Asset Pricing with Epstein-Zin Preferences.*

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COMMENTS WELCOME

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Abstract

Hansen, Heaton and Li (2005) have recently shown, how news about changes in the long-run growth rates of consumption can impact on current asset prices, if preferences are nonseparable over time. This paper provides a complementary approach to theirs. It examines asset pricing with generalized Epstein-Zin preferences, allowing for nonseparabilities between consumption and leisure as well as trend growth in consumption. A log-linear approximation for the asset pricing formula is provided, showing how news about future consumption and leisure changes matter for asset prices. The asset pricing formulas are evaluated empirically.

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JEL codes: E32, G12, E22, E24

1 Introduction

This paper examines asset pricing with Epstein-Zin preferences, allowing for nonseparabilities between consumption and leisure as well as trend growth in consumption. A log-linear approximation for the asset pricing formula is provided, showing how news about future consumption and leisure changes matter for asset prices. The asset pricing formulas are evaluated empirically.

In particular since Mehra and Prescott (1985), many approaches have been undertaken to provide preference-based (or production-based) theories consistent with observed asset-pricing facts. Surveys of the literature are e.g. in Cochrane (2001) or Campbell (2003), and a list of "exotic preferences" generated by the ensuing research can be found in Backus, Routledge and Zin (2004). While there are many routes, those seeking to "reverse-engineer" preferences from asset pricing facts have mainly followed two approaches. The first involves habit formation or "catching up with the Joneses", see e.g. Campbell and Cochrane (1999) for a formulation that matches a number of key asset market facts. The second approach seeks a separation of the intertemporal elasticity of substition from risk aversion, which could be called Porteus-Kreps-Epstein-Zin-Weil preferences[1]. This paper contributes to the latter branch of research.

Most approaches to pricing assets, using Epstein-Zin preferences, typically substitute out some unobserved value function by an expression, involving the value of the market in order to derive testable implications, see in particular Epstein and Zin (1989) and Campbell (1996). This makes it both hard to apply this framework to the case of non-representative agents as well as applying it in the solution of business cycle models, as in e.g. Tallarini (2000). This paper instead follows the lead of Hansen, Heaton and Li (2005) to derive asset pricing equations from a log-linear expansion around some steady state or some well-understood benchmark (such as the case of unitary intertemporal elasticity of substitition). This paper is complementary to theirs. Rather than directly casting the pricing equation as an operator equation and exploiting the framework of Hansen-Scheinkman (2005), we instead derive an approximate infinite-horizon formula for pricing assets, involving future news about leisure and consumption. Combined

[1] These preferences can be reinterpreted within a robust-control perspective, see Hansen, Sargent and Tallarini, 1999.

with assumptions regarding the evolution of the state of the economy, an operator-based pricing equation can then be re-derived and compared to the results in Hansen-Heaton-Li (2005) as well as Campbell (1996). An empirical application is provided.

A number of researchers have stressed that nonseparabilities between consumption and leisure are important to explain key facts regarding labor markets, see e.g. Hall (2006). Likewise, many aggregate models typically feature trending variables, e.g. due to a unit root or trend growth in the log of total factor productivity. In this paper, I therefore seek to develop an asset pricing formula based on Epstein-Zin preferences, which allows for these features. Rather than evaluating the resulting asset pricing equations with the available apparatus of numerical techniques, see e.g. Judd (1998), I provide asset pricing formulas, using a log-linear framework. The log-linear framework has become a useful point of departure for understanding key features of asset pricing more deeply, and is therefore a useful and perhaps even necessary complement to an entirely numerical evaluation. For a number of reasons, in particular the reasons emphasized in Weitzman (2005), the approximation here should be regarded as a "small-shock" approximation, rather than an asset pricing framework that properly handles tail-events.

2 Preferences

I use capital letters to denote the original variables, and small letters to denote log-deviations from a steady-state growth path (unless explicitly stated otherwise). The preferences I wish to examine are given by

$$\begin{aligned} V_t &= (1-\tilde{\beta})U(C_t, \Phi_t L_t; \Phi_t) \\ &\quad +\beta H^{-1}\left(E_t\left[H(V_{t+1})\right]\right) \end{aligned} \tag{1}$$

where C_t denotes consumption and L_t denotes leisure, and where Φ_t is an exogenous and possibly trending variable, and where I assume that $(C_t/\Phi_t, L_t) \in \mathcal{D}$ for some open convex subset $\mathcal{D}$ of $\mathbf{R}^2_{++}$. I make several assumptions.

Assumption A. 1 *$(\Phi_t/\Phi_{t-1}, C_t/\Phi_t, L_t)$ is a stationary and strictly positive stochastic process. Furthermore, the logarithmic means $\bar{\Gamma}, \bar{C}$ and $\bar{L}$ satisfying*

$$\log\left(\bar{\Gamma}\right) = E\left[\log \Phi_t - \log \Phi_{t-1}\right]$$

$$\begin{aligned}\log\left(\bar{C}\right) &= E\left[\log C_t - \log \Phi_t\right] \\ \log\left(\bar{L}\right) &= E\left[\log L_t\right]\end{aligned}$$

are well defined with $(\bar{C}, \bar{L}) \in \mathcal{D}$.

This assumption is satisfied in many stochastic models, where the driving process as well as the resulting economic variables, including consumption and leisure, are stationary and their logs have finite means. It is also satisfied in many models with a stochastically trending total factor productivity, provided $\log \Phi_t$ is cointegrated with the log of total factor productivity. This includes models with preference shocks and stochastic trend growth, where the productivity of leisure grows with the productivity of market labor in the long run. Interestingly, it also includes models with catching-up-with-the-Joneses preferences, where Φ_t is some average of present and past aggregate consumption so that the ratio of private consumption C_t to the aggregation variable Φ_t is stationary.

Assumption A. 2 *$U(\cdot,\cdot;\cdot)$ is twice continously differentiable. It is concave and strictly increasing in its first two arguments.*

The role of the third argument in $U(\cdot,\cdot;\cdot)$ will become clearer below. For the other two arguments, this is a standard assumption.

Assumption A. 3 *The functions obey the following functional form restrictions.*

$$U(\Phi C, L; \Phi) = \Phi^{1-\eta}\left(\tilde{U}(C, L) + \chi\right) - \tilde{\chi} \tag{2}$$

and

$$H(V) = \left((1-\eta)(V+\chi)\right)^{\frac{1-\nu}{1-\eta}} \tag{3}$$

for some function $\tilde{U}(\cdot,\cdot)$ and parameters $\eta > 0, \nu > 0, \chi, \tilde{\chi}$ satisfying

$$\begin{aligned}\nu &> 0 \\ (1-\eta)\left(\tilde{U}(C, L) + \chi\right) &\geq 0, \ \textit{all } (C, L) \in \mathcal{D} \\ \tilde{\beta} &= \beta\bar{\Gamma}^{1-\eta} < 1 \\ \tilde{\chi} &= \frac{1-\beta}{1-\tilde{\beta}}\chi\end{aligned}$$

Note that $\tilde{U}(\cdot,\cdot)$ must be concave.

This may appear at first to be a strange assumption. Intuitively, this assumption assures that the curvatures do not change as the economy grows richer. Instead, risk aversion with respect to relative gambles in consumption remains stationary. Likewise, the tradeoff between leisure and consumption remains stationary (i.e. income and substitution effects balance), when wages grow with consumption. The same considerations have led many researchers to assume CRRA specifications in growing economies with separable preferences: the assumption above constitutes a generalization to the case of nonseparabilities across time. These assertions follow from the following fundamental property of the preference formulation above. Define

$$\begin{aligned}\tilde{C}_t &= \frac{C_t}{\Phi_t} \\ \tilde{V}_t &= \Phi_t^{\eta-1}\left(V_t+\chi\right)-\chi\end{aligned}$$

Proposition 1 *Equation (1) can be rewritten as*

$$\begin{aligned}\tilde{V}_t &= (1-\tilde{\beta})\tilde{U}(\tilde{C}_t,L_t)+ \\ &\qquad \tilde{\beta}H^{-1}\left(E_t\left[\left(\frac{\Phi_{t+1}}{\bar{\Gamma}\Phi_t}\right)^{1-\nu}H(\tilde{V}_{t+1})\right]\right)\end{aligned} \tag{4}$$

Proof: *Note that for any φ and x, one has $\varphi^{1-\eta}\left(H^{-1}(x)+\chi\right) = H^{-1}(\varphi^{1-\nu}x)+\chi$, as long as everything is well-defined. Note that $\chi = (1-\tilde{\beta})\tilde{\chi}+\beta\chi$. Finally, note that*

$$H(V_{t+1}) = \left((1-\eta)\Phi_{t+1}^{1-\eta}(\tilde{V}_{t+1}+\chi)\right)^{\frac{1-\nu}{1-\eta}} = \Phi_{t+1}^{1-\nu}H(\tilde{V}_{t+1})$$

Thus,

$$\begin{aligned}\tilde{V}_t+\chi &= \Phi_t^{\eta-1}(V_t+\chi) \\ &= (1-\tilde{\beta})\Phi_t^{\eta-1}(U+\tilde{\chi}) \\ &\quad +\frac{\tilde{\beta}}{(\bar{\Gamma}\Phi_t)^{1-\eta}}\left(H^{-1}\left(E_t\left[\Phi_{t+1}^{1-\nu}H(\tilde{V}_{t+1})\right]\right)+\chi\right) \\ &= (1-\tilde{\beta})\left(\tilde{U}(\tilde{C}_t,L_t)+\chi\right) \\ &\quad +\tilde{\beta}\left(H^{-1}\left(E_t\left[\left(\frac{\Phi_{t+1}}{\bar{\Gamma}\Phi_t}\right)^{1-\nu}H(\tilde{V}_{t+1})\right]\right)+\chi\right)\end{aligned}$$

Subtract χ to obtain (4).

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Define

$$\bar{V} = \bar{U} = \tilde{U}(\bar{C}, \bar{L})$$

and note that $\bar{V}$ is that value for V_t and V_{t+1}, which satisfies (4) for $\tilde{C}_t = \bar{C}$, $L_t = \bar{L}$, $\Phi_{t+1}/\Phi_t = \bar{\Gamma}$.

I shall also often impose the following, additional assumption.

Assumption A. 4

$$1 = \frac{\tilde{U}_C\left(\bar{C}, \bar{L}\right)\bar{C}}{\bar{U}} \tag{5}$$

This can always be assured by shifting the intercept of the felicity function $\tilde{U}(\cdot,\cdot)$ without affecting economic choices, i.e., this is a normalization of the preference function. The assumption is convenient, since I can then easily calculate random shifts in the value function $\tilde{V}$ in terms of equivalent permanent increases in consumption.

It is time to examine some examples.

Example 1 *Suppose that $H(\cdot)$ is linear, i.e. suppose that $\nu = \eta \neq 0$. In that case, (1) becomes*

$$V_0 = E\left[\sum_{t=0}^{\infty} \beta^t U(C_t, L_t; \Phi_t)\right]$$

i.e., the standard formula for time-separable preferences, except that it more generally also allows for Φ_t to enter the felicity function directly.

