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IDENTIFICATION IN LINEAR SIMULTANEOUS EQUATIONS MODELS WITH COVARIANCE RESTRICTIONS: AN INSTRUMENTAL VARIABLES INTERPRETATION

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Necessary and sufficient conditions for identification with linear coefficient and covariance restrictions are developed in a limited information context. For the limited information case, covariance restrictions aid identification if and only if they imply that a set of endogenous variables is predetermined in the equation of interest (generalizing the idea of recursiveness). Under full information, covariance restrictions imply that residuals from other equations are predetermined in a particular equation and, under certain conditions, can aid in identification. Sufficient conditions for identification are obtained for the hierarchical system in which the identification of a particular equation does not depend upon the identifiability of higher-numbered equations. In the general case, the FIML first order conditions show that if the system of equations is identifiable as a whole, covariance restrictions cause residuals to behave as instruments. In both limited and full information settings, the link between identification and estimation is worked out: restrictions useful for identification yield instruments required for estimation.

1. INTRODUCTION

THE PROBLEMS OF IDENTIFIABILITY and estimation of structural parameters in linear simultaneous equations models are closely related: see, e.g., Richmond [**11**]. When prior information consists of linear restrictions on the structural coefficients (henceforth "coefficient restrictions"), the early work at the Cowles Foundation determined necessary and sufficient conditions for identification and related these to maximum likelihood estimation (Koopmans, Rubin, and Leipnik [**10**], Koopmans [**8**], or Koopmans and Hood [**9**]). For this case (coefficient restrictions), the work was extended to show the relationship between identifiability and instrumental variables estimation: i.e., that the restrictions required for identification give rise to the instrumental variables required for estimation (Fisher [**3**, Theorem 2.7.2]).

This picture is greatly complicated when restrictions on the structural disturbance variances and covariances (henceforth "covariance restrictions") are allowed. Koopmans, Rubin, and Leipnik [**10**] recognized the usefulness of such restrictions for identification and demonstrated their equivalence to bilinear restrictions on the coefficients. This work was pursued by Wegge [**14**], Rothenberg [**12**], and especially Fisher [**1**, **2**], surveyed in Fisher [**3**, Chapters 3 and 4]. Structural estimation, too, is complicated by covariance restrictions, in the sense that system instrumental variables estimators (3SLS) are asymptotically inefficient when covariance restrictions are present; see Rothenberg and Leenders [**13**].

Two features of these results are (i) the absence of useful necessary and sufficient conditions for identifiability in the presence of covariance restrictions,

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and (ii) the disappearance of the link between restrictions required for identification and instrumental variables required for estimation. For the limited information case,[2] we derive simple necessary and sufficient conditions for identifiability in linear simultaneous equations models subject to linear restrictions on the coefficients and covariances. For the full information case,[3] we show through the FIML first order conditions how covariance restrictions affect identification. We do not obtain necessary and sufficient conditions in general, but we derive a useful sufficient condition for the hierarchical case in which the equations can be ordered so that the identifiability of an equation is only affected by the identifiability of lower-numbered equations.

In practical terms, covariance restrictions aid identification in the limited information model if and only if a set of endogenous variables is uncorrelated with the disturbance in the first structural equation and may be treated as predetermined. In the full information model, the covariance restriction $\sigma_{ij} = 0$ is useful whenever it implies that an *identifiable residual* from equation j can be treated as predetermined in equation i and vice-versa. In this sense, identifiability and the existence of sufficient instrumental variables are shown to be equivalent.

2. PRELIMINARIES

Consider the classical simultaneous equations model

$$YB' + Z\Gamma' = U \tag{2.1}$$

in which Y is a $T \times G$ matrix of observations on the G jointly dependent random vectors y_i $(i = 1, \ldots, G)$ and Z is the $T \times K$ matrix of observations on the K variables which are predetermined in *every* equation: i.e., $\text{plim}_{T\to\infty}(1/T)Z'u_i = \mathbf{0}$, where $U = [u_1, \ldots, u_G]$. The structural disturbances u_i are independent and identically distributed but contemporaneously correlated across equations. Thus $\text{cov}(U_j) = \Sigma$ $(j = 1, \ldots, T)$ where U_j is the jth row of U and Σ is a positive semidefinite, symmetric, $G \times G$ matrix, which may be singular to accommodate identities among the structural equations.

Imposing exclusion restrictions and the normalization $\beta_{ii} = 1$, equation (2.1) becomes

$$y_i = X_i\delta_i + u_i \qquad (i = 1, \ldots, G), \tag{2.2}$$

where $X_i = [Y_i Z_i]$ and $\delta' = [\beta_i' \gamma_i']$, the $(g_i - 1) + k_i$ coefficients in the ith structural equation which are unknown, a priori. Without loss of generality, we will be concerned with the identifiability of the unknown parameters $[\beta_1, \gamma_1, \sigma_{11}]$ in the

[2]Prior information is restricted to the coefficients and disturbance covariances of the first structural equation, along with a list of its predetermined explanatory variables.

[3]Prior information on any coefficients or covariances is permitted.

first structural equation

$$y_1 = Y_1\beta_1 + Z_1\gamma_1 + u_1, \qquad \mathrm{var}(u_{11}) = \sigma_{11},$$

which is equivalent to the identifiability of σ_{11} and the first row of $\mathbf{B}$ and $\boldsymbol{\Gamma}$: i.e., of $[\mathbf{B}_1, \boldsymbol{\Gamma}_1, \sigma_{11}]$.

Since $\mathbf{B}$ is nonsingular, equation (2.1) can be solved to yield the reduced form equations:

(2.3) $$Y + Z\boldsymbol{\Gamma}'(\mathbf{B}')^{-1} = U(\mathbf{B}')^{-1}$$

or

$$Y = Z\boldsymbol{\Pi}' + V,$$

where the reduced form parameters $\boldsymbol{\Pi}$ and $\boldsymbol{\Omega} = \mathrm{cov}(V)$ are related to the structural parameters by

(2.4) $$\boldsymbol{\Pi} = -\mathbf{B}^{-1}\boldsymbol{\Gamma},$$

(2.5) $$\boldsymbol{\Omega} = \mathbf{B}^{-1}\boldsymbol{\Sigma}(\mathbf{B}')^{-1}.$$

Since $E(Y|Z) = Z\boldsymbol{\Pi}'$ and $\mathrm{cov}(Y|Z) = \boldsymbol{\Omega}$, the reduced form parameters are identifiable provided that Z is of full rank, which we assume. Moreover, the Jacobian of the transformation from U to V is nonvanishing, so that the structural parameters are identifiable from observations on Y and Z if and only if $(\mathbf{B}, \boldsymbol{\Gamma}, \boldsymbol{\Sigma})$ are uniquely determined in equations (2.4) and (2.5), given $(\boldsymbol{\Pi}, \boldsymbol{\Omega})$ and whatever prior information on $[\mathbf{B}, \boldsymbol{\Gamma}, \boldsymbol{\Sigma}]$ that we are prepared to assume. In particular, for the parameters of the first structural equation to be identifiable, it is necessary and sufficient that equations (2.4) and (2.5) admit of a unique solution for $[\mathbf{B}_1, \boldsymbol{\Gamma}_1, \sigma_{11}]$, given $(\boldsymbol{\Pi}, \boldsymbol{\Omega})$ and prior information on $[\mathbf{B}, \boldsymbol{\Gamma}, \boldsymbol{\Sigma}]$.

Let us now isolate those parts of equations (2.4) and (2.5) which actually restrict the parameters of the first structural equation.

LEMMA 1: *Treating* $(\mathbf{B}, \boldsymbol{\Gamma}, \boldsymbol{\Sigma})$ *as unknowns,*

(2.6) $$\boldsymbol{\Pi}'\mathbf{B}' = -\boldsymbol{\Gamma}', \qquad \boldsymbol{\Omega}\mathbf{B}' = \mathbf{B}^{-1}\boldsymbol{\Sigma}$$

and

(2.7) $$\boldsymbol{\Pi}'\mathbf{B}_1' = -\boldsymbol{\Gamma}_1', \qquad \boldsymbol{\Omega}\mathbf{B}_1' = \mathbf{B}^{-1}\boldsymbol{\Sigma}_1'$$

have the same solutions $(\mathbf{B}_1'^*, \boldsymbol{\Gamma}_1'^*, \boldsymbol{\Sigma}_1'^*)$.

