb{R}^n \to \mathbb{R}$ solves the following equation:*

$$\begin{cases} 0 = I(x)\left((1-\gamma)\mu_C(x) + (1-\gamma)^2 \frac{||\sigma_C(x)||^2}{2} - \beta\theta\right) + \\ \qquad \frac{\mathcal{D}I(x)^\theta}{I(x)^{(\theta-1)}} + (1-\gamma)\theta\sigma_C(x)\sigma_X(x)^\top I_x(x) + \theta + I(x)\lambda(x)\overline{\mathcal{J}}I(x)^\theta & \text{for} \quad \rho,\gamma \neq 1 \\ 0 = I(x)\left((1-\rho)\mu_C(x) - \beta\right) + I(x)\mathcal{D}\log I(x) + 1 + I(x)\lambda(x)\log\left(1+\overline{\mathcal{J}}I(x)\right) & \text{for} \quad \gamma = 1, \rho \neq 1 \\ 0 = I(x)\left((1-\gamma)\mu_C(x) + (1-\gamma)^2\frac{||\sigma_C(x)||^2}{2}\right) + \mathcal{D}I(x) + \\ \qquad (1-\gamma)\sigma_C(x)\sigma_X(x)^\top I_x(x) - \beta I(x)\log I(x) + I(x)\lambda(x)\overline{\mathcal{J}}I(x) & \text{for} \quad \rho = 1, \gamma \neq 1 \end{cases} \tag{A.14}$$

and satisfies the transversality condition ($\lim_{T\to\infty} \mathrm{E}[J(T)] = 0$ for $J(t)$ defined below) then the value function is given by:

$$\begin{cases} J(t) = u_\gamma(C_t)(\beta I(x_t))^\theta & \text{for} \quad \rho,\gamma \neq 1 \\ J(t) = \log(C_t) + \frac{\log(\beta I(x_t))}{1-\rho} & \text{for} \quad \gamma = 1, \rho \neq 1 \\ J(t) = u_\gamma(C_t) I(x_t) & \text{for} \quad \rho = 1, \gamma \neq 1\,. \end{cases} \tag{A.15}$$

The corresponding pricing kernel is:

$$\begin{cases} \Pi(t) = e^{-\int_0^t (\beta\theta + \frac{(1-\theta)}{I(x_s)})ds}(C_t)^{-\gamma}(I(x_t))^{(\theta-1)} & \text{for} \quad \rho,\gamma \neq 1 \\ \Pi(t) = e^{-\int_0^t \frac{\beta}{I(x_s)})ds}\frac{1}{(C_t I(x_t))} & \text{for} \quad \gamma = 1, \rho \neq 1 \\ \Pi(t) = e^{-\int_0^t \beta(1+\log I(x_s))ds}(C_t)^{-\gamma} I(x_t) & \text{for} \quad \rho = 1, \gamma \neq 1\,. \end{cases} \tag{A.16}$$

Proof 1 *We provide the proof for the case $\gamma, \rho \neq 1$. The special cases are treated similarly.*

From its definition

$$J(t) = \mathrm{E}_t\left(\int_t^\infty f(c_s, J(s))\right)\,. \tag{A.17}$$

Thus, $J(x_t, c_t) + \int_0^t f(c_s, J(x_s, c_s))ds$ is a martingale. This observation implies that:

$$\mathrm{E}[dJ(x_t, c_s) + f(c_t, J(x_t, c_t))dt] = 0\,. \tag{A.18}$$

Equivalently:

$$\frac{\mathcal{D}J(c_t, X_t)}{J(c_t, X_t)} + \overline{\mathcal{J}}J(c_t, X_t) + \frac{f(c_t, J(c_t, X_t))}{J(c_t, X_t)} = 0\,. \tag{A.19}$$

To obtain the equation of the proposition, we use our guess ($J(t) = u_\gamma(c_t)\beta^\theta I(x)^\theta$) and apply the Itô-Doeblin formula using the fact that

$$\frac{f(c,J)}{J} = \frac{u_\rho(c_t)}{((1-\gamma)J)^{1/\theta-1}J} - \beta\theta = \frac{\theta}{I(x)} - \beta\theta \tag{A.20}$$

$$\frac{\mathcal{D}J(t)}{J(t)} = (1-\gamma)\mu_c(x) + \frac{1}{2}(1-\gamma)^2||\sigma_c(x)||^2 + \frac{\mathcal{D}I(x)^\theta}{I(x)^\theta} + (1-\gamma)\theta\sigma_c(x)\sigma_I(x)^\top\,, \tag{A.21}$$

where we have defined $\sigma_I(x_t)^\top = \frac{\sigma_X(x_t)^\top I_x(x_t)}{I(x_t)}$.

Now suppose that $I(\cdot)$ solves this equation. Then, applying the Itô-Doeblin formula to our candidate $J(t)$ we obtain

$$\begin{aligned} J(T) &= J(t) + \int_t^T \mathcal{D}J(s)ds + \int_t^T J_c\sigma_c dz_c(s) + \int_t^T J_x\sigma_x dz_x(s) + \int_t^T J(s^-)\mathcal{J}I(X_s)^\theta dN(s) \\ &= J(t) - \int_t^T f(c_s, J_s)ds + \int_t^T J(s)(1-\gamma)\sigma_c(x_s)dz_c(s) + \int_t^T \theta J(s)\sigma_I(x_s)dz_x(s) + \int_t^T dM(s), \end{aligned}$$

where we have defined the pure jump martingale

$$M(t) = \int_0^t J(s^-)\mathcal{J}I(X_s)^\theta \, dN(s) - \int_0^t \lambda(X_{s^-})J(s^-)\overline{\mathcal{J}}I(X_s)^\theta \, ds\,.$$

If the stochastic integral is a martingale,[12] *and if the transversality condition is satisfied, then we obtain the desired result by taking expectations and letting T tend to infinity:*

$$J(t) = \mathrm{E}\left[\int_t^\infty f(c_s, J_s)\, ds\right], \tag{A.22}$$

which shows that our candidate $J(t)$ solves the recursive stochastic differential equation. Uniqueness follows (under some additional technical conditions) from the appendix in Duffie, Epstein, Skiadas (1992).

The next result investigates the property of equilibrium prices.

Proposition 2 *The risk-free interest rate is given by:*

$$\begin{cases} r(x_t) = \beta + \rho(\mu_C(x_t) + \frac{||\sigma_C(x_t)||^2}{2}) - \gamma(1+\rho)\frac{||\sigma_C(x)||^2}{2} - \\ \qquad (1-\theta)\sigma_I(x)^\top(\sigma_C(x_t) + \frac{1}{2}\sigma_I(x_t)) + \lambda(x_t)\left(\frac{\theta-1}{\theta}\overline{\mathcal{J}}I^\theta - \overline{\mathcal{J}}I^{(\theta-1)}\right) & \text{for} \quad \rho \neq 1 \\ r(x_t) = \beta + \mu_C(x_t) + \frac{||\sigma_C(x_t)||^2}{2} - \gamma||\sigma_C(x_t)||^2 & \text{for} \quad \rho = 1\,. \end{cases} \tag{A.23}$$

Further, the value of the claim to aggregate consumption is given by:

$$\begin{cases} V(t) = C(t)I(x_t) & \text{for} \quad \rho \neq 1 \\ V(t) = \frac{C(t)}{\beta} & \text{for} \quad \rho = 1\,. \end{cases} \tag{A.24}$$

Thus

$$\frac{dV_t}{V_t} = \mu_V(x_t)dt + \left(\sigma_C(x_t) + \sigma_I(x_t)\mathbf{1}_{\{\rho\neq1\}}\right)dz(t) + \mathcal{J}I(x_t)dN(t)\,, \tag{A.25}$$

The risk premium on the claim to aggregate consumption is given by

$$\mu_V(x) + \frac{1}{I(x)} - r(x) = (\gamma\sigma_C(x) + (1-\theta)\sigma_I(x))^\top(\sigma_C(x) + \sigma_I(x)) + \lambda(x)\left(\overline{\mathcal{J}}I(x)^\theta - \overline{\mathcal{J}}I(x)^{\theta-1} - \overline{\mathcal{J}}I(x)\right). \tag{A.26}$$

[12] Sufficient conditions are:

$$\mathrm{E}\left[\int_0^T J(s)^2\left(||\sigma_c(x_s)||^2 + ||\sigma_I(x_s)||^2\right) ds\right] < \infty \ \ \forall T > 0\,.$$

Proof 2 *To prove the result for the interest rate, apply the Itô-Doeblin formula to the pricing kernel. It follows from* $r(t) = -\mathrm{E}[\frac{d\Pi(t)}{\Pi(t)}]/dt$ *that:*

$$r(x_t) = \beta\theta + \frac{(1-\theta)}{I(x_t)} + \gamma\mu_c(x_t) - \frac{1}{2}\gamma^2||\sigma_c(x_t)||^2 - \frac{\mathcal{D}I(x_t)^{(\theta-1)}}{I(x_t)^{(\theta-1)}} - \lambda(X_t)\overline{\mathcal{J}}I(x)^{\theta-1}\,. \tag{A.27}$$

Now substitute the expression for $\frac{1}{I(x)}$ *from the equation in (A.14) to obtain the result.*

To prove the result for the consumption claim, define $V(t) = c_t I(x_t)$*. Then using the definition of*

$$\Pi(t) = e^{-\beta\theta t - \int_0^t \frac{(1-\theta)}{I(x_s)}ds} c_t^{-\gamma} I(x_t)^{\theta-1}$$

we obtain:

$$d\left(\Pi(t)\,V(t)\right) = e^{-\beta\theta t - \int_0^t \frac{(\theta-1)}{I(x_s)}ds}\left(dJ(t) - J(t)\left(\beta\theta + \frac{(1-\theta)}{I(x_t)}\right)dt\right). \tag{A.28}$$

Note that by definition we have:

$$\begin{aligned} dJ(t) &= -f(c_t, J)dt + dM_t \\ &= -J(t)\left(\frac{\theta}{I(x_t)} - \theta\beta\right)dt + dM_t \end{aligned} \tag{A.29}$$

for some P*-martingale* M*. Combining this observation with (A.28), we get:*

$$\begin{aligned} d\left(\Pi(t)\,V(t)\right) &= e^{-\beta\theta t - \int_0^t \frac{(1-\theta)}{I(x_s)}ds}\frac{(-J(t))}{I(x_t)}dt + e^{-\beta\theta t - \int_0^t \frac{(1-\theta)}{I(x_s)}ds}dM_t \\ &= -\Pi(t)\,c(t)\,dt + e^{-\beta\theta t - \int_0^t \frac{(1-\theta)}{I(x_s)}ds}dM_t\,. \end{aligned} \tag{A.30}$$

Thus integrating we obtain

$$\Pi(T)V(T) + \int_t^T \Pi(s)c_s ds = \Pi(t)V(t) + \int_t^T e^{-\beta\theta(u-t) - \int_t^u \frac{(1-\theta)}{I(x_s)}ds}\,dM_u\,. \tag{A.31}$$

Taking expectations, letting $T \to \infty$*, and assuming the transversality condition holds (i.e.,* $\lim_{T\to\infty} \mathrm{E}[\Pi(T)V(T)] = 0$*), we obtain the desired result:*

$$\Pi(t)V(t) = \mathrm{E}_t\left[\int_t^\infty \Pi(s)\,c_s\,ds\right]. \tag{A.32}$$

To derive the excess return equation note that the martingale condition implies:

$$\mathrm{E}_t[\frac{d\Pi(t)V(t)}{\Pi(t)V(t)}] + \frac{D(t)}{V(t)}dt = 0 \tag{A.33}$$

Further, Itô's lemma implies:

$$\begin{aligned} \frac{1}{dt}\mathrm{E}_t[\frac{d\Pi(t)V(t)}{\Pi(t)V(t)}] &= \frac{1}{dt}\mathrm{E}_t\left[\frac{d\Pi(t)}{\Pi(t)} + \frac{dV(t)}{V(t)} + \frac{d\Pi(t)}{\Pi(t)}\frac{dV(t)}{V(t)}\right] \\ &= -r + \mu_V + (\gamma\sigma_C(x) + (1-\theta)\sigma_I(x))^\top(\sigma_C(x) + \sigma_I(x)) + \frac{1}{dt}E_t[\mathcal{J}I(x)^{\theta-1}\mathcal{J}I(x)] \\ &= -r + \mu_V + (\gamma\sigma_C(x) + (1-\theta)\sigma_I(x))^\top(\sigma_C(x) + \sigma_I(x)) + \lambda(x)\left(\overline{\mathcal{J}}I(x)^\theta - \overline{\mathcal{J}}I(x)^{\theta-1} - \overline{\mathcal{J}}I(x)\right). \end{aligned} \tag{A.34}$$

Combining equations (A.33) and (A.34) we get the expression for the excess return on the consumption claim given in equation (A.26).

