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Asset Pricing Explorations for Macroeconomics

1. *Introduction and Overview*

Asset market data are often ignored in evaluating macroeconomic models, and aggregate quantity data are often avoided in empirical investigations of asset market returns. While there may be short-term benefits to proceeding along separate lines, we argue that security market data are among the most sensitive and, hence, attractive proving grounds for models of the aggregate economy.

An important strand of research on economic fluctuation uses models without frictions to explain the movements of aggregate quantities (e.g., Kydland and Prescott, 1982; Long and Plosser, 1983). Historically, asset market data have played little, if any, role in assessing the performance of these models. This habit is surprising. The models center on *intertemporal* decisions, and asset prices provide information about intertemporal marginal rates of substitution and transformation. Hence, asset market data should be valuable in assessing alternative model specifications. Once the basic point that equilibrium models *can* explain particular quantity correlations has been made, one would expect extensive use of price data in general and asset price data in particular to sort among the many specifications of preferences and technology that give roughly similar predictions for quantity correlations.

It is sometimes argued that successful models connecting real quantities to security market data may have to feature frictions, such as transactions costs, imperfect markets, liquidity or borrowing constraints, etc. (For an articulation of this view, see Mehra and Prescott, 1985). If, however, marginal rates of substitution are disconnected from asset returns because of frictions, why should one still expect marginal rates of substitution to line up with marginal rates of transformation? If market frictions are necessary to understand asset price data, they have potentially serious implications for the quantity predictions of business cycle models.

Perhaps the most convincing evidence for our view is that an array of researchers have studied new utility functions in an effort to address the dramatic failure of simple log or power utility models to account for basic features of asset pricing data. Although this research was not explicitly motivated by an effort to match correlations among aggregate quantities, the proposed changes in utility functions might substantially alter important dynamic properties of the resulting models, including measures of the welfare effects of interventions or policy experiments.

Not only can security data be informative for macroeconomic modeling, but macroeconomic modeling should be also valuable in interpreting the cross-sectional and time-series behavior of asset returns. A large body of empirical work on asset pricing aims simply at reducing asset valuation to the pricing of a relatively small number of "factors," without explicit reference to the fundamental sources of risk. While these dimensionality-reduction exercises can be quite useful in some contexts, it is difficult, if not impossible, to evaluate the significance of apparent asset-pricing anomalies without specifying an underlying valuation model that ties asset prices to fundamental features of the underlying economic environment, that is, without using some dynamic economic model. For example, the predictability of returns is only an anomaly given evidence that this predictability is at odds with the times series behavior of marginal rates of substitution or transformation. Clearly, documentation that expected returns on some assets vary over time "because" the expected return on the market or some factor portfolio varies over time fails to address this central issue.

As emphasized by Hansen and Richard (1987), *stochastic discount factors* provide a convenient vehicle for summarizing the implications of dynamic economic models for security market pricing. Alternative models can imply differing stochastic discount factors. A primary aim of this paper is to characterize the properties of the discount factors that are consistent with the behavior of asset market payoffs and prices. Such characterizations are useful for a variety of reasons. First, they provide a common set of diagnostics for a rich class of models, including new models that might be developed in the future. Second, they allow one to assess readily the information content of new financial data sets without recomputing a test of each candidate valuation model. Finally, they provide a general way of assessing the magnitude of asset-pricing *puzzles*. As emphasized by Fama (1970, 1991), almost all of the empirical work in finance devoted to the documentation of apparently anomalous behavior of security market payoffs and prices proceeds, implicitly or explicitly, within the context of particular asset pricing models. Characterizations of stochastic discount factors that are consistent with potentially anomalous security market data provide a more flexible way of understanding and interpreting the empirical findings.

The remainder of this paper is organized as follows. We survey Hansen and Jagannathan's (1991) methods for finding feasible regions for means and standard deviations of stochastic discount factors. We then extend these characterizations by exploring additional features of discount factors implied by security market data. For instance, unconditional volatility in discount factors can be attributed to either average conditional volatility or to variability in conditional means. We provide a characterization of this tradeoff as implied by security market data. We also quantify the sense in which candidate discount factors (implied by specific models) must be more volatile when they are less correlated with security market returns. We then apply these characterizations to reexamine a variety of stochastic discount factor models that have been proposed in the literature. Taken together, these exercises constitute Sections 2 and 3 of our paper.

In Section 4 we follow He and Modest (1991) and Luttmer (1991) and investigate the effects of market frictions on the implications of asset market data for analogs to stochastic discount factors. He and Modest (1991) and Luttmer (1991) have considered a variety of frictions such as short-sale constraints, bid/ask spreads or transactions costs, and borrowing constraints. Not surprisingly, these market imperfections tend to *loosen* the link between asset returns and marginal rates of substitution and transformation. However, they do not eliminate this link, and asset returns still provide useful information for dynamic economic models. We focus exclusively on borrowing constraints because of the attention these imperfections have received in the macroeconomics literature and because of their potential importance in welfare analyses.

2. *Interpreting Asset Market Data using the Frictionless Market Paradigm*

To assess the implications of asset market data for economic models and to discuss asset pricing anomalies, one needs some conceptual framework or paradigm. The frictionless market paradigm is by far the most commonly used framework, because it provides a conceptually simple and convenient benchmark. Of course, it is easy to be critical of frictionless markets. Several remarks come to mind under the heading, "the real world is complicated." Obviously, asset markets do not function exactly as described by this paradigm. At some level of inspection, market frictions such as transaction costs, short sale, and borrowing constraints must be important. Later in this essay, we will have more to say about market frictions. But a better understanding of the implications of asset market data viewed through the frictionless markets paradigm is a valuable (and perhaps necessary) precursor to assessing the importance of financial market imperfections.

2.1 STOCHASTIC DISCOUNT FACTORS

Many frictionless-market empirical analyses are conducted with the additional straitjacket of tightly specified models, featuring consumers that aggregate to known, simple utility functions. Among other things, aggregation typically requires that consumers engage in a substantial degree of risk pooling. Decisive empirical evidence obtained within this straitjacket is easily misconstrued as evidence against the frictionless-market paradigm itself. The points of this subsection are: (1) to emphasize that, as long as there are no arbitrage opportunities, we can always interpret asset market data through the frictionless-market paradigm; and (2) to show that the observable implications of frictionless-market asset pricing models can be conveniently understood by characterizing the *stochastic discount factors* through which such models generate asset price predictions.

We begin by developing the frictionless-markets paradigm in a *now* and *then* economy.[1] Trading in securities markets takes place in the *now* time period, and payoffs to holding these securities are received in a subsequent *then* time period. A payoff to a security is a random variable or equivalently a *bundle* of contingent claims in the *then* time period. Consumers/investors in this economy can form portfolios of securities, without transactions costs, short sale constraints, or other impediments to trade.

The *Principle of No-Arbitrage* follows when consumers are not satiated in the *then* time period. Because consumers always want more of the numeraire good, any portfolio with a payoff that is always nonnegative and sometimes positive must have a positive price. Equivalently, any claim contingent on an event that might occur must have a positive price.

The Principle of No-Arbitrage implies that alternative ways of constructing the same payoff must have the same cost or price, as long as there is a nontrivial, nonnegative portfolio payoff. Thus, the Principle

1. Our formulation closely follows the formulations of Ross (1976), Harrison and Kreps (1979), Kreps (1981), Hansen and Richard (1987), and Clark (1990).

of No-Arbitrage implies that each portfolio payoff must have a *unique* price, that is, we obtain the *Law of One Price.*

Consumers/investors can purchase a claim to a linear combination of any two security market payoffs by simply purchasing the corresponding linear combination of the securities. The unique assignment of prices to portfolio payoffs must inherit this linearity. Thus, we can think of asset pricing in arbitrage-free frictionless-markets as a *linear pricing functional* that maps the space of asset payoffs (*then*) into prices (*now*) on the real line.

How can we think about *testing* the frictionless-market paradigm? We could look for two portfolios with the same payoffs, but different prices (i.e., we could test the *Law of One Price*), or we could look for a portfolio with a nonnegative and nontrivial payoff with a nonpositive price (i.e., test the *Principle of No-Arbitrage*). The detection of pure arbitrage opportunities is seldom the aim of empirical work on asset prices, and consequently empirical researchers typically look at security market data sets that do not imply direct violations of the Principle of No-Arbitrage. For such data sets, we can *always* use the frictionless markets paradigm as an interpretive device. Equivalently, we will be able to find a *stochastic discount factor* that will correctly price all of the observed portfolio payoffs.

As an example, suppose we use data on n primitive payoffs in an econometric analysis. For example, we may use data on the measured one-period returns on n assets. Stack these payoffs into an n-dimensional random vector $\mathbf{x}$ with a finite second moment. A common space P of payoffs to use in econometric analyses of such assets consists of constant-weighted portfolios of the primitive payoffs:

$$P \equiv \{p : p = \mathbf{c} \cdot \mathbf{x} \text{ for some } \mathbf{c} \in \mathbb{R}^n\}, \tag{2.1}$$

where $\mathbf{c}$ is a vector of portfolio weights. Let the vector $\mathbf{q}$ denote the prices of the original payoff vector $\mathbf{x}$. When all of the original security payoffs are converted into returns, $\mathbf{q}$ is a vector of ones. We can then construct a candidate price of a portfolio payoff, say $\mathbf{c} \cdot \mathbf{x}$, from prices of the original n payoffs via:

$$\pi(\mathbf{c} \cdot \mathbf{x}) = \mathbf{c} \cdot \mathbf{q}. \tag{2.2}$$

The Law of One Price is simply the implication that this price assignment depends only on the payoff $\mathbf{c} \cdot \mathbf{x}$ itself and not necessarily on the choice of $\mathbf{c}$ used to construct this payoff. If $E(\mathbf{x}\mathbf{x}')$ is nonsingular, there is only one portfolio weight that achieves any attainable payoff. Thus, the Law of One Price is trivially satisfied. Clearly, the use of formula (2.2) to assign prices to payoffs implies that the pricing functional π will be linear on P.

A *stochastic discount factor* is any random variable y that correctly represents the prices of payoffs via the formula:

$$\pi(p) = E(yp) \text{ for all } p \text{ in } P. \tag{2.3}$$

The name is motivated by the fact that y is used to *discount* payoffs differently in alternative states of the world. Using the familiar covariance decomposition: $cov(y, p) = E(yp) - E(y)E(p)$, equation (2.3) is equivalent to

$$\pi(p) = E(y)E(p) + cov(y, p). \tag{2.4}$$

The first term on the right side of equation (2.4) uses $E(y)$ to discount the mean payoff, and the second term adjusts for the riskiness of the payoff.

The Riesz Representation Theorem guarantees the existence of a stochastic discount factor as long as the Law of One Price is satisfied. For our example, it is easy to construct a stochastic discount factor y:

$$y^* = \mathbf{x}'E(\mathbf{xx}')^{-1}\mathbf{q}. \tag{2.5}$$

This is not the only discount factor, however. For instance, choose any random variable e for which $E(e\mathbf{x}) = 0$. Then $y^* + e$ also is a stochastic discount factor. Define $\mathcal{Y}$ to be the family of all stochastic discount factors, that is, the family of all random variables with finite second moments that satisfy (2.3).

One theoretical device for generating a stochastic discount factor from an underlying model is to use the implied intertemporal marginal rate of substitution of consumers in the model economy. For instance, this is the device used in consumption-based or utility-based asset pricing theory. With a time-separable power utility function, the consumers' first-order conditions imply that equation (2.3) is satisfied for a "candidate" stochastic discount factor given by the marginal rate of substitution m:

$$m = \beta \frac{u'(c_{\text{then}})}{u'(c_{\text{now}})} = \beta(c_{\text{then}}/c_{\text{now}})^{-\gamma} \tag{2.6}$$

where $u(c) = [c^{1-\gamma} - 1]/(1 - \gamma)$ is the one-period power utility function, $\gamma \geq 0$, and $\beta > 0$ is a subjective discount factor. Hence, if accurate consumption data are available, the observable implications of this model specification are that m is in the set $\mathcal{Y}$ of admissible stochastic discount factors.

Utility-based models typically generate strictly positive candidates for stochastic discount factors. For example, in equation (2.6), $u'(c_{\text{now}}) > 0$ and $u'(c_{\text{then}}) > 0$ imply $m > 0$. More generally, Kreps (1981) and Clark (1990) show that under the Principle of No-Arbitrage, there will generally exist a *strictly positive* stochastic discount factor. With this in mind, we let $\mathcal{Y}^{++}$ denote the subset of $\mathcal{Y}$ consisting of all stochastic discount factors that are strictly positive. Any of these discount factors could be used to assign arbitrage-free prices to *derivative claims* formed from payoffs in P or formed from other payoffs traded by consumers. Equivalently, they could be used to assign positive prices to any nontrivial event-contingent claim in the *then* time period. Therefore, utility-based models often lead to a model-based way of constructing a strictly positive candidate m in the set $\mathcal{Y}^{++}$.

The stochastic discount factor given in equation (2.5) might well be negative with positive probability depending on the covariance structure of the primitive payoffs and might not be in $\mathcal{Y}^{++}$. Similarly, incomplete market models such as the familiar Capital Asset Pricing Model of Sharpe (1964), Lintner (1965), and Mossin (1968) and linear factor models as suggested by Ross (1976) and Connor (1984) imply candidate stochastic discount factors that need not be strictly positive. The Capital Asset Pricing Model implies a candidate discount factor that is equal to a constant minus a scale multiple of the return on the wealth portfolio. More generally, exact linear factor pricing models imply stochastic discount factors that are linear combinations of the/an underlying collection of "factors," but they do not restrict these linear combinations to be positive. Hence, whether $\mathcal{Y}$ or the smaller set $\mathcal{Y}^{++}$ is the relevant family of stochastic discount factors depends on the economic models being studied.

2.2 MOMENT IMPLICATIONS FOR DISCOUNT FACTORS

A large body of empirical work in asset pricing specifies and tests models with candidate stochastic discount factors. Given a candidate m, a chi-square test is formed using the sample counterpart to the moment restriction:

$$E(m\mathbf{x} - \mathbf{q}) = 0. \tag{2.7}$$

For models that imply a prespecified parametric family of such m's, one conducts the test by minimizing the hypothetical chi-square value and adjusting the degrees of freedom according to the number of estimated parameters (e.g., see Brown and Gibbons, 1985; Cochrane, 1992a; Epstein and Zin, 1991; Hansen, 1982; Hansen and Singleton, 1982; MacKinlay and Richardson, 1991).