PROOF: Rewrite equations (2.6) as

$$\boldsymbol{\Pi}'\mathbf{B}_i' = -\boldsymbol{\Gamma}_i',$$

$$\mathbf{B}\boldsymbol{\Omega}\mathbf{B}_i' = \boldsymbol{\Sigma}_i',$$

for $i = (1, \ldots, G)$, and observe that $[\mathbf{B}_1'^*, \boldsymbol{\Gamma}_1'^*, \boldsymbol{\Sigma}_1'^*]$ solves equation (2.6) if and only if it solves

$$\boldsymbol{\Pi}'\mathbf{B}_1' = -\boldsymbol{\Gamma}_1',$$

$$\mathbf{B}_i\boldsymbol{\Omega}\mathbf{B}_1' = \sigma_{1i} \qquad (i = 1, \ldots, G),$$

which is equivalent to equations (2.7).

The fact that the existence of a unique solution to a set of *linear* equations turns out to characterize the identification problem (despite the fact that equation (2.7) is not linear in $\mathbf{B}_1'$) prompts the following definition. If $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ is a consistent set of linear equations, its solution, $\boldsymbol{x}^*$, is unique if and only if the dimension of the null space of $\boldsymbol{A}$ is zero. Then a restriction on the structural parameters will be said to be *useful* for identification if and only if it reduces the dimension of the null space of $\boldsymbol{A}$ in the relevant system of linear equations.

For future reference, we denote the null space of a matrix $\boldsymbol{A}$ by $\boldsymbol{N}(\boldsymbol{A})$, its column space by $\boldsymbol{C}(\boldsymbol{A})$, the orthogonal projection operator onto $\boldsymbol{C}(\boldsymbol{A})$ by $\boldsymbol{P}_A$, and the projection operator onto $\boldsymbol{N}(\boldsymbol{A})$ by $\boldsymbol{I} - \boldsymbol{P}_A = \boldsymbol{Q}_A$.

3. LIMITED INFORMATION

In this section, we confine our interest to the first structural equation and our prior information to (i) linear restrictions on $[\mathbf{B}_1, \boldsymbol{\Gamma}_1, \boldsymbol{\Sigma}_1]$ and (ii) the classification of $\boldsymbol{Y}$ and $\boldsymbol{Z}$ into variables which are endogenous or predetermined in the first structural equation.[4] Under unrestrictive conditions, this classification, in turn, requires exclusion restrictions on the entire $\mathbf{B}$ matrix in addition to the linear restrictions on $\boldsymbol{\Sigma}_1$. Our use of the term "limited information" is thus nonstandard; however, this usage seems appropriate for identification and estimation of a single equation when covariance restrictions are present.

3.1. *Restrictions on* $[\mathbf{B}_1, \boldsymbol{\Gamma}_1, \boldsymbol{\Sigma}_1]$

The prior information on $[\mathbf{B}_1, \boldsymbol{\Gamma}_1]$ is written as

$$\boldsymbol{\Phi}\begin{bmatrix} \mathbf{B}_1' \\ \boldsymbol{\Gamma}_1' \end{bmatrix} = \boldsymbol{\phi} \tag{3.1}$$

where $\boldsymbol{\Phi}$ is an $r \times (G + K)$ matrix of known constants and $\boldsymbol{\phi}$ is a known r vector. Identification of $[\mathbf{B}_1, \boldsymbol{\Gamma}_1, \sigma_{11}]$ requires a unique solution to equations (2.7), which

[4] Were prior information limited to linear restrictions on $[\mathbf{B}_1, \boldsymbol{\Gamma}_1]$, we would have the classical case, treated at length in Koopmans and Hood [9] and Fisher [3, Chapter 2]. If, in addition, restrictions on $\boldsymbol{\Sigma}_1$ were available, we would have the case of variance restrictions, covered in Fisher [3, Chapter 3]. In both these cases, necessary and sufficient conditions for identifiability are obtained *relative to the prior information they assume*.

we rewrite as

$$\begin{bmatrix} [\Pi' & & I] \\ & \Phi & \\ \Omega & & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{B}_1' \\ \Gamma_1' \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \phi \\ \mathbf{B}^{-1}\Sigma_1' \end{bmatrix}. \tag{3.2}$$

With no prior information to determine $\mathbf{B}^{-1}\Sigma_1'$, only the first two equations restrict solutions $[\mathbf{B}_1'^*, \Gamma_1'^*]$. This is the case analyzed in Koopmans and Hood [**9**], and a necessary and sufficient condition for identifiability is the familiar

$$\operatorname{rank}\begin{bmatrix} \Pi' & I \\ \Phi & \end{bmatrix} = G + K. \tag{3.3}$$

Additional prior information on Σ_1 is useful if it determines $\mathbf{B}^{-1}\Sigma_1'$, but since every element of $\mathbf{B}^{-1}$ is a function of the elements of $\mathbf{B}_i$ $(i = 2, \ldots, G)$ about which we know nothing, this information is useful only if it implies that $\Sigma_1 = \mathbf{0}$. In this case, $\sigma_{11} = 0$ and identification is equivalent to the uniqueness of solutions to

$$\begin{bmatrix} \Pi' & & I \\ & \Phi & \\ \Omega & & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{B}_1' \\ \Gamma_1' \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \phi \\ \mathbf{0} \end{bmatrix}. \tag{3.4}$$

A necessary and sufficient condition for uniqueness is that the rank of the left-hand matrix be $G + K$, which is equivalent to the Generalized Rank Condition of Fisher [**3**, Theorem 3.8.1].

3.2. *Exclusion Restrictions on* B

We now step out of the classical limited information case, and consider prior information about structural coefficients from equations other than the first. As we have seen, without prior information on other structural equations, covariance restrictions, apart from trivial variance restrictions, do not contribute to the identifiability of the parameters of the first equation. Thus to our linear restrictions on $[\mathbf{B}_1, \Gamma_1, \Sigma_1]$, we append knowledge that certain elements of $\mathbf{B}$ are zero.[5] We shall see presently that this information is necessary and sufficient to classify $[Y\, Z]$ into variables which are predetermined or endogenous *in the first structural equation*, which preserves the limited information flavor of our analysis. For limited information *estimation* of the first equation, this extension of limited information is appropriate, since it characterizes the information necessary to calculate an efficient instrumental variables estimator of δ_1 which uses all variables predetermined in the first equation as instruments.

[5] The following results are easily extended to the case of general linear restrictions on both coefficients and covariances.

3.2.1. *Relative Triangularity*

From equation (3.2), it is clear that restrictions on $\mathbf{B}$ and $\mathbf{\Sigma}_1$ that restrict $\mathbf{B}^{-1}\mathbf{\Sigma}_1'$ are useful for identifying $[\mathbf{B}_1, \mathbf{\Gamma}_1, \sigma_{11}]$. Our information about $\mathbf{B}$ is subject to the following assumptions, which will be maintained throughout: (i) $\mathbf{B}$ is nonsingular, $\mathbf{B}_{ii} = 1$; (ii) certain elements of $\mathbf{B}$ are zero and the remainder are unrestricted; and (iii) sums of products of non-zero elements of $\mathbf{B}$ are nonzero.

The first two assumptions are conventional; the last rules out events of measure zero in the parameter space and ensures that minors of $\mathbf{B}$ can be zero only by appropriate zero restrictions in assumption (ii).

DEFINITION: For a $G \times G$ matrix $\mathbf{B}$, a *chain product* corresponding to the ith row and the jth column is a product of no more than $G - 1$ elements of $\mathbf{B}$ of the form

$$\beta_{ia}\beta_{ab}\beta_{bc}\cdots\beta_{fj}$$

where distinct indices are distinct. The set of all such chain products is denoted $\mathbf{B}_{[i,j]}$.