B Application to the three-dimensional Model

Here we apply the general equations given in Appendix A to our three-state variable model, where the state vector is $X_t = (x_t,\, \Omega_t,\, p_t)$, whose dynamics are given in equations (3), (4), and (8).

B.1 Price-Consumption Ratio

The equation for the price-consumption ratio follows immediately from the dynamics of (x_t, Ω_t, p_t) given in equations (3), (4), and (8) and the PDE (14):

$$\begin{aligned} 0 \;=\; & I\left[(1-\gamma)\mu_C + (1-\gamma)x - \frac{\gamma}{2}(1-\gamma)\Omega - \beta\theta\right] - \theta I_x \kappa_x x \\ & + \frac{\theta}{2}\left(\sigma_{x0} + (\sigma_{xc}^2 + \sigma_{x\Omega})\Omega\right)\left[(\theta-1)\left(\frac{I_x}{I}\right)^2 I + I_{xx}\right] + \theta I_\Omega \kappa_\Omega\left(\overline{\Omega} - \Omega\right) \\ & + \frac{\sigma_\Omega^2}{2}\Omega\theta\left(I_{\Omega\Omega} + (\theta-1)\left(\frac{I_\Omega}{I}\right)^2 I\right) + \theta\left[(\theta-1)\left(\frac{I_x}{I}\right)\left(\frac{I_\Omega}{I}\right) I + I_{x\Omega}\right]\sigma_{xc}\sigma_\Omega\rho_{\Omega C}\Omega \\ & + \theta\left[-p\left(\lambda^G - \overline{\lambda}(p)\right) + (1-p)\phi_{BG} - p\phi_{GB}\right] I_p + (1-\gamma)\theta I_x \sigma_{xc}\Omega \\ & + (1-\gamma)\theta I_\Omega \sigma_\Omega \rho_{\Omega C}\Omega + \overline{\lambda}(p) I\, \overline{\mathcal{J}}\left[I^\theta\right] + \theta\,. \end{aligned} \tag{B.1}$$

B.2 Pricing Kernel and Riskfree Rate

When $\rho, \gamma \neq 1$, the pricing kernel in our three factor economy is

$$\Pi(t) \;=\; e^{\int_0^t ds\,[(\theta-1)I(s)^{-1} - \beta\theta]}\,\beta^\theta\, e^{-\gamma c}\, I(t)^{(\theta-1)}. \tag{B.2}$$

Ito's lemma gives

$$\begin{aligned} \frac{d\Pi}{\Pi} \;=\; & -r\,dt + dz_C\left(-\gamma\sqrt{\Omega} + (\theta-1)\left(\frac{I_x}{I}\right)\sigma_{xc}\sqrt{\Omega} + (\theta-1)\left(\frac{I_\Omega}{I}\right)\sigma_\Omega\sqrt{\Omega}\rho_{\Omega C}\right) \\ & + dz_x\left[(\theta-1)\left(\frac{I_x}{I}\right)\sqrt{\sigma_{x0} + \sigma_{x\Omega}\,\Omega}\right] + dz_\Omega\left[(\theta-1)\left(\frac{I_\Omega}{I}\right)\sigma_\Omega\sqrt{\Omega}\sqrt{1-\rho_{\Omega C}^2}\right] \\ & + dN\left[\frac{I^{\theta-1}(x+\widetilde{\nu}, \Omega+\widetilde{M}, p+\Delta p)}{I^{\theta-1}(x,\Omega,p)} - 1\right] - \overline{\lambda}(p)\,dt\,\mathrm{E}\left[\frac{I^{\theta-1}(x+\widetilde{\nu}, \Omega+\widetilde{M}, p+\Delta p)}{I^{\theta-1}(x,\Omega,p)} - 1\right], \end{aligned} \tag{B.3}$$

where the riskfree rate r equals:

$$\begin{aligned} -r(x,\Omega,p) \;=\; & \left[(\theta-1)I^{-1} - \beta\theta\right] - \gamma\left[\left(\mu_C + x - \frac{1}{2}\Omega\right)\right] + \frac{1}{2}\gamma^2\Omega + (\theta-1)\left(\frac{I_x}{I}\right)\left[-\kappa_x x\right] \\ & + \frac{1}{2}(\theta-1)\left[(\theta-2)\left(\frac{I_x}{I}\right)^2 + \left(\frac{I_{xx}}{I}\right)\right]\left(\sigma_{x0} + (\sigma_{xc}^2 + \sigma_{x\Omega})\Omega\right) + (\theta-1)\left(\frac{I_\Omega}{I}\right)\left[\kappa_\Omega\left(\overline{\Omega} - \Omega\right)\right] \\ & + \frac{1}{2}(\theta-1)\left[(\theta-2)\left(\frac{I_\Omega}{I}\right)^2 + \left(\frac{I_{\Omega\Omega}}{I}\right)\right]\sigma_\Omega^2\Omega + \overline{\lambda}(p)\,\mathrm{E}\left[\frac{I^{\theta-1}(x+\widetilde{\nu}, \Omega+\widetilde{M}, p+\Delta p)}{I^{\theta-1}(x,\Omega,p)} - 1\right] \\ & + (\theta-1)\frac{I_p}{I}\left[-p\left(\lambda^G - \overline{\lambda}(p)\right) + (1-p)\phi_{BG} - p\phi_{GB}\right] - \gamma(\theta-1)\left(\frac{I_x}{I}\right)\sigma_{xc}\Omega \\ & - \gamma(\theta-1)\left(\frac{I_\Omega}{I}\right)\sigma_\Omega\rho_{\Omega C}\Omega + (\theta-1)\left[(\theta-2)\left(\frac{I_x}{I}\right)\left(\frac{I_\Omega}{I}\right) + \left(\frac{I_{x\Omega}}{I}\right)\right]\sigma_{xc}\sigma_\Omega\rho_{\Omega C}\Omega. \end{aligned} \tag{B.4}$$

B.3 Price Dividend Ratio and Equity Premium

The price of the stock market portfolio, $S(t)$, satisfies the well-known formula

$$
\begin{aligned}
0 &= \frac{1}{dt}\mathrm{E}_t\left[d\left(\Pi(t)\,S(t)\right)\right]+\Pi(t)D(t) \\
&= \frac{1}{dt}\mathrm{E}_t\left[S(t)\,d\Pi(t)+\Pi(t)\,dS(t)+d\Pi(t)\,dS(t)\right]+\Pi(t)D(t). \qquad \text{(B.5)}
\end{aligned}
$$

Dividing by $\Pi(t)$ and using $\mathrm{E}\left[\frac{d\Pi}{\Pi}\right]=-r\,dt$ yields

$$
0=-rS(t)+\frac{1}{dt}\mathrm{E}_t\left[dS(t)\right]+\frac{1}{dt}\mathrm{E}_t\left[\frac{d\Pi}{\Pi}\,dS(t)\right]+D(t). \qquad \text{(B.6)}
$$

We define the price-dividend ratio $L(x,\Omega,p)$ via $S(x,\Omega,p,D)=DL(x,\Omega,p)$ and substitute in equation (B.6). Then, dividing by D we find

$$
0=-rL+\frac{L}{dt}\mathrm{E}_t\left[\frac{dL}{L}+\frac{dD}{D}+\frac{dD}{D}\frac{dL}{L}\right]+\frac{L}{dt}\mathrm{E}_t\left[\frac{d\Pi}{\Pi}\left(\frac{dL}{L}+\frac{dD}{D}+\frac{dD}{D}\frac{dL}{L}\right)\right]+1. \qquad \text{(B.7)}
$$

To solve this equation, we need the dynamics of $L(x,\Omega,p)$, which we obtain from Ito's lemma and the dynamics of the state vector in equations (3)-(4) and (8). Substituting in equation (B.7) we find

$$
\begin{aligned}
0 \;=\;& 1-rL+\left(\mu_D+\phi x\right)L-\kappa_x xL_x+\frac{1}{2}L_{xx}\left[\sigma_{x0}+(\sigma_{x\Omega}+\sigma_{xc}^2)\Omega\right]+L_\Omega\kappa_\Omega\left(\overline{\Omega}-\Omega\right)+\frac{1}{2}L_{\Omega\Omega}\sigma_\Omega^2\Omega \\
&+L_{x\Omega}\sigma_{xc}\sigma_\Omega\rho_{\Omega C}\Omega+L_p\Big[-p\left(\lambda^G-\overline{\lambda}(p)\right)+(1-p)\phi_{BG}-p\phi_{GB}\Big]+\sigma_D\rho_{DC}\left[L_x\sigma_{xc}+L_\Omega\sigma_\Omega\rho_{\Omega C}\right]\Omega \\
&+\Omega\Bigg[\sigma_D\rho_{DC}L+L_x\sigma_{xc}+L_\Omega\sigma_\Omega\rho_{\Omega C}\Bigg]\left[-\gamma+(\theta-1)\left(\frac{I_x}{I}\right)\sigma_{xc}+(\theta-1)\left(\frac{I_\Omega}{I}\right)\sigma_\Omega\rho_{\Omega C}\right] \\
&+(\theta-1)L_x\left(\frac{I_x}{I}\right)\left(\sigma_{x0}+\sigma_{x\Omega}\Omega\right)+(\theta-1)L_\Omega\left(\frac{I_\Omega}{I}\right)\sigma_\Omega^2\Omega\left(1-\rho_{\Omega C}^2\right) \\
&+\overline{\lambda}(p)\;\mathrm{E}\left[\left(\frac{I^{(\theta-1)}(x+\nu,\,\Omega+M,\,p+\Delta p)}{I^{(\theta-1)}(x,\Omega,p)}\right)\Big(L(x+\nu,\,\Omega+M,\,p+\Delta p)-L(x,\,\Omega,\,p)\Big)\right]. \qquad \text{(B.8)}
\end{aligned}
$$

Next, we derive the equity risk premium, $(\mu-r)$, via the relation $(\mu-r)=-\frac{1}{dt}\mathrm{E}\left[\frac{d\Pi}{\Pi}\frac{dS}{S}\right]$, which yields

$$
\begin{aligned}
(\mu-r) \;=\;& -(\theta-1)\left(\frac{I_x}{I}\right)\left(\frac{L_x}{L}\right)\Bigg(\sigma_{x0}+\sigma_{x\Omega}\Omega\Bigg)-(\theta-1)\left(\frac{I_\Omega}{I}\right)\left(\frac{L_\Omega}{L}\right)\sigma_\Omega^2\Omega\left(1-\rho_{\Omega C}^2\right) \\
&-\Omega\left[-\gamma+(\theta-1)\left(\frac{I_x}{I}\right)\sigma_{xc}+(\theta-1)\left(\frac{I_\Omega}{I}\right)\sigma_\Omega\rho_{\Omega C}\right]\left[\sigma_D\rho_{DC}+\left(\frac{L_x}{L}\right)\sigma_{xc}+\left(\frac{L_\Omega}{L}\right)\sigma_\Omega\rho_{\Omega C}\right] \\
&-\overline{\lambda}(p)\mathrm{E}\left[\left(\frac{I^{(\theta-1)}(x+\nu,\,\Omega+M,\,p+\Delta p)}{I^{(\theta-1)}(x,\Omega,p)}-1\right)\left(\frac{L(x+\nu,\,\Omega+M,\,p+\Delta p)}{L(x,\,\Omega,\,p)}-1\right)\right]. \qquad \text{(B.9)}
\end{aligned}
$$