This approach has been partially successful to date. However, statistical measures of fit such as a chi-square test statistic may not provide the most useful guide to the modifications that will reduce pricing or other specification errors. At times, the parametric approach looks like a fishing expedition without a well-articulated strategy for finding the promising fishing holes. Also, application of the minimum chi-square approach to estimation and inference sometimes focuses too much attention on whether a model is perfectly specified and not enough attention on assessing model performance.

Hansen and Jagannathan (1991) suggested a complementary empirical approach: Instead of proposing alternative parametric models and testing them, begin first by characterizing the set $\mathcal{Y}$ or $\mathcal{Y}^{++}$ of stochastic discount factors consistent with asset pricing data and divorced from a parametric specification.

To review the simplest characterizations obtained by Hansen and Jagannathan (1991), we study a regression of a discount factor y onto a constant and the vector $\mathbf{x}$ of asset payoffs observed by an econometrician

$$y = a + \mathbf{x}'\mathbf{b} + e, \tag{2.8}$$

where a is a constant term, $\mathbf{b}$ is a vector of slope coefficients, and e is the regression error. The standard least-squares formula for the regression coefficients gives:

$$\mathbf{b} \equiv [cov(\mathbf{x}, \mathbf{x})]^{-1} cov(\mathbf{x}, y) \tag{2.9}$$

$$a \equiv Ey - E\mathbf{x}'\mathbf{b}.$$

Without direct data on the stochastic discount factor y, these regression coefficients cannot be estimated in the usual fashion. Instead, we can exploit the fact that y must be a valid discount factor to infer them. The pricing relation $\mathbf{q} = E(y\mathbf{x})$ implies

$$cov(\mathbf{x}, y) = \mathbf{q} - E(y)E(\mathbf{x}). \tag{2.10}$$

Substituting equation (2.10) into equation (2.9), we obtain

$$\mathbf{b} = [cov(\mathbf{x}, \mathbf{x})]^{-1} [\mathbf{q} - E(y)E(\mathbf{x})]. \tag{2.11}$$

Hence, asset information alone can be used to construct the regression coefficients **b**, given $E(y)$.

Because the right-hand side variables of a regression are uncorrelated with residuals by construction,

$$var(y) = var(\mathbf{x}'\mathbf{b}) + var(e). \tag{2.12}$$

It follows that $var(\mathbf{x}'\mathbf{b})^{1/2}$ gives a lower bound on the standard deviation of y. Thus, we have a lower bound on the standard deviation of *all* admissible stochastic discount factors y in $\mathcal{Y}$ with the prespecified mean, Ey.

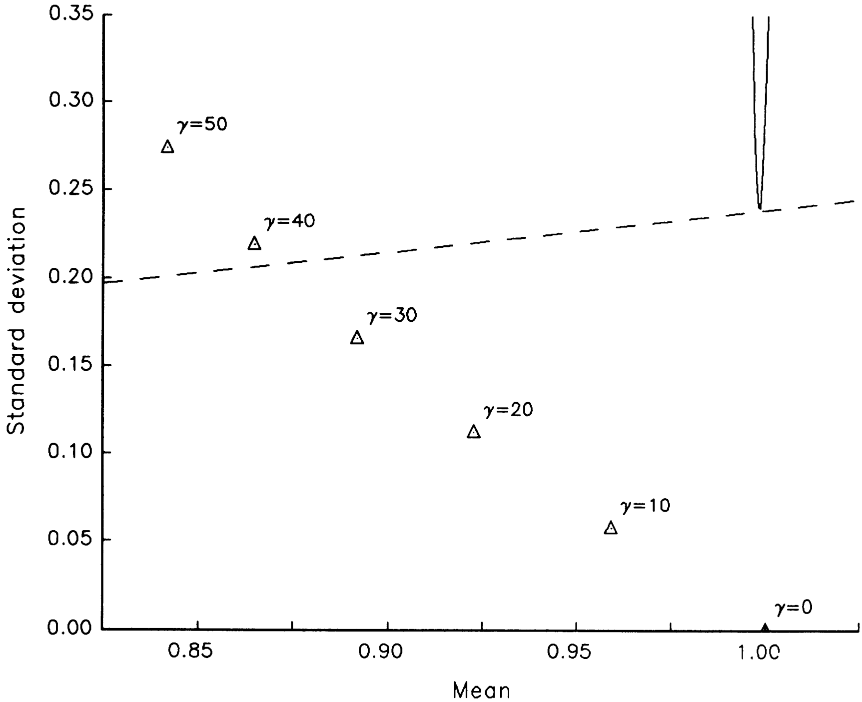
In our construction of a volatility bound, we considered the typical case in which no linear combination of the vector **x** of asset payoffs used in an econometric analysis is identically equal to one, i.e., there is no real risk-free interest rate. As a consequence, the price of a unit payoff is not known, and Ey cannot be inferred from the asset market data. Instead, we must calculate the lower bound on the standard deviation of y for each possible value of the mean. This computation leads to the lower envelope of the set of means and standard deviations of admissible discount factors (in $\mathcal{Y}$), which we denote $\mathcal{S}$.

2.3 ASSET PRICING PUZZLES

Feasible regions for mean-standard deviation pairs of stochastic discount factors can be used to summarize *asset pricing anomalies*. Figure 1 plots two such regions. The regions were constructed using quarterly data on the real value-weighted NYSE portfolio and the 3-month Treasury-bill returns, from 1947 to 1990. In computing the boundaries of these regions, we approximated population moments using their sample counterparts. To justify this use of time series data to approximate population moments, we presume that the now-and-then economy is replicated in a stationary fashion, at least asymptotically (e.g., see Hansen and Richard 1987).

The cup-shaped region in Figure 1 shows how much volatility in stochastic discount factors is implied by two returns often used in empirical analyses of the utility-based intertemporal asset pricing model. The minimum standard deviation of a discount factor y is about 0.25. Because the mean discount factor is near one, and because discount

Figure 1 BOUND ON THE STANDARD DEVIATION OF STOCHASTIC DISCOUNT FACTORS AND EQUITY PREMIUM PUZZLE



Solid line: Minimum standard deviation of discount factors y that satisfy $1 = E(y\mathbf{x})$ for given $E(y)$, where $\mathbf{x}$ = value-weighted NYSE return and Treasury Bill return. Quarterly data, 1947–1990.

Dashed line: Bound calculated from excess return, value-weighted NYSE return *minus* T-bill return.

Triangles: Mean and standard deviation of marginal rate of substitution generated by power utility, using quarterly nondurable and services consumption per capita,

$m_{t+1} = (c_{t+1}/c_t)^{-\gamma}$.

factors have the units of inverse gross returns, this is a substantial standard deviation. Figure 1 also shows us that the mean discount factor (equal to the average of the inverse of the risk-free return if there is one) must be very near 0.998, unless we are willing to accept a dramatically higher standard deviation of the discount factor.

The boundary of the second region is depicted by the dashed line in Figure 1. This boundary was computed using the excess return of stocks over bonds. Hence, it was constructed with a single security payoff with a zero price. To differentiate this region from the initial return region, we will refer to it as the *Equity-Premium Region* $\mathfrak{X}$. In general, the boundary of a feasible region for means and standard deviations constructed from a vector $\mathbf{z}$ of excess returns is a ray from the origin with slope $[E\mathbf{z}'cov(\mathbf{z}, \mathbf{z})^{-1}E\mathbf{z}]^{1/2}$ for positive values of Ey. This slope is just the "price of risk" or the asymptotic slope of the mean-standard deviation for the asset market returns used in an econometric analysis. When $\mathbf{z}$ is a scalar, as in our illustration, the formula for the slope collapses to the ratio of the absolute value of the mean excess return to its standard deviation. Of course, the Equity-Premium Region $\mathfrak{X}$ always contains the original return region $\mathcal{S}$; however, as illustrated in Figure 1, the boundaries touch at one point.

It is not readily apparent that the region $\mathcal{S}$ (or for that matter $\mathfrak{X}$) is "puzzling." Clearly, there *exist* stochastic discount factors that correctly price both securities on average. It only makes sense to use the term *puzzle* once we have narrowed the class of asset valuation models. In other words, we cannot say that the *volatility bounds* for stochastic discount factors are excessively large without knowing how large the volatility is of candidate discount factors implied by particular models.

For a point of reference, and as a diagnostic for a commonly used model, we computed sample means and standard deviations implied by representative consumer models with power utility functions. The triangles in Figure 1 give the mean-standard deviation pair for a candidate discount factor m constructed using formula (2.6) and aggregate quarterly per capita nondurable and services consumption data from 1947 to 1990. These calculations assume that $\beta = 1$ and the indicated range of the curvature coefficients γ. Alternative choices of β can be inferred by making proportional shifts in the means and standard deviations.

Our statement of the *Equity-Premium Puzzle* is that curvature coefficients γ of at least 40 are required to generate the variance of discount factors implied by the equity-premium region $\mathfrak{X}$ (for the triangles to lie over the dashed line). Furthermore, even if we are willing to admit curvature coefficients of 40 or more, the resulting mean-standard deviation pairs still do not lie in the cup because of their low means (the candidates have means $Em < .85$). Recall that Em is the predicted average price of a unit payoff. When the riskless return is equal to this average, the riskfree rate is in excess of 17% per quarter. In effect, there is more than just an *Equity-Premium Puzzle*, but also a *Riskfree-Rate Puzzle* (see also Weil, 1989).[2]

2. Kocherlakota (1990) argued that increasing the subjective discount factor to values greater than one is not implausible and can be consistent with existence of equilibrium in a growing economy with infinitely lived consumers. Increasing β helps to "resolve" the Riskfree-Rate Puzzle but not the Equity-Premium Puzzle.

These statements of the puzzles do *not* involve the specific assumptions of the Mehra and Prescott (1985) model, including a two-state Markov approximation to the distribution of consumption growth, an endowment economy, the identification of a stock index as a claim to aggregate consumption, use of Treasury bills as a proxy for a real risk-free bond, etc. They are not specific to this particular set of assets,[3] nor to postwar data. Thus, our formulation suggests that attempts to resolve the puzzle by allowing levered equity, accounting for the monetary mispricing of Treasury bills, or permitting a more general Markov structure for the endowment shock are not likely to be productive.

2.4 STATISTICAL INFERENCES

In our discussion so far, we have treated sample moments as if they were equal to the underlying population moments. That is, we abstracted from sampling error. It is interesting to know whether the *Equity-Premium Puzzle* and the *Riskfree-Rate Puzzle* still have content once we account for sampling error. To answer this question, we use statistical methods proposed by Hansen, Heaton, and Jagannathan (1992). In the nonparametric spirit of this exercise, we use large sample central limit approximations in making probability assessments.

To test whether sampling error can account for the violation of the volatility bounds, it is convenient to derive equivalent second moment bounds. Note that the orthogonality of the regression residual to the right-hand side variables in the regression implies that the random variable $a + \mathbf{b}'\mathbf{x}$ must satisfy the pricing formula (2.3) and, hence, is a stochastic discount factor in $\mathcal{Y}$. For a prespecified mean Ey, $a + \mathbf{b}'\mathbf{x}$ also must assign a price Ey to a unit payoff. Combining these equations, we have that

$$E\left\{\begin{bmatrix}1\\ \mathbf{x}\end{bmatrix}[1\ \mathbf{x}']\begin{bmatrix}a\\ \mathbf{b}\end{bmatrix} - \begin{bmatrix}Ey\\ \mathbf{q}\end{bmatrix}\right\} = 0. \tag{2.13}$$

By premultiplying equation (2.13) by the row vector $[a, \mathbf{b}']$, we obtain the following formula for the second moment of $a + \mathbf{x}'\mathbf{b}$:

$$E[(a + \mathbf{x}'\mathbf{b})^2] = [Ey\ \mathbf{q}']\begin{bmatrix}a\\ \mathbf{b}\end{bmatrix}. \tag{2.14}$$

3. Formally, one gets roughly similar bounds even if one does not use Treasury-bill data, because many other sets of assets imply about the same slope of the mean-standard deviation frontier.

This formula turns out to be quite useful for econometric inference, because it says that the second moment bound is just a linear combination of the regression coefficients.

Given a candidate discount factor m, we combine relations (2.13) and (2.14) into a composite set of moment restrictions:

$$E\left\{\begin{bmatrix}1\\ \mathbf{x}\end{bmatrix}[1\ \mathbf{x}']\begin{bmatrix}a\\ \mathbf{b}\end{bmatrix} - \begin{bmatrix}m\\ \mathbf{q}\end{bmatrix}\right\} = 0 \tag{2.15}$$

$$E\left\{[m\ \mathbf{q}']\begin{bmatrix}a\\ \mathbf{b}\end{bmatrix} - m^2\right\} \leq 0.$$

For instance, m might be constructed via the power utility formula (2.6). The first set of moment implications requires that $a + \mathbf{x}'\mathbf{b}$ have mean Em and correctly price the payoffs $\mathbf{q}$. The last moment inequality requires that the candidate m satisfies the second moment bound associated with Em. In contrast to the moment restrictions (2.7), the restrictions (2.15) do *not* require the candidate m to price assets correctly on average.

As is clear from our previous discussion, the parameters a and $\mathbf{b}$ can be identified and estimated using only the moment conditions in equation (2.13). We use such estimates to approximate the asymptotic covariance matrix for the composite moment relations in (2.15) and to account for sampling variability when testing inequality (2.14).[4,5] Because of the one-sided nature of the restriction, the probability values of the resulting test statistics are one-half those of a chi-square random variable with one degree of freedom.

In Table 1 we present results for the Volatility Test just described. We report test statistics obtained using the two original returns (value-weighted NYSE and Treasury bill) and using the single excess return. The first group of test statistics pertains to the original region $\mathcal{S}$, while

4. This strategy is very similar to one proposed by Burnside (1991) and Cecchetti, Lam, and Mark (1992).
5. From Hansen (1982) we know that the asymptotic covariance matrix can be interpreted as a spectral density matrix at frequency zero. In our empirical analysis, we followed Newey and West (1987) and used Bartlett weights to estimate this density matrix. To implement the volatility test, we transformed the sample counterparts to the moment conditions using a lower triangular decomposition of an estimate of the inverse of the asymptotic covariance matrix. The last transformed moment condition should hold with an inequality. We obtain our test statistic by minimizing the quadratic form in the transformed moment conditions by choice of a and $\mathbf{b}$ where the last moment condition only contributes to the sample criterion when the inequality is violated.