A chain product has a number of useful properties: (i) If $\mathbf{B}_{(ji)}$ denotes the submatrix of $\mathbf{B}$ obtained by deleting the jth row and the ith column, all elements of $\mathbf{B}_{(ji)}$ and only elements of $\mathbf{B}_{(ji)}$ appear among the products of $\mathbf{B}_{[i,j]}$. (ii) No element of $\mathbf{B}_{[i,j]}$ contains a shorter element of $\mathbf{B}_{[i,j]}$, since no elements come from the jth row of $\mathbf{B}$. (iii) In a given chain product, each index (except i and j) appears exactly once as a row index and once as a column index.

For example, for a 4×4 matrix $\mathbf{B}$,

$$\mathbf{B}_{[2,4]} = \{\beta_{24}, \beta_{21}\beta_{14}, \beta_{23}\beta_{34}, \beta_{21}\beta_{13}\beta_{34}, \beta_{23}\beta_{31}\beta_{14}\}.$$

The notion of a chain product is central to the idea of relative triangularity, which turns out to be the feature of the $\mathbf{B}$ matrix which is relevant for identification.

DEFINITION: Equations (i, j) are *relatively triangular* if and only if $\mathbf{B}_{[i,j]} = \{0\}$.[6]

The name is motivated by the observation that $\mathbf{B}$ is triangular if equations (i, j) are relatively triangular for all $i < j$.

Relative triangularity and the intuitive notion of feedback are closely connected. Consider the following example:

$$\begin{aligned} y_1 &= \beta_{12} y_2 + \beta_{13} y_3 + Z\mathbf{\Gamma}_1' + u_1, \\ y_2 &= \qquad\qquad\;\; \beta_{23} y_3 + Z\mathbf{\Gamma}_2' + u_2, \\ y_3 &= \beta_{32} y_2 \qquad\qquad\;\; + Z\mathbf{\Gamma}_3' + u_3, \end{aligned}$$

[6]Note that equations (i, j) relatively triangular does not imply that equations (j, i) are relatively triangular.

where the coefficients of the exogenous variables are unrestricted. The first equation has a peculiar relationship to the rest of the system, in the sense that the variables (y_2, y_3) are simultaneously determined in the second and third equations with no direct feedback from the first. Indeed, given $\mathbf{B}$ above, one can readily check that $\mathbf{B}_{[3,1]} = \mathbf{B}_{[2,1]} = \{0\}$ so that equations (3, 1) and (2, 1) are relatively triangular. The set $\mathbf{B}_{[i,j]}$ details the paths by which a shock to $\boldsymbol{y}_j$ is transmitted to $\boldsymbol{y}_i$. In the previous example, $\mathbf{B}_{[1,3]} = \{\beta_{13}, \beta_{12}, \beta_{23}\}$, so that a shock to $\boldsymbol{y}_3$ perturbs $\boldsymbol{y}_1$ in the first equation in two ways: directly, since $\beta_{13} \neq 0$ and indirectly through the second equation since both β_{12} and $\beta_{23} \neq 0$.

Somewhat surprisingly, the relative triangularity of equations (i, j) is precisely equivalent to a zero in the (i, j)th position of $\mathbf{B}^{-1}$, which we denote by $\mathbf{B}_{ij}^{-1}$.

LEMMA 2: $\mathbf{B}_{ij}^{-1} = 0$ *if and only if equations* (i, j) *are relatively triangular.*

The proof appears in the Appendix. In the previous example,

$$\mathbf{B}^{-1} = \frac{1}{1-\beta_{32}\beta_{23}} \begin{bmatrix} 1-\beta_{32}\beta_{23} & \beta_{12}+\beta_{13}\beta_{32} & \beta_{13}+\beta_{12}\beta_{23} \\ 0 & 1 & \beta_{23} \\ 0 & \beta_{32} & 1 \end{bmatrix},$$

which has zeros in positions (2, 1) and (3, 1).

The relative triangularity of equations $(i, 1)$ is obviously a necessary condition for $\boldsymbol{y}_i$ to be uncorrelated with $\boldsymbol{u}_1$: a shock to $\boldsymbol{u}_1$ perturbs $\boldsymbol{y}_1$, but there is no direct path from $\boldsymbol{y}_1$ to $\boldsymbol{y}_i$ if equations $(i, 1)$ are relatively triangular. However, as in the case of full triangularity, something further must be said about the disturbance covariances before this information is useful for identification or before $\boldsymbol{y}_i$ can be treated as predetermined in the first equation.

3.2.2. *Relative Recursiveness*

Zero restrictions on $\mathbf{\Sigma}_1$ and rows of $\mathbf{B}$ other than the first restrict solutions to

$$\begin{bmatrix} [\mathbf{\Pi}' & & I] \\ & \mathbf{\Phi} & \\ \mathbf{\Omega} & & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{B}_1' \\ \mathbf{\Gamma}_1' \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \boldsymbol{\phi} \\ \mathbf{B}^{-1}\mathbf{\Sigma}_1' \end{bmatrix} \tag{3.5}$$

only through the third equation. If elements of the vector $\mathbf{B}^{-1}\mathbf{\Sigma}_1'$ are known, they must be zero under our current assumptions, since we have ruled out the possibility that sums of nonzero terms are zero. If elements of $\mathbf{B}^{-1}\mathbf{\Sigma}_1'$ are unknown, then only the top two (block) rows of equation (3.5) are relevant for identification, and we are back in the world of Section 3.1.[7] Formalizing this, let the selection matrix $\mathbf{\Psi}$ be the appropriate $s \times G$ submatrix of the identity $\boldsymbol{I}_G$.

[7] This is proved in the Appendix.

LEMMA 3: *Zero restrictions on* $(\mathbf{B}, \boldsymbol{\Sigma}_1)$ *are useful for identification if and only if they imply*

$$\boldsymbol{\Psi}\mathbf{B}^{-1}\boldsymbol{\Sigma}_1' = \mathbf{0}$$

for some selection matrix $\boldsymbol{\Psi}$.

In order for the ith element of the vector $\mathbf{B}^{-1}\boldsymbol{\Sigma}_1'$, denoted $[\mathbf{B}^{-1}\boldsymbol{\Sigma}_1']_i$, to be zero, zeros are required in the appropriate places of $\mathbf{B}^{-1}$ and $\boldsymbol{\Sigma}_1'$. Assuming from now on that $\sigma_{11} \neq 0$, we have the following proposition.

PROPOSITION 1: $[\mathbf{B}^{-1}\boldsymbol{\Sigma}_1']_i = 0$ *if and only if* (i) *equations* $(i, 1)$ *are relatively triangular, and* (ii) $\boldsymbol{u}_1$ *is uncorrelated with every* $\boldsymbol{u}_k$ *for which equations* (i, k) *are not relatively triangular.*

PROOF: Let $\mathbf{B}_{[i,k]} \otimes \sigma_{k1}$ denote the set of products of each element of $\mathbf{B}_{[i,k]}$ with σ_{k1}, for $k = 1, \ldots, G$. The Proposition can then be restated as $[\mathbf{B}^{-1}\boldsymbol{\Sigma}_1']_i = 0$ if an only if $\mathbf{B}_{[i,k]} \otimes \sigma_{k1} = \{0\}$. Now $[\mathbf{B}^{-1}\boldsymbol{\Sigma}_1']_i$ is the inner product of the ith row of $\mathbf{B}^{-1}$ with $\boldsymbol{\Sigma}_1'$, and since sums of nonzero terms are nonzero, $[\mathbf{B}^{-1}\boldsymbol{\Sigma}_1']_i = 0$ if and only if $[\mathbf{B}^{-1}]_{ik} \neq 0$ implies $\sigma_{k1} = 0$ for all $k = 1, \ldots, G$. This is equivalent to $\mathbf{B}_{[i,k]} \otimes \sigma_{k1} = 0$ by Lemma 2.