B.4 Risk-Neutral Dynamics

From the pricing kernel, we identify the following risk-neutral dynamics:

$$
dc \;=\; \left[\mu_C+x-\frac{1}{2}\Omega+\Omega\left(-\gamma+(\theta-1)\left(\frac{I_x}{I}\right)\sigma_{xc}+(\theta-1)\left(\frac{I_\Omega}{I}\right)\sigma_\Omega\rho_{\Omega C}\right)\right]dt+\sqrt{\Omega}\,dz_C^Q
$$

$$\begin{aligned}
\frac{dD}{D} &= \left[\mu_D + \phi\, x + \rho_{DC}\sigma_D\Omega\left(-\gamma + (\theta-1)\left(\frac{I_x}{I}\right)\sigma_{xc} + (\theta-1)\left(\frac{I_\Omega}{I}\right)\sigma_\Omega\rho_{\Omega C}\right)\right] dt \\
&\quad + \sigma_D\sqrt{\Omega}\left(\rho_{DC}\, dz_C^Q + \sqrt{1-\rho_{DC}^2}\, dz_D^Q\right) \\
dx &= \left[-\kappa_x x + \sigma_{xc}\Omega\left(-\gamma + (\theta-1)\left(\frac{I_x}{I}\right)\sigma_{xc} + (\theta-1)\left(\frac{I_\Omega}{I}\right)\sigma_\Omega\rho_{\Omega C}\right)\right] dt \\
&\quad + \left[(\sigma_{x0} + \sigma_{x\Omega}\,\Omega)\,(\theta-1)\left(\frac{I_x}{I}\right)\right] dt + \sigma_{xc}\sqrt{\Omega}\, dz_c^Q + \sqrt{\sigma_{x0} + \sigma_{x\Omega}\,\Omega}\; dz_x^Q + \widetilde{\nu}\, dN \\
d\Omega &= \left[\kappa_\Omega\left(\overline{\Omega} - \Omega\right) + \sigma_\Omega\rho_{\Omega C}\Omega\left(-\gamma + (\theta-1)\left(\frac{I_x}{I}\right)\sigma_{xc} + (\theta-1)\left(\frac{I_\Omega}{I}\right)\sigma_\Omega\rho_{\Omega C}\right)\right. \\
&\quad \left. + \sigma_\Omega^2(1-\rho_{\Omega C}^2)\Omega(\theta-1)\left(\frac{I_\Omega}{I}\right)\right] dt + \sigma_\Omega\sqrt{\Omega}\left(\rho_{\Omega C}\, dz_C^Q + \sqrt{1-\rho_{\Omega C}^2}\, dz_\Omega^Q\right) + \widetilde{M}\, dN \\
\overline{\lambda}^Q(p) &= \overline{\lambda}^P(p)\mathrm{E}\left[\frac{I^{(\theta-1)}(x+\nu,\,\Omega+M,\,p+\Delta p)}{I^{(\theta-1)}(x,\,\Omega,\,p)}\right], \qquad \text{(B.10)}
\end{aligned}$$

where

$$\pi^Q(\tilde{\nu}=\nu,\,\tilde{M}=M) = \pi^P(\tilde{\nu}=\nu,\,\tilde{M}=M) * \frac{I^{(\theta-1)}(x+\nu,\,\Omega+M,\,p+\Delta p)}{\mathrm{E}\left[I^{(\theta-1)}(x+\nu,\,\Omega+M,\,p+\Delta p)\right]}. \qquad \text{(B.11)}$$

C Affine Approximation to the Model

To solve the model, we approximate the price-consumption ratio with an exponential affine function,

$$I(p,x,\Omega) = e^{A(p)+B(p)x+F(p)\Omega}. \qquad \text{(C.1)}$$

Plugging the approximation of I into equation (B.1), and dividing by I, we find

$$\begin{aligned}
0 &= (1-\gamma)\mu_c + (1-\gamma)x - \frac{\gamma}{2}(1-\gamma)\Omega - \beta\theta - \theta B\kappa_x x + \frac{\theta^2}{2}\left[\sigma_{x0} + \left(\sigma_{xc}^2 + \sigma_{x\Omega}\right)\Omega\right]B^2 \\
&\quad + \theta F\kappa_\Omega\left(\overline{\Omega} - \Omega\right) + \left(\frac{\sigma_\Omega^2}{2}\right)\theta^2F^2\Omega + \theta^2 BF\sigma_{xc}\sigma_\Omega\rho_{\Omega c}\Omega \\
&\quad + \theta\left[-p\left(\lambda^G - \overline{\lambda}(p)\right) - p\phi_{GB} + (1-p)\phi_{BG}\right]\left[A_p + xB_p + \Omega F_p\right] \\
&\quad + (1-\gamma)\theta B\sigma_{xc}\Omega + (1-\gamma)\theta F\sigma_\Omega\rho_{\Omega c}\Omega + \frac{\theta}{I} - \overline{\lambda}(p) \\
&\quad + \overline{\lambda}(p)\; e^{\theta\left[A(p\lambda^G/\overline{\lambda}(p)) - A(p)\right]}\; e^{\theta x(B(p\lambda^G/\overline{\lambda}(p)) - B(p))}\; e^{\theta\Omega(F(p\lambda^G/\overline{\lambda}(p)) - F(p))} \\
&\quad \times e^{\theta B(p\lambda^G/\overline{\lambda}(p))(\overline{\nu} + \frac{\alpha}{\xi}) + \frac{\sigma_\nu^2\theta^2}{2}B(p\lambda^G/\overline{\lambda}(p))^2}\left(\frac{\xi}{\xi + \alpha\theta B(p\lambda^G/\overline{\lambda}(p)) - \theta F(p\lambda^G/\overline{\lambda}(p))}\right). \qquad \text{(C.2)}
\end{aligned}$$

To solve equation (C.2), we apply a continuous-time analog of the Campbell and Shiller (1988) log-linear approximation; see, e.g., Campbell and Viceira (2002) and Chacko and Viceira (2005). We use Taylor's formula to expand the exponential terms in x and Ω around the points $x_0 \equiv \left(\frac{\overline{\nu}}{\kappa_x}\right)\overline{\lambda}(p)$ and $\Omega_0 \equiv \overline{\Omega} + \left(\frac{1}{\xi\kappa_\Omega}\right)\overline{\lambda}(p)$. We then collect terms linear in x, linear in Ω, and independent of x and Ω to obtain a system of three equations that define the functions $A(p)$, $B(p)$, and $F(p)$:

$$0 = (1-\gamma)\mu_c - \beta\theta + \frac{\theta^2}{2}\sigma_{x0}B^2 + \theta F\kappa_\Omega\overline{\Omega} - \overline{\lambda}(p) + \theta e^{-(A + x_0 B + \Omega_0 F)}\left(1 + x_0 B + \Omega_0 F\right)$$

$$
\begin{aligned}
& +\zeta_1(p)\left[1-\theta x_0\left(B(p\lambda^G/\overline{\lambda}(p))-B(p)\right)-\theta\Omega_0\left(F(p\lambda^G/\overline{\lambda}(p))-F(p)\right)\right] \\
& +\theta\left[-p(\lambda^G-\overline{\lambda}(p))-p\phi_{GB}+(1-p)\phi_{BG}\right]A_p \\
0 \;=\; & (1-\gamma)-\theta B\kappa_x-\theta Be^{-(A+Bx_0+F\Omega_0)}+\zeta_1(p)\theta\left[B(p\lambda^G/\overline{\lambda}(p))-B(p)\right] \\
& +\theta\left[-p(\lambda^G-\overline{\lambda}(p))-p\phi_{GB}+(1-p)\phi_{BG}\right]B_p \\
0 \;=\; & -\left(\frac{\gamma}{2}\right)(1-\gamma)+\frac{\theta^2}{2}\left(\sigma_{xc}^2+\sigma_{x\Omega}\right)B^2-\theta F\kappa_\Omega+\left(\frac{\sigma_\Omega^2}{2}\right)\theta^2F^2+\theta^2BF\sigma_{xc}\sigma_\Omega\rho_{\Omega c} \\
& +(1-\gamma)\theta B\sigma_{xc}+(1-\gamma)\theta F\sigma_\Omega\rho_{\Omega c}-\theta Fe^{-(A+Bx_0+F\Omega_0)}+\zeta_1(p)\theta\left[F(p\lambda^G/\overline{\lambda}(p))-F(p)\right] \\
& \theta\left[-p(\lambda^G-\overline{\lambda}(p))-p\phi_{GB}+(1-p)\phi_{BG}\right]F_p, \qquad \text{(C.3)}
\end{aligned}
$$

where we have defined the function:

$$
\begin{aligned}
\zeta_1(p) \;\equiv\; & \overline{\lambda}(p)\, e^{\theta\left[A(p\lambda^G/\overline{\lambda}(p))-A(p)\right]}e^{\theta(\overline{\nu}+\frac{\alpha}{\xi})\,B(p\lambda^G/\overline{\lambda}(p))+\frac{\sigma_\nu^2\theta^2}{2}B^2(p\lambda^G/\overline{\lambda}(p))}e^{\theta x_0\left[B(p\lambda^G/\overline{\lambda}(p))-B(p)\right]} \\
& \times e^{\theta\Omega_0\left[F(p\lambda^G/\overline{\lambda}(p))-F(p)\right]}\left(\frac{\xi}{\xi+\alpha\theta B(p\lambda^G/\overline{\lambda}(p))-\theta F(p\lambda^G/\overline{\lambda}(p))}\right). \qquad \text{(C.4)}
\end{aligned}
$$

We approximate the functions $A(p)$, $B(p)$, and $F(p)$ with a linear combination of general Chebyshev polynomials, and determine the coefficients of the approximation via least square minimization of the approximation error (e.g., Judd (1998)). We extend the approximation to include Chebyshev polynomials up to order 20 (adding higher-order polynomials does not change the solution). This approach gives us a semi-closed form solution to the model, which facilitates the analysis greatly. To check the accuracy of this approach, we also solve the model via fixed-point iterations over the price-dividend ratio I. Albeit considerably slower, this alternative method converges to a nearly identical solution.