Example 2 *A standard specification is*

$$\begin{aligned} U(C, L; \Phi) &= \frac{C^{1-\eta}}{1-\eta} = \tilde{U}(C, L) \\ 0 &= \chi = \tilde{\chi} \end{aligned}$$

Note that $U(C, L; \Phi)$ does not depend[2] *on L. In that case, equation (1) reads*

$$V_t = \frac{1}{1-\eta}\left(C^{1-\eta} + \beta\left(E\left[((1-\eta)V_{t+1})^{\frac{1-\nu}{1-\eta}}\right]\right)^{\frac{1-\eta}{1-\nu}}\right)$$

which is a standard specification for Epstein-Zin preferences. It is easy to verify assumption 3. Note that $U(\cdot,\cdot;\cdot)$ does not depend on its third argument. Note, though, that assumption 4 is violated.

Example 3 *Let*

$$\tilde{U}(C, L) \;=\; \frac{(Cv(L))^{1-\eta}}{1-\eta} - \chi^* \tag{6}$$

for some strictly positive and strictly increasing function $v(L)$ so that $\tilde{U}(\cdot,\cdot)$ is concave. In order to achieve (5), I need

$$\chi^* = \frac{\eta}{1-\eta}\left(\bar{C}v(L)\right)^{1-\eta} \tag{7}$$

Consequently,

$$\chi^* = \frac{\eta}{1-\eta}\bar{U} \tag{8}$$

To assure $(1-\eta)(\tilde{U}(C, L)+\chi) > 0$, as required in (3), one needs χ to satisfy

$$(1-\eta)(\chi^* - \chi) \geq 0$$

The simplest assumption is to set

$$\chi = \chi^* \tag{9}$$

This assumption has an additional role, see the remarks following proposition 2. With this and assumption 3,

$$U(C_t, L_t; \Phi_t) = \frac{(C_t v(L_t))^{1-\eta} - \frac{1-\beta}{1-\tilde{\beta}}\eta\left(\bar{C}v(L)\right)^{1-\eta}}{1-\eta}$$

and $U(\cdot,\cdot;\cdot)$ again does not depend upon its third argument. For $v(L)$ constant, one obtains a version of example 2, additionally satisfying assumption 4.

[2]Strictly speaking, this violates assumption 2 and my theory below does not apply to this case. However, it is easy to see that a simplified version of that theory, dropping all terms involving leisure, applies here as well.

Example 4 *With the definition of the previous example 3, let* $v(L) = L^{\theta}$, $\theta > 0$. *Then,*

$$\tilde{U}(C, L; \eta) \to \log(C) + \theta \log(L) - \left(\log(\bar{C}) + \theta \log(\bar{L})\right) + 1 \tag{10}$$

as $\eta \to 1$. *This thus delivers the preference specification in Tallarini (2000) except for the constant intercept. The intercept is due to my normalization (5), which one can also check directly. Furthermore*

$$H^{-1}\left(E[H(V_{t+1}]\right) \to \frac{1}{1-\nu} \log\left(E\left[\exp\left((1-\nu)V_{t+1}\right)\right]\right)$$

One way to check this is to define $f(\eta) = (1-\eta)\chi(\eta) = \eta\left(\bar{C}^{1-\alpha}\bar{L}^{\alpha}\right)^{1-\eta}$. *Note that* $f(1) = 1$ *and write* $(1-\eta)(V_{t+1} + \chi(\eta)) \approx 1 + (1-\eta)(V_{t+1} - f'(1)) \approx \exp((1-\eta)(V_{t+1} - f'(1)))$ *to see that* $H(V_{t+1}; \eta) \to \exp((1-\nu)(V_{t+1} - f'(1)))$. *Likewise,* $H^{-1}(x; \eta) \to \log(x)/(1-\nu) + f'(1)$. *Combining, the terms* $f'(1)$ *drop out.*

Example 5 *For a case where* Φ_t *will enter the utility function* $U(\cdot, \cdot; \cdot)$, *consider the GHH-preferences, see Greenwood, Hercowitz and Huffman (1988),*

$$\tilde{U}(C, L) = \frac{\left(C - \kappa(A-L)^{1+\phi}\right)^{1-\eta}}{1-\eta} - \chi \tag{11}$$

where A *is the total time endowment and thus,* $A - L$ *is working time, and* κ, ϕ, η *are parameters and where*

$$\chi = -\frac{\left(C - \kappa(A-L)^{1+\phi}\right)^{-\eta}}{1-\eta} \kappa(A-L)^{1+\phi}$$

in order to fulfill assumption (4). Now,

$$U(C_t, L_t; \Phi_t) = \frac{\left(C_t - \Phi_t \kappa(A-L_t)^{1+\phi}\right)^{1-\eta}}{1-\eta} - \chi$$

which amounts to letting the "productivity" of leisure grow as the economy grows, and which is familiar from the literature employing GHH preferences.

Example 6 *For a simple catching-up-with-the-Joneses example, consider* $\Phi_t = C_t^{aggr}$ *to be aggregate consumption at date* t*, and* $\tilde{U}(\cdot,\cdot)$ *any of the specifications above. Note though, that for* Φ_t *to enter the original problem, one needs a specification such that* Φ_t *does not drop from* $U(C_t, L_t; \Phi_t)$*. E.g. with (11) of example 5, one obtains*

$$V_t = (1-\tilde{\beta})\frac{\left(C_t - C_t^{aggr}\kappa(A-L_t)^{1+\phi}\right)^{1-\eta}}{1-\eta} - \chi + \beta H^{-1}\left(E_t\left[H(V_{t+1})\right]\right)$$

3 Second-Order Characterizations

Introduce

$$\eta_{cc} = -\frac{\tilde{U}_{CC}(\bar{C},\bar{L})\bar{C}}{\tilde{U}_C(\bar{C},\bar{L})} \tag{12}$$

$$\eta_{ll} = -\frac{\tilde{U}_{LL}(\bar{C},\bar{L})\bar{L}}{\tilde{U}_L(\bar{C},\bar{L})} \tag{13}$$

$$\eta_{cl,c} = \frac{\tilde{U}_{CL}(\bar{C},\bar{L})\bar{C}}{\tilde{U}_L(\bar{C},\bar{L})} \tag{14}$$

$$\eta_{cl,l} = \frac{\tilde{U}_{CL}(\bar{C},\bar{L})\bar{L}}{\tilde{U}_C(\bar{C},\bar{L})} \tag{15}$$

which characterize the curvature properties of the felicity function $\tilde{U}$. Note that $\eta_{cc} \geq 0$ is the usual risk aversion with respect to consumption and $\eta_{ll} \geq 0$ is risk aversion with respect to leisure. Due to the Epstein-Zin formulation, e.g. the role for η_{cc} will be the characterization of intertemporal substitution, rather than risk aversion.

Define

$$\kappa = \frac{\tilde{U}_L\bar{L}}{\tilde{U}_C\bar{C}}$$

If preferences are nonseparable in consumption and leisure, then $\eta_{cl,c} \neq 0$ and consequently

$$\kappa = \frac{\eta_{cl,l}}{\eta_{cl,c}}$$

and hence, κ can be calculated from $\eta_{cl,l}$ and vice versa, given a value for $\eta_{cl,c}$. To provide some further intuition on κ, consider a stochastic neoclassical growth model with a Cobb-Douglas production function, where wage times labor equals the labor share $(1-\theta)$ times output Y_t. The usual first-order condition with respect to leisure then shows κ to be the ratio of the expenditure shares for consumption to leisure, and is equal to

$$\kappa = \frac{\bar{L}}{(1-\bar{L})} \frac{(1-\theta)\bar{Y}}{\bar{C}}$$

The following proposition may be useful, if one wishes to avoid dependency of $U(\cdot,\cdot;\cdot)$ on Φ.

Proposition 2 *Do not necessarily impose assumption 4. $U(C,L;\Phi)$ does not depend on Φ for all (C,L,Φ), iff*

$$(1-\eta)(\tilde{U}(C,L)+\chi) \equiv \tilde{U}_C(C,L)C \tag{16}$$

$U(C,L;\Phi)$ does not depend on Φ for (C,L) locally around $(\Phi\bar{C},\bar{L})$ up to a second-order approximation, iff

$$(1-\eta)(\bar{U}+\chi) = \tilde{U}_C(\bar{C},\bar{L})\bar{C} \tag{17}$$

$$\eta = \eta_{cc} \tag{18}$$

$$1-\eta = \eta_{cl,c} \tag{19}$$

Proof: *For equation (16), differentiate the right-hand side of the relationship between $U(\cdot,\cdot;\cdot)$ and $\tilde{U}(\cdot,\cdot)$ in assumption 3 with respect to Φ. For the local approximation, differentiate again with respect to C, with respect to L and with respect to Φ. Evaluate these as well as (16) at the steady state. Noting that one equation is implied by the three others, one obtains (17) to (19).* •

When assumption 4 is imposed, equation (17) can be rewritten as

$$\chi = \frac{\eta}{1-\eta}\bar{U} \tag{20}$$

Note that this coincides with (8), provided (9) holds. Thus, (9) is necessary in example 3 to assure that $U(C,L;\Phi)$ does not depend on Φ. Equation (18)

links the relative risk aversion with respect to consumption of the auxiliary felicity function $\tilde{U}(\cdot,\cdot)$ to the risk aversion parameter η of the functional form assumption 3. It may be natural to impose this condition anyhow. Equation (18) shows, that using the same η in (6) as in (2) was necessary for that example in order for $U(\cdot,\cdot;\cdot)$ not to depend on Φ. Equation (19) is an equation effectively familiar from imposing the equality of income and substitution effects in balanced growth models, see King and Plosser(1989). It is easy to verify directly that (18) and (19) are satisfied in example 3.

Introduce

$$\zeta = -\frac{H''(\bar{V})\bar{V}}{H'(\bar{V})} = \frac{\nu-\eta}{1-\eta}\frac{\bar{V}}{\bar{V}+\chi} \tag{21}$$

as the elasticity of the function $H(\cdot)$, measuring the degree of curvature in departing from the benchmark expected discounted utility framework. Note that $\zeta = 0$ iff $H(\cdot)$ is linear, i.e., if the benchmark expected discounted utility framework applies. If the the normalization assumption 4 together with local independence of $U(C,L;\Phi)$ around the steady state path is imposed, then

$$\zeta = \nu - \eta \tag{22}$$

as can be seen from equation (20) together with $\bar{V} = \bar{U}$.

The next proposition shows, that these values together with some steady state values completely determine our preference specification up to a second order approximation.

Proposition 3 *1. Assume values for $\bar{V} > 0$, $\bar{C} > 0$, $\bar{L} > 0$, $\eta > 0$, $\eta_{cc} \geq 0$, $\eta_{ll} \geq 0$, $\kappa > 0$, $\eta_{cl,c}$, ζ and χ. Iff these values satisfy*

$$0 < (1-\eta)\left(1+\frac{\chi}{\bar{V}}\right)\zeta + \eta \tag{23}$$

$$0 < (1-\eta)(\bar{V}+\chi) \tag{24}$$

$$0 \leq \eta_{cc}\eta_{ll} - \eta_{cl,c}^2\kappa \tag{25}$$

then there is a concave utility function $\tilde{U}(C,L)$, $U(C,L,\Phi)$ defined on some open domain $\mathcal{D} \subset \mathbf{R}^2_{++}$ containing $(\bar{C},\bar{L})$ as well as $\nu > 0$, $H(\cdot)$ satisfying (21), (12) to (15) as well as assumptions 2, 3 and 4. The function $\tilde{U}(C,L)$, $U(C,L,\Phi)$ is unique up to a second order approximation.