Note that the condition on $\boldsymbol{\Sigma}_1'$ is less restrictive than $\sigma_{1j} = 0$ $(j = 2, \ldots, G)$, which, for the entire system, would imply diagonality of $\boldsymbol{\Sigma}$. In particular, σ_{1j} need not be zero to obtain $[\mathbf{B}^{-1}\boldsymbol{\Sigma}_1']_i = 0$, provided that equations (i, j) are relatively triangular. Of course, since $\beta_{ii} = 1$, σ_{i1} must equal zero for $[\mathbf{B}^{-1}\boldsymbol{\Sigma}_1']_i = 0$. If equations (i, j) are such that $[\mathbf{B}^{-1}\boldsymbol{\Sigma}_j']_i = 0$ for all $i < j = 2, \ldots, G$, the system of equations is fully recursive in the sense of Wold **[15]**, which motivates the following definition.

DEFINITION: Equations (i, j) are *relatively recursive* if and only if $[\mathbf{B}^{-1}\boldsymbol{\Sigma}_j']_i = 0$.

Intuitively, equations (i, j) are relatively recursive if and only if there are no paths by which a shock to $\boldsymbol{u}_j$ can be transmitted to $\boldsymbol{y}_i$. In our previous example, equations (2, 1) and (3, 1) were relatively triangular. If $\sigma_{12} = 0$ but $\sigma_{13} \neq 0$, shocks to $\boldsymbol{u}_1$ reach $\boldsymbol{y}_3$ directly through the correlation with $\boldsymbol{u}_3$ and reach $\boldsymbol{y}_2$ indirectly since σ_{13} and β_{23} differ from zero. Thus unless both σ_{12} and σ_{13} are zero, equations (2.1) and (3.1) are relatively triangular but not relatively recursive.

Since

$$\left[\mathbf{B}^{-1}\boldsymbol{\Sigma}_j'\right]_i = \left[\mathbf{B}^{-1}\boldsymbol{\Sigma}\right]_{ij} = \left[\operatorname*{plim}_{T\to\infty} \frac{1}{T} \boldsymbol{Y}'\boldsymbol{U}\right]_{ij},$$

$[\mathbf{B}^{-1}\boldsymbol{\Sigma}_1']_i = 0$ if and only if $\operatorname{plim}_{T\to\infty}(1/T)\boldsymbol{y}_i'\boldsymbol{u}_1 = 0$. This proves the following.

PROPOSITION 2: *Equations* $(i, 1)$ *are relatively recursive if and only if* y_i *is predetermined in the first equation.*

Moreover, if y_i is predetermined in the first equation, some additional structure is imposed on the system of simultaneous equations:

COROLLARY 2.1: *If* y_i *is predetermined in the first equation, then every endogenous variable in the ith structural equation* (y_j) *for which* $\sigma_{j1} = 0$ *is predetermined in the first structural equation.*

PROOF: As shown, y_i predetermined in the first equation is equivalent to $\mathbf{B}_{[i,k]} \otimes \sigma_{k1} = \{0\}$, for $k = 1, \ldots, G$. Now $\mathbf{B}_{[i,k]} = \mathbf{B}_{[i,j]} \otimes \mathbf{B}_{[j,k]}$, so that

$$\left[\mathbf{B}_{[i,j]} \otimes \mathbf{B}_{[j,k]}\right] \otimes \sigma_{k1} = \{0\}$$

by hypothesis. Endogenous variables in the ith structural equation have coefficients $\beta_{ij} \neq 0$; thus the above equation implies $\mathbf{B}_{[j,k]} \otimes \sigma_{k1} = \{0\}$, for all $k \neq j$. Provided $\sigma_{j1} = 0$, this is equivalent to y_j being predetermined in the first structural equation.

Note that in the case of a diagonal disturbance covariance matrix, an endogenous variable predetermined in the first equation gives rise to a very special structure. If y_i is predetermined in the first equation, every endogenous variable in the ith equation is also predetermined in the first equation. And every endogenous variable in their respective equations is predetermined in the first equation. Thus in the diagonal $\mathbf{\Sigma}$ case, a relatively recursive situation is one in which a *set* of endogenous variables determines itself independently from the first endogenous variable, and all elements of the set are predetermined in the first equation.

Summarizing to this point, we have shown that zero restrictions on $(\mathbf{B}, \mathbf{\Sigma}_1)$ are useful for identification if and only if they imply that a set of endogenous variables is predetermined in the first equation. Before formally analyzing identifiability in these circumstances, we might well ask where this type of prior information is likely to come from.

As pictured by Fisher [**3**, p. 101], the list of exogenous variables $(\boldsymbol{Z})$ in our model is derived from the block-recursive structure of the "universe-embracing" equation system that determines all economic variables. If it is reasonable to possess the knowledge that $\boldsymbol{Z}$ is predetermined in *every* equation in the model, it is surely no less reasonable to know that an endogenous variable is predetermined in a particular equation in the model. The analysis in this section extends the block-recursive structure used to justify the exogeneity of $\boldsymbol{Z}$ to a relatively recursive structure which justifies treating y_j as predetermined in the first equation. A priori, there is no reason why the first type of information would be reasonable but the second arcane.

Secondly, for over-identified equations, a number of tests of exogeneity have been proposed, e.g., Wu [**16**], Hausman [**4**], and Hausman and Taylor [**5**]. These tests determine whether a specified set of variables (endogeneous, exogenous, included, or excluded) can be treated as predetermined in a given equation, which gives rise to information of the form $\text{plim}(1/T)y_i'u_1 = 0$. This is not, of course, *prior* information, but the preceding analysis shows what parametric restrictions are actually under test in exogeneity tests and how, if correct, they affect the identification question.

3.3 *Identification with Exclusion Restrictions*

Having zero restrictions on $[\mathbf{B}, \mathbf{\Sigma}_1]$, identifiability of the parameters of the first equation reduces to the uniqueness of solutions $[\mathbf{B}_1', \mathbf{\Gamma}_1', \sigma_{11}]$ to

$$\mathbf{\Pi}'\mathbf{B}_1' = -\mathbf{\Gamma}_1', \tag{3.6}$$

$$\mathbf{\Omega}\mathbf{B}_1' = \mathbf{B}^{-1}\mathbf{\Sigma}_1', \tag{3.7}$$

given

$$\mathbf{\Phi}\begin{bmatrix}\mathbf{B}_1' \\ \mathbf{\Gamma}_1'\end{bmatrix} = \phi, \tag{3.8}$$

$$\mathbf{\Psi}\mathbf{\Omega}\mathbf{B}_1' = \mathbf{\Psi}\mathbf{B}^{-1}\mathbf{\Sigma}_1' = \mathbf{0}. \tag{3.9}$$

We wish to treat this as a system of *linear* equations in $[\mathbf{B}_1', \mathbf{\Gamma}_1', \mathbf{\Sigma}_1']$ subject to linear constraints; however, the presence of $\mathbf{B}^{-1}$ in equations (3.7) and (3.9) appears to rule this out. On the contrary, solution(s) $\mathbf{B}_1'$ to the problem

$$\mathbf{\Psi}\mathbf{\Omega}\mathbf{B}_1' = \mathbf{0}$$

$$\text{subject to} \quad \mathbf{\Psi}\mathbf{B}^{-1}\mathbf{\Sigma}_1' = \mathbf{0}$$

are identical to the unrestricted solution(s) $\mathbf{B}_1'$ to

$$\mathbf{\Psi}\mathbf{\Omega}\mathbf{B}_1' = \mathbf{0}.$$

Formally:

LEMMA 4: *For any* $\mathbf{\Sigma}_1'$, $\mathbf{B}^{-1}\mathbf{\Sigma}_1' = \mathbf{0}$ *is consistent with any value of* $\mathbf{B}_1$.