Table 1 VOLATILITY TESTS USING T-BILL AND VALUE-WEIGHTED RETURNS

	Returns		*Excess returns*	
γ	*Statistic*	*p-value*	*Statistic*	*p-value*
1	2.19	.069	2.68	.051
5	4.93	.013	2.43	.060
10	4.90	.013	1.47	.113
15	4.76	.014	0.75	.193
20	4.61	.016	0.36	.274
30	4.30	.019	0.05	.412
40	3.99	.023	0.00	.500
50	3.66	.027	0.00	.500

The Volatility Test is a test of the moment conditions

$$E\left\{\begin{bmatrix}1\\ \mathbf{x}\end{bmatrix}[1\,\mathbf{x}']\begin{bmatrix}a\\ \mathbf{b}\end{bmatrix} - \begin{bmatrix}m\\ \mathbf{q}\end{bmatrix}\right\} = 0 \text{ and } E\left\{[m\,\mathbf{q}']\begin{bmatrix}a\\ \mathbf{b}\end{bmatrix} - m^2\right\} \leq 0$$

where $m = (c_{\text{then}}/c_{\text{now}})^{-\gamma}$ and $\mathbf{x}$ = value-weighted NYSE and T-bill returns. The asymptotic covariance matrix was estimated by weighting autocovariance j with the Bartlett weight $(\tau - |j|)/\tau$ for $|j| < \tau$ and adding. The results reported are for $\tau = 10$.[6]

the second group pertains to the equity premium region $\mathcal{X}$. As is evident from Table 1, sampling error does not appear to be the explanation for the *puzzles* displayed in Figure 1.[7] In comparing the two columns of test statistics in Table 1, recall that raising γ increases volatility of the implied discount factors but has an adverse effect on the mean. The adverse mean effect is evident in test statistics based on the two returns but absent in the test statistics constructed using only the excess return. Interestingly, the smallest probability value for the return-based tests occurs at $\gamma = 1$.

Table 2 reports a chi-square test of the pricing relation (2.7). We in-

6. The magnitude of the test statistics turned out to be sensitive to the choice of τ. We also tried values of $\tau = 5$, 15, and 20. In the case of $\gamma = 1$, the test statistics range from 1.63 to 2.72 when both returns were used and from 2.21 to 2.79 when the single excess return was used. In the case of $\gamma = 50$, the test statistics ranged from 4.63 to 2.69 when both returns were used. Overall, the test based on both returns turned out to be more sensitive to the choice of τ as might be expected because of the serial dependence in the real T-bill return.
7. When m is constant, the limiting distribution no longer applies because the only solution to the moment conditions is $a = m$ and $\mathbf{b} = 0$. Hence, for m's with very low variation (values of γ near zero), the performance of the volatility test statistic might be poor.

clude this Pricing-Error Test for the sake of comparison and to emphasize that volatility bounds are *not* a substitute for directly testing the pricing implications of a model. It is necessary that a correctly specified asset pricing model satisfy the bounds, but not sufficient. The point of the Volatility Tests is to assess whether the volatility bounds are robust to sampling error. For this particular data set, the two sets of test statistics seem to convey very similar messages, although the probability values tend to be smaller for the Pricing-Error Tests.

2.5 INCORPORATING POSITIVITY

As we discussed previously, the Principle of No-Arbitrage implies the existence of a strictly positive stochastic discount factor. Furthermore, utility-based candidate discount factors are strictly positive by construction. For these reasons, it is interesting to look at volatility bounds for positive stochastic discount factors (or, more conveniently, for nonnegative stochastic discount factors). Unfortunately, we can no longer appeal to least-squares regression theory to derive these bounds. However, there is a useful representation result that can be applied instead. As long as (π, P) satisfies the Principle of No-Arbitrage, Hansen and Jagannathan (1991) showed that there exists a payoff p^* in P such that

$$\pi(p) = E[max\{p^*, 0\}p] \text{ for all } p \text{ in } P. \quad (2.16)$$

Table 2 PRICING-ERROR TESTS USING T-BILL AND VALUE-WEIGHTED RETURNS

	Returns		*Excess returns*	
γ	$\chi^2(2)$	*p-value*	$\chi^2(1)$	*p-value*
1	17.65	<.001	10.30	.001
5	58.20	<.001	5.51	.002
10	62.92	<.001	6.83	.009
15	61.87	<.001	4.38	.036
20	59.97	<.001	2.79	.094
30	55.61	<.001	1.26	.262
40	51.04	<.001	0.66	.417
50	46.46	<.001	0.38	.537

The Pricing-Error Test is a test of the moment conditions

$$E(m\mathbf{x}) = \mathbf{q}$$

where $m = (c_{then}/c_{now})^{-\gamma}$ and $\mathbf{x}$ = value-weighted NYSE and T-bill returns. We used the sample covariance matrix as an estimate of the asymptotic covariance matrix.

Thus, the pricing functional π can always be represented using an option on a payoff in P with a zero strike price. Clearly, $max\{p^*, 0\}$ is a nonnegative random variable. Hansen and Jagannathan (1991) verified that this also has the smallest second moment among nonnegative stochastic discount factors.

This representation leads to a characterization of the feasible region, $\mathcal{S}^+$, of mean-standard deviations for nonnegative discount factors. To apply it, we add a unit payoff to P and make up alternative prices for that payoff. Specifying a price for a unit payoff is equivalent to assigning a mean to y. This assignment is not arbitrary because the arbitrage bounds from the literature on options pricing impose limits on the range of possible prices of a unit payoff consistent with the absence of arbitrage opportunities (see Hansen and Jagannathan, 1991, for details). For each price assignment within these bounds, we find the option on a payoff in the augmented space that satisfies the counterpart to equation (2.16). The lower envelope of $\mathcal{S}^+$ is constructed by computing the standard deviations of each such option.

Figure 2 gives a comparison of the boundaries of regions $\mathcal{S}$ and $\mathcal{S}^+$, without and with positivity, constructed using the real value-weighted and T-bill returns as in Figure 2. Notice that the boundaries agree for ranges of Ey for which the volatility bounds are small. Once the volatility bounds get larger than about 0.7, the boundaries start to depart. This pattern is easy to explain: as the standard deviation of y, whose mean is near one, rises past 0.7, the frontier y's in $\mathcal{Y}$ are more likely to be negative in some states of the world and, hence, omitted from $\mathcal{Y}^{++}$. As one might therefore expect, this pattern is quite common across data sets on various assets. Hence, exploiting nonnegativity tends to be an important refinement when the original volatility bounds (for $\mathcal{Y}$) are already quite substantial.

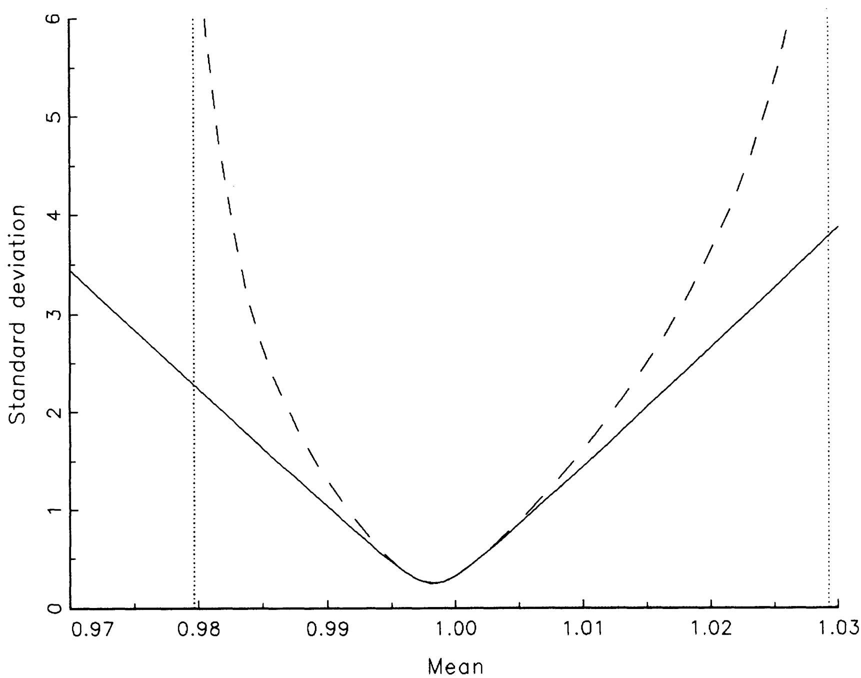
The vertical lines in Figure 2 are used to denote the upper and lower arbitrage bounds on the mean of y. By eliminating discount factors that are negative in some states of the world, we gain considerably more information about the means of the remaining nonnegative discount factors. Where before we could only quantify a dramatic increase in the standard deviation of discount factors associated with mean discount factors far from 0.998, now we can *rule out* mean discount factors below about 0.98 or above 1.03. In this way, positivity makes the Riskfree-Rate Puzzle more dramatic.

Finally, for this particular data set, the corresponding excess return region $\mathcal{X}^+$ coincides with the original excess return region $\mathcal{X}$. Consequently, positivity has no impact on the Equity-Premium Puzzle.

2.6 LENGTHENING THE INVESTMENT HORIZON

Next we explore the sensitivity of our findings to the "investment horizon" between the *now* and *then* periods. In the calculations reported so far, we used quarterly data with returns measured over the quarter. Hence, the investment horizon coincided with the sampling interval. We now expand the investment horizon to be 1 year, 2 years, and 5 years. We have (at least) three reasons for doing this. First, other empirical investigations have focused on annual data to incorporate prewar data (e.g., see Grossman and Shiller, 1981; Hansen and Jagannathan, 1991; Mehra and Prescott, 1985). By including annual investment horizons, we will facilitate comparisons to that previous work. Second, using widely separated quarterly consumption data to measure a long-horizon marginal rate of substitution may mitigate time-

Figure 2 VOLATILITY BOUND WITH POSITIVITY



Solid line: Volatility bound generated by real value-weighted NYSE and T-bill returns.

Dashed line: Bound on the standard deviation of *nonnegative* discount factors generated by real value-weighted NYSE and T-bill returns.

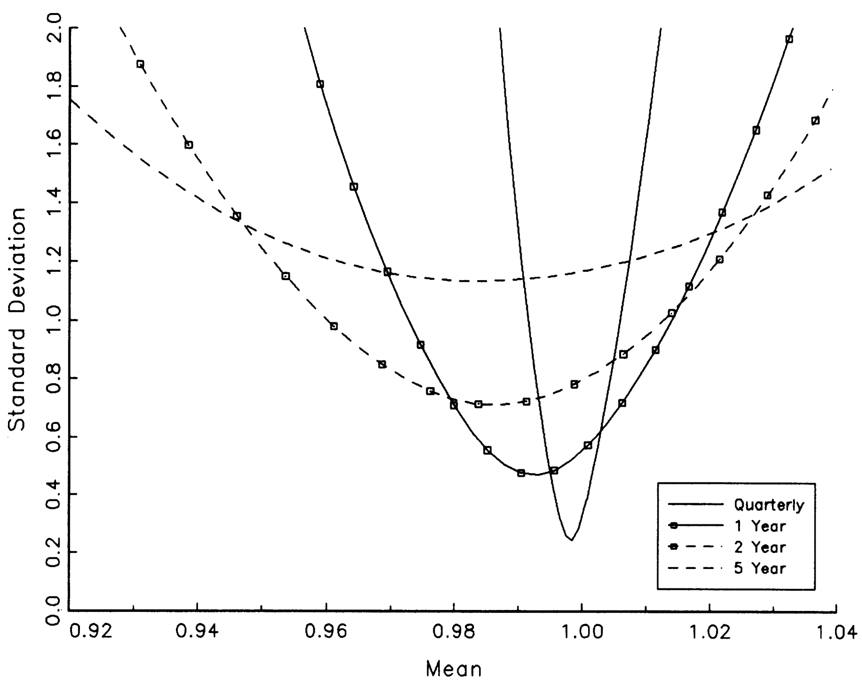
Dotted vertical lines: Arbitrage bounds on minimum and maximum mean discount factor.

aggregation biases. Finally, to help think about solutions to asset pricing puzzles, it may be useful to assess whether these puzzles are less pronounced at longer horizons.

Figure 3 reports the regions for 1-, 2-, and 5-year horizons along with the previously reported quarterly horizon. The asset return data are compounded quarterly value-weighted NYSE and T-bill returns. All regions include the positivity restriction.

Discount factors at different investment horizons are different objects, so we expect the feasible regions for means and standard deviations to be altered as we change horizon. A two-period stochastic discount factor is a product of two consecutive one-period discount factors, so we might expect the mean of a two-period discount factor to be lower and its variance to be higher than that of a one-period discount factor. As seen in Figure 4, the bottom of the mean-standard deviation frontier shifts up and to the left as we increase the investment horizon. While the volatility implications of the long-horizon returns are more dramatic,

Figure 3 VOLATILITY BOUNDS FOR LONG-HORIZON RETURNS

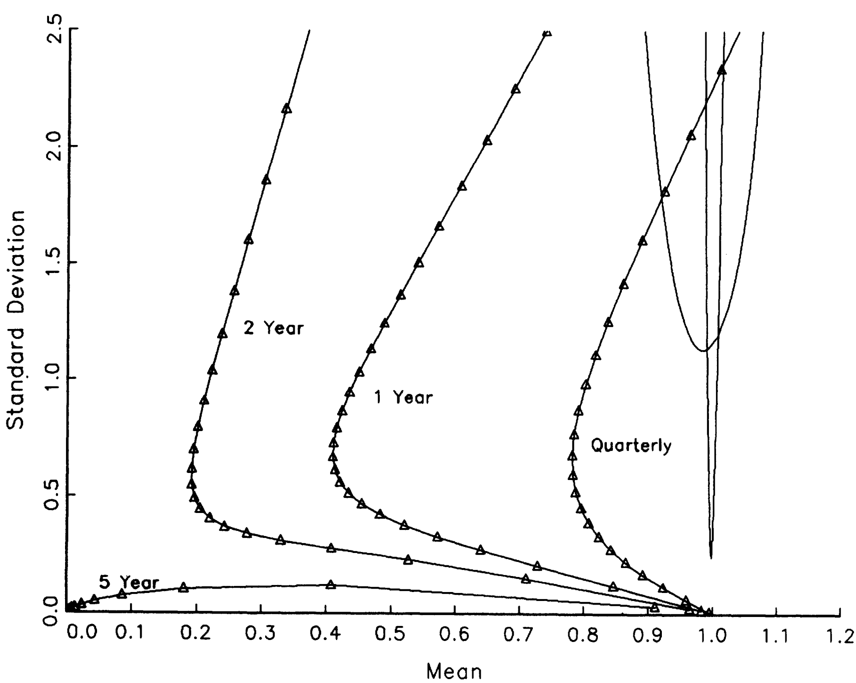


Each line gives the volatility bound on nonnegative discount factors generated by the real value-weighted NYSE and T-bill returns at the indicated horizon. Long-horizon returns are computed by compounding quarterly returns.

there is less information about the mean of longer horizon discount factors, as reflected by the horizontal expansion of the regions.