We continue our approximation by looking for a price-dividend ratio of the form

$$
L(x,\Omega,p)=e^{A^L(p)+B^L(p)x+F^L(p)\Omega}. \qquad \text{(C.5)}
$$

We plug this expression into the price-dividend ratio equation (B.8), divide by L, and Taylor expand the exponential terms to be linear in x and Ω around the points x_0 and Ω_0. We then collect terms to obtain a system of three equations that define the functions $A^L(p)$, $B^L(p)$, and $F^L(p)$:

$$
\begin{aligned}
0 \;=\; & A_p^L\left[-p\left(\lambda^G-\overline{\lambda}(p)\right)+(1-p)\phi_{BG}-p\phi_{GB}\right] \\
& +e^{-A^L(p)-B^L(p)x_0-F^L(p)\Omega_0}(1+B^L(p)x_0+F^L(p)\Omega_0) \\
& +(\theta-1)\,e^{-A-Bx_0-F\Omega_0}(1+Bx_0+F\Omega_0)-\beta\theta-\gamma\mu_C+\frac{1}{2}(\theta-1)^2B^2\sigma_{x0} \\
& +(\theta-1)F\kappa_\Omega\overline{\Omega}-\overline{\lambda}(p) \\
& +(\theta-1)A_p(p)\left[-p\left(\lambda^G-\overline{\lambda}(p)\right)+(1-p)\phi_{BG}-p\phi_{GB}\right] \\
& +\mu_D+\frac{(B^L)^2}{2}\sigma_{x0}+F^L\kappa_\Omega\overline{\Omega}+(\theta-1)B^L\,B\sigma_{x0} \\
& +\zeta_2(p)\;\left[1-\left[(\theta-1)\left[B(p\lambda^G/\overline{\lambda}(p))-B(p)\right]+B^L(p\lambda^G/\overline{\lambda}(p))-B^L(p)\right]x_0\right. \\
& \qquad\left.-\left[(\theta-1)\left[F(p\lambda^G/\overline{\lambda}(p))-F(p)\right]+F^L(p\lambda^G/\overline{\lambda}(p))-F^L(p)\right]\Omega_0\right]
\end{aligned}
$$

$$
\begin{aligned}
0 &= B^L_p \Big[-p\left(\lambda^G - \overline{\lambda}(p)\right) + (1-p)\phi_{BG} - p\phi_{GB}\Big] \\
&\quad - e^{-A^L(p) - B^L(p)x_0 - F^L(p)\Omega_0} B^L(p) - (\theta-1)\, e^{-A - Bx_0 - F\Omega_0} B - \gamma - (\theta-1)B\kappa_x \\
&\quad + (\theta-1)B_p(p) \Big[-p\left(\lambda^G - \overline{\lambda}(p)\right) + (1-p)\phi_{BG} - p\phi_{GB}\Big] \\
&\quad + \phi - \kappa_x B^L \\
&\quad + \zeta_2(p)\left[(\theta-1)\left[B(p\lambda^G/\overline{\lambda}(p)) - B(p)\right] + B^L(p\lambda^G/\overline{\lambda}(p)) - B^L(p)\right] \\
0 &= F^L_p \Big[-p\left(\lambda^G - \overline{\lambda}(p)\right) + (1-p)\phi_{BG} - p\phi_{GB}\Big] \\
&\quad - e^{-A^L(p) - B^L(p)x_0 - F^L(p)\Omega_0} F^L(p) - (\theta-1)\, e^{-A - Bx_0 - F\Omega_0} F \\
&\quad + \frac{\gamma(1+\gamma)}{2} + \frac{1}{2}(\theta-1)^2 B^2(\sigma^2_{xc} + \sigma_{x\Omega}) - (\theta-1)F\kappa_\Omega + \frac{1}{2}(\theta-1)^2F^2\sigma^2_\Omega \\
&\quad + (\theta-1)F_p(p) \Big[-p\left(\lambda^G - \overline{\lambda}(p)\right) + (1-p)\phi_{BG} - p\phi_{GB}\Big] \\
&\quad - \gamma(\theta-1)B\sigma_{xc} - \gamma(\theta-1)F\sigma_\Omega\rho_{\Omega C} \\
&\quad + (\theta-1)^2 BF\sigma_{xc}\sigma_\Omega\rho_{\Omega C} + \frac{(B^L)^2}{2}(\sigma_{x\Omega} + \sigma^2_{xc}) - F^L\kappa_\Omega + \frac{(F^L)^2}{2}\sigma^2_\Omega \\
&\quad + B^L F^L \sigma_{xc}\sigma_\Omega\rho_{\Omega C} + \sigma_D\rho_{DC}\left[B^L\sigma_{xc} + F^L\sigma_\Omega\rho_{\Omega C}\right] \\
&\quad + \left[\sigma_D\rho_{DC} + B^L\sigma_{xc} + F^L\sigma_\Omega\rho_{\Omega C}\right]\left[-\gamma + (\theta-1)B\sigma_{xc} + (\theta-1)F\sigma_\Omega\rho_{\Omega C}\right] \\
&\quad + (\theta-1)B^L B\sigma_{x\Omega} + (\theta-1)F^L F\sigma^2_\Omega\left(1 - \rho^2_{\Omega C}\right) \\
&\quad + \zeta_2(p)\left[(\theta-1)\left[F(p\lambda^G/\overline{\lambda}(p)) - F(p)\right] + F^L(p\lambda^G/\overline{\lambda}(p)) - F^L(p)\right], \qquad \text{(C.6)}
\end{aligned}
$$

where we have defined the function ζ_2:

$$
\begin{aligned}
\zeta_2(p) &= \overline{\lambda}(p) e^{(\theta-1)\left[A(p\lambda^G/\overline{\lambda}(p)) - A(p) + \left[B(p\lambda^G/\overline{\lambda}(p)) - B(p)\right]x_0 + \left[F(p\lambda^G/\overline{\lambda}(p)) - F(p)\right]\Omega_0\right]} \\
&\quad * e^{A^L(p\lambda^G/\overline{\lambda}(p)) - A^L(p) + \left[B^L(p\lambda^G/\overline{\lambda}(p)) - B^L(p)\right]x_0 + \left[F^L(p\lambda^G/\overline{\lambda}(p)) - F^L(p)\right]\Omega_0} \\
&\quad * \chi(B^L(p\lambda^G/\overline{\lambda}(p)) + (\theta-1)\, B(p\lambda^G/\overline{\lambda}(p)), F^L(p\lambda^G/\overline{\lambda}(p)) + (\theta-1)\, F(p\lambda^G/\overline{\lambda}(p))), \qquad \text{(C.7)}
\end{aligned}
$$

with $\chi(B,F) \equiv \mathrm{E}\left[e^{B\widetilde{\nu} + F\widetilde{M}}\right] = e^{B(\overline{\nu} + \frac{\alpha}{\xi}) + \frac{\sigma^2_\nu B^2}{2}} \frac{\xi}{\xi + \alpha B - F}$. Similar to $A(p)$, $B(p)$, and $F(p)$, we approximate $A^L(p)$, $B^L(p)$, and $F^L(p)$ with a linear combination of general Chebyshev polynomials of order 20, and determine the coefficients of this approximation via least square minimization of the approximation error.

D Pre- and Post-Crash Implied Volatility Patterns

Figure 1 shows the permanent regime shift in pre- and post-1987 market crash implied volatilities for S&P 500 options. The plot in Panel A depicts the spread between implied volatilities for S&P 500 options that have a strike-to-price ratio $X = K/S - 1 = -10\%$ and at-the-money implied volatilities. The plot in panel B depicts the spread between implied volatilities for options that have a strike-to-price ratio $X = K/S - 1 = 2.5\%$ and at-the-money implied volatilities.

D.1 American Options on the S&P 500 Futures

We construct implied volatility functions from 1985 to 1995 by using transaction data on American options on S&P 500 futures. As in Bakshi et al. (1997), prior to analysis we eliminate observations that have price lower than \$(3/8) to mitigate the impact of price discreteness on option valuation. Since near-maturity options are typically illiquid we also discard observations with time-to-maturity lower than 10 calendar days. For the same reason we do not use call and put contracts that are more than 3% in-the-money. Finally, we disregard observations on options that allow for arbitrage opportunities, e.g., calls with a premium lower than the early exercise value.

We consider call and put transaction prices with the three closest available maturities. For each contract we select the transaction price nearest to the time of the market close and we pair it with the nearest transaction price on the underlying S&P 500 futures. This approach typically results in finding a futures price that is time stamped within 6 seconds from the time of the option trade. We approximate the risk-free rate with the three-month Treasury yield and we compute implied volatilities using the Barone-Adesi and Whaley (1987) pricing formula for American options.

At each date and for each of the three closest maturities we interpolate the cross section of implied volatilities with a parabola. This approach is similar to the one used in Shimko (1993). In doing so we require that we have at least three implied volatility observations, one of which with a strike-to-price ratio $X = K/S - 1$ no higher than -9%, one with X no lower than 1.5%, and one in between these two extremes. We record the interpolated implied volatility at $X = 0$ and the implied volatility computed at the available X-values closest to -10% and 2.5%.

Then at each date and for each of the three X choices we interpolate the implied volatility values across the three closest maturities using a parabola. We use the fitted parabola to obtain the value of implied volatility at 30 days to maturity. If only two maturities are available, we replace the parabola with a linear interpolation. If only one maturity is available we retain the value of implied volatility observed at that maturity provided that such maturity is within 20 to 40 days.

Trading in American options on the S&P 500 futures contract began on January 28, 1983. Prior to 1987, only quarterly options maturing in March, June, September, and October were available. Additional serial options written on the next quarterly futures contracts and maturing in the nearest two months were introduced in 1987 (e.g., Bates (2000)). This data limitation, combined with the relatively scarce size and liquidity of the option market in early years, renders it difficult to obtain smirk observations at the 30-day maturity with -10% moneyness. Therefore, we start the plot in December 1985. After this date we find implied volatility values with the desired parameters for most trading days. Relaxing the time-to-maturity and moneyness requirements results in longer implied volatility series going back to January 1983. Qualitatively, the plot during the period from January 1983 to December 1985 remains similar to that for the period from December 1983 to October 1987 (see, e.g., Bates (2000)).

D.2 European Options on the S&P 500 Index

After April 1996, we use data on S&P 500 index European options. We obtain daily SPX implied volatilities from April 1996 to April 2006 from the Optionmetrics database. Similar to what discussed in Section D.1, we exclude options with price lower than \$(3/8), time-to-maturity lower than 10 calendar days, and contracts that are more than 3% in-the-money.

At each date and for each of the three closest maturities we interpolate the cross-section of implied volatilities using a parabola. We have also considered a spline interpolation, which has produced similar results. We use the fitted parabola to compute the value of implied volatilities for strike-to-index-price ratios $X = K/S - 1 = -10\%$, zero, and 2.5%. Finally, we interpolate implied volatilities at each of these three levels of moneyness across the three closest maturities. We use the fitted parabola to compute the value of implied volatility at the 30-day maturity.

References

Aït-Sahalia, Yacine, and Andrew W. Lo, 2000, Nonparametric Risk Management and Implied Risk Aversion, *Journal of Econometrics* 94, 9-51

Attanasio, Orazio P., and Guglielmo Weber, 1989, Intertemporal Substitution, Risk Aversion and the Euler Equation for Consumption, *Economic Journal* 99, 59-73.

Boudoukh, Jacob, Roni Michaely, Matthew Richardson, and Michael R. Roberts, 2007, On the Importance of Measuring Payout Yield: Implications for Empirical Asset Pricing, *Journal of Finance* 62, 877-915.

Bakshi, Gurdip, Charles Cao, and Zhiwu Chen, 1997, Empirical Performance of Alternative Option Pricing Models, *Journal of Finance* 52, 2003-2049.

Bakshi, Gurdip, Charles Cao, and Zhiwu Chen, 2000, Pricing and Hedging Long-Term Options, *Journal of Econometrics* 94, 277-318.

Bakshi, Gurdip, Nikunj Kapadia, and Dilip Madan, 2003, Stock Return Characteristics, Skew Laws, and the Differential Pricing of Individual Equity Options, *Review of Financial Studies* 16, 101-143.

Bansal, Ravi, A. Ronald Gallant, and George Tauchen, 2007, Rational Pessimism, Rational Exuberance, and Asset Pricing Models, *Review of Economic Studies* 74, 1005-1033.

Bansal, Ravi, Thomas D. Tallarini Jr., and Amir Yaron, 2006, The Return to Wealth, Asset Pricing, and the Intertemporal Elasticity of Substitution, Working Paper, Duke University, Board Of Governors, and Wharton School.