PROOF: $[\mathbf{B}^{-1}\mathbf{\Sigma}_1']_i = 0$ if and only if $\mathbf{B}_{[i,k]} \otimes \sigma_{k1} = \{0\}$ by Proposition 1. Now $\mathbf{B}_{[i,1]}$ contains no elements of $\mathbf{B}_1$, so $\mathbf{B}_{[i,1]} = \{0\}$ does not restrict $\mathbf{B}_1$. Similarly, for $k \neq 1$, $\mathbf{B}_{[i,k]} \otimes \sigma_{k1} = \{0\}$ does not restrict any element of $\mathbf{B}_1$, since any element of $\mathbf{B}_1$ in a $\mathbf{B}_{[i,k]}$ chain product must be preceded by a $\mathbf{B}_{[i,1]}$ chain product which we know to be zero.

Intuitively, the coefficients of the first equation are irrelevant in determining if equations $(i, 1)$ are relatively recursive. We may thus treat $\mathbf{\Psi}\mathbf{\Omega}\mathbf{B}_1' = \mathbf{0}$ as a linear function of $\mathbf{B}_1'$ and ask if the coefficient restrictions $\mathbf{\Phi}$ and the covariance restrictions $\mathbf{\Psi}$ are adequate to determine $(\mathbf{B}_1, \mathbf{\Gamma}_1)$ uniquely.

Rewriting the system as

$$\begin{bmatrix} \boldsymbol{\Pi}' & & I \\ & \boldsymbol{\Phi} & \\ \boldsymbol{\Psi\Omega} & & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{B}_1' \\ \boldsymbol{\Gamma}_1' \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \boldsymbol{\phi} \\ \mathbf{0} \end{bmatrix}, \tag{3.10}$$

we obtain necessary and sufficient conditions for the identifiability of the parameters of the first equation, given linear restrictions on $(\mathbf{B}_1, \boldsymbol{\Gamma}_1)$ and zero restrictions on $(\mathbf{B}, \boldsymbol{\Sigma}_1)$. Note that the "necessity" part of the result depends upon our restriction to limited information at the beginning of this Section. If prior information is allowed on other rows of $\boldsymbol{\Gamma}$ or $\boldsymbol{\Sigma}$, the conditions below are no longer necessary for identification.

PROPOSITION 3 (Rank): *A necessary and sufficient condition for the identifiability of* $(\mathbf{B}_1, \boldsymbol{\Gamma}_1)$ *in the limited information case is*

$$\operatorname{rank}\begin{bmatrix} \boldsymbol{\Pi}' & & I \\ & \boldsymbol{\Phi} & \\ \boldsymbol{\Psi\Omega} & & \mathbf{0} \end{bmatrix} = G + K.$$

The three block rows of the matrix in question have dimensions $K \times (G+K)$, $r \times (G+K)$, and $s \times (G+K)$ respectively. For the matrix to have rank $(G+K)$, $K+r+s$ must be greater than or equal to $G+K$.

COROLLARY 3.1 (Order): *A necessary condition for identifiability in the limited information case is that the number of endogenous variables not exceed the number of restrictions*: $r+s \geq G$.

Specializing these results to the familiar case of exclusion restrictions, we obtain

$$\begin{bmatrix} \boldsymbol{\Pi}_{12}' \\ \boldsymbol{\Psi\Omega}_1 \end{bmatrix} \boldsymbol{\beta}_1 = \mathbf{0},$$

where $\boldsymbol{\Pi}_{12}'$ is the submatrix of $\boldsymbol{\Pi}'$ relating the included endogenous variables $\boldsymbol{Y}_1$ with the excluded exogenous variables, and $\boldsymbol{\Omega}_1$ is the submatrix of $\boldsymbol{\Omega}$ representing $\operatorname{cov}(\boldsymbol{Y}, \boldsymbol{Y}_1 \mid \boldsymbol{Z})$.

COROLLARY 3.2 (Rank): *The parameters of the first structural equation are identifiable in the limited information case under exclusion restrictions if and only if*

$$\operatorname{rank}\begin{bmatrix} \boldsymbol{\Pi}_{12}' \\ \boldsymbol{\Psi\Omega}_1 \end{bmatrix} = g_1 - 1.$$

COROLLARY 3.3 (Order): *A necessary condition for identifiability is* $(K - k_1) + s \geq g_1 - 1$: *that the number of explanatory endogenous variables not exceed the number of excluded exogenous variables plus the number of covariance restrictions.*

A similar interpretation in terms of instruments for the case of general linear coefficient restrictions is easily obtained. The coefficient restrictions $\boldsymbol{\Phi}$ effectively reduce by r the dimension of the space spanned by the explanatory variables corresponding to unconstrained coefficients (see Hausman and Taylor [**6**, Appendix B]). The restrictions $\boldsymbol{\Psi}$ add s columns of $\boldsymbol{Y}$ to the K columns of $\boldsymbol{Z}$ which are predetermined in the first equation. In order to have as many instruments as explanatory variables, we must have r and s restrictions such that $K + G - r \leq K + s$. Using Corollary 3.1, we obtain the following proposition:

PROPOSITION 4 (Order): *A necessary condition for identifiability in this case is that the number of unconstrained coefficients in the first equation not exceed the number of predetermined variables for the first equation.*

For the corresponding rank condition, we show that identification is equivalent to the existence of the 2SLS estimator, using all exogenous and predetermined endogenous variables as instruments. Limiting ourselves to exclusion restrictions without loss of generality, let the matrix of instruments for the first equation be

$$\boldsymbol{W} = [\boldsymbol{Y}\boldsymbol{\Psi}' \; \boldsymbol{Z}],$$

and recall that the 2SLS estimator must invert the matrix $\boldsymbol{X}_1'[\hat{\boldsymbol{Y}}_1 \; \boldsymbol{Z}_1]$ where $\hat{\boldsymbol{Y}}_1 = \boldsymbol{P}_W \boldsymbol{Y}_1$ and $\boldsymbol{Z}_1 = \boldsymbol{P}_W \boldsymbol{Z}_1$.

PROPOSITION 5 (Rank): *The parameters of the first structural equation are identifiable in this case if and only if the* 2SLS *estimator is well-defined, using* $\boldsymbol{W}$ *as the matrix of instruments.*

PROOF: Note first that

$$\operatorname*{plim}_{T\to\infty} \frac{1}{T} \boldsymbol{W}'\boldsymbol{V}_1 = \begin{bmatrix} \operatorname*{plim}_{T\to\infty} \frac{1}{T}\boldsymbol{\Psi}\boldsymbol{Y}'\boldsymbol{V}_1 \\ \operatorname*{plim}_{T\to\infty} \frac{1}{T}\boldsymbol{Z}'\boldsymbol{V}_1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Psi}\boldsymbol{\Omega}_1 \\ \boldsymbol{0} \end{bmatrix},$$

where $\boldsymbol{V}_1$ is the reduced form disturbance corresponding to $\boldsymbol{Y}_1$:

$$\boldsymbol{Y}_1 = \boldsymbol{Z}_1\boldsymbol{\Pi}_{11}' + \boldsymbol{Z}_2\boldsymbol{\Pi}_{12}' + \boldsymbol{V}_1.$$

Projecting this onto $C(\boldsymbol{W})$, we obtain

$$\boldsymbol{P}_W\boldsymbol{Y}_1 = \boldsymbol{Z}_1\boldsymbol{\Pi}_{11}' + \boldsymbol{Z}_2\boldsymbol{\Pi}_{12}' + \boldsymbol{P}_W\boldsymbol{V}_1. \tag{3.11}$$

Now suppose the rank condition of Corollary 3.2 fails, so that there exists a

$g_1 - 1$ vector $\boldsymbol{\lambda} \neq \mathbf{0}$ such that

$$\begin{bmatrix} \boldsymbol{\Pi}'_{12} \\ \boldsymbol{\Psi}\boldsymbol{\Omega}_1 \end{bmatrix} \boldsymbol{\lambda} = \mathbf{0}.$$