In Figure 4 we report the two extreme boundaries (quarterly and five-year investment horizons) together with the mean-standard deviation pairs for the candidate discount factors constructed using power utility functions. Figure 4 extends the range of the power γ beyond the values explored in Figure 1. Note that there now *is* a value of the power at which the quarterly candidates enter the region. However, the power is extreme, $\gamma \cong 210$. One possible reason for entertaining large values of γ follows from the work of Constantinides and Duffie (1991). They gave an illustration of a model with incomplete markets in which the

Figure 4 VOLATILITY BOUNDS AND LONG-HORIZON MARGINAL RATES OF SUBSTITUTION



Solid lines: Volatility bounds calculated from one-quarter and 5-year real value-weighted NYSE and T-bill returns, as in Figure 3.

Lines with triangles: Mean and standard deviation of marginal rates of substitution generated by power utility,

$$m_{t+k} = (c_{t+k}/c_t)^{-\gamma},$$

$k = 1$ (quarterly), $k = 4$ (1 year), $k = 8$ (2 year) and $k = 20$ (5 year). Symbols plotted at γ increments of 10.

implied power for the aggregate intertemporal marginal rate of substitution is a *mongrel* of the underlying preference parameter and the parameters governing heterogeneity in the endowments across individuals. In their illustration, large values of γ need not reflect high values of the curvature parameters in the individual preferences. (See also Mankiw, 1986, and Scheinkman, 1989, for similar observations.)

Raising consumption ratios to extremely large negative powers results in large measures of marginal rates of substitution when the consumption ratios are less than one. In effect, large values of γ magnify the effect of "bad events" for the purposes of asset pricing (see also Rietz, 1988). The mean of the power utility candidates (triangles) starts to increase when these negative growth rate observations start to dominate the sample moments of m. Because the sample moments are dominated by a few data points, the calculations for large values of γ may reflect very poor estimates of the population moments. For this reason, interpreting our large γ results may be treacherous.

As the investment horizon increases, the Equity-Premium and Risk-free-Rate Puzzles do not vanish, but instead appear to be more pronounced. For instance, larger values of γ are required to enter the feasible regions. This occurs because there are fewer and fewer consumption growth observations less than one at longer horizons.[8] In the extreme case of a 5-year investment horizon, the mean discount factor always declines as γ increases, and the standard deviation never approaches the bounds. In this case, there are no 5-year consumption ratios that are less than one in our sample. Of course, the sample information for the longer investment horizons is likely to be quite weak.

In comparing 1-year investment horizon results to those of Hansen and Jagannathan (1991), the postwar data used in constructing Figure 5 looks *more* puzzling because of the absence of the depression data points with pronounced negative consumption growth rates. These extra prewar data points dramatically increase the standard deviation of the model-based candidate discount factors. The turning point for the 1-year "triangle" curve occurs at $\gamma \cong 15$, and the 1-year discount factors enter the region at $\gamma \cong 30$ when prewar data is included.

2.7 USING CONDITIONAL INFORMATION TO DECOMPOSE UNCONDITIONAL VARIATION

The predictability of returns is another apparent puzzle that has received a lot of attention in the finance literature (see Fama, 1991, for a review). For this reason, we follow Hansen and Richard (1987) by

8. The number of negative consumption growth rate observations in our sample is 33 for a one-quarter, 20 for a 1-year, 7 for a 2-year, and 0 for a 5-year horizon.

introducing formally conditioning information into our setup. Let $\mathcal{G}$ denote a conditioning information set available to economic agents and to econometricians in the *now* period, which naturally includes the prices of securities.[9] Asset prices must obey

$$\mathbf{q} = E(y\mathbf{x} \mid \mathcal{G}). \tag{2.17}$$

There are a variety of ways in which we can exploit conditioning information in $\mathcal{G}$. For instance, conditioning information can be used to sharpen the unconditional volatility bounds. Alternatively, conditioning information can be used to split the unconditional variance into two components: the average conditional variance and the variance of the conditional mean:

$$var(y) = E[var(y \mid \mathcal{G})] + var[E(y \mid \mathcal{G})]. \tag{2.18}$$

If returns were unpredictable, all unconditional variance would be due to conditional variance, and none to variation in the conditional mean. Knowledge of the split between the two components would help us to understand better the information in asset market data about conditional moments of the stochastic discount factors.

Asset market data turns out to contain information about how to make this split in variance. In light of relation (2.18), this split also has implications for the *unconditional* variance of stochastic discount factors implied by the *conditional* moments of asset returns. We describe briefly how to form feasible regions for the pair $\{E[var(y \mid \mathcal{G})], var[E(y \mid \mathcal{G})]\}$. More details are provided in Appendix 1. First, we use the fact that any y on the $\{E[var(y \mid \mathcal{G})], var[E(y \mid \mathcal{G})]\}$ frontier must also be on the conditional (on $\mathcal{G}$) mean-standard deviation frontier for y. If not, one could lower conditional variance with no effect on conditional mean. Gallant, Hansen, and Tauchen (1990) provided a two-(conditional) dimensional characterization of the latter frontier. Using their characterization, we find the frontier for $\{E[var(y \mid \mathcal{G})], var[E(y \mid \mathcal{G})]\}$ by solving a constrained minimization problem: choose a y on the conditional mean-standard deviation frontier to minimize $E[var(y \mid \mathcal{G})]$ given $var[E(y \mid \mathcal{G})]$.

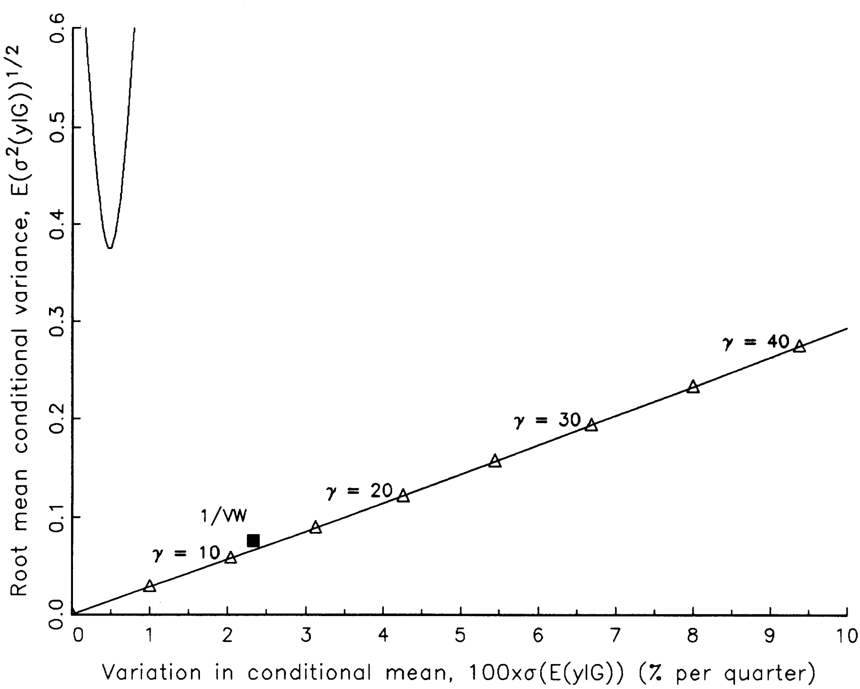
To compute such frontiers, we need a model of the first and second conditional moments of asset returns and the candidates. Our model is formed from regressions of the log returns and log consumption ratios on the value-weighted dividend/price ratio, the lagged log T-bill return, and the term premium. For simplicity, we assume the conditional covariance matrix for log returns and log consumption growth is constant,

9. The analysis permits consumers to observe more information than econometricians.

and estimate it as the residual covariance matrix in the forecasting regression. We then infer the first two conditional moments of the *levels* of consumption growth and returns assuming lognormality. (Incorporating more sophisticated models of conditional heteroskedasticity or more flexible laws of motion might lead to valuable improvements on these calculations.)

Figure 5 presents our results. Notice that most of the unconditional variance of discount factors comes from conditional variance; only a narrow range of variation in the conditional mean discount factor is consistent with the data. This makes sense, because the real return on Treasury-bills is nearly riskless and is nearly constant over time. Also, notice that the unconditional standard deviation bound for discount

Figure 5 CONDITIONAL MOMENT BOUNDS



Solid line: Bound on the root mean conditional variance versus standard deviation of conditional mean discount factor.

Line with symbols: Marginal rates of substitution generated by power utility,

$m_{t+1} = (c_{t+1}/c_t)^{-\gamma}$.

All calculations are based on regressions of the log value weighted return, log T-bill return and log consumption growth on the term premium, value weighted dividend/price ratio, and lagged log T-bill return.

factors is about .38, which is higher than the bound of about .24 that we encountered previously. The reason for the increase in the bound is that we have now incorporated conditioning information embedded in the conditional first and second moments of returns to sharpen the unconditional volatility bounds as in Gallant, Hansen, and Tauchen (1990).

Figure 5 also includes the corresponding conditioning information decomposition for power utility functions. For low values of the power γ, the candidate discount factors have about the right predictability of conditional means, but only slight predictability is required (or allowed). However, these discount factors do not have enough conditional volatility on average. As the curvature rises, the conditional variation increases, but the unconditional volatility attributed to the conditional mean becomes too extreme. Thus, the models predict dramatically too much variation in the price of a unit payoff (the reciprocal of the riskfree return).

The solid square in Figure 5 gives the volatility split for the reciprocal of the value-weighted return on the NYSE. This candidate m can be justified under an assumption of logarithmic utility where the value-weighted return is used as a measure of the return on the wealth portfolio (see Rubinstein, 1976, and Epstein and Zin, 1991). This candidate also suffers from too much variation in the conditional mean and too little conditional variation (on average).

2.8 OTHER PUZZLES

Despite the widespread attention the *Equity-Premium Puzzle* has received, other data sets can imply much sharper restrictions on the family of feasible stochastic discount factors. Hansen and Jagannathan (1991) found dramatic bounds implied by quarterly holding-period returns on Treasury-bills of varying maturity. Knez (1991) found a *Default-Premium Puzzle*, sharp bounds implied by a data set that includes corporate and government bounds of similar maturity. In both cases, the apparent presence of near arbitrage opportunities—highly correlated returns with similar standard deviations and slightly different means—makes the bounds dramatic, especially when positivity is incorporated. Bekaert and Hodrick (1992) found sharp volatility bounds for stochastic discount factors implied by security payoffs and prices constructed from data on foreign exchange and international equity markets. Cochrane (1992b) constructed volatility bounds implied by a linearized present value model. In addition, these techniques can elegantly address many empirical questions relating to traditional factor pricing models in finance. For example, Snow (1991) recast the *Small Firm Effect* as set of implications for a variety of moments of stochastic discount factors.

3. *Other Candidate Discount Factors*

The equity premium and related puzzles has given rise to an industry of "solutions." One class of "solutions" preserves the frictionless-markets framework, but modifies preferences or technology to produce the appropriate mean and standard deviation of discount factors. Space does not allow us a complete review of all the preferences that have been proposed, but these may give some of the flavor.

3.1 HABIT PERSISTENCE

Constantinides (1990), Heaton (1991), and Ferson and Constantinides (1991) have looked at implications of models in which consumers' preferences display habit persistence. In these preference orderings, a high value of consumption yesterday raises the marginal utility of consumption today. For instance, the time t period utility function now depends on $c_t - \theta c_{t-1}$ instead of just c_t where θ is positive. As a result of the positive value of θ, a given series on consumption is transformed into a more volatile marginal rate of substitution series. The marginal rate of substitution for this utility function can be expressed as

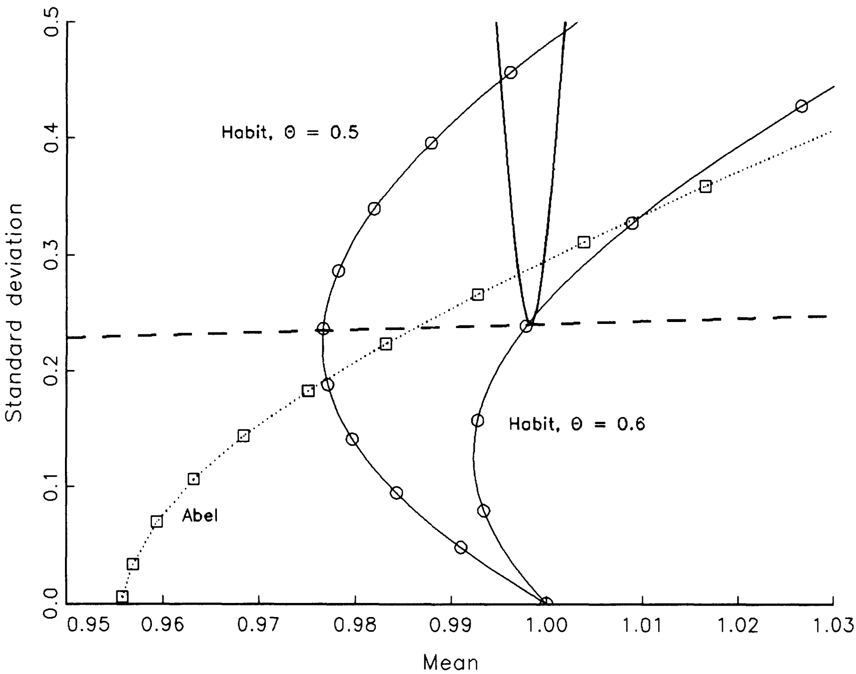
$$m_{t+1} = \beta(\Delta c_t)^{-\gamma} \frac{(\Delta c_{t+1} - \theta)^{-\gamma} - \beta\theta(\Delta c_{t+1})^{-\gamma} E_{t+1}(\Delta c_{t+2} - \theta)^{-\gamma}}{(\Delta c_t - \theta)^{-\gamma} - \beta\theta(\Delta c_t)^{-\gamma} E_t(\Delta c_{t+1} - \theta)^{-\gamma}}, \tag{3.1}$$

where $\Delta c_t = c_t/c_{t-1}$ and E_t is the expectation operator conditioned on time t information.