2. *Assume values for* $\bar{V} > 0$, $\bar{C} > 0$, $\bar{L} > 0$, $\eta > 0$, $\eta_{ll} \geq 0$, $\kappa > 0$ *as well as* ζ. *Iff these values satisfy*

$$\begin{aligned} 0 &< \eta + \zeta && (26) \\ 0 &\leq \eta\eta_{ll} - (1-\eta)^2\kappa && (27) \end{aligned}$$

then there is a concave utility function $\tilde{U}(C, L)$, $U(C, L; \Phi)$ *defined on some open domain* $\mathcal{D} \subset \mathbf{R}^2_{++}$ *containing* $(\bar{C}, \bar{L})$ *as well as* $\nu > 0$, $H(\cdot)$ *satisfying (21), (12) to (15) as well as assumptions 2, 3 and 4 such that* $U(C, L; \Phi)$ *does not depend on* Φ *locally around* $(\Phi\bar{C}, \bar{L})$. *The function* $\tilde{U}(C, L)$, $U(C, L, \Phi)$ *is unique up to a second order approximation.*

Proof:

1. *Suppose such functions exist. Then* $\nu > 0$ *implies (23), concavity of* $\tilde{U}(\cdot, \cdot)$ *implies (25) and the positivity of* $(1-\eta)(\tilde{U}(C, L) + \chi)$ *evaluated at* $(\bar{C}, \bar{L})$ *implies (24).*

   Conversely, suppose these conditions hold. Let

$$\nu = (1-\eta)\left(1 + \frac{\chi}{\bar{V}}\right)\zeta + \eta$$

   and define $H(V)$ *per the functional form in assumption 3. Define*

$$\tilde{U}_C(\bar{C}, \bar{L}) = \frac{\bar{V}}{\bar{C}} > 0$$

   exploiting assumption 4. Note that

$$\tilde{U}_L(\bar{C}, \bar{L}) = \kappa\frac{\bar{C}}{\bar{L}}\tilde{U}_C(\bar{C}, \bar{L}) > 0$$

   Define

$$\hat{C} = \frac{C - \bar{C}}{\bar{C}},\ \hat{L} = \frac{L - \bar{L}}{\bar{L}}$$

   The second-order approximation of $\tilde{U}$ *around* $(\bar{C}, \bar{L})$ *must be the following quadratic function, which I shall conversely use for providing a*

constructive example,

$$\begin{aligned}\frac{\tilde{U}(C,L)}{\bar{U}} &= 1+\frac{\tilde{U}_C}{\bar{U}}(C-\bar{C})+\frac{\tilde{U}_L}{\bar{U}}(L-\bar{L}) \\ &\quad +\frac{1}{\bar{U}}\left(\frac{1}{2}\tilde{U}_{CC}(C-\bar{C})^2+\tilde{U}_{CL}(C-\bar{C})(L-\bar{L})+\frac{1}{2}\tilde{U}_{LL}(L-\bar{L})^2\right) \\ &= 1+\hat{C}+\kappa\,\hat{L}-\frac{1}{2}\eta_{cc}\hat{C}^2+\eta_{cl,l}\hat{C}\hat{L}-\frac{1}{2}\kappa\,\eta_{ll}\hat{L}^2 \qquad (28)\end{aligned}$$

exploiting the normalization of assumption 4. The assumptions are now satisfied. Tracing through this construction, one can see that there is no choice, demonstrating uniqueness.

2. *Suppose, the conditions hold. Define*

$$\begin{aligned}\nu &= \zeta+\eta \\ \chi &= \frac{\eta}{1-\eta}\bar{V} \\ \eta_{cc} &= \eta \\ \eta_{cl,c} &= 1-\eta\end{aligned}$$

Complete the construction as in the previous step. Note that

$$(1-\eta)(\bar{V}+\chi)=\bar{V}>0$$

Per proposition 2 and equation 22, the result now follows. The converse is now easily established as well.

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Since $\eta_{cl,l}=\kappa\eta_{cl,c}$, the concavity condition (25) can be rewritten as

$$0\leq\eta_{cc}\eta_{ll}-\eta_{cl,c}\eta_{cl,l} \qquad (29)$$

which may be slightly more reminiscent of the formula for a determinant. Define $c=\log(C)-\log(\bar{C})$, $l=\log(L)-\log(\bar{L})$ and $u=\log(\tilde{U}(C,L))-\log(\bar{U})$ to be the log-deviations, and notice that $c\approx\hat{C}$ up to first order, etc.. It may

be tempting to directly replace these terms in (28). Since that is a second-order approximation, however, one needs to be more careful. A second-order Taylor expansion of $f(c,l) = \log \tilde{U}(\bar{C}\exp(c), \bar{L}\exp(l))$ yields instead

$$u \approx c + \kappa l - \frac{1}{2}\eta_{cc}c^2 + (\eta_{cl,l} - \kappa)cl - \frac{1}{2}\kappa\left(\eta_{ll} + 1 - \kappa\right)l^2$$

Example 7 *To be specific, example 3 gives*

$$\begin{aligned}
\eta_{cc} &= \eta \\
\eta_{cl,c} &= 1 - \eta \\
\kappa &= \frac{v'(\bar{L})\bar{L}}{v(\bar{L})} \\
\eta_{cl,l} &= (1-\eta)\kappa \\
\eta_{ll} &= \eta\kappa - \frac{v''(\bar{L})\bar{L}}{v'(\bar{L})}
\end{aligned}$$

For the logarithmic case, i.e. for example 4, one obtains

$$\begin{aligned}
\eta_{cc} &= 1 \\
\eta_{cl,c} &= 0 \\
\kappa &= \theta \\
\eta_{cl,l} &= 0 \\
\eta_{ll} &= 1
\end{aligned}$$

which one could have also obtained from the previous set of equations, noting e.g. that $\eta_{ll} = \eta\theta - (\theta - 1) = 1$.

4 The Investment Problem

To proceed towards asset pricing, consider the investment problem of an agent maximizing V_0 subject to the evolution of preferences, (1) as well as a recursively defined budget constraint of the form

$$C_t + S_t + \ldots = R_t S_{t-1} + \ldots$$

where S_t is the wealth invested in some asset with a gross return (measured in consumption units) of R_t from period $t-1$ to t. Let Λ_t be the Lagrange

multiplier on the budget constraint, and let Ω_t be the Lagrange multiplier on (1). We obtain the standard Lucas (1978) asset pricing equation,

$$\Lambda_t = \beta E_t[\Lambda_{t+1} R_{t+1}] \tag{30}$$

as well as two further first-order condition from differentiation with respect to V_t and with respect to C_t.

A "period" here shall be interpreted to be the relevant investment horizon. For example, while trading costs (and, in some countries, Tobin taxes) probably are a major friction for short investment horizons such as a few months, they presumably matter less, if the horizon is several years. Thus, I shall abstract from trading costs, despite the considerable attention they have attracted, see e.g. Luttmer (1999), and instead investigate a variety of investment horizons. A further reason for considering different investment horizons is the return predictability, which has been observed at longer rather than shorter horizons.

Since these variables are trending, it is more convenient to restate the investment problem in terms of the detrended variables[3]. Define

$$\begin{aligned} \Gamma_t &= \frac{\Phi_t}{\Phi_{t-1}} \\ \tilde{S}_t &= \frac{S_t}{\Phi_t} \end{aligned}$$

The maximization problem now reads

$$\begin{aligned} &\max V_0 \;\; s.t. \\ \tilde{V}_t &= (1-\tilde{\beta})\tilde{U}(\tilde{C}_t, L_t) + \\ &\qquad \tilde{\beta} H^{-1}\left(E_t\left[\left(\frac{\Gamma_{t+1}}{\bar{\Gamma}}\right)^{1-\nu} H(\tilde{V}_{t+1})\right]\right) \end{aligned} \tag{31}$$

$$\tilde{C}_t + \tilde{S}_t + \ldots = \frac{R_t}{\Gamma_t}\tilde{S}_{t-1} + \ldots \tag{32}$$

Let $\tilde{\Omega}_t$ be the Lagrange multiplier for the first constraint (31) and $\tilde{\Lambda}_t$ be the Lagrange multiplier for the second constraint (32). The first-order conditions

[3]Equivalently, take the first-order condition from the original problem, and detrend them.

are

$$\frac{\partial}{\partial \tilde{V}_t}: \quad \tilde{\Omega}_t = \tilde{\Omega}_{t-1}\left(H^{-1}\right)'\left(E_{t-1}\left[\left(\frac{\Gamma_t}{\bar{\Gamma}}\right)^{1-\nu} H(\tilde{V}_t)\right]\right)\left(\frac{\Gamma_t}{\bar{\Gamma}}\right)^{1-\nu} H'(\tilde{V}_t) \tag{33}$$

$$\frac{\partial}{\partial \tilde{C}_t}: \quad \tilde{\Lambda}_t = (1-\tilde{\beta})\tilde{\Omega}_t \tilde{U}_1(\tilde{C}_t, L_t) \tag{34}$$

$$\frac{\partial}{\partial \tilde{S}_t}: \quad \tilde{\Lambda}_t = \tilde{\beta} E_t\left[\frac{\tilde{\Lambda}_{t+1}}{\Gamma_{t+1}} R_{t+1}\right] \tag{35}$$

All variables in these equations are now stationary, allowing the possibility for approximation around some fixed value. While this can in principle be done with high-powered numerical tools such as Judd (1998), we seek to understand the implications of these equations, using loglinearization here, in order to obtain some "first-order, small-noise" insights.

The four equations (31) and (33) to (35) are the key equation for asset pricing. Equation (35) is the standard asset pricing equation (30), modified with a term due to detrending. To make practical use of this equation, I need to rewrite $\tilde{\Lambda}_t$ in terms of observables. Equation (34) relates $\tilde{\Lambda}_t$ to the slope of the felicity function $\tilde{U}(\cdot,\cdot)$, evaluated at observed consumption and leisure, which suffices, if preferences are time-separable. Here, however, there is an additional term indicated from $\tilde{\Omega}_t$, if $H(\cdot)$ is nonlinear. These can be obtained from (33) in principle, except that now unobservables in terms of the value function $\tilde{V}$ seem to arise, which is given by equation (31). This has led researchers in the past to seek ways to substitute out $\tilde{V}$ using observables such as wealth or some proxy thereof. Here, I pursue a different route. As shall be shown below for loglinearization, (33) can be related back to observables on consumption and leisure with an additional parameter characterizing preferences.

5 Loglinearization

As before, let the log of variables with a bar denote the expectation of the log of the corresponding stochastic variables. I.e., introduce also

$$\begin{aligned} \log(\bar{\Omega}) &= E[\log \Omega_t] \\ \log(\bar{\Lambda}) &= E[\log \Lambda_t] \\ \log(\bar{R}) &= E[\log R_t] \end{aligned}$$

where I assume from now on, that R_t is strictly positive. Note that $\Omega_t \equiv 0$, if $H(\cdot)$ is linear, i.e, strictly speaking, the calculations below only apply to the case of non-separability over time. It is easy to infer the appropriate equations, if there is separability, and I shall include comments to that end, when appropriate.

