Notes on Epstein-Zin Asset Pricing
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Asset pricing with Kreps-Porteus preferences, starting with theoretical results from Epstein and Zin (Econometrica 1989, JPE 1991) and moving on to log-linear log-normal approximations that we can use to interpret Bansal-Yaron, Lettau-Ludvigson-Wachter, Hansen-Heaton-Li, etc. No guarantees of accuracy or sense.

Basics

Environment. The setting is a Lucas exchange economy: a “tree” generates a dividend each period equal to output $y$, which in equilibrium equals the consumption of the single representative agent. The growth rate $x$ (of the dividend/output/consumption) follows a stationary Markov process based on some as yet unspecified definition of the state. Preferences are homothetic, which generates a stationary price-dividend ratio $Q$. If $q = Qy$ is the price, the (gross)

$$r_{pt+1} = \frac{q_{t+1} + yt+1}{qt} = \left(\frac{Q_{t+1} + 1}{Q_t}\right) x_{t+1},$$

where $x_{t+1} = yt+1/yt$.  

Pricing relation. In this or any other arbitrage-free environment, the return $r_i$ on any tradeable asset $i$ satisfies

$$1 = E_t(m_{t+1} r_{it+1}),$$

for some positive pricing kernel $m$. Epstein and Zin propose preferences characterized by the time aggregator

$$U_t = [(1 - \beta) c^\rho + \beta \mu_t(U_{t+1})]^{1/\rho}$$

and the (expected utility) certainty equivalent function

$$\mu_t(z_{t+1}) = \left[E_t(z_{t+1}^\alpha)\right]^{1/\alpha}$$

for some random variable $z$. Here $\rho < 1$ captures time preference (the intertemporal elasticity of substitution is $1/(1 - \rho)$) and $\alpha < 1$ captures risk aversion (the coefficient of relative risk aversion is $1 - \alpha$). The innovation relative to additive utility is that $\rho$ and $\alpha$ need not be equal. We refer to these preferences as Kreps-Porteus to distinguish them from other preferences described by Epstein and Zin (Econometrica, 1989).

With these preferences and the pure exchange environment (both are necessary), the pricing kernel is

$$m_{t+1} = \beta^\gamma x_{t+1}^{\gamma(\rho-1)} r_{pt+1}^{\gamma-1},$$
where \( \gamma = \alpha / \rho \). If \( \gamma = 1 \) \((\alpha = \rho)\) this reduces to the traditional additive model in which \( m_{t+1} = \beta x_t^{\rho - 1} = \beta x_t^{\alpha - 1} \).

Solution method. In the additive model, the process for \( m \) follows directly from that of \( x \). Here we need to find \( r_p \) first. We do this in the following steps: (i) Apply the pricing relation (2) to \( r_t = r_p \) to find the price-dividend ratio \( Q \):

\[
Q_t^\gamma = E_t [\beta^\gamma x_t^\rho (Q_{t+1} + 1)^\gamma] = E_t \left( [\beta x_t^\rho (Q_{t+1} + 1)]^\gamma \right). \tag{6}
\]

Through this equation, a process for \( x \) implies a process for \( Q \). (ii) Given processes for \( x \) and \( Q \) we use (1) to compute the return \( r_p \). (iii) Given \( r_p \) we use (5) to compute the pricing kernel, which allows us to price any asset we like.

For future reference, note that \( Q \) is constant (independent of the state) if \( \rho = 0 \) (log time aggregator) or \( x \) is iid (the same distribution in all states).

Log-linear log-normal approximation

Log-normal dividend process. We can get a sense of how this works by considering a log-normal environment. Let us say that the dividend growth rate follows the infinite moving average process

\[
\log x_t = \bar{x} + \sum_{j=0}^{\infty} \chi_j \varepsilon_{t-j}, \tag{7}
\]

with \( \{\varepsilon_t\} \sim \text{NID}(0, 1) \) and \( \sum_j \chi_j^2 < \infty \) ("square summable"). This is general enough to allow a wide variety of growth rate dynamics.