Bansal, Ravi, and Amir Yaron, 2004, Risks for the Long Run: A Potential Resolution of Asset Pricing Puzzles, *Journal of Finance* 59, 1481-1509.

Barone-Adesi, Giovanni, and Robert E. Whaley, 1987, Efficient Analytic Approximation of American Option Values, *Journal of Finance* 42, 301-320.

Barro, Robert J., 2006, Rare Disasters and Asset Markets in the Twentieth Century, *Quarterly Journal of Economics* 121, 823-866.

Bates, David S., 1996, Jumps and Stochastic Volatility: Exchange Rate Processes Implicit in Deutsche Mark Options, *Review of Financial Studies* 1, 69-107.

Bates, David S., 2000, Post-'87 Crash Fears in the S&P 500 Futures Option Market, *Journal of Econometrics* 94, 181-238.

Bates, David S., 2008, The Market for Crash Risk, *Journal of Economic Dynamics and Control* 32, 2291-2321.

Beeler, Jason, and John Y. Campbell, 2009 The Long-Run Risks Model and Aggregate Asset Prices: An Empirical Assessment, Working Paper, Harvard University.

Benzoni, Luca, 2002, Pricing Options under Stochastic Volatility: An Empirical Investigation, Working Paper, University of Minnesota.

Bollen, Nicolas P. B., and Robert E. Whaley, 2004, Does Net Buying Pressure Affect the Shape of Implied Volatility Functions? *Journal of Finance* 59, 711-753.

Bollerslev, Tim, George Tauchen, and Hao Zhou, 2009, Expected Stock Returns and Variance Risk Premia, *Review of Financial Studies*, forthcoming.

Bondarenko, Oleg, 2003, Why are Put Options So Expensive?, Working Paper, University of Illinois at Chicago.

Broadie, Mark, Mikhail Chernov, and Michael S. Johannes, 2007, Model Specification and Risk Premia: Evidence from Futures Options, *Journal of Finance* 62, 1453-1490.

Brown, David P., and Jens C. Jackwerth, 2004, The Pricing Kernel Puzzle: Reconciling Index Option Data and Economic Theory, Working Paper, UW Madison and University of Konstanz.

Buraschi, Andrea, and Alexei Jiltsov, 2006, Model Uncertainty and Option Markets with Heterogeneous Beliefs, *Journal of Finance* 61, 2841-2897.

Campbell, John Y., and Robert J. Shiller, 1988, The Dividend-Price Ratio and Expectations of Future Dividends and Discount Factors, *Review of Financial Studies* 1, 195-228.

Campbell, John Y., and Luis M. Viceira, 2002, Strategic Asset Allocation: Portfolio Choice for Long-Term Investors, Oxford University Press

Chacko, George, and Luis M. Viceira, 2005, Dynamic Consumption and Portfolio Choice with Stochastic Volatility in Incomplete Markets, *Review of Financial Studies* 18, 1369-1402.

Chernov, Mikhail, and Eric Ghysels, 2000, A Study Towards a Unified Approach to the Joint Estimation of Objective and Risk Neutral Measures for the Purpose of Options Valuation, *Journal of Financial Economics* 56, 407-458.

Chabi-Yo, Fousseni, René Garcia, and Eric Renault, 2008, State Dependence Can Explain the Risk Aversion Puzzle, *Review of Financial Studies* 21, 973-1011.

Collin-Dufresne, Pierre, Robert S. Goldstein, and Jean Helwege, 2002, Is Credit Event Risk Priced? Modeling Contagion via the Updating of Beliefs, Working Paper, UC Berkeley, University of Minnesota, and University of Arizona.

Constantinides, George M., Jens C. Jackwerth, and Stylianos Perrakis, 2009, Mispricing of S&P 500 Index Options, *Review of Financial Studies* 22, 1247-1277.

Coval, Joshua D., and Tyler Shumway, 2001, Expected Option Returns, *Journal of Finance* 56, 983-1009.

David, Alexander, 1997, Fluctuating Confidence in Stock Markets: Implications for Returns and Volatility, *Journal of Financial and Quantitative Analysis* 32, 427-462.

David, Alexander, and Pietro Veronesi, 2002, Option Prices with Uncertain Fundamentals, University of Calgary and University of Chicago.

David, Alexander, and Pietro Veronesi, 2009, Macroeconomic Uncertainty and Fear Measures Extracted From Index Options, University of Calgary and University of Chicago.

Dennis, Patrick, and Stewart Mayhew, 2002, Risk-Neutral Skewness: Evidence from Stock Options, *Journal of Financial and Quantitative Analysis* 37, 471-493.

Drechsler, Itamar, and Amir Yaron, 2008, What's Vol Got To Do With It, Working Paper, Wharton and NYU.

Driessen, Joost, and Pascal Maenhout, 2007, An Empirical Portfolio Perspective on Option Pricing Anomalies, *Review of Finance* 11, 561-603.

Duffie, Darrell, 2001, *Dynamic Asset Pricing Theory*, Princeton University Press, Princeton and Oxford.

Duffie, Darrell, and Larry G. Epstein, 1992a, Stochastic Differential Utility, *Econometrica* 60, 353-394.

Duffie, Darrell, and Larry G. Epstein, 1992b, Asset Pricing with Stochastic Differential Utility, *Review of Financial Studies* 5, 411-436.

Duffie, Darrell, Jun Pan, and Kenneth Singleton, 2000, Transform Analysis and Asset Pricing for Affine Jump Diffusions, *Econometrica* 68, 1343-137.

Duffie, Darrell, and Costis Skiadas, 1994, Continuous-Time Security Pricing: A Utility Gradient Approach, *Journal of Mathematical Economics* 23, 107-132.

Epstein, Larry G., and Stanley E. Zin, 1989, Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework, *Econometrica* 57, 937-969.

Eraker, Bjørn, 2004, Do Stock Prices and Volatility Jump? Reconciling Evidence from Spot and Option Prices, *Journal of Finance* 59, 1367-1404.

Eraker, Bjørn, 2008, Affine General Equilibrium Models, *Management Science* 54, 2068-2080.

Eraker, Bjørn, and Ivan Shaliastovich, An Equilibrium Guide to Designing Affine Pricing Models, Mathematical Finance 18, 519-543

Figlewski, Stephen, 1989, Options Arbitrage in Imperfect Markets, *Journal of Finance* 44, 1289-1311.

Fisher, Mark, and Christian Gilles, 1999, Consumption and Asset Prices with Homothetic Recursive Preferences, Working Paper, Federal Reserve Bank of Atlanta.

Fama, Eugene F., and Kenneth R. French, 1988, Dividend yields and expected stock returns, *Journal of Financial Economics* 22, 325.

Garcia, René, Richard Luger, and Eric Renault, 2001, Asymmetric Smiles, Leverage Effects and Structural Parameters, Working Paper, Université de Montréal and Bank of Canada.

Garcia, René, Richard Luger, and Eric Renault, 2003, Empirical Assessment of an Intertemporal Option Pricing Model with Latent Variables, *Journal of Econometrics* 116, 49-83.

Gârleanu, Nicolae B., Lasse H. Pedersen, and Allen M. Poteshman, 2009, Demand-Based Option Pricing, *Review of Financial Studies* 22, 4259-4299.

Green, T. Clifton, and Stephen Figlewski, 1999, Market Risk and Model Risk for a Financial Institution Writing Options *Journal of Finance* 54, 1465-1499.

Guvenen, Fatih, 2001, Mismeasurement of the Elasticity of Intertemporal Substitution: The Role of Limited Stock Market Participation. Working Paper, University of Rochester.

Hall, Robert E., 1988, Intertemporal Substitution in Consumption *Journal of Political Economy* 96, 339-357.

Han, Bing, 2008, Investor Sentiment and Option Prices *Review Financial Studies* 21, 387-414.

Hansen, Lars Peter, and Thomas J. Sargent, 2006, Fragile beliefs and the Price of Model Uncertainty, Working Paper, University of Chicago and NYU.

Hansen, Lars P., and Kenneth J. Singleton, 1982, Generalized Instrumental Variables Estimation of Nonlinear Rational Expectations Models, *Econometrica* 50, 1269-1286.

Heston, Steven L., 1993, A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options, *Review of Financial Studies* 6, 327-343.

Huang, Jing-zhi, and Liuren Wu, 2004, Specification Analysis of Option Pricing Models Based on Time-Changed Levy Processes, *Journal of Finance* 59, 1405-1440.

Hugonnier, Julien, Dmitry Kramkov, and Walter Schachermayer, 2005, On Utility Based Pricing of Contingent Claims in Incomplete Markets, *Mathematical Finance* 15, 203-212.

Jackwerth, Jens C., 2000, Recovering Risk Aversion from Option Prices and Realized Returns, *Review of Financial Studies* 13, 433-451.

Jackwerth, Jens C., and Mark Rubinstein, 1996, Recovering Probability Distributions from Option Prices, *Journal of Finance* 51, 1611-1631.

Jones, Christopher S., 2003, The Dynamics of Stochastic Volatility: Evidence from Underlying and Options Markets, *Journal of Econometrics* 116, 181-224.

Judd, Kenneth L., 1998, Numerical Methods in Economics, MIT Press.

Kleidon, Allan W., 1992, Arbitrage, Nontrading, and Stale Prices: October 1987, *Journal of Business* 65, 483-507.

Kreps, David M., and Evan L. Porteus, 1978, Temporal Resolution of Uncertainty and Dynamic Choice Theory, *Econometrica* 46, 185-200.

Liu, Jun, and Francis A. Longstaff, 2004, Losing Money on Arbitrage: Optimal Dynamic Portfolio Choice in Markets with Arbitrage Opportunities, *Review of Financial Studies* 17, 611-641.

Liu, Jun, and Jun Pan, 2003, Dynamic Derivative Strategies, *Journal of Financial Economics* 69, 401-430.

Liu, Jun, Jun Pan, and Tan Wang, 2005, An Equilibrium Model of Rare-Event Premia and Its Implication for Option Smirks, *Review of Financial Studies* 18, 131-164.

Longstaff, Francis A., 1995, Option Pricing and the Martingale Restriction, *Review of Financial Studies* 8, 1091-1124.

Mehra, Rajnish, and Edward C. Prescott, 1985, The Equity Premium: A Puzzle, *Journal of Monetary Economics* 15, 145-161.

Pan, Jun, 2002, The Jump-Risk Premia Implicit in Options: Evidence from an Integrated Time-Series Study, *Journal of Financial Economics* 63, 3-50.

Rosenberg, Joshua V., and Robert F. Engle, 2002, Empirical Pricing Kernels, *Journal of Financial Economics* 64, 341-372.

Rubinstein, Mark, 1994, Implied Binomial Trees, *Journal of Finance* 49, 771-818.

Santa-Clara, Pedro, and Alessio Saretto, 2009, Option strategies: Good deals and margin calls, *Journal of Financial Markets* 12, 391-417.

Schroder, Mark, and Costis Skiadas, 1999, Optimal Consumption and Portfolio Selection with Stochastic Differential Utility, *Journal of Economic Theory* 89, 68-126.

Schroder, Mark, and Costis Skiadas, 2003, Optimal Lifetime Consumption-Portfolio Strategies under Trading Constraints and Generalized Recursive Preferences, *Stochastic Processes and their Applications* 108, 155-202.

Shephard, Neil G., Andrew C. Harvey, 1990, On the Probability of Estimating a Deterministic Component in the Local Level Model, *Journal of Time Series Analysis* 11, 339-347.

Shleifer, Andrei, and Robert W. Vishny, 1997, The Limits of Arbitrage, *Journal of Finance* 52, 35-55.

Skiadas, Costis, 2003, Robust Control and Recursive Utility, *Finance and Stochastics* 7, 475-489.