Use small letters to denote the loglinear deviation of some variable from its steady state. It is not hard to see[4], that (31) loglinearizes to

$$v_t = (1-\tilde{\beta})(c_t + \kappa l_t) + \tilde{\beta} E_t \left[\frac{\nu - \eta}{\zeta} \gamma_{t+1} + v_{t+1} \right] \tag{36}$$

exploiting (5). Note that the coefficient on γ_{t+1} is equal to 1 if (22) holds. I.e., risk-aversion or intertemporal substitution does not enter this equation. Rather, it converts temporary changes in consumption and leisure and expected preference shifts into shifts of the value function. In particular, note that a permanent increase in consumption compared to the steady state, $c_s \equiv \bar{c}, s \geq t$, and with all other variables equal to zero, results in $v_t = \bar{c}$. I.e., v_t measures the shift in welfare in terms of an equivalent permanent percentage increase in consumption.

Equation (36) shows, that v_t can be related back to observables, i.e., to c_t, l_t, as well as γ_t, as long as the parameters η,ν and ζ characterizing preferences are known. The parameter γ_t can be inferred either from the first-order condition with respect to leisure or - if e.g. denoting total factor productivity or a smoothed version thereof - observed directly (to the extent that total factor productivity is observable).

Equation (33) loglinearizes to

$$\begin{aligned} \omega_t - \omega_{t-1} &= -\zeta \left(v_t - E_{t-1}[v_t] \right) \\ &\quad + (1-\nu) \left(\gamma_t - E_{t-1}[\gamma_t] \right) + (1-\eta) E_{t-1}[\gamma_t] \end{aligned} \tag{37}$$

where I have sorted the terms conveniently. Note that the change in the Lagrange multiplier on the value function equation is driven by news about the value function and the preference shift parameter. Additionally, predictable movements in the preference shift parameter lead to changes in the multiplier

[4]This can be shown by noting that, generally for variables $Y_t = f(X_t)$, one has $y_t = (f'(\bar{X})\bar{X})/(f(\bar{X})x_t$. Calculate $(H'(\bar{V})\bar{V})/(H(\bar{V})) = ((1-\nu)/(1-\eta))(\bar{V}/(\bar{V}+\chi)) = ((1-\nu)/(\nu-\eta))\zeta$, and likewise for $(H')^{-1}(\cdot)$. Also, compare the result to directly loglinearizing (eq:logprefs), noting that $\bar{U} = \bar{V} = 1-\alpha$ there.

ω_t, if the intertemporal elasticity of substitution η is different from unity. If $\zeta = 0$, which is the benchmark case of welfare as the discounted sum of expected utilities, and if there are no preference shocks, $\gamma_t \equiv 0$, then $\omega_t \equiv 0$, starting at the steady state $\omega_{-1} = 0$.

Finally, equations (34) and (35) loglinearize to

$$\begin{aligned} \lambda_t - \omega_t &= -\eta_{cc} c_t + \eta_{cl,l} l_t && (38) \\ 0 &= E_t \left[\lambda_{t+1} - \lambda_t + r_{t+1} - \gamma_{t+1} \right] && (39) \end{aligned}$$

In a model without stochastic long-run growth, i.e. where $\gamma_t \equiv 0$, the four equations (36) to (39) simplify to

$$\begin{aligned} v_t &= (1 - \tilde{\beta})(c_t + \kappa l_t) + \tilde{\beta} E_t \left[v_{t+1} \right] \\ \omega_t - \omega_{t-1} &= -\zeta \left(v_t - E_{t-1}[v_t] \right) \\ \lambda_t - \omega_t &= -\eta_{cc} c_t + \eta_{cl,l} l_t \\ 0 &= E_t \left[\lambda_{t+1} - \lambda_t + r_{t+1} \right] \end{aligned}$$

6 Asset price implications

6.1 Preliminaries

Introduce the abbreviation

$$m_{t+1} = \lambda_{t+1} - \lambda_t - \gamma_{t+1} \tag{40}$$

for the log-deviation of the stochastic discount factor

$$M_{t+1} = \tilde{\beta} \, \frac{\tilde{\Lambda}_{t+1}}{\tilde{\Lambda}_t \Gamma_{t+1}}$$

from its nonstochastic counterpart, $\bar{M} = \tilde{\beta} / \bar{\Gamma}$.

Rewrite the Lucas asset pricing equation (35) as

$$0 = \log \left(\bar{M} \bar{R} \right) + \log \left(E_t \left[\exp \left(m_{t+1} + r_{t+1} \right) \right] \right) \tag{41}$$

Note that there is no approximation involved so far.

Assume that, conditionally on information at date t, m_{t+1} and r_{t+1} are jointly lognormally distributed, conditional on information up to and including t. Let

$$\mathrm{Cov}_t(X_{t+1}, Y_{t+1}) = E_t[(X_{t+1} - E_t[X_{t+1}])(Y_{t+1} - E_t[Y_{t+1}])]$$

denote covariances, conditional on information up to and including t. Introduce the abbreviated notation

$$\begin{aligned} \mathrm{cov}_{m,r,t} &= \mathrm{Cov}_t(m_{t+1}, r_{t+1}) \\ \sigma^2_{m,t} &= \mathrm{Cov}_t(m_{t+1}, m_{t+1}) \\ \sigma^2_{r,t} &= \mathrm{Cov}_t(r_{t+1}, r_{t+1}) \\ \rho_{m,r,t} &= \frac{\mathrm{cov}_{m,r,t}}{\sigma_{m,t}\,\sigma_{r,t}} \end{aligned}$$

These variances, covariances and correlations may generally depend on time, as indicated above.

Using the standard formula for the expectation of lognormally distributed variables, equation (41) can be rewritten as

$$0 = \log\left(\bar{M}\bar{R}\right) + E_t[m_{t+1}] + E_t[r_{t+1}] + \frac{1}{2}\left(\sigma^2_{m,t} + \sigma^2_{r,t} + 2\rho_{m,r,t}\sigma_{m,t}\sigma_{r,t}\right) \quad (42)$$

For the risk-free rate

$$r^f_t = \log R^f_{t+1} = \log \bar{R}^f + r_{t+1} = \log \bar{R}^f + E_t\left[r_{t+1}\right]$$

i.e. for an asset with $\sigma^2_r = 0$, I have

$$r^f_t = -\log\left(\bar{M}\right) - E_t[m_{t+1}] - \frac{1}{2}\sigma^2_{m,t} \quad (43)$$

As usual, the risk-free rate varies over time either due to variations in the expected growth rate of the shadow value of wealth, $E_t[m_{t+1}]$, or its conditional variance, $\sigma^2_{m,t}$.

For any risky asset, note that

$$\log E_t[R_{t+1}] = E_t[r_{t+1}] + \frac{1}{2}\sigma^2_{r,t}$$

Let $\mathcal{SR}_t$ denote the Sharpe ratio of that asset, calculated as the ratio of the risk premium or equity premium and the standard deviation of the log return, expressed in terms of log-returns rather than percent returns,

$$\mathcal{SR}_t = \frac{\log E_t[R_{t+1}] - r_t^f}{\sigma_{r,t}}$$

The Sharpe ratio is the "price for risk", and generally a more useful number than the equity premium itself, see Lettau and Uhlig (2002) for a detailed discussion. Since it is the difference of the log returns that matters, it usually does not much matter whether both returns are calculated in real terms or in nominal terms. The calculations in nominal terms are usually easier due to the availability of suitable data. Obviously, if r_t^f is a safe nominal return, it will not be a safe real return. This matters if unpredictable inflation volatility is substantial: the Sharpe ratio would then not fully reflect the excess return of a risky over a safe asset. I find that

$$\mathcal{SR}_t = -\rho_{m,r,t}\sigma_{m,t} \tag{44}$$

Moreover, the maximally possible Sharpe ratio $\mathcal{SR}_t^{\max}$ for any asset is

$$\mathcal{SR}_t^{\max} = \sigma_{m,t} \tag{45}$$

This expression only depends on elements of the preference specification, i.e. preference parameters as well as data on consumption, leisure and growth, but not on the underlying asset structure.

6.2 Consumption, leisure and growth

Up to this point, the asset pricing calculations above were exact, given the assumption of lognormal distributions. I shall now proceed to use the loglinear approximations to the first order conditions, i.e. equations (36) to (39). I assume that all logdeviations have a joint normal distribution, conditional on information available at date t. I also assume that all variables dated t are in the information set at date t.

I now apply this standard logic to the preference specification above. Since the model was formulated such that there is a steady state, the results above stay valid, if I replace the logarithms of the Lagrange multiplier with

the log-deviations, etc.., except that for comparison to the data, one ought to keep in mind (and possibly correct the formulas with) the average expected consumption growth rate.

Use (38) to replace λ_{t+1} in the expression (40) for m_{t+1},

$$m_{t+1} = -\eta_{cc} c_{t+1} + \eta_{cl,l} l_{t+1} - \gamma_{t+1} + \omega_{t+1} - \lambda_t \tag{46}$$

and rederive the expression for the risk free rate (43) and the Sharpe ratio (44) in terms of the individual components of m_{t+1}. I obtain

Proposition 4 *To a first-order approximation*

$$r_t^f = -\log\left(\bar{M}\right) + \eta_{cc} E_t[c_{t+1} - c_t] - \eta_{cl,l} E_t[l_{t+1} - l_t] + \eta E_t[\gamma_{t+1}] - \frac{1}{2}\sigma_{m,t}^2 \tag{47}$$

and

$$\mathcal{SR}_t = \eta_{cc}\rho_{c,r,t}\sigma_{c,t} - \eta_{cl,l}\rho_{l,r,t}\sigma_{l,t} + \rho_{\gamma,r}\sigma_{\gamma,t} - \rho_{\omega,r,t}\sigma_{\omega,t} \tag{48}$$

One needs to be careful, how the term "first-order approximation" is to be understood in this proposition. The equation is exact, when all variables are jointly and conditional log-normal and the equation (38) holds exactly, i.e. not just as an approximation. However, if (38) is a first-order approximation to (34), then joint log-normality of the log-deviations with the log return would imply, that the first-order approximation is applied globally on the entire real line for $\log C_t$, $\log L_t$ and $\log \Gamma_{t+1}$, not just locally around $\bar{C}$, $\bar{L}$ and $\bar{\Gamma}$, as pointed out by Samuelson (1970). Judd and Guu have shown that the first-order approximation above holds, as the standard deviations of the shocks converge to zero, provided the underlying asset pricing equation is analytic, which essentially means that it can be represented by an infinite-order Taylor expansion. Whether analyticity can be established, if the underlying random variables have unbounded support (as is the case for a normal distribution) or whether weaker but provable conditions exist, which validate the mean-variance analysis here for lognormally distributed shocks, as the volatility of shocks converge to zero, is an as-of-yet unsolved problem. Analyticity can be established, if the random variables have bounded support, however. Thus, the "first-order approximation" means that it holds, as the variance of the underlying shocks and the diameter of their support set converges to zero, while at the same time approach a normal distribution,

when comparing the demeaned shocks, divided by their standard deviation. A precise statement will be included in a future version of this paper.

For a practical procedure to calculate $\sigma_{m,t}^2$ in equation (47), see equation (59) below. If m_t is homoskedastic, then the risk-free rate is only related to the predictable growth rates in consumption and leisure, as usual. As a consequence and if e.g. $\eta_{cc} = \eta$, then its inverse is the intertemporal elasticity of substitution.

Equation (73) is a key equation for asset pricing, which I shall examine more closely. First, it is useful to consider some special cases.