Log-linear approximation. The problem is that the return \( r_p \) isn’t log-normal: the \((Q + 1)\) term in (6) isn’t log-linear in \( Q \), so \( Q \) isn’t exactly log-normal — nor is \( r_p \). But we might guess that it’s approximately log-normal, a guess we’ll make here without further verification. A linear approximation \( \log(Q + 1) \) [in \( \log Q \)] around an arbitrary point \( \log \bar{Q} \) is

\[
\log(Q + 1) \cong \kappa_0 + \kappa_1 \log Q \tag{8}
\]

where \( \kappa_1 = \bar{Q}/(Q + 1) < 1 \) and \( \kappa_0 = \log(\bar{Q} + 1) - \kappa_1 \log \bar{Q} \). [Note: these aren’t free parameters — they should be implied by the model via \( \bar{Q} \). More later.]

Solution. With this approximation, we conjecture an infinite MA process for \( \log Q \) and use it to find the kernel:

\[
\log Q_t = \bar{Q} + \sum_{j=0}^{\infty} \theta_j \varepsilon_{t-j} \tag{9}
\]

with \( \sum_j \theta_j^2 < \infty \). We start by evaluating (6):

\[
\log \left[ \beta^\gamma x_t^{\rho \gamma} (Q_{t+1} + 1)^\gamma \right] = \gamma \left( \log \beta + \rho \log x_{t+1} + \log(1 + Q_{t+1}) \right)
\]

\[
= \gamma (\log \beta + \rho \bar{x} + \kappa_0 + \kappa_1 \bar{Q}) + \gamma \sum_{j=0}^{\infty} (\rho \chi_j + \kappa_1 \theta_j) \varepsilon_{t-1-j}. \]
To compute the conditional expectation, recall that if \( x \sim N(a,b) \), then \( \log E(x) = a + b/2 \). Applying that here, we have

\[
\gamma \log Q_t = \gamma (\bar{Q} + \sum_{j=0}^{\infty} \theta_j \varepsilon_{t-j}) \\
= \gamma (\log \beta + \rho \bar{x} + \kappa_0 + \kappa_1 \bar{Q}) + \gamma^2 (\rho \chi_0 + \kappa_1 \theta_0)^2/2 + \gamma \sum_{j=0}^{\infty} (\rho \chi_{j+1} + \kappa_1 \theta_{j+1}) \varepsilon_{t-j}.
\]

Lining up terms, we see:

\[
(1 - \kappa_1) \bar{Q} = (\log \beta + \rho \bar{x} + \kappa_0) + \gamma (\rho \chi_0 + \kappa_1 \theta_0)^2/2 \\
\theta_j = \rho \chi_{j+1} + \kappa_1 \theta_{j+1}
\]

with the second equation holding for \( j \geq 0 \).

It takes some effort to find the \( \theta \)'s. If we solve (13) for \( \theta_{j+1} \) and substitute repeatedly, we find

\[
\theta_j = \kappa_1^{-j} \left( \theta_0 - \rho \sum_{i=1}^{j} \kappa_1^{i-1} \chi_i \right).
\]

The square summability condition requires \( \lim_{j \to \infty} \theta_j^2 = 0 \), which implies

\[
\kappa_1 \theta_0 = \rho \sum_{j=1}^{\infty} \kappa_1^j \chi_j = \rho X_0.
\]

(This condition isn’t enough for square summability, but gives us \( \theta_0 \) if it does.) Given \( \theta_0 \), we then use (10) to fill out the sequence. With (11) we can refine our solution of the price process:

\[
(1 - \kappa_1) \bar{Q} = (\log \beta + \rho \bar{x} + \kappa_0) + \alpha \rho (\chi_0 + X_0)^2/2 \\
\theta_j = \rho \kappa_1^{-j} \sum_{i=j+1}^{\infty} \kappa_1^i \chi_i = \rho \kappa_1^{-j} X_j.
\]

for \( X_j = \sum_{i=j+1}^{\infty} \kappa_1^i \chi_i \) and \( j \geq 0 \). For future reference, note that \( (1 - \kappa_1) \bar{Q} = \kappa_1 \), which we could use later to eliminate \( \kappa_1 \) from our expressions. [Recall: \( \kappa_1 \) is not a primitive parameter and should, in principle, derived from the parameters governing preferences and the growth rate process.]