Veronesi, Pietro, 1999, Stock Market Overreaction to Bad News in Good Times: A Rational Expectations Equilibrium Model, *Review of Financial Studies* 12, 975-1007.

Veronesi, Pietro, 2000, How Does Information Quality Affect Stock Returns?, *Journal of Finance* 55, 807-837.

Tables and Figures

Table 1: Model coefficients

The table reports the coefficients for our baseline model calibration, expressed with yearly decimal scaling.

Preferences					
$\gamma = 10$	$\Psi = 2$	$\beta = 0.0176$			
Consumption and dividends					
$\mu_C = 0.018$	$\mu_D = 0.025$	$\phi = 2.6050$	$\sigma_D = 5.3229$	$\rho_{DC} = 0.2523$	
Predictable mean component, x					
$\kappa_x = 0.2785$	$\sigma_{xc} = 0.1217$	$\sigma_{x0} = 0$	$\sigma_{x\Omega} = 0.1301$		
Economic uncertainty, Ω					
$\kappa_\Omega = 1.0484$	$\overline{\Omega} = 0.0006$	$\sigma_\Omega = 0.004$	$\rho_{\Omega C} = -0.6502$		
Jumps					
$\overline{\nu} = -0.035$	$\sigma_\nu = 0.0216$	ξ =2100	$\alpha = 3$	$\lambda^G = 0.0005$	$\lambda^B = 0.035$
Transition probabilities					
$\phi_{GB} = 0.0025$	$\phi_{BG} = 0.025$				
Idiosyncratic return shocks					
$\sigma_i = 0.3137$	$\mu_{\nu_i} = 0.0036$	$\sigma_{\nu_i} = 0.0632$	$\lambda_i = 5$		

Table 2: How well does the model match underlying fundamentals?

We compare sample moments of yearly growth rates in aggregate consumption and dividends to the values predicted by the baseline calibration of the model. The measure for aggregate consumption is the real yearly series of per-capita consumption expenditures in nondurable goods and services (source: NIPA tables, Bureau of Economic Analysis). We obtain monthly dividends from returns, with and without dividends, on the CRSP value-weighted market index (e.g., Fama and French (1988)). We sum the monthly dividends to obtain the yearly dividend series, and we deflate that series using CPI data. We report empirical results for two sample periods. The first spans 80 years of data, from 1929 to 2008, the second spans the post World War II period, 1946-2008. Standard errors estimates, in brackets, are robust with respect to both autocorrelation and heteroskedasticity. Next, we simulate 10,000 samples of monthly consumption and dividend data from the model, each spanning a period of 80 years (same as the length of the 1929-2008 sample period). We aggregate the monthly series to obtain series of yearly growth rates $\frac{\Delta C}{C}$ and $\frac{\Delta D}{D}$. For each of the 10,000 simulated samples, we compute summary statistics for these series. We report the mean value of these statistics, as well as the 5th, 50th, and 95th percentiles. We repeat the analysis with two simulation schemes. In the first set of simulations, we initialize the Markov chain for the λ process at $\lambda(t=0)=\lambda^G$. In the second set, we initialize $\lambda(t=0)=\lambda^B$.

	Data		Model, w/ initial cond. $\lambda(t=0)=\lambda^G$				Model, w/ initial cond. $\lambda(t=0)=\lambda^B$			
	1929-2008	1946-2008	Mean	5%	50%	95%	Mean	5%	50%	95%
Panel A: Consumption growth, $\frac{\Delta C}{C}$										
Mean	0.0189 (0.0031)	0.0190 (0.0021)	0.0166	0.0077	0.0166	0.0253	0.0147	0.0048	0.0148	0.0242
Std. dev.	0.0221 (0.0052)	0.0129 (0.0014)	0.0245	0.0206	0.0243	0.0288	0.0252	0.0209	0.0250	0.0303
Skewness	-1.2447 (0.7983)	-0.5378 (0.5905)	-0.0505	-0.5225	-0.0470	0.4074	-0.0981	-0.6251	-0.0838	0.3828
Kurtosis	7.7297 (1.8802)	3.3963 (0.7489)	2.9146	2.2735	2.8256	3.8531	2.9735	2.2853	2.8589	4.0348
$AC(1)$	0.4218 (0.1153)	0.3631 (0.0917)	0.4185	0.2335	0.4223	0.5905	0.4385	0.2466	0.4412	0.6181
$AC(2)$	0.1268 (0.1569)	0.0735 (0.1448)	0.1720	-0.0627	0.1725	0.4012	0.1932	-0.0443	0.1930	0.4291
$AC(5)$	-0.0123 (0.1285)	-0.0573 (0.0656)	0.0470	-0.1752	0.0453	0.2759	0.0552	-0.1714	0.0525	0.2872
$AC(10)$	0.0405 (0.0950)	0.0626 (0.1017)	-0.0268	-0.2500	-0.0269	0.1974	-0.0260	-0.2541	-0.0254	0.2060

Table 2, continued

	Data		Model, w/ initial cond. $\lambda(t=0)=\lambda^G$				Model, w/ initial cond. $\lambda(t=0)=\lambda^B$			
	1929-2008	1946-2008	Mean	5%	50%	95%	Mean	5%	50%	95%
Panel B: Dividend growth, $\frac{\Delta D}{D}$										
Mean	0.0173 (0.0114)	0.0224 (0.0094)	0.0205	-0.0101	0.0201	0.0508	0.0157	-0.0168	0.0156	0.0476
Std. dev.	0.1105 (0.0244)	0.0659 (0.0096)	0.1125	0.0958	0.1122	0.1298	0.1135	0.0965	0.1131	0.1314
Skewness	0.3822 (0.7873)	1.0729 (0.4959)	0.2388	-0.2034	0.2272	0.7149	0.2251	-0.2351	0.2133	0.7110
Kurtosis	9.0564 (2.4074)	5.3017 (0.8784)	3.0179	2.2896	2.8937	4.1583	3.0289	2.3014	2.9045	4.1497
$AC(1)$	0.1877 (0.1398)	0.2482 (0.1141)	0.2793	0.0969	0.2818	0.4514	0.2887	0.1028	0.2917	0.4650
$corr(\Delta C/C, \Delta D/D)$	0.5923 (0.1905)	0.1809 (0.1238)	0.3647	0.1746	0.3690	0.5378	0.3807	0.1880	0.3851	0.5556

Table 3: How well does the model match basic asset prices moments?

We compare sample asset prices moments to the values predicted by the baseline calibration of the model. The total real yearly market return, $\frac{\Delta S}{S} + \frac{D}{S}$, is the yearly return, inclusive of all distributions, on the CRSP value-weighted market index, adjusted for inflation using the CPI. The real risk-free rate r_f is the inflation-adjusted three-month rate from the 'Fama Risk Free Rates' database in CRSP. In computing the logarithmic price-dividend ratio, $log(S/D)$, we consider two measures of dividends. The first is the real dividend on the CRSP value-weighted index. The second is the real dividend on the CRSP value-weighted market index, adjusted to include share repurchases (Boudoukh et al. (2007)). Standard errors estimates, in brackets, are robust with respect to both autocorrelation and heteroskedasticity. Next, we simulate 10,000 samples of monthly stock market returns, riskfree rates, dividends, and stock market portfolio prices, each spanning a period of 80 years. We aggregate the monthly series at the yearly frequency. For each of the 10,000 simulated samples, we compute summary statistics for these series. We report the mean value of these statistics, as well as the 5th, 50th, and 95th percentiles. We repeat the analysis with two simulation schemes. In the first set of simulations, we initialize the probability process p at $p(t=0) = p^{Pre}$, which corresponds to the pre-crash economy. In the second set, we initialize $p(t=0) = p^{Post}$, which corresponds to the post-crash economy.

	Data		Model (pre-crash)				Model (post-crash)			
	1929-2008	1946-2008	Mean	5%	50%	95%	Mean	5%	50%	95%
Panel A: Total real market return, $\frac{\Delta S}{S} + \frac{D}{S}$										
Mean	0.0737 (0.0199)	0.0760 (0.0211)	0.0685	0.0380	0.0683	0.0994	0.0728	0.0423	0.0726	0.1040
Std. dev.	0.2032 (0.0177)	0.1809 (0.0143)	0.1618	0.1395	0.1614	0.1861	0.1628	0.1402	0.1623	0.1874
Std. dev. (monthly obs)	0.1903 (0.0179)	0.1477 (0.0070)	0.1518	0.1448	0.1513	0.1611	0.1522	0.1447	0.1514	0.1628
Skewness	-0.2086 (0.3233)	-0.3199 (0.4835)	0.3293	-0.1561	0.3156	0.8580	0.3178	-0.1834	0.3074	0.8487
Kurtosis	2.7813 (0.2419)	2.9173 (0.3479)	3.1954	2.3282	3.0154	4.6450	3.2097	2.3381	3.0288	4.6942
Skewness (monthly obs)	0.3309 (0.7081)	-0.6008 (0.2583)	-0.0843	-0.9511	0.0090	0.1514	-0.1205	-1.0525	0.0019	0.1495
Kurtosis (monthly obs)	11.3264 (2.3856)	4.9747 (0.8870)	4.0033	2.7948	3.0485	11.3500	4.2790	2.7986	3.0620	12.5785
Panel B: Real risk-free rate, r_f										
Mean	0.0064 (0.0073)	0.0069 (0.0063)	0.0107	0.0072	0.0108	0.0139	0.0087	0.0051	0.0089	0.0120
Std. dev.	0.0386 (0.0072)	0.0316 (0.0082)	0.0062	0.0046	0.0061	0.0082	0.0061	0.0045	0.0060	0.0081

Table 3, continued

	Data		Model (pre-crash)				Model (post-crash)			
	1929-2008	1946-2008	Mean	5%	50%	95%	Mean	5%	50%	95%
Panel C: Equity premium, $\frac{\Delta S}{S} + \frac{D}{S} - r_f$										
Mean	0.0673 (0.0214)	0.0691 (0.0195)	0.0578	0.0289	0.0575	0.0871	0.0641	0.0350	0.0638	0.0934
Std. dev.	0.2052 (0.0222)	0.1742 (0.0143)	0.1606	0.1384	0.1602	0.1845	0.1615	0.1391	0.1610	0.1858
Skewness	-0.2692 (0.3584)	-0.2705 (0.4980)	0.3360	-0.1465	0.3212	0.8610	0.3253	-0.1693	0.3117	0.8517
Kurtosis	3.0852 (0.3140)	3.0060 (0.4180)	3.1953	2.3307	3.0128	4.6461	3.2083	2.3397	3.0277	4.6654
Skewness (monthly obs)	0.2203 (0.6756)	-0.5836 (0.2622)	-0.0817	-0.9357	0.0103	0.1528	-0.1174	-1.0376	0.0031	0.1509
Kurtosis (monthly obs)	10.9809 (2.1599)	4.9824 (0.8981)	3.9883	2.7948	3.0486	11.2290	4.2599	2.8008	3.0621	12.4307
Panel D: Logarithmic price-dividend ratio, $log\left(\frac{S}{D}\right)$										
Mean	3.3450 (0.1069)	3.4346 (0.1207)	3.1630	3.0625	3.1720	3.2127	3.1149	2.9791	3.1306	3.1726
Mean (w/ shares rep.)	3.1519 (0.0698)	3.2027 (0.0814)								
Std. dev.	0.4239 (0.0699)	0.4249 (0.0733)	0.0879	0.0594	0.0799	0.1600	0.0863	0.0587	0.0791	0.1458
Std. dev. (w/ shares rep.)	0.2915 (0.0248)	0.2930 (0.0287)								