Example 8 *Consider first the benchmark case of time-separable preferences without trend growth* $\Phi_t \equiv \bar{\Phi}$ *and a constant-relative-risk-aversion utility function in consumption only,*

$$U(C_t, L_t; \bar{\Phi}) = \frac{C_t^{1-\eta}}{1-\eta}$$

In that case, $\eta_{cc} = \eta$, $\eta_{cl,l} = 0$ *and (73) reads*

$$\mathcal{SR}_t = \eta \rho_{c,r,t} \sigma_{c,t} \tag{49}$$

This equation has been emphasized by Lettau and Uhlig (2002) and provides a summary statement of the equity premium observation of Mehra and Prescott (1985). Assuming t *to denote years, and given observations of the Sharpe ratio* $sr_t \approx 0.3$ *(which is lower than the typical number given in the literature, since I am using log-returns here), a conditional standard deviation for aggregate annual consumption growth of 0.015 and a conditional correlation between stock returns and consumption of near* 0.4 *implies* $\eta = 50$*. For a more detailed discussion, see section 8 or e.g. Cochrane (2001).*

Example 9 *Consider the same specification, but now allow for stochastic consumption growth as well. Consider rederiving equation (73) directly from the original Lucas asset pricing equation (30), noting that*

$$\begin{aligned} C_t &= \Phi_t \bar{C} \exp(c_t) \\ C_{t+1} &= \Phi_t \bar{\Gamma} \bar{C} \exp(c_{t+1} + \gamma_{t+1}) \end{aligned}$$

One obtains

$$\mathcal{SR}_t = \eta \rho_{c,r,t} \sigma_{c,t} + \eta \rho_{\gamma,r,t} \sigma_{\gamma,t} \tag{50}$$

In particular, both terms contain the relative risk aversion η as factor, because c_{t+1} and γ_{t+1} both enter the asset pricing equation in the same way. It therefore may appear puzzling, that the term involving γ_{t+1} only enters with a unitary coefficient in (73). There is no contradiction here, however. With time-separability, $\zeta = 0$ and $\nu = \eta$, so that (37) becomes

$$\omega_{t+1} - \omega_t = (1-\eta)(\gamma_{t+1} - E_t[\gamma_{t+1}]) + (1-\eta)E_t[\gamma_{t+1}]$$

and hence,

$$\begin{aligned} \sigma_{\omega,t} &= (1-\eta)\sigma_{\gamma,t} \\ \rho_{\omega,r,t} &= \rho_{\gamma,r,t} \end{aligned}$$

Replacing these terms in (73) delivers equation (50).

Example 10 *Consider time-separable preferences without trend growth $\Phi_t \equiv \bar{\Phi}$, but where the utility function is not separable in consumption and leisure, i.e., where $\eta_{cl,l} \neq 0$. Suppose further, that $\rho_{l,r,t}\sigma_{l,t} \neq 0$. Then, given observations on correlations, standard deviations and the Sharpe ratio, for any value for $\eta_{cc} > 0$, there is a value $\eta_{cl,l}$ solving (73),*

$$\eta_{cl,l} = \frac{\eta_{cc}\rho_{c,r,t}\sigma_{c,t} - \mathcal{SR}_t}{\rho_{l,r,t}\sigma_{l,t}} \tag{51}$$

Thus, there is a large class of preferences which deliver the observed Sharpe ratio for any given value of $\eta_{cc} > 0$, given the observations on consumption and leisure and returns, see proposition 3: there is no need to impose $\eta_{cc} = 50$. While one can assume low values for the relative risk aversion with respect to gambles in consumption, risk aversion is not really gone per se: it is just shifted to leisure instead. Note in particular, that the relative risk aversion for leisure needs to satisfy

$$\eta_{ll} \geq \frac{\eta_{cl,c}^2 \kappa}{\eta_{cc}} \tag{52}$$

due to concavity of preferences, see equation (25). This is discussed in greater detail in section 8 below.

These examples show, that the Sharpe ratio is related to news about the economic variables during the holding period for the asset, but not beyond

that. This changes with nonseparable preferences. To see this and to derive a Sharpe ratio formula in terms of observables, I need a bit more algebra. For any variable x, define the date-t news about x_{t+j} per

$$\epsilon_{x,j,t} = E_t[x_{t+j}] - E_{t-1}[x_{t+j}]$$

With (36), note that

$$\begin{aligned}\epsilon_{v,j,t} &= E_t[v_{t+j}] - E_{t-1}[v_{t+j}] \\ &= (1-\tilde{\beta})\left(\epsilon_{c,j,t} + \kappa\epsilon_{l,j,t}\right) + \tilde{\beta}\frac{\nu-\eta}{\zeta}\epsilon_{\gamma,j,t+1} + \tilde{\beta}\epsilon_{v,j+1,t}\end{aligned}$$

With this and equation (37), note now that

$$\begin{aligned}\omega_{t+1} - E_t[\omega_{t+1}] &= (1-\nu)\epsilon_{\gamma,0,t+1} - \zeta\epsilon_{v,0,t+1} \\ &= (1-\eta)\epsilon_{\gamma,0,t+1} \\ &\quad - \sum_{j=0}^{\infty}\tilde{\beta}^j\left(\zeta(1-\tilde{\beta})\epsilon_{c,j,t+1} + \kappa\zeta(1-\tilde{\beta})\epsilon_{l,j,t+1} + (\nu-\eta)\epsilon_{\gamma,j,t+1}\right)\end{aligned}$$

where I have telescoped out the previous equation and split the $\epsilon_{\gamma,0,t+1}$-term. Note that the first term $(1-\eta)\epsilon_{\gamma,0,t+1}$ is a term which already arises with separable preferences, and which was crucial in example 9. The other terms only arise due to nonseparabilities across time, however, i.e. only if $\nu \neq \eta$. One can see, that current news about future consumption fluctuations and, in particular, current news about future changes in the growth rate of consumption impact on the surprise in ω_{t+1} and therefore impact on asset pricing. It is this component which is the center of attention in Hansen, Heaton and Li (2005): (future) consumption strikes back.

For a given asset with returns r_{t+1}, define the covariances between future consumption-, leisure- and growth-changes with the next period-return,

$$\begin{aligned}\tau_{c,r,t} &= \sum_{j=0}^{\infty}\tilde{\beta}^j E_t\left[\epsilon_{c,j,t+1}\epsilon_{r,0,t+1}\right] \\ \tau_{l,r,t} &= \sum_{j=0}^{\infty}\tilde{\beta}^j E_t\left[\epsilon_{l,j,t+1}\epsilon_{r,0,t+1}\right] \\ \tau_{\gamma,r,t} &= \sum_{j=0}^{\infty}\tilde{\beta}^j E_t\left[\epsilon_{\gamma,j,t+1}\epsilon_{r,0,t+1}\right]\end{aligned}$$

The Sharpe ratio equation (73) can then be restated as follows.

Proposition 5 *To a first-order approximation*

$$\begin{aligned} \mathcal{SR}_t &= \eta_{cc}\rho_{c,r,t}\sigma_{c,t} - \eta_{cl,l}\rho_{l,r,t}\sigma_{l,t} + \eta\rho_{\gamma,r}\sigma_{\gamma,t} \\ &\quad + \frac{1}{\sigma_{r,t}}\left(\zeta(1-\tilde{\beta})\tau_{c,r,t} + \zeta\kappa(1-\tilde{\beta})\tau_{l,r,t} + (\nu-\eta)\tau_{\gamma,r,t}\right) \end{aligned} \tag{53}$$

For the interpretation of "first-order approximation", see the discussion following proposition 4. There are obviously alternative restatements of (53). Let e.g.

$$\begin{aligned} \sigma^2_{\epsilon,c,j,t} &= \mathrm{Cov}_t(\epsilon_{c,j,t+1}, \epsilon_{c,j,t+1}) = E_t[\epsilon^2_{c,j,t+1}] \\ \rho_{\epsilon,c,j,r,t} &= \frac{\mathrm{Cov}_t(\epsilon_{c,j,t+1}, \epsilon_{r,j,t+1})}{\sigma_{\epsilon,c,j,t}\sigma_{r,t}} = \frac{E_t[\epsilon_{c,j,t+1}\epsilon_{r,t+1}]}{\sigma_{\epsilon,c,j,t}\sigma_{r,t}} \end{aligned}$$

If $\sigma_{\epsilon,c,j,t} \neq 0$ for all j, then one can rewrite equation (53) by replacing the left-hand-side with the right-hand-side of the equation

$$\frac{\tau_{c,r,t}}{\sigma_{r,t}} = \sum_{j=0}^{\infty} \tilde{\beta}^j \rho_{\epsilon,c,j,r,t}\sigma_{\epsilon,c,j,t}$$

and thereby express this term in the same manner as the first three terms in (53).

The derivations above were always concerning a particular asset. One can equally well find out the maximally possible Sharpe ratio per (45). Note that

$$\begin{aligned} \epsilon_{m,0,t+1} &= -\eta_{cc}\epsilon_{c,0,t+1} + \eta_{cl,l}l_{t+1} - \eta\gamma_{t+1} \\ &\quad - \sum_{j=0}^{\infty} \tilde{\beta}^j \left(\zeta(1-\tilde{\beta})\epsilon_{c,j,t+1} + \kappa\zeta(1-\tilde{\beta})\epsilon_{l,j,t+1} + (\nu-\eta)\epsilon_{\gamma,j,t+1}\right) \end{aligned}$$

To calculate the maximal Sharpe ratio, one needs to calculate the variance of $\epsilon_{m,0,t+1}$, which involves not only the variances of the terms in (54), but also their covariances. I provide a tractable expression in (59) below.

7 A VAR-Approach

These quantities can be calculated from the data by calculating the news about future variables from the impulse responses of a vector autoregression.

This also provides for a link to the operator-approach in Hansen, Heaton and Li (2005).