Notice that formula (3.1) requires the evaluation of some conditional expectations. To get a rough idea of how the resulting stochastic discount factor behaves, we made the simplifying assumption that consumption growth rates are independent and identically distributed over time. This allowed to us to approximate conditional expectations by their unconditional counterparts. For a more serious investigation of the properties of the implied stochastic discount factors, a reader should consult Gallant, Hansen, and Tauchen (1990) and Heaton (1991). Among other things, Heaton's analysis includes an explicit model of consumption growth at finer than quarterly frequencies and addresses the issue of time-aggregation biases.

Figure 6 includes calculations of the mean and standard deviation of this candidate discount factor, using habit parameters $\theta = 0.5$ and $\theta = 0.6$. Figure 6 contains two curves indexed by the choice of θ. As Figure 6 shows, the effect of habit persistence is to raise both the mean and the standard deviation of the discount factor for a given power coefficient γ. In comparing models with habit persistence to the time separable models, notice that the two habit persistence curves in Figure 6 enter the feasible region $\mathcal{S}$ at considerably lower power parameters, $\gamma = 12.5$ and $\gamma = 7.5$ respectively. Hence, Figure 6 demonstrates how habit-persistence can be used as a substitute for extremely large curvature parameters as a device for increasing the volatility of candidate discount factors. Ferson and Constantinides (1991) entertained considerably larger values of the habit persistence parameter θ. While these values

Figure 6 NON-TIME-SEPARABLE UTILITY



Solid, dashed line: Volatility bound calculated from value-weighted NYSE return and T-bill return, and excess return, respectively.

Habit: Habit persistence marginal rates of substitution

$$m_{t+1} = (\Delta c_t)^{-\gamma}\frac{(\Delta c_{t+1} - \theta)^{-\gamma} - \beta\theta(\Delta c_{t+1})^{-\gamma}E(\Delta c_{t+2} - \theta)^{-\gamma}}{(\Delta c_t - \theta)^{-\gamma} - \beta\theta(\Delta c_t)^{-\gamma}E(\Delta c_{t+1} - \theta)^{-\gamma}};\ \Delta c_t \equiv \frac{c_t}{c_{t-1}}.$$

Symbols plotted at γ increments of 2.5.

Abel: Catch up with the Joneses marginal rate of substitution

$$m_{t+1} = \left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} \Big/ \left(\frac{c_t}{c_{t-1}}\right)^{1-\gamma}.$$

Symbols plotted at γ increments of 5.

of θ further increase the volatility of the candidate m, values of θ close to one can lead to the consumers being "satiated" in numeraire consumption good, i.e., the numerator and denominator terms of (3.1) can be negative (see Heaton, 1991, for some examples).

Abel (1990) argued for a form of habit persistence he calls "catch up with the Joneses" utility, in which the time t utility function of an individual consumer depends on the ratio c_t/c^a_{t-1} and c^a_{t-1} is aggregate consumption in the previous time period. Individual consumers treat the aggregate parametrically, so they presume that their own consumption behavior cannot influence the aggregate. The idea is that you only care how well you do relative to everyone else. In the equilibrium $c^a_t = c_t$, this specification of preferences leads to a stochastic discount factor:

$$m_{t+1} = \beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} \Big/ \left(\frac{c_t}{c_{t-1}}\right)^{1-\gamma}. \tag{3.2}$$

Note that the conditional expectations that enter equation (3.1) are absent from equation (3.2). The marginal rate of substitution enters the feasible region at a value of γ around 40. Because Em now always increases with γ, the difference between the equity-premium region $\mathscr{X}$ and the original return region $\mathscr{S}$ is less critical in assessing the model. In other words, while there still seems to be an *Equity-Premium Puzzle*, the *Riskfree-Rate Puzzle* is much less evident with this preference specification.

3.2 NONEXPECTED UTILITY

Epstein and Zin (1991) used a recursive utility formulation that relaxes the usual assumption of separability across states (see also Weil, 1989). An interesting feature of their specification is that the intertemporal marginal rates of substitution depend on powers of consumption ratios in adjacent time periods and the return on the wealth portfolio. The formula for the resulting discount factor is

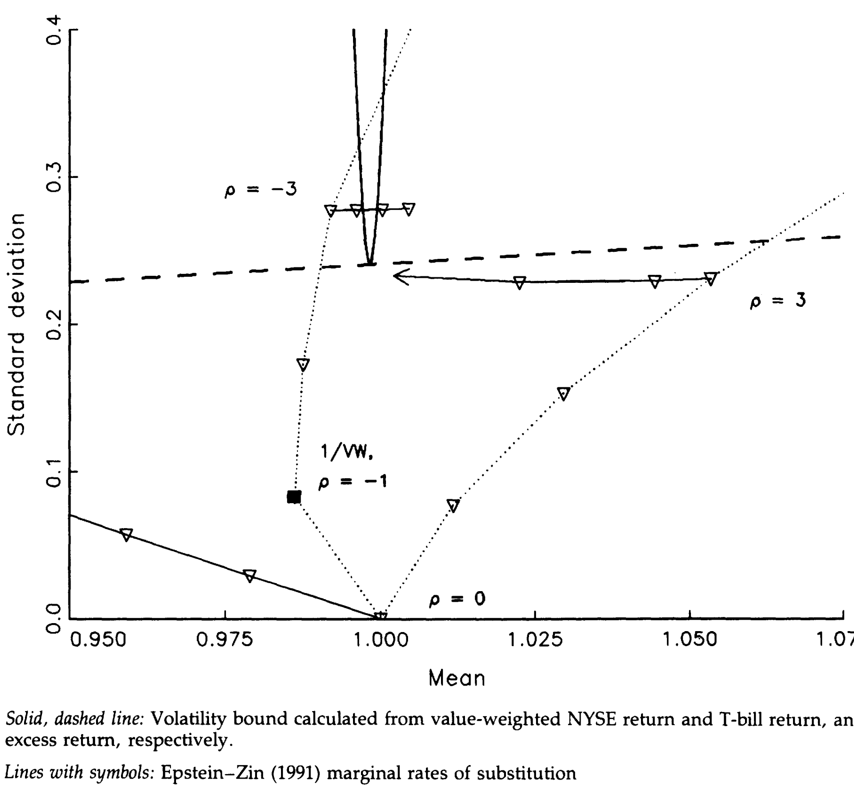
$$m_{t+1} = \beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} \left(r_{w,t+1}\right)^{\rho}, \tag{3.3}$$

where $r_{w,t+1}$ is the gross return on the wealth portfolio, β is positive, ρ is unrestricted, γ has the same sign as $\rho + 1$ and $\rho < \gamma$. When ρ is minus one, γ is zero. (We modified the notation used by Epstein and

Zin so that distinct parameters are used to capture the separate contributions of the consumption growth and the return on the wealth portfolio.) The reason that market-wealth return enters in equation (3.3) is that Epstein and Zin wanted an "observable" proxy for the innovation in the equilibrium utility index. They derived such a proxy by "inverting" the pricing formula for market-wealth return.

Figure 7 presents the means and standard deviations of the Epstein–Zin (1991) marginal rates of substitution for several different parameter configurations. Following Epstein and Zin, we measured the return on the wealth portfolio by the value-weighted NYSE return. In all cases we

Figure 7. NON-STATE-SEPARABLE UTILITY



Solid, dashed line: Volatility bound calculated from value-weighted NYSE return and T-bill return, and excess return, respectively.

Lines with symbols: Epstein–Zin (1991) marginal rates of substitution

$$m_{t+1} = \left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} (r_{VWt+1})^{\rho}$$

Solid lines connect $\gamma = 0, -1, -2, -3$ for $\rho = -3$, $\gamma = 0,5,10$ for $\rho = 0$, $\gamma = 3,5,10$ for $\rho = 3$. For $\rho = -1$ only $\gamma = 0$ is allowed. *Dotted lines* connect $\gamma = 0$ for $\rho \leq 0$ and $\gamma = \rho$ for $\rho \geq 0$.

Solid square: log utility, $m = 1/r_{VW}$

set $\beta = 1$ as a benchmark. As before, it is easy to see how changes in β alter the mean-standard deviation pair. Figure 8 gives four curves depicted by solid lines and indexed by four different values of ρ: $\rho = -3, -1, 0$, and 3. Movements along each curve corresponds to changes in γ. When $\rho = -1$, the curve is reduced to a single point, and the resulting m is just the reciprocal of the return on the wealth portfolio. When $\rho = 0$, the curve becomes the power utility curve depicted in previous figures.

As is evident from this picture, variability in the candidate m is enhanced by increasing $|\rho|$, that is, having the market enter with higher (absolute) powers. Changing γ has a relatively greater impact on the mean discount factor than its standard deviation. Thus, most of the ability of this model to generate volatile discount factors comes from the contribution of the proxy for the wealth return, rather than from the contribution of consumption.

3.3 PRODUCTION-BASED MODELS

One can also build models of stochastic discount factors by exploiting intertemporal production functions. To this end, Cochrane (1991, 1992a) and Braun (1991) showed how to construct a time series of the (marginal) physical returns to investment from production data given a specification of an intertemporal production function and its parameters. In a frictionless-markets setting, these investment returns should obey the same pricing relations as returns constructed from security market data. One could therefore check whether physical returns to investment are priced compatibly with security market returns. However, our earlier comments about the flexibility of the frictionless-markets paradigm also apply to this question of pricing compatibility. Thus, it does not seem to us to be fruitful to devise a formal "test" of pricing compatibility without narrowing the class of valuation models.

Investment returns can be used more judiciously in the context of particular valuation models that express a stochastic discount factor as a function of investment returns. For instance, Cochrane (1991) constructed and studied a broadly based measure of the aggregate return to investment derived from an adjustment cost model of the aggregate intertemporal production technology. Such a return might provide a more comprehensive measure of the return to the wealth portfolio than the value-weighted return on the NYSE. It could be used to replace the market return in a test of the Epstein and Zin (1991) model or in tests of the traditional linear capital asset pricing model.

Alternatively, Cochrane (1992a) constructed an exact factor pricing model using returns to residential and nonresidential fixed investments as factors. By design factor models provide additional flexibility for satisfying pricing relations because of the freedom to select factor loadings. For instance, the factor loadings and technology parameters in Cochrane's model can be chosen to exactly satisfy the sample moment conditions for the value-weighted NYSE and T-bill returns used in this section and, hence, to satisfy the volatility bounds. However, because of the selection of factor loadings, diagnostics focusing on only the first two moments of stochastic discount factors that we study in this section may not be particularly illuminating for assessing the performance of factor models.

3.4 CORRELATION OF DISCOUNT FACTORS WITH ASSET RETURNS

As we saw in Section 2, stochastic discount factors on the mean-standard deviation frontier (on the frontier of the region $\mathcal{S}$) are linear combinations of the payoff vector $\mathbf{x}$ and a unit payoff. In terms of the least squares regression (2.8), they are given by $a + \mathbf{b}'\mathbf{x}$. Thus, the least volatile stochastic discount factors are *perfectly* correlated with a payoff of a portfolio of the assets used in an econometric analysis. In other words, the R^2 obtained by regressing a frontier y onto $\mathbf{x}$ and a constant is one. Candidate discount factors implied by alternative economic models often produce regression R^2 that are substantially less than one.

Rearranging the definition of R^2, we obtain

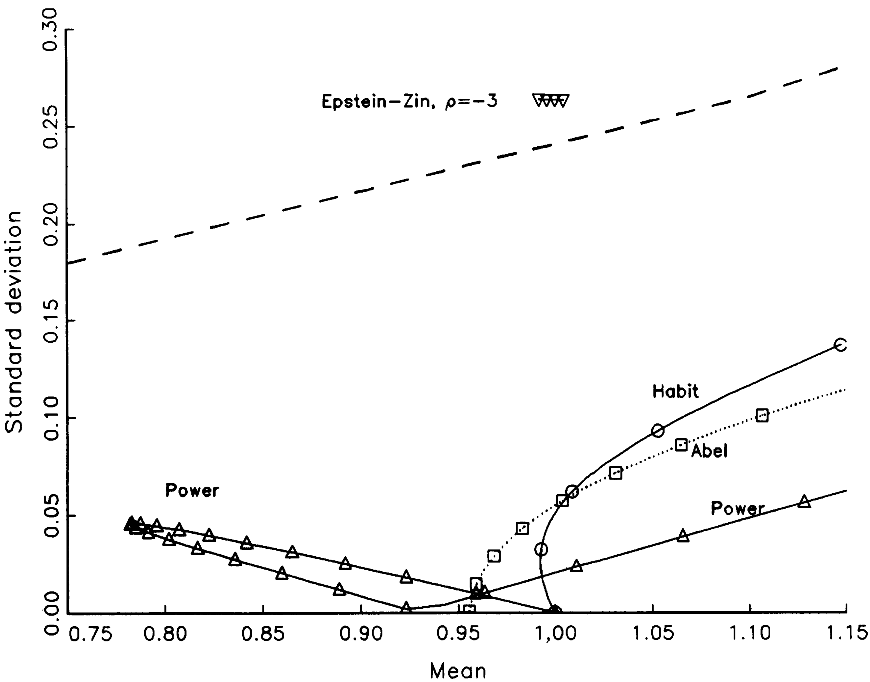
$$var(y) = [var(\mathbf{x}'\mathbf{b})]/R^2. \tag{3.4}$$

This formula can be used to construct *iso*-R^2 contours that lie above the boundary of $\mathcal{S}$. These contours are obtained by magnifying the original standard deviation bounds by the square root of the reciprocal of the R^2. If a candidate discount factor is not perfectly correlated with the asset payoff vector $\mathbf{x}$, it must be more volatile than the bounds derived in Section 2.