Next, we use the solution to find the return \( r_p \) on the aggregate portfolio and the pricing kernel \( m \). From (1), the return is

\[
\log r_{pt+1} = \log (Q_{t+1} + 1) - \log Q_t + x_{t+1} \\
= [\kappa_0 - (1 - \kappa_1) \bar{Q} + \bar{x}] + (\chi_0 + \kappa_1 \theta_0) \varepsilon_{t+1} + \sum_{j=0}^{\infty} (\chi_{j+1} + \kappa_1 \theta_{j+1} - \theta_j) \varepsilon_{t-j} \\
= - \log \beta + (1 - \rho) \bar{x} - \alpha \rho (\chi_0 + X_0)^2/2 + (\chi_0 + \rho X_0) \varepsilon_{t+1} + (1 - \rho) \sum_{j=0}^{\infty} \chi_{j+1} \varepsilon_{t-j}.
\]
When $\rho \neq 0$, the dynamics of the return differ from those of the growth rate in the initial term (apart from scaling). The risk aversion parameter $\alpha$ plays no role in this, although it does affect the mean. From (5), the pricing kernel is

$$
\log m_{t+1} = \gamma \log \beta + \gamma (\rho - 1) \log x_{t+1} + (\gamma - 1) \log r_{pt+1}
$$

$$
= [\log \beta + (\rho - 1) \bar{x} - \alpha (\alpha - \rho) (\chi_0 + X_0)^2 / 2]
+ [(\rho - 1) \chi_0 + (\alpha - \rho) (\chi_0 + X_0)] \varepsilon_{t+1} + (\rho - 1) \sum_{j=0}^{\infty} \chi_{j+1} \varepsilon_{t-j}.
$$

Unlike the additive case, the moving average coefficients of the pricing kernel differ from those of the growth rate in the first term. How much depends on $X_0$, the (weighted) cumulative sum of moving average coefficients from next period on. Note, too, that in the iid case ($X_0 = 0$),

$$
\log m_{t+1} = [\log \beta + (\rho - 1) \bar{x} - \alpha (\alpha - \rho) (\chi_0)^2 / 2] + (\alpha - 1) \chi_0 \varepsilon_{t+1}.
$$

The model is then observationally equivalent to one with additive utility and a different discount factor (Kotcherlakota, JF, 1990).

**Finding $\kappa_1$.** There’s no obvious simple substitution to get rid of $\kappa_1$. We could iterate once we have everything else and make sure it satisfies its definition. Stan’s suggestion is to approximate at the solution to the iid case, where (12) becomes

$$
\kappa_1 = (1 - \kappa_1) \bar{Q} = (\log \beta + \rho \bar{x} + \kappa_0) + \alpha \rho (\chi_0)^2 / 2
$$

It’s a little ugly, but with our expression for $\kappa_0$ we could solve this for $\kappa_1$ and $\bar{Q}$.

**Utility-based approach**

This starts with an idea we got from Hansen-Heaton-Li (“Consumption strikes back,” October 2005): to do the log-linear approximation directly on the recursive representation of utility. They note that the pricing kernel can be represented by

$$
m_{t+1} = \beta x_{t+1}^{\rho - 1} \left( \frac{x_{t+1} v_{t+1} + \frac{1}{\mu_t(x_{t+1} v_{t+1})}}{x_{t+1}} \right)^{\alpha - \rho}.
$$

Here the trick is to evaluate the second term.