Postmultiplying equation (3.11) by $\boldsymbol{\lambda}$, we see that

$$\boldsymbol{P}_W \boldsymbol{Y}_1 \boldsymbol{\lambda} = \boldsymbol{Z}_1 \boldsymbol{\Pi}'_{11} \boldsymbol{\lambda} + \mathbf{0}$$

so that $\hat{\boldsymbol{Y}}$ $(= \boldsymbol{P}_W \boldsymbol{Y}_1)$ and $\boldsymbol{Z}_1$ are linearly dependent. In the other direction, suppose there exist $(g_1 - 1)$ and k_1 vectors ξ_1, ξ_2 such that $\hat{\boldsymbol{Y}}_1 \xi_1 + \boldsymbol{Z}_1 \xi_2 = 0$. Since $\boldsymbol{P}_W = \boldsymbol{I} - \boldsymbol{Q}_W$,

$$\boldsymbol{Y}_1 \xi_1 + \boldsymbol{Z}_1 \xi_2 - \boldsymbol{Q}_W \boldsymbol{V}_1 \xi_1 = 0,$$

where $\boldsymbol{Q}_W \boldsymbol{Y}_1 = \boldsymbol{Q}_W \boldsymbol{V}_1$ since $\boldsymbol{Z}_1$ and $\boldsymbol{Z}_2$ are elements of $\boldsymbol{C}(\boldsymbol{W})$. Adding this to the first structural equation, we obtain

$$\boldsymbol{y}_1 = \boldsymbol{Y}_1(\xi_1 + \boldsymbol{\beta}_1) + \boldsymbol{Z}_1(\xi_2 + \gamma_1) + \boldsymbol{u}_1 - \boldsymbol{Q}_W \boldsymbol{V}_1 \xi_1$$

which is observationally equivalent to the first equation, so that the parameters $(\boldsymbol{\beta}_1, \gamma_1, \sigma_{11})$ are not identifiable.

The intuitive correspondence between restrictions for identification and instruments for estimation is stated precisely in Proposition 5. Fisher [**3**, Theorem 2.7.2] showed that for coefficient restrictions, identification is equivalent to the existence of the 2SLS estimator, using all exogenous variables as instruments. Our Lemma 3 shows that zero restrictions on $(\mathbf{B}, \boldsymbol{\Sigma}_1)$ are useful for identification if and only if they imply that certain endogenous variables are predetermined in the first equation. Proposition 5 then shows that identification is equivalent to the existence of the 2SLS estimator which uses all exogenous variables plus endogenous variables predetermined in the first equation as instruments. In this sense, identification in the limited information case is equivalent to the existence of the appropriate 2SLS estimator.

3.4. *A Limited Information Example*

Recall the example of Section 3.2.1, with the added restrictions that $\beta_{23} = \beta_{13} = \sigma_{12} = 0$:

$$\begin{aligned} \boldsymbol{y}_1 &= \beta_{12}\boldsymbol{y}_2 + \boldsymbol{Z}\boldsymbol{\Gamma}'_1 + \boldsymbol{u}_1, \\ \boldsymbol{y}_2 &= \qquad\quad \boldsymbol{Z}\boldsymbol{\Gamma}'_2 + \boldsymbol{u}_2, \\ \boldsymbol{y}_3 &= \beta_{32}\boldsymbol{y}_2 + \boldsymbol{Z}\boldsymbol{\Gamma}'_3 + \boldsymbol{u}_3, \end{aligned}$$

where the coefficients of the exogenous variables are still unrestricted. Here,

$$\mathbf{B}^{-1}\boldsymbol{\Sigma}'_1 = \begin{bmatrix} \sigma_{11} \\ 0 \\ \sigma_{13} \end{bmatrix}$$

so that equations (2, 1) are relatively recursive and y_2 is predetermined in the first equation. The order condition (Corollary 3.3) is fulfilled for the first equation, since there is one explanatory endogenous variable and one restriction. To verify the rank condition, note that $\mathbf{\Pi}'_{12} = \mathbf{0}$ and that the reduced form covariance matrix is given by

$$\mathbf{\Omega} = \begin{bmatrix} \sigma_{11} + \beta_{12}^2\sigma_{22} & - & - \\ \beta_{12}\sigma_{22} & \sigma_{22} & - \\ \sigma_{13} + \beta_{12}\beta_{32}\sigma_{22} + \beta_{12}\sigma_{23} & \sigma_{23} + \beta_{32}\sigma_{22} & \beta_{32}^2\sigma_{22} + 2\beta_{32}\sigma_{23} + \sigma_{33} \end{bmatrix};$$

thus

$$\mathbf{\Omega}_1 = \begin{bmatrix} \beta_{12}\sigma_{22} \\ \sigma_{22} \\ \sigma_{23} + \beta_{32}\sigma_{22} \end{bmatrix}$$

and $\mathbf{\Psi} = (0 \;\; 1 \;\; 0)$. The rank condition is satisfied, provided $\sigma_{22} \neq 0$, since

$$\operatorname{rank}\begin{bmatrix} \mathbf{\Pi}'_{12} \\ \mathbf{\Psi\Omega}_1 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} \mathbf{0} \\ \sigma_{22} \end{bmatrix} = 1 = g_1 - 1.$$

If we treat y_2 as predetermined in the first equation, the 2SLS estimate of β_{12} corresponds to the least squares estimate:

$$\hat{\beta}_{12} = \frac{y_2' Q_Z y_1}{y_2' Q_Z y_2}.$$

Since $(1/T)y_j Q_Z y_i$ is consistent for ω_{ij}, $i, j = 1, 2, 3$, this estimator is equivalent to

$$\hat{\beta}_{12} \xrightarrow{p} \frac{\omega_{21}}{\omega_{22}} = \beta_{12}.$$

Thus 2SLS using y_2 as an instrument yields the estimator of β_{12} obtained from solving the reduced form.

4. FULL INFORMATION

In the previous section, we showed that for zero restrictions on $(\mathbf{B}, \mathbf{\Sigma}_1)$ to be useful for identifying $(\mathbf{B}_1, \mathbf{\Gamma}_1)$ under limited information, they must restrict elements of the vector $\mathbf{B}^{-1}\mathbf{\Sigma}'_1$ to be zero. If we have additional prior information, enough to identify the parameters of equations other than the first, we can use the fact that

$$\mathbf{B}_j \mathbf{B}^{-1}\mathbf{\Sigma}'_1 = \sigma_{1j}$$

to obtain a homogeneous restriction on $\mathbf{B}^{-1}\mathbf{\Sigma}'_1$, provided $\sigma_{1j} = 0$, and $\mathbf{B}_j$ is known or estimable. This case requires more prior information but is undoubtedly more relevant in practice than the relatively recursive structure of Section 3.

Note that in this section, the identifiability of the ith structural equation may depend upon whether the parameters of other equations are identifiable. Single equation criteria for identification are inappropriate here, and we must consider the question of identification of the entire *set* of equations.

4.1. *Residuals as Instruments*

With unrestricted prior information on $[\mathbf{B}, \boldsymbol{\Gamma}, \boldsymbol{\Sigma}]$, we can partition the set of G structural equations into two sets: those which are identifiable by coefficient restrictions alone [indexed by $j \in \boldsymbol{J}$], and those which are not [indexed by $i \in \boldsymbol{I}$]. Since $\mathbf{B}_j$ and $\boldsymbol{u}_j$ are now known or estimable, we can use either $\boldsymbol{Y}\hat{\mathbf{B}}_j'$ or $\hat{\boldsymbol{u}}_j$ as an instrument in any equation for which $\sigma_{ij} = 0$.