Rather than report regression R^2's for alternative candidate discount factors and trace out the corresponding *iso*-R^2 contours, it is more convenient to study the volatility of the least squares projection of m onto $\mathbf{x}$ and a constant. It follows from the analysis in Section 2 that if a candidate m is a valid discount factor, then so is its least-squares projection onto $\mathbf{x}$ and a constant. Hence, the mean and standard deviation of this projection should satisfy the original bounds derived in Section 2. Clearly the standard deviation of this projection can be low, even for highly volatile candidate discount factors, if the candidates are poorly correlated with asset returns.

Figure 8 presents results obtained by initially regressing a variety of

Figure 8. VOLATILITY BOUNDS, WITH STANDARD DEVIATION OF REGRESSIONS OF CANDIDATE DISCOUNT FACTORS ON EXCESS RETURN



Dashed line: Bound calculated from value-weighted NYSE return and T-bill return, and excess return, respectively.

Lines with symbols: In each case, we run a regression

$$m_t = a + bz_t + e_t$$

where z_t is the excess return, $VW_t - TB_t$. The lines with symbols report the standard deviation of fitted values. They should intersect the $VW - TB$ bound. "Habit" reports $\theta = 0.5$, "Epstein–Zin" reports $\rho = -3$. Symbols are plotted at γ increments of 10 for "Power" and "Abel," at γ increments of 5 for "Habit," and at γ increments of 1 for "Epstein and Zin."

candidate discount factors onto a constant and the excess return of stocks over bonds. Hence, the feasible region of interest is the excess return region $\mathfrak{X}$. The figure reports the means and standard deviations of the fitted projections.[10]

As is evident from the figure, the power utility and habit persistence candidate discount factors are poorly correlated with the excess return

10. We could have regressed the candidates onto a constant and both returns. This would increase variability of the projection but shrink the feasible region. A version of Figure 8 constructed with two returns looks somewhat different than Figure 8. We present the excess return version because it was more "puzzling."

and, hence, their volatility is substantially reduced by the initial regression. In contrast, versions of the Epstein–Zin discount factor retain a high degree of variability once correlation with the excess return is taken into account. This is not surprising because these discount factors are constructed using (a nonlinear function of) the value-weighted NYSE return to proxy for the return on the wealth portfolio. (We have not attempted to account for sampling error. As before, we are not proposing these calculations as substitutes for formal statistical testing.)

The Epstein–Zin calculations are likely to be misleading because the value-weighted return on the NYSE may behave quite differently than true wealth portfolio returns. For instance, the aggregate investment return constructed in Cochrane (1991) is less correlated with the excess return of stocks over bonds. Furthermore, recall that the market return enters the Epstein–Zin candidate discount factors as a proxy for the innovation in the recursive utility index valuated at the equilibrium consumption process. Hence, an alternative strategy to construct the implied stochastic discount factor is to use an estimated law of motion for consumption to infer the innovation. This approach would avoid the implicit assumption that consumption coincides with dividends, and it is also likely to result in lower correlation with the excess return.[11]

The inclusion of the R^2 dimension to the stochastic discount factor characterization adds an extra challenge to proponents of market incompleteness as a source of discount rate variability. Incomplete markets may still be frictionless. In this case, *each* consumer's marginal rate of substitution m^i should still satisfy the asset pricing equation $\mathbf{q} = E(m^i\mathbf{x})$, so each consumer's marginal rate of substitution should satisfy the volatility bounds. Thus, incomplete market models must generate individual consumption growth series that are not only highly volatile, but that are also better correlated with asset payoffs than is aggregate consumption growth. But a model whose main assumption is that individual incomes *cannot* be insured in formal security markets seems designed to generate individual consumption variability that is *uncorrelated* with payoffs on traded assets.[12]

4. *Implications for Models with Borrowing Constraints*

In this section we consider the implications of asset market data for models in which some consumers face borrowing constraints. Our dis-

11. This second approach could also be used to provide an information-based decomposition of the unconditional volatility described in Section 2.7 applied to the Epstein–Zin (1991) model.
12. It is *possible* to create such models. For example, Constantinides and Duffie (1992) showed how to construct examples of incomplete market economies in which individual intertemporal marginal rates of substitution are valid stochastic discount factors.

cussion illustrates how *market frictions* can loosen the link between asset markets and measured intertemporal marginal rates of substitution based on aggregate data. Of course, borrowing constraints are only one form of market friction that might be quantitatively important. Other frictions include incomplete markets, proportional transactions costs such as bid-ask spreads, and budget constraint kinks due to taxation. We have already commented on models with incomplete markets, and we will comment briefly on transactions costs in our concluding subsection. One reason we focus on models with borrowing constraints is that, in contrast to some other forms of transactions costs, their quantitative impact is not likely to be confined to high-frequency movements in the time series data. Quite the contrary, borrowing constraints, if important, should distort the pricing links to intertemporal marginal rates of substitution at low frequencies as well.

4.1 ISSUES IN MODEL FORMULATION

It is straightforward to see how an *individual's* Euler equation is modified by a borrowing constraint: The Euler equality is replaced by an Euler inequality, reflecting the presence of nonnegative Kuhn–Tucker multipliers on the constraints. More thought is required to relate asset prices to economic *aggregates*, because aggregate consumption sums over individuals who are constrained and others who are not.

For this reason, we sketch a simple model with borrowing constraints. The setup is taken from Townsend (1980) and is a simple version of one used by Bewley (1980). There are two consumer types, A and B. Their endowments of a nonstorable good oscillate between c_{high} and c_{low}. Consumers of type A begin with c_{high} and B consumers begin with c_{low}. There is no uncertainty and no variation in the aggregate endowment. Consumers have the same time-separable power utility function.

In the absence of impediments to communication, agents would borrow and lend to achieve constant (Pareto optimal) consumption profiles. We suppose instead that consumers are not allowed to borrow. Townsend (1980) gave a "turnpike version" of this model to justify formally the imposition of borrowing constraints through a physical impediment to communication.

In the presence of borrowing constraints, there is no trade, and the equilibrium interest rates are set so that the *unconstrained* consumers are content to consume their endowments. In other words, the price q of the riskless bond is given by

$$q = m^u \equiv \beta(c_{\text{low}}/c_{\text{high}})^{-\gamma}, \tag{4.1}$$

where m^u is the intertemporal marginal rate of substitution of the unconstrained consumer. Clearly, m^u is greater than the intertemporal marginal rate of substitution of the constrained consumer, that is, is greater than $\beta(c_{\text{high}}/c_{\text{low}})^{-\gamma}$.

We want to know what happens when an econometrician uses *aggregate* data to measure the intertemporal marginal rate of substitution. In this simple illustration, there is no aggregate variation so that the measured aggregate intertemporal marginal rate of substitution, denoted m^a, is just β. Consequently, the econometrician constructs a candidate that is *less* than the discount factor:

$$m^u \geq m^a = \beta. \tag{4.2}$$

This downward bias in m^a is inherited from the distortion in the intertemporal marginal rate of substitution of the constrained consumers. The candidate stochastic discount factor based on aggregate data implies a lower price for a one-period bond and, hence, a higher interest rate. While this "incorrect" use of aggregate data leads to a "pricing error," the price is biased in a predictable direction.

Next we modify this setup by introducing in turn two alternative means for consumers to substitute consumption over time: valued-fiat money and a storage technology. Consider first a version of this economy with valued-fiat money. If the consumers with low endowments have money, they will exchange this money for goods as long as m^u in (4.1) measured at the pretrade endowment position is greater than one. Townsend (1980) showed that in a setup with a constant (noninterventionist) money supply, the equilibrium consumption sequences of each agent still oscillate and that nonnegativity constraints on money bind in alternating periods. A version of equations (4.1) and (4.2) still hold for this economy with the appropriate alterations. In particular, the equilibrium q is one in Townsend's monetary economy because the real rate of return to holding money is zero, and c_{low} and c_{high} now denote equilibrium rather than endowment consumption levels. The allocation associated with the monetary equilibrium Pareto dominates that of autarky; however, because the real return to holding money is less than β^{-1}, it is still not Pareto optimal.

A storage technology with zero depreciation leads to very similar implications. In such an economy, the intertemporal marginal rate of transformation pins down the single-period asset return, so the real interest rate is zero. One can thus reinterpret Townsend's monetary economy as one in which the equilibrium consumption oscillates because consumers' nonnegativity constraints on storage bind every other time period. Relations (4.1) and (4.2) continue to apply for q equal to the intertemporal marginal rate of transformation determined by the storage technology.

For reasons of empirical plausibility, we are interested in observable implications that can accommodate much more general endowment patterns than the ones specified by the previous illustrations. For instance, it is potentially important to accommodate stochastic endowments that grow over time and stochastic technologies for transferring consumption from one period to the next. (Bewley, 1980, allowed for stochastic endowments in his general setup and Deaton, 1991, incorporated endowment growth into a stochastic environment.) Nevertheless, the example economies we discussed illustrate the following related features that occur more generally: (1) some consumers are up against borrowing constraints in equilibrium, (2) the market discount factor for pricing assets equals the intertemporal marginal rate of substitution of unconstrained consumers, and (3) an intertemporal marginal rate of substitution measured using aggregate data is less than or equal to the market discount factor and the marginal rate of substitution of unconstrained consumers.

Feature (1) does not always emerge because consumers may save to avoid the borrowing constraint in some models. For instance, by making the storage technology in the previous example productive so that the stored good is interpreted as a capital stock, the zero interest rate implication is avoided. If this technology is too productive (relative to β^{-1}) or if the utility cost to being constrained is too severe, then the borrowing constraints may not bind in equilibrium (e.g., see Bewley, 1977; Deaton, 1991; Scheinkman and Weiss, 1986). On the other hand, Deaton (1991) emphasized that endowment growth may make borrowing constraints more likely to bind in equilibrium.

Feature (2) has led some empirical researchers to attempt to identify a sample of "unconstrained" consumers (e.g., see Hayashi, 1987; Mankiw and Zeldes, 1991; Runkle, 1991; Zeldes, 1989) and examine asset pricing implications for these individuals. In line with the calculations reported in the previous sections, we are primarily interested in asset pricing implications for aggregate time series data on consumption. Furthermore, as is manifested in implication (3), the measured marginal rates of substitution using aggregate data should still be informative even though they are "contaminated" by the consumption of the constrained consumers. Recall that this inequality implication results because aggregate data combines the consumption of the constrained and unconstrained individuals.

Our empirical analysis in this section focuses on implication (3): the downward distortion of the measured intertemporal marginal rate of substitution obtained using aggregate consumption data. Not surprisingly, this implication can be obtained for much more general specifications of economies with borrowing constraints. However, once uncertainty is incorporated, there are alternative ways of formalizing the notion of a borrowing constraint (e.g., see Hindy, 1992; Zeldes, 1989), and we will investigate two such alternatives.

Suppose that an individual i faces a sequence of one-period budget constraints of the form:

$$c^i + E(yp^i|\mathcal{G}) = e^i \tag{4.3}$$

where c^i is consumption of person i in the current time period, y is the market-determined stochastic discount factor for pricing one-period securities, p^i is the payoff in the subsequent time period of securities purchased in the current time period, and e^i is income including an exogenous endowment and the security market payoff in the current time period. Hence, $E(yp^i|\mathcal{G})$ is the market value of payoff p^i, and the budget constraint says that consumption plus the value of securities purchased must equal an endowment plus the payoffs of securities previously purchased. In addition, we restrict the payoff p^i to be in an information set $\mathcal{G}'$ available in the subsequent time period.

Following Luttmer (1991), one of the forms of a borrowing constraint we consider is referred to as a *solvency* constraint:[13]

$$p^i \geq 0. \tag{4.4}$$

That is, any contingent contract that includes debt in some states is prohibited. A solvency constraint can be motivated by the severe limits on communication in Townsend's (1980) turnpike setup when uncertainty is explicitly introduced.[14]

A second weaker notion of a borrowing constraint is a restriction that the current-period value of the portfolio payoff be nonnegative:

$$E(yp^i|\mathcal{G}) \geq 0. \tag{4.5}$$

13. Our use of the term *solvency* constraint is different from that of Hindy (1992). For Hindy (1992), a solvency constraint encompasses a broad class of borrowing constraints including short-sale constraints and a market-wealth constraint as special cases.

14. Less severe impediments to communication have been explored by Townsend and Wallace (1987) and Manuelli and Sargent (1992). In their environments, private debt sometimes circulates in equilibrium.

It states that the value of the consumer's portfolio *today* must be nonnegative, and we will refer to it as a *market-wealth constraint*. It does not preclude p^i from being negative in some states of the world. This restriction has been used by Zeldes (1989), He and Modest (1991), Hindy (1992), and Santos and Woodford (1992), among others, and is motivated by a restriction that a consumer is prohibited from borrowing against future endowments (or sometimes labor income) to support consumption today.[15]

Because we are interested in arbitrage-free models, or equivalently, models in which all nontrivial event contingent claims have strictly positive prices ($\mathcal{P}r\{y > 0\} = 1$), both versions of borrowing constraints eliminate pure debt-contingent contracts (nontrivial choices of p^i that are less than or equal to zero). This is the sense in which both constraints eliminate pure borrowing. The *market-wealth* constraint is less restrictive because it permits p^i to be negative on some nontrivial event as long as p^i is positive on other events so that its market value is nonnegative. As we will see, because the *solvency* constraint represents a *more* severe limitation on portfolio choices, it leads to *weaker* empirical implications for marginal rates of substitution than the *market-wealth* constraint.

To ascertain which constraint is better justified would require a more serious modeling endeavor that examines what impediments to communication underlie the constraints. The solvency constraint may be problematic because of the potential difficulty in practice of establishing whether a complicated security market transaction does indeed result in a limited liability payoff. On the other hand, the *market-wealth* constraint may be hard to justify because it does not eliminate consumers' ability to engage in extreme short-selling strategies as a device for smoothing consumption across states in subsequent time periods.