Let Y_t be a vector of variables and their lags up to some maximal lag length, containing in particular the log of the ratio of consumption to some chosen trend variable $\log(C_t/\Phi_t)$, log leisure $logL_t$, the log of the growth rate of the chosen trend variable $\log \Gamma_t = \log \Phi_t - \log \Phi_{t-1}$ and log returns $logR_t$ as the first, second, third and forth variable. Suppose that one can summarize the correlation structure in form of a VAR,

$$Y_t = BY_{t-1} + u_t,\ E_{t-1}[u_t u_t'] = \Sigma_{t-1} \tag{54}$$

where the u_t are uncorrelated across time and where one lag in the VAR suffices due to "stacking" lags of the variables into the vector Y_t. I assume that $\tilde{\beta}B$ has all its eigenvalues inside the unit circle. Note that I used the somewhat unconventional date $t-1$ for the variance-covariance matrix of the innovation dated t: that way, the VAR notation is consistent with the notation for conditional covariances above. Note that I allow for heteroskedasticity in the innovations but not in the VAR coefficients. The news about consumption (detrended with Φ_t), leisure and growth is now given by

$$\begin{aligned}
\epsilon_{c,j,t+1} &= \left(B^j u_{t+1}\right)_1 \\
\epsilon_{l,j,t+1} &= \left(B^j u_{t+1}\right)_2 \\
\epsilon_{\gamma,j,t+1} &= \left(B^j u_{t+1}\right)_3
\end{aligned}$$

The covariance with the news about returns is now

$$\begin{aligned}
E_t[\epsilon_{c,j,t+1}\epsilon_{r,0,t+1}] &= \left(B^j \Sigma_t\right)_{14} \\
E_t[\epsilon_{l,j,t+1}\epsilon_{r,0,t+1}] &= \left(B^j \Sigma_t\right)_{24} \\
E_t[\epsilon_{\gamma,j,t+1}\epsilon_{r,0,t+1}] &= \left(B^j \Sigma_t\right)_{34}
\end{aligned}$$

Summing up, one now obtains

$$\begin{aligned}
\tau_{c,r,t} &= \left(\left(\sum_{j=0}^{\infty} \tilde{\beta}^j B^j\right)\Sigma_t\right)_{14} \\
&= Q(\tilde{\beta}, t)_{14} \\
\tau_{l,r,t} &= Q(\tilde{\beta}, t)_{24} \\
\tau_{\gamma,r,t} &= Q(\tilde{\beta}, t)_{34}
\end{aligned}$$

where

$$Q(\tilde{\beta},t)=\left(I-\tilde{\beta}B\right)^{-1}\Sigma_t$$

and where the dependence on the preference parameter $\tilde{\beta}$ and time t has been indicated via the argument. Alternatively, it is useful to rewrite (46) as

$$\epsilon_{m,0,t+1}=\vec{\mathbf{a}}'u_{t+1}-\vec{\mathbf{b}}'\left(I-\tilde{\beta}B\right)^{-1}u_{t+1} \tag{55}$$

where the vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$ and, additionally, the vector $\vec{\mathbf{e}}_4$, are defined as

$$\vec{\mathbf{a}}=\begin{bmatrix}-\eta_{cc}\\ \eta_{cl,l}\\ -\eta\\ 0\\ \vdots\\ 0\end{bmatrix},\ \vec{\mathbf{b}}=\begin{bmatrix}\zeta(1-\beta)\\ \zeta(1-\beta)\kappa\\ \nu-\eta\\ 0\\ \vdots\\ 0\end{bmatrix},\ \vec{\mathbf{e}}_4=\begin{bmatrix}0\\ 0\\ 0\\ 1\\ 0\\ \vdots\\ 0\end{bmatrix} \tag{56}$$

Together with proposition (5), the following proposition follows.

Proposition 6 *Given the VAR representation (54) of the data, to a first-order approximation*

$$\begin{aligned}\mathcal{SR}_t\sigma_{r,t}&=\eta_{cc}\left(\Sigma_t\right)_{14}-\eta_{cl,l}\left(\Sigma_t\right)_{24}+\eta\left(\Sigma_t\right)_{34}\\ &+\zeta(1-\tilde{\beta})Q(\tilde{\beta},t)_{14}+\zeta\kappa(1-\tilde{\beta})Q(\tilde{\beta},t)_{24}+(\nu-\eta)Q(\tilde{\beta},t)_{34}\end{aligned} \tag{57}$$

where

$$\sigma_{r,t}=\sqrt{\left(\Sigma_t\right)_{44}}$$

Alternatively,

$$\mathcal{SR}_t\sigma_{r,t}=-\vec{\mathbf{a}}'\Sigma_t\vec{\mathbf{e}}_4+\vec{\mathbf{b}}'Q(\tilde{\beta},t)\vec{\mathbf{e}}_4 \tag{58}$$

To calculate the maximal Sharpe ratio per equation (45), use the following proposition.

Proposition 7 *To a first-order approximation,*

$$\left(\mathcal{SR}_t^{\max}\right)^2=\sigma_{m,t}^2=\vec{\mathbf{a}}'\Sigma_t\vec{\mathbf{a}}-2\vec{\mathbf{b}}'Q(\tilde{\beta},t)\vec{\mathbf{a}}+\vec{\mathbf{b}}'Q(\tilde{\beta},t)\left(I-\tilde{\beta}B'\right)^{-1}\vec{\mathbf{b}} \tag{59}$$

7.1 k-Period Asset Holdings

Since there may be frictions and preventive trading costs in adjusting portfolios on a quarterly basis, it may be more sensible to apply the asset pricing formulas to a holding period of k periods rather than one period, i.e. for a longer period than the time distance between subsequent observations of the data.

The derivation is quite similar, and a sketch suffices. The Lucas asset pricing formula is

$$1 = E_t \left[\frac{\tilde{\Lambda}_{t+k}}{\tilde{\Lambda}_t} \frac{R_{t+1}}{\Gamma_{t+1}} \frac{R_{t+2}}{\Gamma_{t+2}} \cdots \frac{R_{t+k}}{\Gamma_{t+k}} \right]$$

Define the compounded log-deviation of the stochastic discount factor as

$$m_{t,t+k} = \sum_{i=1}^{k} m_{t+i} = \lambda_{t+k} - \lambda_t - \sum_{i=1}^{k} \gamma_{t+i}$$

Define the averaged[5] compounded log-deviation of the return as

$$r_{t,t+k} = \frac{1}{k} \sum_{i=1}^{k} r_{t+i} \tag{60}$$

Similarly, let $r^f_{t,t+k}$ be the log-deviation for the $k-$period risk free at date t. Since there generally is unexpected future variation in the risk free rate, it is not the case that (60) (or a version, where expectations are taken on the right hand side) holds for the k-period and 1-period risk-free rates . Put differently, the compounding of the one-period risk-free rate is a risky return, to which the asset pricing formulas below apply. For the k-period risk-free rate itself, the following generalization of (47) is obtained immediately.

Proposition 8 *To a first-order approximation*

$$\begin{aligned} r^f_{t,t+k} &= -\log\left(\bar{M}\right) \\ &+ \frac{1}{k}\left(\eta_{cc} E_t[c_{t+k} - c_t] - \eta_{cl,l} E_t[l_{t+k} - l_t] + \eta E_t[\sum_{i=1}^{k} \gamma_{t+i}] - \frac{1}{2}\sigma^2_{m_{t,t+k},t}\right) \end{aligned} \tag{61}$$

[5] I average the return rather than add them up, since that corresponds more closely to the way returns are usually reported.

For the calculation of $\sigma^2_{m_{t,t+k},t}$, see proposition 10 below.

Define the normalized Sharpe ratio as

$$\mathcal{SR}_{k,t} = \frac{E_t[r_{t,t+k}] - r^f_{t,t+k}}{\sqrt{k}\,\sigma_{r_{t,t+k},t}}$$

where and $\sigma_{r_{t,t+k},t}$ is the conditional standard deviation of the difference between $r_{t,t+k}$ and $r^f_{t,t+k}$.

Note that if the expected k-period log excess return can be written as a sum of per-period excess returns r^e_{t+i},

$$\left(\sum_{i=1}^{k} r_{t+i}\right) - r^f_{t,k} = \sum_{i=1}^{k} r^e_{t+i}$$

in such a way that the per-period excess returns are iid, then $\mathcal{SR}_{k,t} = \mathcal{SR}_{1,t} = \mathcal{SR}_t$, i.e. in that case, the normalized Sharpe ratio is independent of the holding horizon. Since excess returns are known to be somewhat predictable, and since there are unpredictable variations in the safe rate, one would not generally expect this independence from the horizon k to hold exactly, but it provides a benchmark for comparison.

For the general case and under joint log-normality, equation (44) now becomes

$$\sqrt{k}\,\mathcal{SR}_{t,k} = -\rho_{m_{t,t+k},r_{t,t+k},t}\sigma_{m_{t,t+k},t} \tag{62}$$

or, alternatively,

$$\sqrt{k}\,\mathcal{SR}_{t,k}\sigma_{r_{t,t+k},t} = -\mathrm{cov}_t(m_{t,t+k}, r_{t,t+k}) \tag{63}$$

Note that

$$\begin{aligned} m_{t,t+k} - E_t[m_{t,t+k}] &= \lambda_{t+k} - E_t[\lambda_{t+k}] - \sum_{i=1}^{k}(\gamma_{t+i} - E_t[\gamma_{t+i}]) \\ &= -\eta_{cc}(c_{t+k} - E_t[c_{t+k}]) + \eta_{cl,l}(l_{t+k} - E_t[l_{t+k}]) \\ &\quad - \sum_{i=1}^{k}(\gamma_{t+i} - E_t[\gamma_{t+i}]) + \omega_{t+k} - E_t[\omega_{t,t+k}] \end{aligned} \tag{64}$$

The decomposition of the compounded log returns and the $k-$period change in log consumption and leisure into news is

$$c_{t+k} - E_t[c_{t+k}] = \sum_{i=1}^{k} \epsilon_{c,k-i,t+i}$$

$$
\begin{aligned}
l_{t+k} - E_t[l_{t+k}] &= \sum_{i=1}^{k} \epsilon_{l,k-i,t+i} \\
\sum_{i=1}^{k} (\gamma_{t+i} - E_t[\gamma_{t+i}) &= \sum_{i=1}^{k} \sum_{j=0}^{k-i} \epsilon_{\gamma,j,t+i} \\
r_{t,t+k} - E_t[r_{t,t+k}] &= \sum_{i=1}^{k} (r_{t+i} - E_t[r_{t+i}]) \\
&= \sum_{i=1}^{k} \sum_{j=0}^{k-i} \epsilon_{r,j,t+i}
\end{aligned}
$$

Furthermore

$$
\begin{aligned}
\omega_{t+k} - E_t[\omega_{t+k}] &= \sum_{i=1}^{k} (\omega_{t+i} - \omega_{t+i-1}) - \sum_{i=1}^{k} E_t\left[\omega_{t+i} - \omega_{t+i-1}\right] \\
&= -\zeta \sum_{i=1}^{k} \epsilon_{v,0,t+i} \\
&\quad +(1-\nu) \sum_{i=1}^{k} \epsilon_{\gamma,0,t+i} \\
&\quad +(1-\eta) \sum_{i=1}^{k} \left(E_{t+i-1}[\gamma_{t+i}] - E_t[\gamma_{t+i}]\right) \\
&= -\sum_{i=1}^{k} \sum_{j=0}^{\infty} \tilde{\beta}^j \left(\zeta(1-\tilde{\beta})\epsilon_{c,j,t+i} + \kappa\zeta(1-\tilde{\beta})\epsilon_{l,j,t+i} + (\nu-\eta)\epsilon_{\gamma,j,t+i}\right) \\
&\quad +(1-\eta) \sum_{i=1}^{k} \left(\gamma_{t+i} - E_t[\gamma_{t+i}]\right)
\end{aligned}
$$

Note that the terms $\epsilon_{\gamma,0,t+i}$ stemming from $\epsilon_{v,0,t+i}$ have been split across the last two last lines in this formula.