PROPOSITION 6: *If* $\sigma_{ij} = 0$ *and* $\mathbf{B}_j$ *is known or estimable, then* $\boldsymbol{Y}\hat{\mathbf{B}}_j'$ *or* $\hat{\boldsymbol{u}}_j$ *is predetermined in the ith structural equation.*

The proof follows from the consistency of $\hat{\mathbf{B}}_j$:

$$\operatorname*{plim}_{T\to\infty} \frac{1}{T}\hat{\mathbf{B}}_j \boldsymbol{Y}'\boldsymbol{u}_i = \mathbf{B}_j\mathbf{B}^{-1}\boldsymbol{\Sigma}_i' = \sigma_{ij} = 0$$

and

$$\operatorname*{plim}_{T\to\infty} \frac{1}{T}\hat{\boldsymbol{u}}_j'\boldsymbol{u}_i = \operatorname*{plim}_{T\to\infty} \frac{1}{T}\hat{\mathbf{B}}_j \boldsymbol{Y}'\boldsymbol{u}_i + \operatorname*{plim}_{T\to\infty} \frac{1}{T}\hat{\boldsymbol{\Gamma}}_j \boldsymbol{Z}'\boldsymbol{u}_i = \sigma_{ij} = 0.$$

Provided all the exogenous variables $\boldsymbol{Z}$ are used as instruments, $\boldsymbol{Y}\hat{\mathbf{B}}_j'$ and $\hat{\boldsymbol{u}}_j$ are equivalent instruments in the sample, since they differ only by $\boldsymbol{Z}\hat{\boldsymbol{\Gamma}}_j'$.

The role of residuals as instruments in the presence of covariance restrictions is determined by the form of the first order conditions of the FIML estimator:

$$-\begin{bmatrix} \mathbf{B}^{-1}\boldsymbol{\Gamma}\boldsymbol{Z}' + \tilde{\boldsymbol{U}}'\boldsymbol{b}' \\ \boldsymbol{Z}' \end{bmatrix}(\boldsymbol{Y}\mathbf{B}' + \boldsymbol{Z}\boldsymbol{\Gamma}')\boldsymbol{\Sigma}^{-1} \overset{u}{=} 0,$$

$$T\boldsymbol{\Sigma} - (\boldsymbol{Y}\mathbf{B}' + \boldsymbol{Z}\boldsymbol{\Gamma}')'(\boldsymbol{Y}\mathbf{B}' + \boldsymbol{Z}\boldsymbol{\Gamma}') \overset{u}{=} 0,$$

where $\tilde{\boldsymbol{U}}'\boldsymbol{b}'$ represent residuals uncorrelated with the corresponding disturbance multiplied by the appropriate elements of $\mathbf{B}'^{-1}$, and the symbol $\overset{u}{=}$ denotes equality imposed on the *unknown* parameters only.[8] (See Hausman–Taylor [7, Equation (3.13)].) The first set of equations determines the unknown slope parameters $(\beta_{ij}, \gamma_{ij})$, while the third set of equations determines the unknown covariance parameters σ_{ij}. From the top equation, it is clear that every covariance restriction $\sigma_{ij} = 0$ gives rise to two (potential) instruments: $\hat{\boldsymbol{u}}_i$ in equation (j) and $\hat{\boldsymbol{u}}_j$ in equation (i).

There are several circumstances in which using residuals as instruments does not contribute to either estimation or identification:

[8]The first line is derived from differentiation of the log likelihood function with respect to the unknown elements of $\mathbf{B}'$. The term $\tilde{\boldsymbol{U}}'\boldsymbol{b}'$ represents that part of the reduced form disturbance that can be used (by virtue of the covariance restrictions) to form a predicted $\hat{\boldsymbol{Y}}$.

PROPOSITION 7: *If y_i is predetermined in the jth equation, $\hat{u}_j$ cannot be used as an instrument in the ith equation.*

PROOF: From the ith structural equation,

$$\hat{u}_j' y_i = \hat{u}_j' X_i \delta_i + \hat{u}_j' u_i$$

so that $\text{plim}(1/T)\hat{u}_j' y_i = 0$ and $\sigma_{ij} = 0$ implies $\text{plim}(1/T)\hat{u}_j' X_i = \mathbf{0}$. The instrument $\hat{u}_j$ is orthogonal to the explanatory variables in the ith equation and thus fails to contribute to the appropriate rank condition. To see this explicitly, note that

$$\text{plim}\,\frac{1}{T}\, y_i'\hat{u}_j = 0 \quad \text{implies} \quad \mathbf{B}_j \mathbf{\Omega}_i = \mathbf{0},$$

so that knowledge of $\mathbf{B}_j$ does not contribute to the rank condition in (say) Corollary 3.2.

COROLLARY 7.1: *If y_i is predetermined in the jth equation, $\hat{u}_j$ cannot be used as an instrument for y_i in any equation.*

The proof follows from the fact that $\text{plim}(1/T)y_i' u_j = 0$.

Finally, note that a covariance restriction cannot be used twice in the following sense:

PROPOSITION 8: *Suppose $\sigma_{jk} = 0$ and equation k is just-identified using $\hat{u}_j$ as an instrument. Then the resulting $\hat{u}_k$ cannot be used as an instrument in equation j.*

PROOF: Let W_k denote the matrix of observations on the instruments which just-identify equation k:

$$W_k = \left[\hat{u}_j \;\; Z\right],$$

where the residual $\hat{u}_j = y_j - X_j\hat{\delta}_j$ and $\hat{\delta}_j$ is an arbitrary consistent estimate of δ_j. The resulting $\hat{u}_k$ is orthogonal to the vector $y_j - X_j\hat{\delta}_j$ in the sample:

$$\begin{aligned}\hat{u}_k'\left[y_j - X_j\hat{\delta}_j\right] &= u_k'\left[I - W_k(X_k' W_k)^{-1}X_k'\right](y_j - X_j\hat{\delta}_j) \\ &= u_k'\left[I - W_k(X_k' W_k)^{-1}X_k'\right]W_k\Phi_k = 0,\end{aligned}$$

where Φ_k is the selection matrix for which $W_k\Phi_k = [y_j - X_j\hat{\delta}_j]$. Now the 2SLS estimate of δ_j is the solution to

$$P_{W_j}\left[y_j - X_j\delta_j\right] = \mathbf{0} \tag{4.1}$$

where $W_j = [\hat{u}_k \;\; Z]$. The orthogonality result implies that adding $\hat{u}_k$ to the instruments used in the initial $\hat{\delta}_j$ will have no effect on the solution to equation (4.1).

4.2. *Identification under Full Information*

There is no difficulty in using residuals as instrumental variables in other equations, and indeed, the FIML first order conditions show that *every* covariance restriction of the form $\sigma_{jk} = 0$ causes $\hat{\boldsymbol{u}}_k$ to behave as an instrument in equation j and $\hat{\boldsymbol{u}}_j$ to behave as an instrument in equation k. If the system of equations as a whole is identifiable, then the FIML estimator can always be regarded as an instrumental variable estimator, using the appropriate residuals as instruments: see Hausman and Taylor [7].

However, among equations which are not identifiable by coefficient restrictions alone, this analysis will not distinguish residuals which are identifiable from those which are not.[9] We must therefore consider the identifiability of the parameters of the system of equations as a whole. Recall our partition of the system into equations which are identifiable by coefficient restrictions alone, indexed by $j \in \boldsymbol{J}$, and those which are not, indexed by $i \in \boldsymbol{I}$. From our previous considerations, we can state sufficient conditions for the identifiability of equation i, $i \in \boldsymbol{I}$, based on coefficient restrictions and an ordering of equations so that residuals necessary for instruments are known to be previously identified.

PROPOSITION 9: *A sufficient condition for the identification of the parameters of the ith structural equation* ($i \in \boldsymbol{I}$) *is that*

$$\operatorname{rank}(\boldsymbol{X}_i' \boldsymbol{P}_{\boldsymbol{W}_i} \boldsymbol{X}_i) = \operatorname{rank} \boldsymbol{X}_i$$

where

$$\boldsymbol{W}_i = [\hat{\boldsymbol{u}}_j \; \boldsymbol{Y}\boldsymbol{\Psi}_i \; \boldsymbol{Z}],$$

$\boldsymbol{Y}\boldsymbol{\Psi}_i$ *are endogenous variables predetermined in equation* i, $\sigma_{ij} = 0$, *and equations indexed by* $j \in \boldsymbol{J}$ *are previously identifiable.*

This result provides an instrumental variable interpretation of a sufficient condition for identification based on a hierarchical ordering of restrictions similar to Wegge [**14**, Theorem IV]. Wegge requires sufficient restrictions on the parameters of each equation and an ordering of equations so that restrictions pertaining to equation i all reference lower-numbered equations j. Our result is expressed in terms of the instruments such restrictions create.