In many setups, borrowing constraints are imposed simultaneously with other forms of market incompleteness. For instance, if there is a small collection of underlying limited liability securities, an alternative more severe notion of a *solvency* constraint is a set of short sale constraints on the individual securities. Similarly, the market wealth constraint is often coupled with other forms of market incompleteness (e.g., see Bewley, 1977; Santos and Woodford, 1992; Scheinkman and Weiss, 1986).

15. Because constraint (4.5) is stated in terms of market wealth and, hence, permits p^i to be negative with positive probability, we have implicitly ruled out security payoffs whose market value is ambiguous. That is, we have eliminated payoffs whose negative part has a market value $-\infty$, even though it is offset by a positive part with a market value $+\infty$.

In our analysis, we will follow Luttmer (1991) and consider the case in which security markets are complete except for the borrowing constraint. That is, consumers are permitted to write general contingent contracts that respect the borrowing constraint and are verifiable in the subsequent time period (have payoffs in $\mathcal{G}'$). Consequently, we do not consider the potentially interesting interaction between borrowing constraints and market incompleteness.[16] A benefit to permitting a rich array of security market transactions is that we can apply an aggregation result of Luttmer (1991) to characterize the behavior of the intertemporal marginal rate of substitution constructed from aggregate time series data.

4.2 OBSERVABLE IMPLICATIONS

In the periodic endowment model discussed previously, we saw how a simple model with a borrowing constraint led to the implication (3) that the measured intertemporal marginal rate of substitution using aggregate data is less than or equal to the market discount factor. Luttmer (1991) showed that the same relation applies in stochastic environments and with a solvency constraint.

To understand this restriction better, we briefly sketch the argument used by Luttmer (1991). Let C^+ denote the *cone* of payoffs that can be obtained from one-period security market transactions consistent with the solvency constraint. The payoffs in C^+ include all random variables that are nonnegative and in the set $\mathcal{G}'$ of information available in the subsequent *then* time period. The first-order conditions for consumer i facing a solvency constraint can be characterized conveniently as

$$E(m^i p|\mathcal{G}) \leq E(yp|\mathcal{G}) \qquad \text{for all } p \text{ in } C^+. \tag{4.6}$$

In effect, the usual Euler equality is replaced by an inequality because of the presence of a Kuhn–Tucker multiplier from the solvency constraint. Given the complete markets construction of the cone C^+ and the fact that m^i and y are in the cone, it follows that

$$m^i \leq y. \tag{4.7}$$

Furthermore, Luttmer (1991) showed that, if all consumers have identical power utility functions with a common subjective discount factor,

16. For instance, in the incomplete markets equilibrium of Scheinkman and Weiss' (1986) model, even though the borrowing constraints are slack, their presence has a nontrivial impact on the competitive equilibrium consumption allocation over time and across consumers.

the intertemporal marginal rate of substitution m^a inherits the downward bias of the individual intertemporal marginal rate of substitution:

$$m^a \leq y. \tag{4.8}$$

Thus, implication (3) continues to hold in this more general stochastic setting.

Next we follow He and Modest (1991) in our consideration of the less restrictive market wealth constraint (4.5). Consumers can now form portfolio payoffs in addition to those in the cone C^+. For instance, let Z denote the set of one-period security market payoffs with zero market prices, i.e., excess returns. Any payoff in Z clearly satisfies the market-wealth constraint. As emphasized by He and Modest (1991), for these portfolio payoffs we still obtain the usual Euler equality:

$$E(m^i z|\mathcal{G}) = E(yz|\mathcal{G}) = 0 \qquad \text{for all } z \text{ in } Z. \tag{4.9}$$

Note that the payoff $m^i - yE(ym^i|\mathcal{G})/E(y^2|\mathcal{G})$ has a zero market price and, hence, is in Z. Using this payoff as z in equation (4.9), it follows that

$$m^i = \psi^i y \text{ for } \psi^i \equiv E(ym^i|\mathcal{G})/E(y^2|\mathcal{G}). \tag{4.10}$$

Furthermore, equation (4.8) implies that $0 < \psi^i \leq 1$. The random variable ψ^i captures the distortion in the marginal rate of substitution caused by the presence of the market-wealth constraint, and $1 - \psi^i$ can be viewed as the "shadow value" of the borrowing constraint in terms of consumption in the *now* time period. In summary, the less restrictive market-wealth constraint implies a more stringent implication that the intertemporal marginal rate of substitution is *proportional* to y (conditioned on $\mathcal{G}$).

Again, a version of Luttmer's aggregation result for power utility functions applies. The market-wealth constraint is related to an economic aggregate measure of the intertemporal marginal rate of substitution via:

$$m^a = \psi^a y. \tag{4.11}$$

Mimicking the usual aggregation arguments for power utility functions, it is straightforward to show that

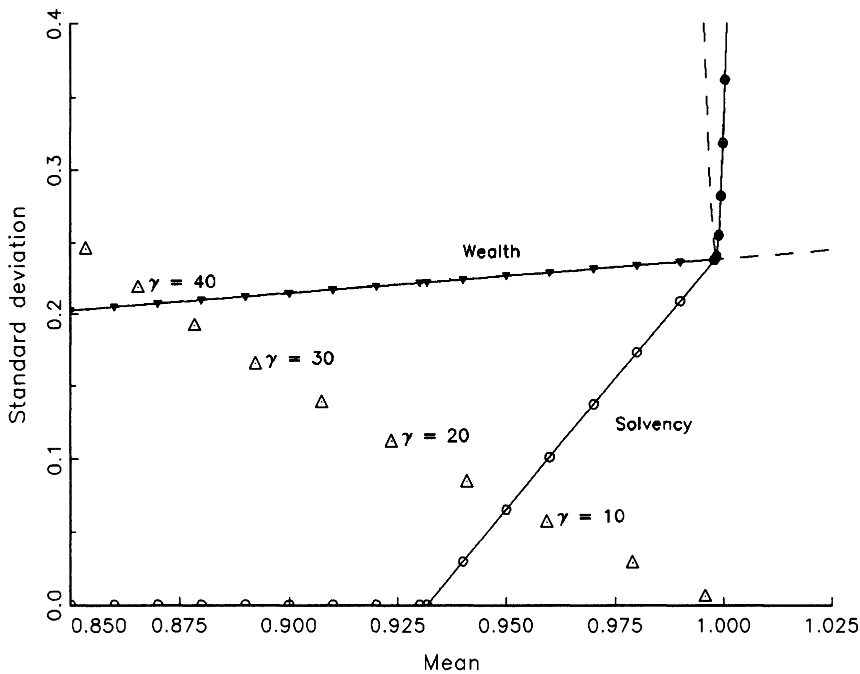
$$\psi^a = \left[\sum_i (\psi^i)^\alpha (c^i/c^a)\right]^{1/\alpha}, c^a = \sum_i c^i, \alpha = -1/\gamma, \tag{4.12}$$

and c^i is consumption of person i in the *now* time period. Hence, the distortion factor ψ^a that emerges in the aggregate relation is a geometric weighted average of the individual ψ^i's with consumption weights in the conditioning information set $\mathcal{G}$. Therefore, like ψ^i, ψ^a is in the conditioning information set $\mathcal{G}$ and is between zero and one.

4.3 RESULTS

In Figure 9 we report the boundaries of three feasible sets of mean-standard deviation pairs. First we reproduce the region $\mathcal{S}^+$, described in Section 2, of mean-standard deviation pairs for nonnegative market

Figure 9. VOLATILITY BOUNDS WITH BORROWING CONSTRAINTS



Solid line with circles (Solvency): Volatility bound with *solvency constraint* that investors may not hold portfolios that could have a negative payoff, using value-weighted return and 3-month T-bill return.

Solid line with solid triangles (Wealth): Volatility bound with *market-wealth* constraint, that the value of investor's portfolios must be positive at the initial date, using value-weighted return and T-bill return.

Dashed lines: Regular volatility bounds using value-weighted NYSE return and T-bill return, and excess return, respectively.

Triangles: Power utility marginal rates of substitution,

$m_{t+1} = (c_{t+1}/c_t)^{-\gamma}$.

discount factors y. The second region, denoted $\mathcal{B}^+$, is the feasible set for the aggregate measures m^a of intertemporal marginal rates of substitution that satisfy restriction (4.8) implied by the solvency constraint. The third region, denoted $\mathcal{W}^+$, is the feasible set for random variables m^a that also satisfy the proportionality restriction (4.11). The regions are constructed using results from Luttmer (1991) and He and Modest (1991), and the mechanics underlying these constructions are provided in the next subsection. All three regions were constructed using the same quarterly value-weighted and T-bill return series used in many of our previous figures. Of course, both $\mathcal{B}^+$ and $\mathcal{W}^+$ are expanded versions of the set $\mathcal{S}^+$, and the feasible set $\mathcal{W}^+$ is smaller than $\mathcal{B}^+$ because it incorporates the additional proportionality restriction.

As before, we use triangles to record the mean-standard deviation pairs of aggregate measures of intertemporal marginal rates of substitution for several choices of the power $-\gamma$. Abstracting from sampling error, values of γ in the vicinity of 40 are necessary to get into the more restrictive market-wealth region $\mathcal{W}^+$, whereas values as low as 15 enter the solvency region $\mathcal{B}$. Reducing the subjective discount factor β to values less than one increases the range of γ's that are inside the respective regions. Fairly sizable curvature values are still required to "resolve" the *Equity-Premium Puzzle* for either version of a borrowing constraint, especially for the market-wealth constraint. However, the *Riskfree-Rate Puzzle* now disappears.

In the case of the market-wealth constraint and say, $\gamma = 40$, the distortion factor $E\psi^a$ required for (4.10) to be satisfied is on average about .87, which may seem implausibly large. Recall that $1 - \psi^i$ is shadow value of the market-wealth constraint for person i. Hence, $1 - E\psi^a$ is a (somewhat complicated) "average" shadow value where averaging takes place across consumers and over states of the world. When $\gamma = 40$, this "average" shadow value is about 0.13 for quarterly data. Consequently, the presence of a market-wealth constraint eliminates the *Riskfree-Rate Puzzle* by presuming there is a high average shadow value for the constraint. Subsection 4.5 below includes a related reservation about the solvency constraint.

4.4 COMPUTATION

We now describe how to compute the boundaries of the solvency region $\mathcal{B}^+$ and market-wealth region $\mathcal{W}^+$ reported in Figure 9. As indicated previously, Luttmer (1991) provided a general algorithm for calculating the lower envelope of $\mathcal{B}^+$. The algorithm is fairly easy to describe when there are only two original securities, each with a limited liability. Recall in our application, we used the value-weighted and T-bill returns as our payoffs, both of which we interpret as having limited liability.

Let **x** denote a random vector formed by stacking these two returns, and let P^+ denote the cone of limited-liability payoffs that can be constructed from constant-weighted portfolios:

$$P^+ \equiv \{p : p = \mathbf{c} \cdot \mathbf{x} \text{ for } \mathbf{c} \in \mathbb{R}^2, p \geq 0\}. \tag{4.13}$$

Any m^a that is less than or equal to y will assign a price to a payoff in P^+ that is less than or equal to its market price:

$$E(m^a p) \leq E(yp) \qquad \text{for all } p \text{ in } P^+. \tag{4.14}$$

Because portfolio payoffs outside of P^+ are sometimes negative, the price distortion for these payoffs is ambiguous in sign.

Next, note that in exploring the ramifications of equation (4.14), it suffices to look at the two *edge* portfolios, say, p_1 and p_2. Any other payoff in P^+ is a convex combination of these edges with nonnegative portfolio weights. Because the original two securities have nonnegative payoffs, each edge has a positive portfolio weight on one of these securities and a nonpositive weight on the other.[17] We normalize these edge payoffs so that their price is one, that is, they are returns, and we number them so that

$$Ep_1 \geq Ep_2. \tag{4.15}$$

It turns out the boundary of $\mathcal{B}^+$ has three segments. First, there is a horizontal segment at $\sigma(m^a) = 0$ from $E(m^a) = 0$ to $E(m^a) = 1/Ep_1$. This segment is present because for any constant m^a between zero and $1/Ep_1$, inequality (4.14) is satisfied for both p_1 and p_2. Furthermore, as long as the constant is strictly less than $1/Ep_1$, the inequalities will be strict. When the constant is equal to $1/Ep_1$, (4.14) will hold with equality by construction for edge payoff p_1. This is the point at which the second segment begins.

The second segment of the boundary of $\mathcal{B}^+$ coincides with a segment

17. The payoffs we used are strictly positive for all dates in the sample. We approximated the edges by initially holding fixed the positive weight and adjusting the negative weight until the resulting portfolio payoff is zero at one sample point and positive at all other points. In practice, *approximation* of edges using the empirical distribution may be poor because upper and lower endpoints of these distributions may be hard to estimate.

of the boundary of feasible mean-standard deviation region $\mathcal{S}_1^+$ for stochastic discount factors that correctly price p_1 (but not necessarily p_2). The boundary of $\mathcal{S}_1^+$ touches the horizontal axis at the point $(1/Ep_1, 0)$ because as was just mentioned, a constant discount factor $1/Ep_1$ prices p_1 correctly. Furthermore, any other frontier discount factor for $\mathcal{S}_1^+$ will also be a frontier random variable for $\mathcal{B}^+$ as long as (4.14) is satisfied for p_2. Hence, to construct the second segment of the boundary of $\mathcal{B}^+$, we follow the right boundary of $\mathcal{S}_1^+$ until we obtain a frontier discount factor that also just prices p_2 correctly.

The third segment of the boundary of $\mathcal{B}^+$ coincides with a segment of the boundary of the region $\mathcal{S}^+$, of stochastic discount factors that price correctly all constant-weighted portfolios of value-weighted and T-bill returns. Such discount factors satisfy (4.14) with equality for all admissible payoffs.[18]

Consider next the boundary of $\mathcal{W}^+$. This construction follows He and Modest (1991) and is included here for completeness. Multiplying both sides of (4.11) by **x**, taking expectations (first conditioned on $\mathcal{G}$ and then unconditionally), we have

$$E(m^a \mathbf{x}) = (E\psi^a)\mathbf{q} \quad , \text{ where } 0 < E\psi^a \leq 1. \tag{4.16}$$

Consequently, for any m^a satisfying equation (4.11), we can find a stochastic discount factor $m^a/E\psi^a$ that prices the payoffs in **x** correctly.