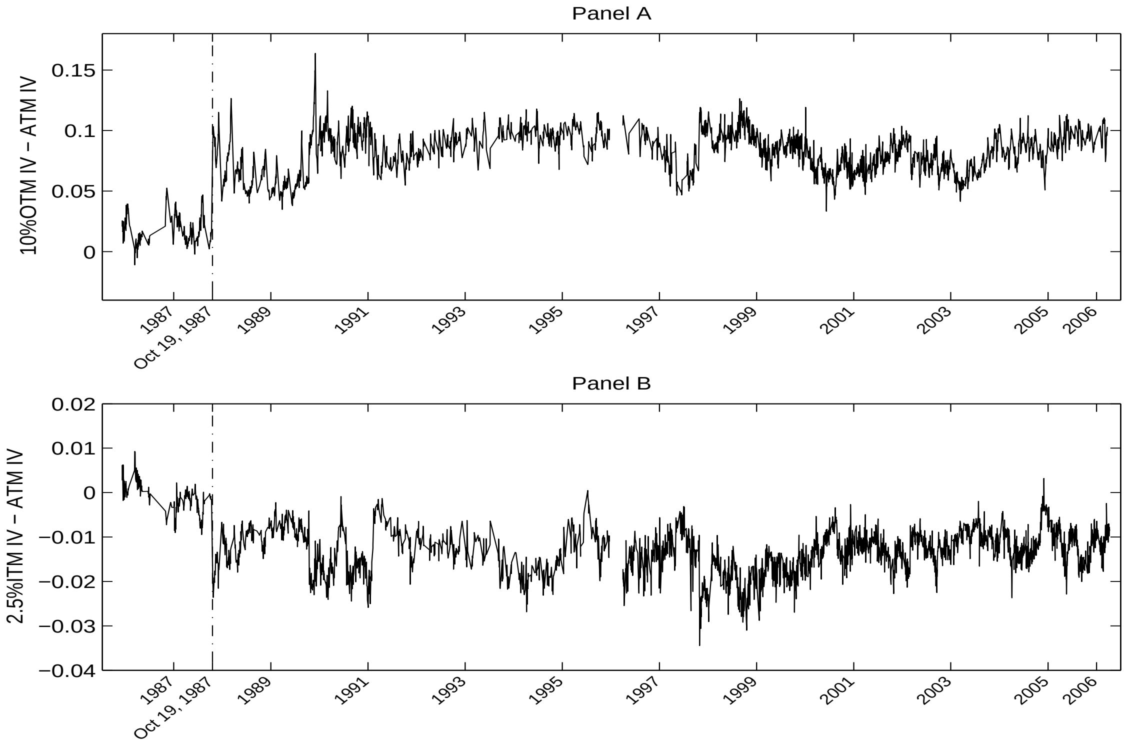

Figure 1: Pre- and Post-Crash Implied Volatility Smirk for S&P 500 Options with One Month to Maturity. The plot in Panel A depicts the spread between implied volatilities for S&P 500 options with a strike-to-price ratio $X = K/S - 1 = -10\%$ and at-the-money implied volatilities. The plot in Panel B depicts the spread between implied volatilities for options with a strike-to-price ratio $X = K/S - 1 = 2.5\%$ and at-the-money implied volatilities. Appendix D explains how we constructed the implied volatility series.

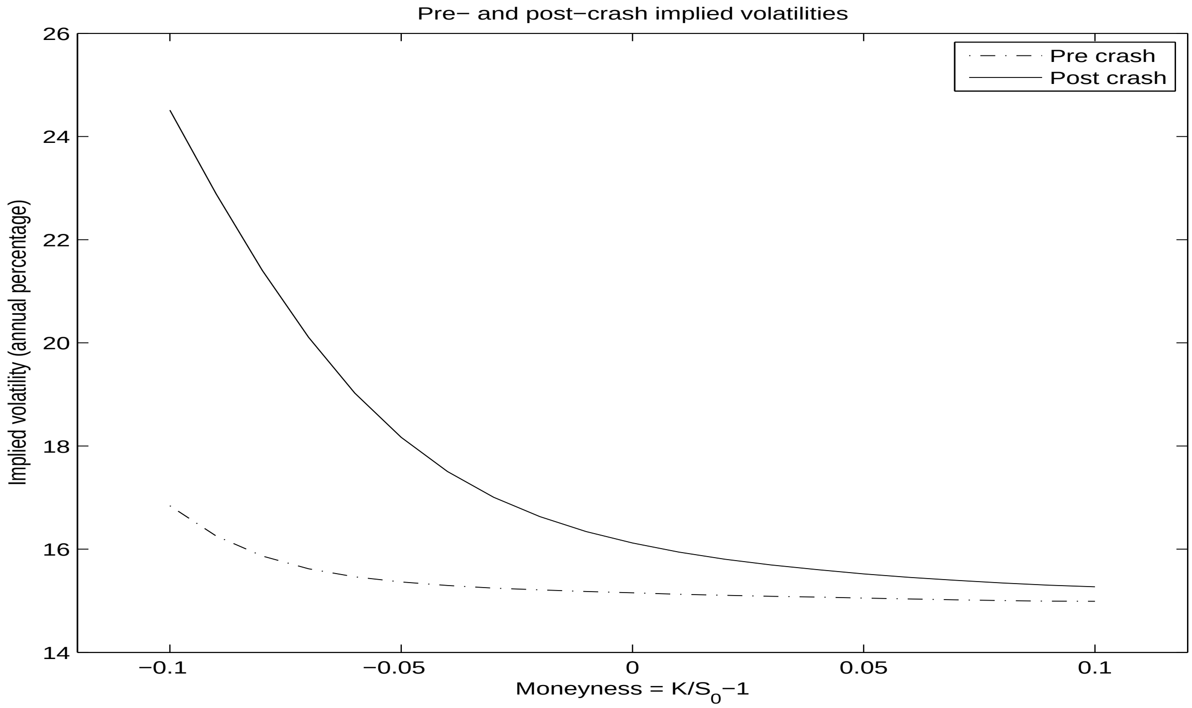

Figure 2: The plot depicts the implied volatility smirk pre- and post-1987 market crash. Implied volatilities are from S&P 500 options with one month to maturity. The model coefficients are set equal to the baseline values.

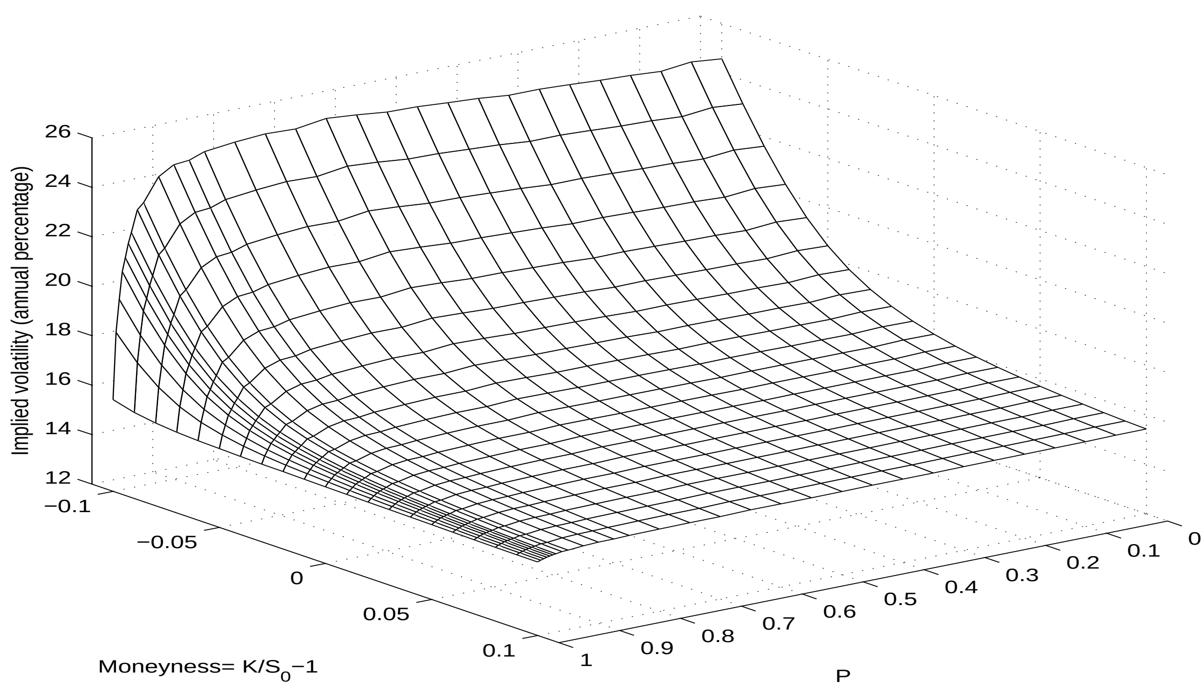

Figure 3: The plot depicts the implied volatility smirk as a function of the probability P the agent assign to be in the low-crash-intensity economy. Implied volatilities are from S&P 500 options with one month to maturity. The model coefficients are set equal to the baseline values.

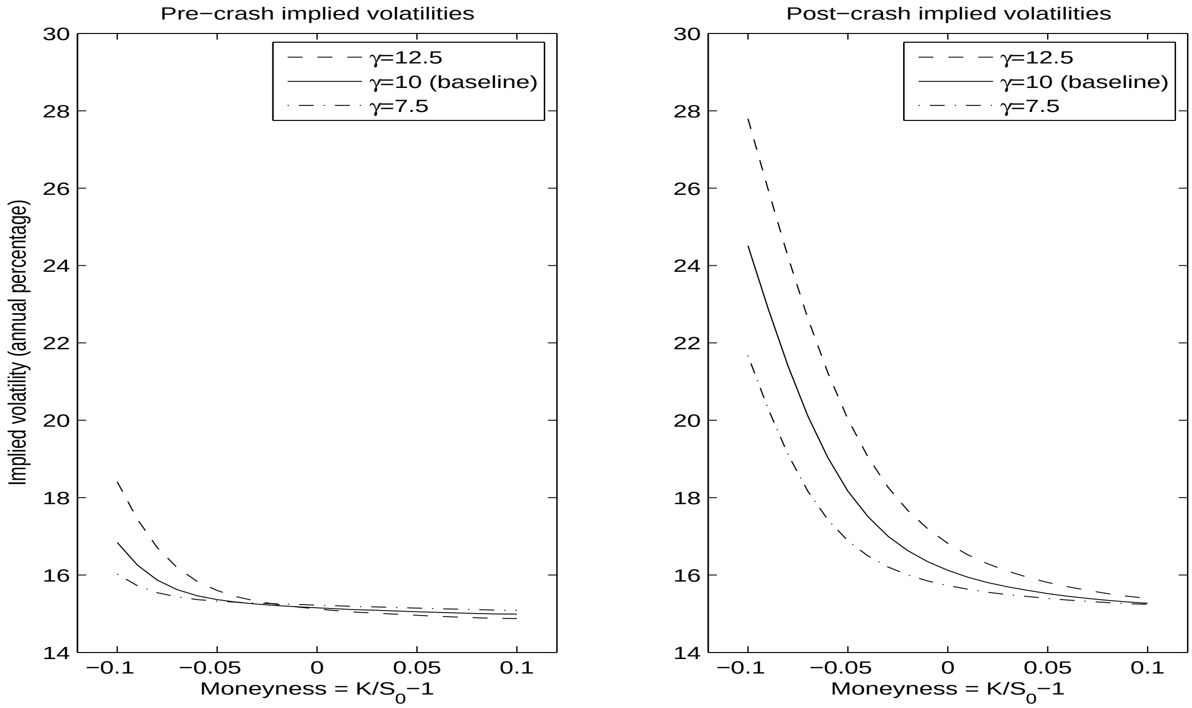

Figure 4: The plot illustrates the sensitivity of the implied volatility smirk to the elasticity of relative risk aversion coefficient γ. Implied volatilities are for S&P 500 options with one month to maturity.