As in section 7, suppose that the dynamics of the data can be summarized by the VAR in equation (54) with the conventions adopted there. Introduce the following matrices

$$
\begin{aligned}
\Sigma_{t,i} &= E_t[\Sigma_{t+i}] \\
Q(\tilde{\beta}, t, k) &= (1 - \tilde{\beta}B)^{-1} \sum_{i=1}^{k} \sum_{j=0}^{k-i} \Sigma_{t,i-1} (B')^j
\end{aligned}
$$

$$
\begin{aligned}
S(t,k) &= \sum_{i=1}^{k}\sum_{j=0}^{k-i} B^{k-i}\Sigma_{t,i-1}(B')^{j} \\
P(t,k) &= \sum_{i=1}^{k}\sum_{j_1=0}^{k-i}\sum_{j_2=0}^{k-i} B^{j_1}\Sigma_{t,i-1}(B')^{j_2}
\end{aligned}
$$

Recall the definition of $\vec{\mathbf{b}}$ and $\vec{\mathbf{e}}_4$ in equation (56). Define vectors $\vec{\mathbf{a}}^*$, $\vec{\mathbf{e}}_3$ as follows:

$$
\vec{\mathbf{a}}^* = \begin{bmatrix} -\eta_{cc} \\ \eta_{cl,l} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ \vec{\mathbf{e}}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{65}
$$

Calculation of the conditional covariances now yields the following generalization of proposition 6:

Proposition 9 *Given the VAR representation (54) of the data, to a first-order approximation*

$$
\begin{aligned}
k^{3/2}SR_{t,k}\sigma_{r_{t,t+k},t} &= \eta_{cc}\left(S(t,k)\right)_{14} - \eta_{cl,l}\left(S(t,k)\right)_{24} + \eta\left(P(t,k)\right)_{34} \\
&+\zeta(1-\tilde{\beta})Q(\tilde{\beta},t,k)_{14} + \zeta\kappa(1-\tilde{\beta})Q(\tilde{\beta},t,k)_{24} + (\nu-\eta)Q(\tilde{\beta},t,k)_{34}
\end{aligned} \tag{66}
$$

or alternatively

$$
k^{3/2}SR_{t,k}\sigma_{r_{t,t+k},t} = \left(-\vec{\mathbf{a}}^{*\prime}S(t,k) + \eta\,\vec{\mathbf{e}}_3' P(t,k) + \vec{\mathbf{b}}'Q(\tilde{\beta},t,k)\right)\vec{\mathbf{e}}_4 \tag{67}
$$

where

$$
\sigma_{r_{t,t+k},t} = \frac{\sqrt{\left(P(t,k)\right)_{44}}}{k}
$$

To similarly calculate the maximal Sharpe ratio, rewrite (64) as

$$
m_{t,t+k} - E_t[m_{t,t+k}] = \sum_{i=1}^{k} \vec{c}_i' u_{t+i} \tag{68}
$$

where

$$
\vec{c}_i' = \vec{\mathbf{a}}^{*\prime}B^{k-i} - \eta\,\vec{\mathbf{e}}_3'\left(\sum_{j=0}^{k-i} B^j\right) - \vec{\mathbf{b}}'(I-\tilde{\beta}B)^{-1} \tag{69}
$$

The following proposition follows.

Proposition 10 *To a first-order approximation,*

$$\mathcal{SR}_{t,k}^{\max} = \frac{1}{\sqrt{k}}\sigma_{m_{t,t+k},t}$$

where

$$\sigma^2_{m_{t,t+k},t} = \sum_{i=1}^{k} \vec{c}_i' \Sigma_{t,i-1} \vec{c}_i \tag{70}$$

8 An empirical implementation

This section will be redone in a future version of this paper. For now, there is just some rough birds-eye perspective on the data, and some conclusions for time-separable preferences, when there are nonseparabilities between consumption and leisure.

8.1 Data

Let us investigate the data on the correlations of log leisure, log consumption and log excess returns. Here, log leisure is taken to be the negative of log labor, calculated from the time series AWHI, and log consumption is calculated from the time series PCENDC96, both available from the St. Louis Federal Reserve Bank. To calculate log excess returns $r_{t+1} - r_t^f$, we used the time series TRSP500, which is the total value of a S&P500 portfolio, with dividends reinvested, took logs and quarterly averages, and subtracted from this series the log of the value of a "safe portfolio of compounded quarterly interest rates, taken from the 1-year treasury bill rate. Of this series, we took k-th differences to vary the length of the asset holding period, and likewise for log leisure and log consumption. This leaves in some predictable movements, which can and should be taken out, using the VAR approach of section 7. I.e., in principle, one should also subtract out the part of the excess return which is predictable with e.g. current price-dividend ratios, in order to calculate conditional correlations and standard deviations. The same is true for consumption and leisure. In these calculations, we thus "pretend", that these k-th differences are not predictable and calculate their raw, unconditional correlations. For a first look at the data, differencing may suffice.

The asset market results are in table 1, whereas the standard deviations and correlations with leisure and consumption are in table 2. The time period is 1970:1 to 2003:4. Note that the Sharpe ratio appears to be lower by nearly a factor of two compared to the usual numbers: this is to some degree due to using log returns, which "worsens" negative stock market returns, and "lessens" positive returns, as is necessary for calculating compounded returns (i.e. geometric averages), although that does not appear to explain it entirely.

What one can see in tables 1 and 2 is the following. First, there are no surprises as far as the market price for risk is concerned, as one varies the horizon: the annualized Sharpe ratio remains fairly constant at around 0.3. Second, the correlation between leisure and excess returns over a short holding period of one quarter is very low and too low to be of much help in helping with high consumption risk aversion to explain the equity premium observation.

Third, and more interestingly, the picture does change at longer holding horizons. For example, at a holding period of one year or four quarters, the correlation between leisure and excess returns is already -.21, at eight quarters, it is -.39, and generally exceeds the correlation of consumption with excess returns at horizons above two years.

Finally, the correlation between leisure and stock returns is negative, i.e. stocks provide "insurance against fluctuations in leisure. This is intuitively not surprising, since one expects stocks to do well in booms, which are precisely the times when hours and output are high. Since the Sharpe ratio is determined by the cross derivative term $\eta_{cl,l}$ and not the relative risk aversion with respect to leisure, this insurance aspect is not a problem for the preference-based asset pricing framework: we shall examine the precise implications in the following subsection. If relative risk aversion in consumption is not alone to explain the observed Sharpe ratio, then (73) and the negative correlation between leisure and stock returns implies that one needs $\eta_{cl,l} > 0$, i.e. one needs that leisure and consumption are complements.

The asset pricing formulas above in principle allow for time variation in the volatilities. To generate a time-varying volatility series for leisure, I have calculated the GARCH process

$$\sigma_{l,t}^2 = (1-\phi)\sigma_{l,t-1}^2 + \phi(l_t - l_{t-1} - E[l_t - l_{t-1}])^2$$

initializing the process with the unconditional variance of leisure. I have

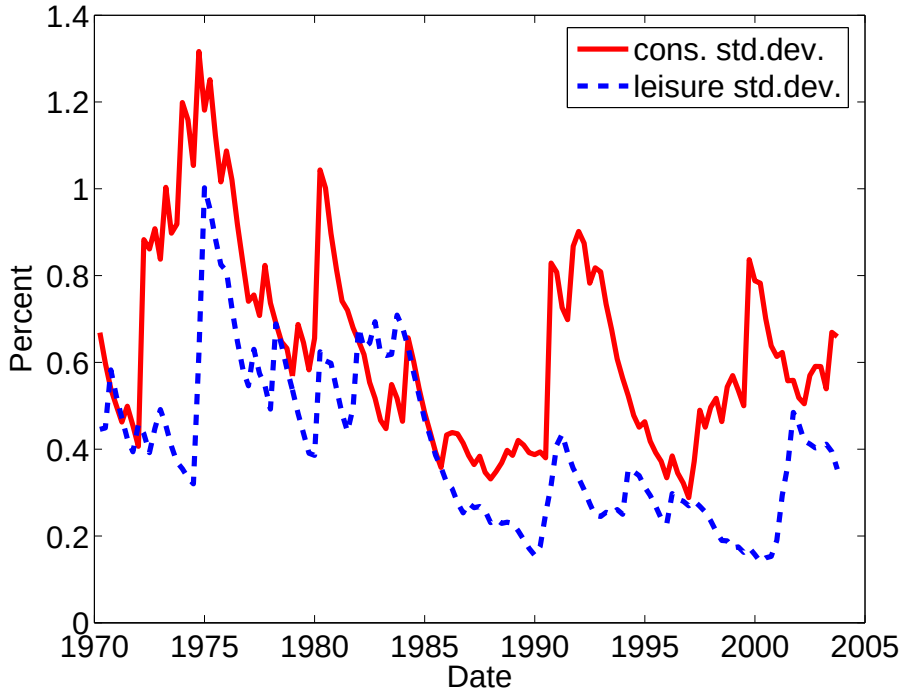

Figure 1: *The time-varying volatilies of leisure and consumption.*

likewise proceeded for consumption. A plot of the two series is in figure 1.

Equation (73) suggests that changing volatities induce changes in the Sharpe ratio. For example, assuming time-separability as well constant correlations, I find

$$\Delta \mathcal{SR}_{t+1} = \eta_{cc}\rho_{c,r}\Delta\sigma_{c,t+1} - \eta_{cl,l}\rho_{l,r}\Delta\sigma_{l,t+1} \tag{71}$$

Assuming furthermore, that stock market volatility stays constant as well, a surprise decrease in the Sharpe ratio implies an extra positive surprise in stock returns. Keeping in mind the negative correlation $\rho_{l,r} < 0$ and the positive value for $\eta_{cl,l}$, equation (71) therefore predicts a negative correlation between stock returns and changes in the volatilies of consumption as well as leisure. Table 3 investigates this issue. Indeed, and in particular at longer horizons, we see that the correlation is negative indeed, in particular between the volatility for leisure and stock returns. I.e., decreases in business cycle uncertainty increase stock returns: this makes a lot of intuitive sense. Figure

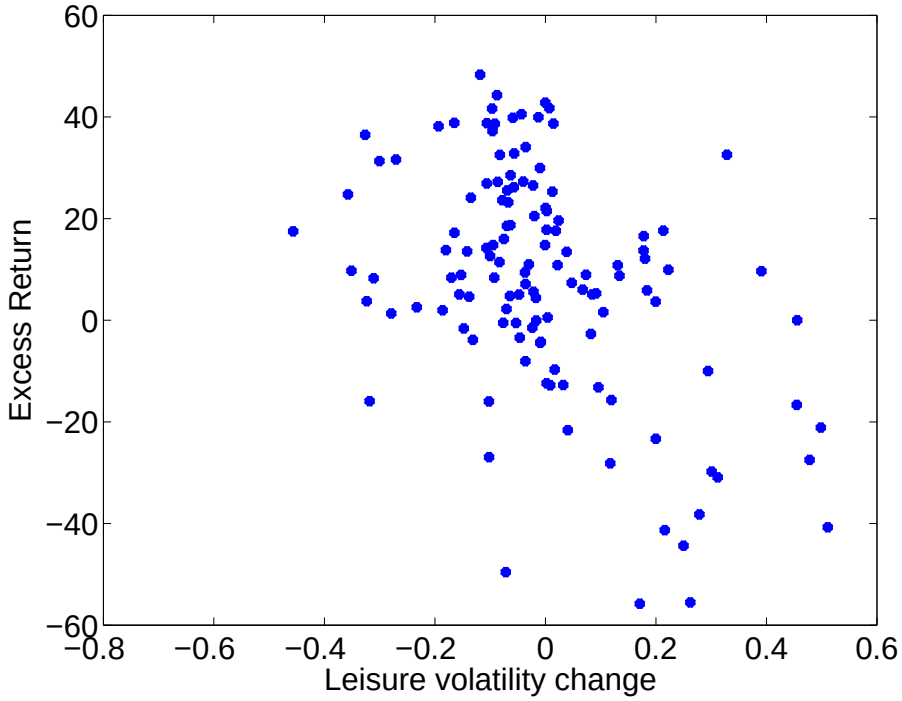

Figure 2: *The correlation between changing leisure volatility and excess stock returns for a holding period of* $k = 8$ *quarters*

2 shows that negative correlation for a holding period of $k = 8$ quarters.