Step 1. Since preferences are homogeneous of degree one, we can divide (3) by $c_t$ to get

$$
v_t = [(1 - \beta) + \beta \mu_t(v_{t+1} x_{t+1})^\rho]^{1/\rho},
$$

where $v_t = U_t / c_t$. We’ll now do a log-linear approximation of this, which serves the same purpose as the Campbell-Shiller log-linear approximation of $\log(Q + 1)$ in equation (8). Taking logs, let

$$
\log v_t = \rho^{-1} \log [(1 - \beta) + \beta \mu^\rho]
= \rho^{-1} \log [(1 - \beta) + \beta \exp(\rho u_t)],
$$
where \( u_t = \log \mu_t \). A first-order approximation of the rhs around \( u = 0 \) is
\[
\log v_t = \beta u_t = \beta \log \mu_t(v_{t+1}x_{t+1}). \tag{14}
\]
If we approximate around an arbitrary value \( \bar{u} \), then we get
\[
\log v_t = \rho^{-1} \log [(1 - \beta) + \beta \exp(\rho \bar{u})] + \left( \frac{\beta \exp(\rho \bar{u})}{1 - \beta + \beta \exp(\rho \bar{u})} \right) (u_t - \bar{u})
= \kappa_0 + \kappa_1 \log \mu_t(v_{t+1}x_{t+1}).
\]
The parameters \((\kappa_0, \kappa_1)\) may be different from those used earlier. HHL start with \( \rho = 0 \), which gives you a discount factor of \( \beta \) regardless.

Step 2. Now it’s the usual guess and verify. Guess
\[
\log v = \bar{v} + \sum_{j=0}^\infty \nu_j \varepsilon_{t-j}
\]
for parameters to be determined. Evaluate the certainty equivalent [equation (4)]:
\[
\log \mu_t(v_{t+1}x_{t+1}) = \bar{v} + \bar{x} + \alpha(v_0 + \chi_0)^2/2 + \sum_{j=0}^\infty (\nu_{j+1} + \chi_{j+1}) \varepsilon_{t-j}
\]
Then the recursion (14) implies
\[
\bar{v} = \kappa_0 + \kappa_1 (\bar{v} + \bar{x}) + \kappa_1 \alpha(v_0 + \chi_0)^2/2
\]
\[
\nu_j = \kappa_1 (\nu_{j+1} + \chi_{j+1}), \ j \geq 0.
\]
Solving forward, we find
\[
\nu_j = \sum_{i=1}^\infty \kappa_1^i \chi_{j+i}
\]
\[
\nu_j + \chi_j = \sum_{i=0}^\infty \kappa_1^i \chi_{j+i} \equiv Z_j.
\]
This allows us to express \( \log v \) in terms of primitives.

Step 3. A slight variant of the mrs formula is
\[
m_{t+1} = \beta x_t^{\alpha-1} v_{t+1}^{\alpha-\rho} \mu_t(x_{t+1}v_{t+1})^{\rho-\alpha}.
\]
Line up terms:
\[
\log x_{t+1} = \bar{x} + \sum_{j=0}^\infty \chi_j \varepsilon_{t+1-j}
\]
\[
\log v_{t+1} = \bar{v} + \sum_{j=0}^\infty \nu_j \varepsilon_{t+1-j}
\]
\[
\log \mu_t = \bar{v} + \bar{x} + \alpha Z_0^2/2 \sum_{j=0}^\infty Z_{j+1} \varepsilon_{t+1-j}.
\]
That gives us

\[
\log m_{t+1} = \log \beta + (\rho - 1)\bar{x} + (\rho - \alpha)\alpha Z_0^2/2
+ [(\rho - 1)\chi_0 + (\alpha - \rho)Z_0]\varepsilon_{t+1} + (\rho - 1)\sum_{j=0}^{\infty} \chi_j \varepsilon_{t-j},
\]

which is similar to what we had before. [Needs to be checked.] Note that the discounting in the sums of depends on the point around which we approximate, since that affects \( \kappa_1 \).