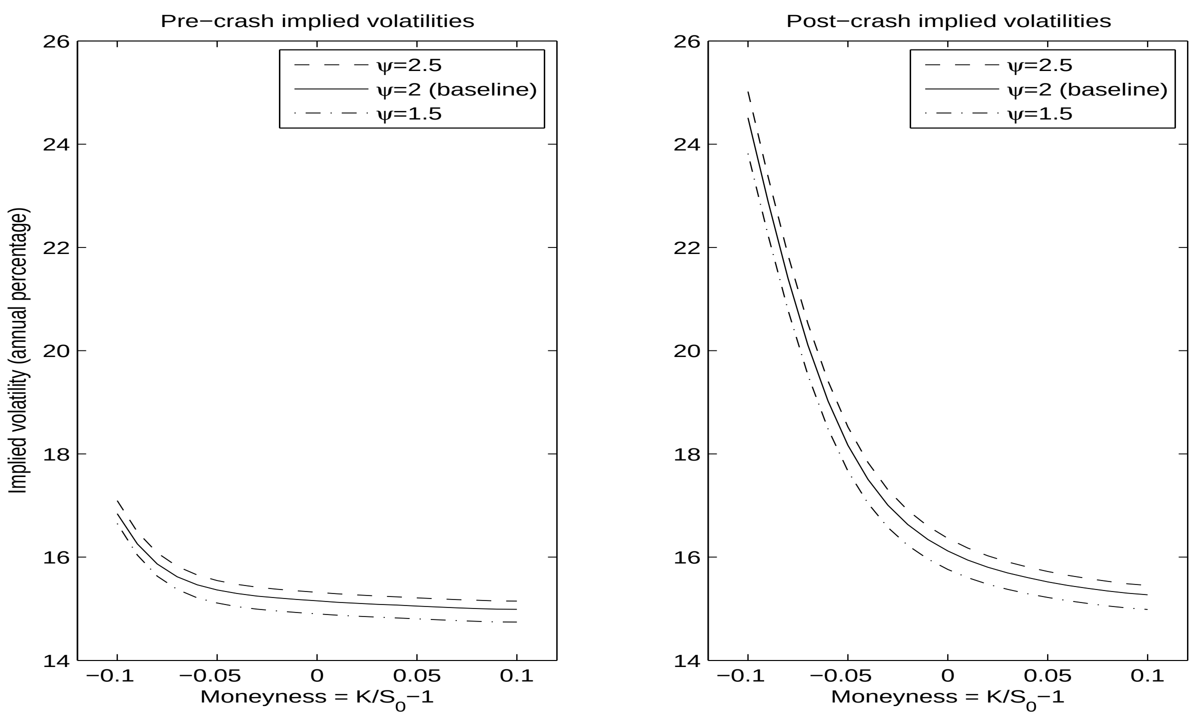

Figure 5: The plot illustrates the sensitivity of the implied volatility smirk to the elasticity of intertemporal substitution coefficient $\Psi = \frac{1}{\rho}$. Implied volatilities are for S&P 500 options with one month to maturity.

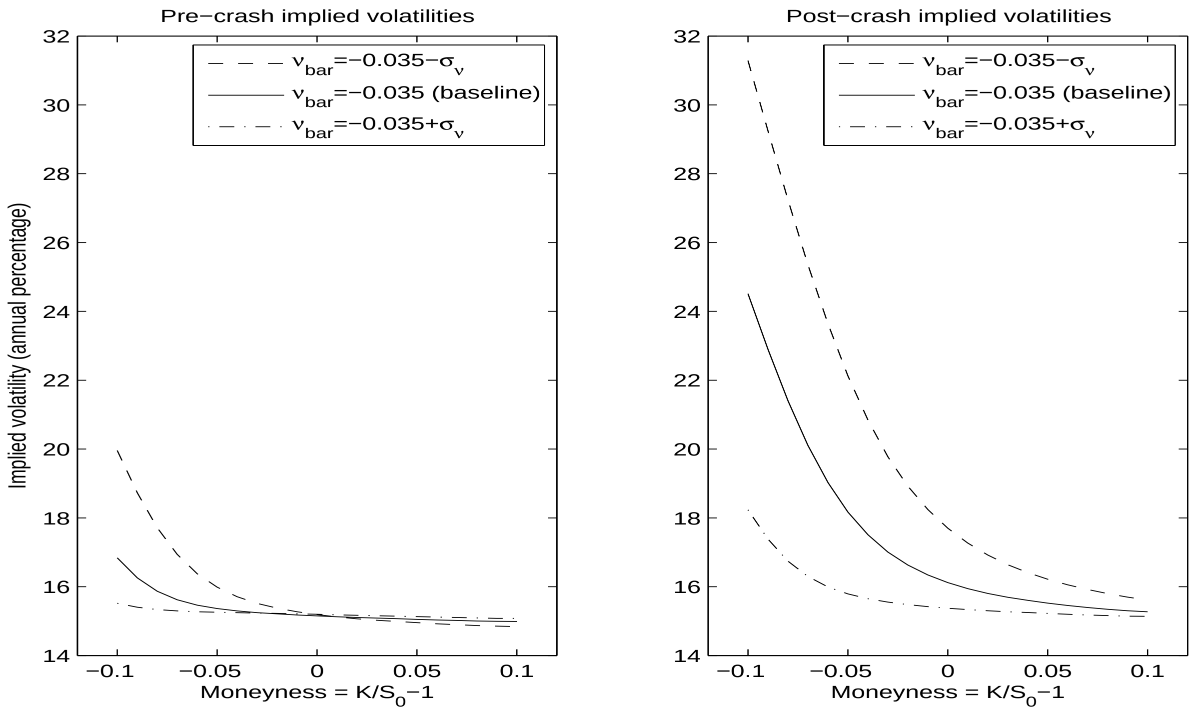

Figure 6: The plot illustrates the sensitivity of the implied volatility smirk to the jump coefficient $\bar{\nu}$. Implied volatilities are for S&P 500 options with one month to maturity.

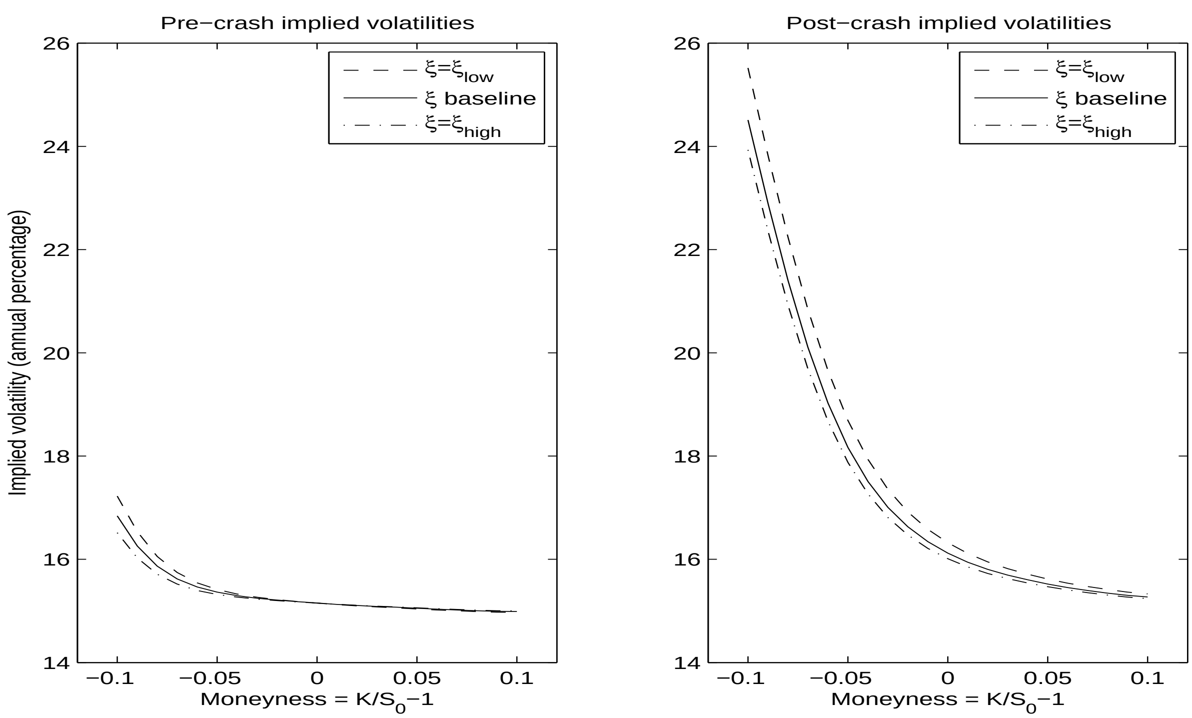

Figure 7: The plot illustrates the sensitivity of the implied volatility smirk to the jump coefficient ξ. Implied volatilities are for S&P 500 options with one month to maturity.

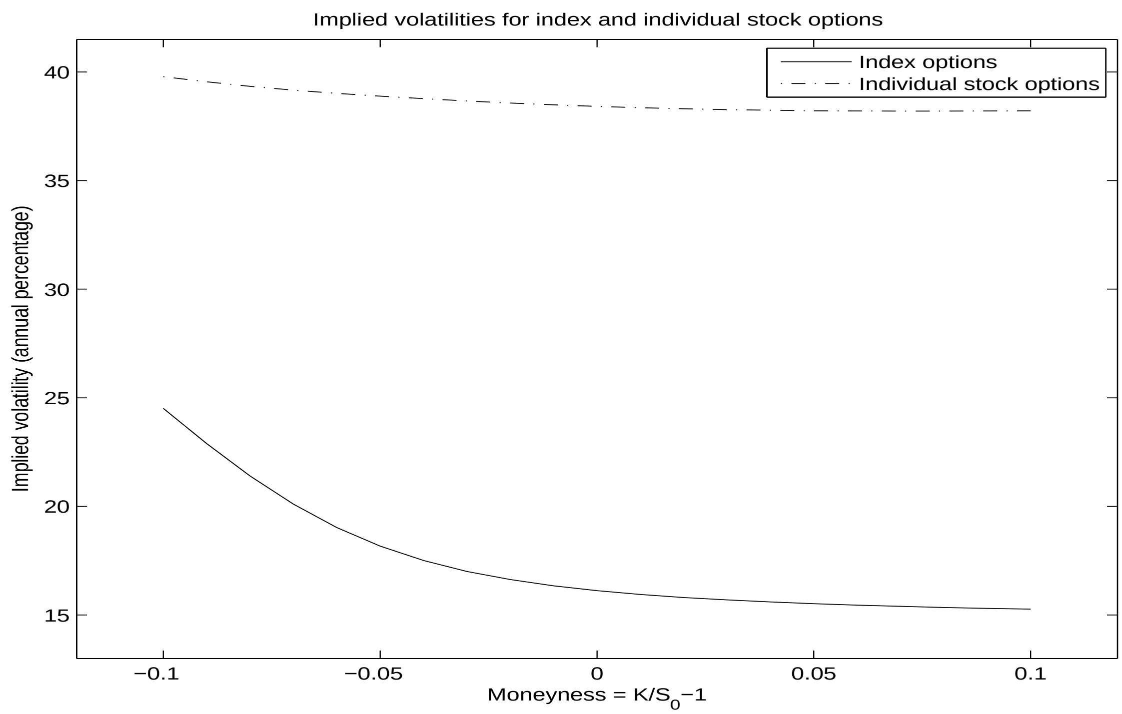

Figure 8: The plot contrasts the implied volatility function for individual stock options to the volatility smirk for S&P 500 index options with one month to maturity. The model coefficients are set equal to the baseline values.