8.2 Implications for preferences

We now use these observations to draw out implications for preferences, assuming now that volatilities and correlations stay constant. The standard case, on which practically the entire asset pricing literature has focussed, is the case $\eta_{cl,l} = 0$. If additionally, preferences are time separable, i.e. $\nu = \eta$, then (73) implies

$$\eta_{cc} = \frac{\mathcal{SR}}{\rho_{c,r}\sigma_c} \tag{72}$$

Horizon k (Quarters)	std.dev. of r_{t+1}	Sharpe ratio	Annualized Sharpe ratio, $SR\sqrt{4/j}$
1	6.87	0.15	0.30
2	10.37	0.21	0.29
3	13.18	0.24	0.28
4	15.40	0.27	0.27
5	17.51	0.29	0.26
6	19.32	0.31	0.25
7	20.96	0.33	0.25
8	22.21	0.36	0.26
9	23.34	0.39	0.26
10	24.66	0.42	0.26
11	25.81	0.44	0.27
12	26.75	0.47	0.27
13	27.69	0.50	0.28
14	28.42	0.54	0.29
15	29.01	0.58	0.30
16	29.47	0.63	0.31
17	29.99	0.67	0.33
18	30.75	0.71	0.33
19	31.17	0.76	0.35
20	31.41	0.82	0.37

Table 1: *Properties of excess returns, when varying the holding horizon.*

Horizon k (Quarters)	std.dev. of leis., σ_l	std.dev. of cons., σ_c	corr(c,l)	corr(l,r)	corr(c,r)
1	0.45	0.67	-0.33	-0.07	0.27
2	0.80	1.04	-0.42	-0.08	0.34
3	1.11	1.33	-0.51	-0.15	0.37
4	1.36	1.64	-0.55	-0.21	0.39
5	1.58	1.90	-0.58	-0.28	0.39
6	1.78	2.10	-0.61	-0.33	0.40
7	1.95	2.27	-0.62	-0.36	0.41
8	2.10	2.42	-0.62	-0.39	0.42
9	2.23	2.52	-0.61	-0.42	0.40
10	2.32	2.60	-0.62	-0.45	0.37
11	2.40	2.67	-0.63	-0.47	0.36
12	2.46	2.73	-0.62	-0.50	0.34
13	2.50	2.80	-0.62	-0.52	0.35
14	2.51	2.87	-0.60	-0.54	0.36
15	2.51	2.95	-0.59	-0.56	0.37
16	2.49	3.01	-0.57	-0.58	0.39
17	2.47	3.06	-0.55	-0.60	0.41
18	2.45	3.09	-0.53	-0.60	0.41
19	2.42	3.12	-0.51	-0.60	0.41
20	2.39	3.11	-0.48	-0.59	0.41

Table 2: *Variances and correlations of leisure and consumption with excess returns.*

Horizon k (Quarters)	std.dev. of leis.vol.	std.dev. of cons.vol.	$corr(\sigma_c, \sigma_l)$	$corr(\sigma_l, r)$	$corr(\sigma_c, r)$
1	0	0.01	0.18	0.06	0.00
2	0.01	0.02	0.22	-0.01	-0.00
3	0.02	0.02	0.24	-0.13	-0.01
4	0.02	0.03	0.21	-0.23	-0.00
5	0.03	0.04	0.21	-0.28	0.01
6	0.03	0.05	0.18	-0.32	0.02
7	0.03	0.06	0.17	-0.38	0.02
8	0.03	0.07	0.17	-0.46	0.02
9	0.04	0.07	0.18	-0.50	-0.00
10	0.04	0.08	0.18	-0.52	-0.04
11	0.04	0.09	0.20	-0.52	-0.06
12	0.04	0.10	0.24	-0.53	-0.07
13	0.04	0.10	0.28	-0.53	-0.08
14	0.04	0.11	0.31	-0.53	-0.10
15	0.05	0.11	0.35	-0.51	-0.11
16	0.05	0.11	0.38	-0.52	-0.11
17	0.05	0.11	0.41	-0.54	-0.13
18	0.05	0.11	0.44	-0.54	-0.12
19	0.05	0.10	0.45	-0.53	-0.10
20	0.05	0.10	0.43	-0.52	-0.09

Table 3: *Variances and correlations of the volatility of leisure, the volatility of consumption and excess returns.*

for the level of relative risk aversion in consumption. Using an annual holding period, $k = 4$, and the data of the tables above, one obtains

$$\eta_{cc} = \frac{0.27}{1.64\% * 0.39} = 42$$

Even assuming perfectly positive correlation, one needs $\eta_{cc} = 16.5$. Other authors typically find even much higher values, see Campbell (2004). These values seem high on a priori grounds and incompatible with standard macroeconomic models.

With nonseparabilities between consumption and leisure, however, lower values for η_{cc} are possible, when the value of the cross-derivative is changed simultaneously as well. To that end, rewrite equation (73) as

$$\eta_{cl,l} = \frac{\mathcal{SR} - \eta_{cc}\rho_{c,r}\sigma_c}{-\rho_{l,r}\sigma_l} \tag{73}$$

For the macroeconomic implications, and since leisure is fairly volatile, it is desirable to pick the relative risk aversion with respect to leisure as low as possible. We thus assume that equation (**??**) holds with equality,

$$\eta_{ll} = \frac{\kappa\eta_{cl,l}^2}{\eta_{cc}}$$

For holding periods of one year, $k = 4$ and two years, $k = 8$, table 4 as well as figures 3 and 4 show the resulting values as a function of the relative risk aversion for consumption, η_{cc}.

We see that explaining the Sharpe ratio remains hard: low values for the relative risk aversion in consumption require dramatically high values for the relative risk aversion in leisure. It is some progress that one can explain the observed Sharpe ratio at levels of relative risk aversion below 20, even when taking account the correct correlations, using the calculations based on a holding period of $k = 8$ quarters. Obviously, these are still fairly high numbers.

9 Conclusions

This paper has provided a loglinear framework for pricing assets, using a generalization of Epstein-Zin preferences, allowing for nonseparabilities between

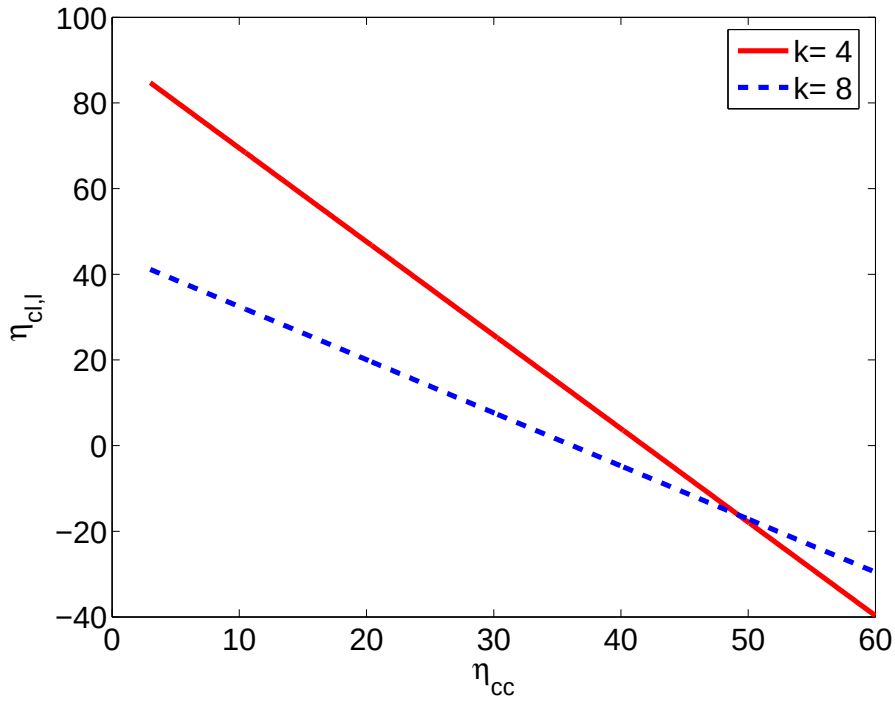

Figure 3: *The implied value for the cross-derivative $\eta_{cl,l}$, when varying the relative risk aversion for consumption between 3 and 60.*

η_{cc}	$\eta_{cl,l}$		η_{ll}	
	k=4	k=8	k=4	k=8
3.0	84.7	41.1	1389.2	327.5
5.0	80.4	38.7	749.8	173.5
10.0	69.5	32.5	280.0	61.2
15.0	58.5	26.3	132.6	26.7
20.0	47.6	20.1	65.8	11.7
30.0	25.8	7.7	12.9	1.1
40.0	4.0	-4.7	0.2	0.3
50.0	-17.9	-17.1	3.7	3.4

Table 4: *Implied values for the cross-derivative term $\eta_{cl,l}$ and the minimal relative risk aversion in leisure η_{ll}, when varying the relative risk aversion in consumption η_{cc}.*

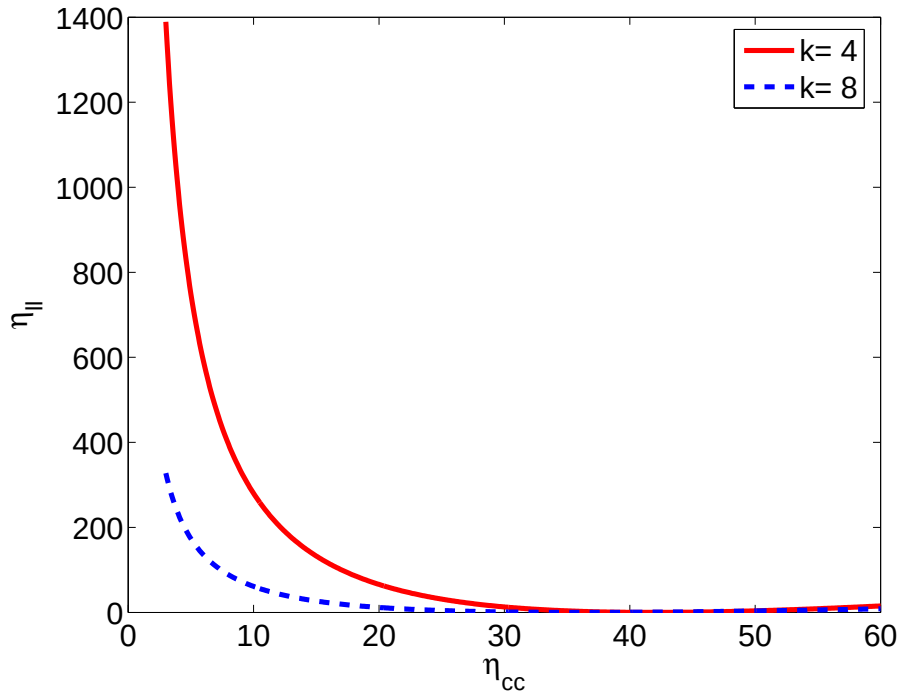

Figure 4: *The implied value for the minimal relative risk aversion in leisure η_{ll}, when varying the relative risk aversion for consumption between 3 and 60.*

consumption and leisure. As in Hansen, Heaton and Li (2005), I find that news about future consumption (and leisure) matters for current asset prices, if preferences are non-separable across time. The relationship between future news and the current Sharpe ratio has been calculated, using a log-normal framework. A VAR formulation has been provided, allowing the calculation of the news component based on current innovations.

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