Because the means and standard deviations of scale multiples of random variables simply inherit the same scaling, the link between $\mathcal{S}^+$ and $\mathcal{W}^+$ is particularly simple. Scale all of the points in $\mathcal{S}^+$ by arbitrary numbers between zero and one, that is, for any point in $\mathcal{S}^+$, the ray from the origin to this point is in $\mathcal{W}^+$. Notice from Figure 9 that the boundary of $\mathcal{W}^+$ has two segments. One coincides with a portion of the boundary of $\mathcal{S}^+$, and the other coincides with a portion of the boundary of the analogous region constructed using the excess return of stocks over bonds.

4.5 USING AGGREGATES TO SHARPEN BOUNDS ON MARKET DISCOUNT FACTORS

There is a different way to use the information contained in the marginal rates of substitution based on aggregate data in the presence of borrowing constraints. Rather than weakening the volatility implications to

18. With more than two securities, one follows a natural generalization of the above procedure. One first locates the *edges* of the cone P^+. Once they are located, one can use Luttmer's more general algorithm for computing the bounds with the short-sale constraints imposed on the *edge* payoffs.

accommodate measured marginal rates of substitution using aggregate data, we now use the aggregate data to *sharpen* the implications for stochastic discount factors that *correctly* price the asset payoffs. For instance, suppose a marginal rate of substitution m^a constructed from aggregate data is less than or equal to the market discount factor y as is implied by both versions of borrowing constraints. Then m^a can be used to sharpen volatility bounds for y and, hence, for the intertemporal marginal rates of substitution of unconstrained consumers.

Calculation of the mean-standard deviation region for such discount factors is an easy extension of our previous analysis. In Section 2, we reviewed Hansen and Jagannathan's (1991) construction of least volatile stochastic discount factors that price assets and are greater than zero. Now we want discount factors that price assets and are greater than a given y, that is, we want y's that satisfy

$$E(y\mathbf{x}) = \mathbf{q} \text{ and } y \geq m^a. \tag{4.17}$$

As Hansen and Jagannathan (1991) find that frontier discount factors are the larger of some portfolio payoff $a + \mathbf{x}'\mathbf{b}$ and zero, now frontier discount factors have the form

$$y = \begin{cases} a + \mathbf{x}'\mathbf{b} & \text{if } a + \mathbf{x}'\mathbf{b} \geq m^a \\ m^a & \text{if } a + \mathbf{x}'\mathbf{b} \leq m^a \end{cases} \tag{4.18}$$

for some two-dimensional vector $\mathbf{b}$ of portfolio weights and some real number a. A brief description of how we computed a and $\mathbf{b}$ in practice is provided in Appendix 2.

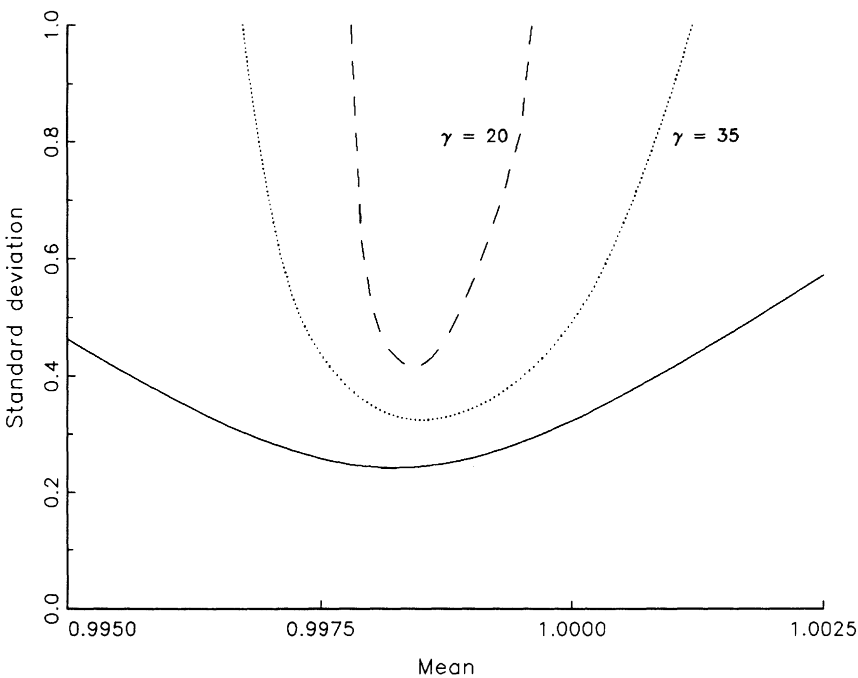
In Figure 10 we report calculations for two different specifications of m^a constructed using aggregate consumption data and values of $\gamma = 20$ and 35. For comparative purposes, we also include the bounds computed imposing only nonnegativity ($y \geq 0$). For values of $\gamma < 18$, the feasible region is empty, because m^a fails to satisfy the pricing inequalities (4.14). (Candidates m^a with lower values of γ entered the solvency constraint region of Figure 9. They had the same mean and standard deviation as a random variable that satisfied the pricing inequalities, but these candidates did not themselves satisfy the inequalities.) As γ increases, the aggregate data is less informative about the feasible set of stochastic discount factors.

Notice that for $\gamma = 20$, the volatility bounds are at least doubled, and there is considerably more information about the feasible range of the mean of stochastic discount factors. Thus, although solvency constraints loosen the implications of security market data for marginal rates of substitution computed from aggregate data, the data still pose a striking puzzle for the behavior of unconstrained consumers. Thus, even though models with solvency constraints weaken substantially the link between aggregate consumption data and asset market returns, taken together the consumption and asset return data still imply rather startling implications for the intertemporal marginal rates of substitution of the unconstrained consumers.

4.6 EXTENSIONS

Recently, there has been a variety of work in asset pricing that seems well suited to guide empirical analyses of other market frictions. For instance, Prisman (1986), Jouni and Kallal (1991), and Luttmer (1991)

Figure 10 VOLATILITY BOUNDS FOR DISCOUNT FACTORS GREATER THAN A CANDIDATE



Solid line: Volatility bound using value-weighted NYSE and T-bill returns, for reference.

Dashed and dotted lines: Minimum standard deviation of discount factors that (1) price the real value-weighted NYSE and T-bill returns (satisfy $1 = E(y\mathbf{x})$), (2) are always greater than a candidate based on aggregate data, $y_{t+1} \geq (c_{t+1}/c_t)^{-\gamma}$; and (3) have the indicated mean.

showed how to define a meaningful notion of the Principle of No-Arbitrage in the presence of transactions costs and short-sale constraints. In addition, Jouni and Kallal (1991), building upon Harrison and Kreps (1979), Kreps (1981), and Clark (1990), established that this extended Principle of No-Arbitrage is equivalent to the existence of a counterpart to a strictly positive stochastic discount factor. He and Modest (1991) and Luttmer (1991) showed how to adapt some of the apparatus described in Section 2 to accommodate these market frictions.

While market frictions loosen the implications of asset pricing data for candidate discount factors, the important issues for empirical research are to assess the magnitude and direction of the alterations and to determine the extent to which asset pricing puzzles disappear when market frictions are accommodated. For example, Luttmer (1991) showed that by introducing bid-ask spreads into the analysis, the *Term-Premium Puzzle* implied by the holding period returns on short-term Treasury-bills is reduced to about same order of magnitude as the *Equity-Premium Puzzle.*

A possible shortcoming of this approach is that the imperfections are imposed directly on the security markets. Although the solvency-constraint model presumably could be justified in environments like Townsend's (1980) in which there are explicit impediments to communication, these impediments are extreme and cannot be used to rationalize the market-wealth constraint. Other restrictions on information flows and communication are known to have important implications for the optimal allocation of resources (e.g., see Atkeson and Lucas, 1992; Green, 1987; Phelan and Townsend, 1991; Prescott and Townsend, 1984; Townsend, 1987). However, to date this literature has not provided an alternative tractable vehicle for extracting information from asset market returns in building models of dynamic economies.

5. *Concluding Remarks*

This paper continues a line of research that seeks a better understanding of the implications of security market data for building dynamic economic models. More precisely, we have surveyed and extended a literature that characterizes stochastic discount factors and provides information from asset market data about the properties of marginal rates of substitution and transformation.

Our first extension was to consider the impact of the correlation of discount factors with asset returns. A successful discount factor must be either highly correlated with asset returns, or have even higher variance than indicated by the original bounds derived in Hansen and Jagannathan (1991). Our second extension used conditioning information to split the unconditional variance of discount factors into two components: average conditional variance and variation in conditional means.

These two extensions refine previous characterizations of stochastic discount factors. Quarterly discount factors should be highly volatile (standard deviation at least .24 based on unconditional moments, .38 based on conditional moments), they should have a mean of about .998, they should be highly correlated with asset returns or have even higher variance, and most of their unconditional variation must be accounted for by average conditional variation rather than variation of conditional means. These characterizations may help in the further refinement of frictionless representative consumer models. Alternatively, we may be led to consider models with market frictions.

While market frictions loosen the implications of asset pricing data for candidate discount factors, the important issues for empirical research are to assess the magnitude and direction of the alterations and to determine the extent to which asset pricing puzzles disappear when market frictions are accommodated. Our analysis focused on two forms of frictions: solvency constraints and market-wealth constraints. These seemed to alleviate the puzzles to some extent. However, it appears that our estimated "average" shadow value of the market-wealth constraint is implausibly high, and that the solvency constraint implies extreme volatility for the intertemporal marginal rates of substitution of the unconstrained consumers.

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APPENDIX 1

In this appendix we provide more information about our method for computing Figure 5. The aim is to minimize $E[var(y|\mathcal{G})]$ subject to a constraint that $var[E(y|\mathcal{G})] = c$ for some valid positive number c. First, write the expected conditional variance of y as

$$E[var(y|\mathcal{G})] = E(y^2) - E\{[E(y|\mathcal{G})]^2\}, \tag{A.1}$$

and the variance of $E(y|\mathcal{G})$ as

$$var[E(y|\mathcal{G})] = E[E(y|\mathcal{G})^2] - (Ey)^2. \tag{A.2}$$

Then the associated Lagrangian is given by

$$\mathcal{L} = E(y^2) - (1 + \lambda)E\{[E(y|\mathcal{G})]^2\} + \lambda(Ey)^2. \tag{A.3}$$

Finally, as argued in the text, use the representation of the conditional mean-standard deviation frontier for stochastic discount factors given in Gallant, Hansen, and Tauchen (1990):

$$y_w = p^* + we^*, \tag{A.4}$$

where $w \in \mathcal{G}$, p^* is the minimum second moment stochastic discount factor and e^* is the error in the conditional projection of a unit payoff onto the vector $\mathbf{x}$ of payoffs used in an econometric analysis. Explicit formulas are given in Gallant, Hansen, and Tauchen. In solving the constrained minimization problem it suffices to look at discount factors of the form (A.4). Hence, we only need to choose w.

To derive the first-order conditions, we follow the usual practice of perturbing the optimal w in any arbitrary direction in the conditioning information set $\mathcal{G}$. The optimal (real) coefficient on that perturbation must be zero for all such directions. This results in the following first-order conditions for the optimization:

$$E(y_w e^*|\mathcal{G}) - (1 + \lambda)E(y_w|\mathcal{G})E(e^*|\mathcal{G}) + \lambda(Ey_w)E(e^*|\mathcal{G}) = 0. \tag{A.5}$$

Because e^* is the conditional projection error of a unit payoff onto the vector $\mathbf{x}$ of payoffs, the corresponding projection is $1 - e^*$. Hence, e^* is conditionally orthogonal to $1 - e^*$, and

$$E[(e^*)^2|\mathcal{G}] = E(e^*|\mathcal{G}). \tag{A.6}$$

It follows from (A.4) and (A.6) that

$$\begin{aligned} E(y_w e^*|\mathcal{G}) &= wE[(e^*)^2|\mathcal{G}] \\ &= wE(e^*|\mathcal{G}). \end{aligned} \tag{A.7}$$

Substituting equation (A.7) into equation (A.5) and dividing by $E(e^*|\mathcal{G})$ gives

$$w - (1 + \lambda)E(y_w|\mathcal{G}) + \lambda(Ey_w) = 0 \tag{A.8}$$

Solving equations (A.8) and (A.4) for w gives the solution to the problem.

APPENDIX 2

In this appendix we briefly describe an algorithm that we used to construct the volatility bounds reported in Figure 10. The algorithm is very similar but not identical to one suggested by Hansen and Jagannathan (1992) for computing specification error bounds.

Step 1: Transform prices so that the new pricing formula is

$$E[(y - m^a)\mathbf{x}] = \hat{\mathbf{q}} \qquad \text{where } \hat{\mathbf{q}} \equiv \mathbf{q} - E(m^a\mathbf{x}).$$

Step 2: Find the arbitrage bounds for pricing unit payoff with the $\hat{\mathbf{q}}$ prices.

Step 3: Augment $\mathbf{x}$ with a unit payoff and augment $\hat{\mathbf{q}}$ with a corresponding ˆ price of a unit payoff within the arbitrage bounds. Setting a ˆ price for a unit payoff determines the mean of y via:

$$Ey = \hat{\pi}(1) + Em^a.$$

Step 4: Find the arbitrage bounds for pricing m^a using the augmented payoffs and prices. For each price, find the minimum norm nonnegative stochastic discount factor by solving the dual problem in Hansen and Jagannathan (1991). Note that

$$E(y^2) = E[(y - m^a)^2] + 2E[(y - m^a)m^a] + E[(m^a)^2].$$

By fixing the price assignment to m^a, we fix the middle term, and the third term is fixed by our measure of m^a. Hence, we minimize the left side by minimizing the first term. The dual problem in Hansen and Jagannathan (1991) is designed to minimize $E[(y - m^a)^2]$ by choice of a nonnegative value of $(y - m^a)$ subject to the pricing restriction. Finally, minimize $E(y^2)$ by choice of the price assignment to m^a.

Notice that step 3 provides a mean and step 4 the corresponding second moment bound. Since the mean is fixed in step 4, we obtain a corresponding volatility bound. Steps 3 and 4 must be repeated for all ˆ price assignments to a unit payoff within the arbitrage bounds.

Comment

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This paper has all the merits, and some of the disadvantages, of a systematic methodology and a tight focus. It reminds me of a famous poem