4.3. *Examples*

The simplest example is the supply and demand model

$$y_1 = \beta_{12} y_2 + \gamma_{11} z + \boldsymbol{u}_1,$$

$$y_2 = \beta_{21} y_1 \qquad\qquad + \boldsymbol{u}_2,$$

[9] The residual $\hat{\boldsymbol{u}}_j$ is said to be identifiable if the parameters of the jth structural equation are identifiable.

where $\sigma_{12} = 0$. Here the second equation is identifiable by coefficient restrictions alone, and $\hat{u}_2$ is a legitimate instrument in the first equation. The reduced form parameters are given by

$$\Pi = \frac{1}{1 - \beta_{12}\beta_{21}} \begin{bmatrix} \gamma_{11} \\ \beta_{21}\gamma_{11} \end{bmatrix},$$

$$\Omega = \frac{1}{(1 - \beta_{12}\beta_{21})^2} \begin{bmatrix} \sigma_{11} + \beta_{12}^2\sigma_{22} & \beta_{21}\sigma_{11} + \beta_{12}\sigma_{22} \\ \beta_{21}\sigma_{11} + \beta_{12}\sigma_{22} & \beta_{21}^2\sigma_{11} + \sigma_{22} \end{bmatrix} = \begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{bmatrix},$$

in standard notation, and we can recover all of the structural parameters from consistent estimates of the reduced form:

$$\hat{\beta}_{21} = \hat{\Pi}_2/\hat{\Pi}_1,$$

$$\hat{\beta}_{12} = \frac{\hat{\beta}_{21}\hat{\omega}_{11} - \hat{\omega}_{12}}{\hat{\beta}_{21}\hat{\omega}_{12} - \hat{\omega}_{22},}$$

$$\hat{\gamma}_{11} = \hat{\Pi}_1(1 - \hat{\beta}_{12}\hat{\beta}_{21}).$$

Note that $\hat{\beta}_{12}$ can be written as an instrumental variable estimator:

$$\hat{\beta}_{12} = \frac{(y_2 - \hat{\beta}_{21}y_1)'Q_z y_1}{(y_2 - \hat{\beta}_{21}y_1)'Q_z y_2,}$$

where $\hat{u}_2 \equiv y_2 - \hat{\beta}_{21}y_1$ behaves as an instrument in the first equation. Note that the second equation is still just-identified, despite the fact that $\hat{u}_1$ is now available as an additional instrument:

$$\hat{u}_1 = Q_z(y_1 - y_2\hat{\beta}_{12}) = Q_z y_1 - Q_z y_2\left(\frac{\hat{u}_2' Q_z y_1}{\hat{u}_2' Q_z y_2}\right),$$

so that

$$\tilde{\beta}_{21} = \frac{\hat{u}_1' y_2}{\hat{u}_1' y_1}$$

$$= \left[y_1' Q_z y_1 - y_1' Q_z y_2\left(\frac{\hat{u}_2' Q_z y_1}{\hat{u}_2' Q_z y_2}\right)\right]^{-1}\left[y_1' Q_z y_2 - y_2' Q_z y_2\left(\frac{\hat{u}_2' Q_z y_1}{\hat{u}_2' Q_z y_2}\right)\right]$$

where

$$\hat{u}_2' = y_2 - \left(\frac{z' y_2}{z' y_1}\right) y_1 .$$

Simplifying,

$$\tilde{\beta}_{21} = \frac{y_2'Q_zy_2\,y_1'Q_zy_1 - y_1'Q_zy_2\,y_1'Q_zy_2}{-y_1'Q_zy_2\,y_2'Q_zy_1 + y_2'Q_zy_2\,y_1'Q_zy_1}\left(\frac{z'y_2}{z'y_1}\right)$$

$$= \left(\frac{z'y_2}{z'y_1}\right) = \hat{\beta}_{21},$$

so that using $\hat{\boldsymbol{u}}_1$ as an instrument in Equation 2 yields the same IV estimator as before. Equation 2 is still just-identified, as we knew from Proposition 8.

To illustrate Proposition 7, let

$$\begin{aligned} y_1 + \qquad \beta_{13}y_3 + Z\Gamma_1' &= u_1, \\ \beta_{21}y_1 + y_2 + \beta_{23}y_3 + Z\Gamma_2' &= u_2, \\ \beta_{31}y_1 + \qquad y_3 + Z\Gamma_3' &= u_3, \end{aligned}$$

and Σ is diagonal. Using Proposition 1, y_1 and y_3 are predetermined in the second equation and no other y_i is predetermined in any equation. If their respective equations are identifiable, $\hat{\boldsymbol{u}}_2$ is predetermined in the first and third equations and $(\hat{\boldsymbol{u}}_1, \hat{\boldsymbol{u}}_3)$ are predetermined in the second equation. However, since (y_1, y_3) are predetermined in the second equation, $\hat{\boldsymbol{u}}_2$ cannot be used as an instrument in the first and third equations by Proposition 7. Thus the parameters of the second equation are the only identifiable parameters of the system, despite the fact that the order condition (Corollary 3.3) holds in the first and third equations. Observe that the IV estimator of β_{13} using $\hat{\boldsymbol{u}}_2$ as an instrument is given by

$$\hat{\beta}_{13} = \frac{\hat{u}_2'Q_Zy_1}{\hat{u}_2'Q_Zy_3} \to \frac{\hat{\beta}_{21}\omega_{11} + \omega_{12} + \hat{\beta}_{23}\omega_{13}}{\hat{\beta}_{21}\omega_{13} + \omega_{23} + \hat{\beta}_{23}\omega_{33}},$$

and that both the numerator and denominator is zero. Thus from Corollary 3.2,

$$\operatorname{rank}\begin{bmatrix} \Pi_{12}' \\ \Psi\Omega_1 \end{bmatrix} = \operatorname{rank}(\hat{\mathbf{B}}_2\Omega_1) = \operatorname{rank}(\hat{\beta}_{21}\omega_{11} + \omega_{12} + \hat{\beta}_{23}\omega_{13})$$

$$= 0 < g_1 - 1 = 1,$$

so that the failure of $\hat{\boldsymbol{u}}_2$ as an instrument in the first equation implies the failure of the rank condition in Corollary 3.2. Despite the fulfillment of the order condition, the parameters of the first equation are not identifiable.

It is not *necessary* for identification that there exist a set of equations identifiable by coefficient restrictions alone, although this is the case to which our sufficient conditions (Proposition 9) pertain. Consider the following three equation example,[10] in which no equation is identifiable by coefficient restric-

[10] This was provided by L. Wegge in private communication.

tions alone:

$$y_1 = \beta_{12} y_2 + u_1 ,$$
$$y_2 = \beta_{23} y_3 + u_2 ,$$
$$y_3 = \beta_{31} y_1 + u_3 ,$$

and $\mathbf{\Sigma}$ is diagonal. Examination of the quadratic equation

$$\mathbf{B}\hat{\mathbf{\Omega}}\mathbf{B}' = \mathbf{\Sigma} \tag{4.2}$$

shows the existence of a unique solution for $(\beta_{12}, \beta_{23}, \beta_{31}, \sigma_{11}, \sigma_{22}, \sigma_{33})$ in terms of the six observable elements of $\mathbf{\Omega}$.

For our purposes, assign as an instrument (i) $\hat{u}_1$ to the third equation, (ii) $\hat{u}_2$ to the first equation, and (iii) $\hat{u}_3$ to the second equation; and consider IV estimation of β_{12} in the first equation. Potential instruments are $\hat{u}_2$ and $\hat{u}_3$, where

$$\hat{u}_3 = y_3 - y_1\left(\frac{\hat{u}_1' y_3}{\hat{u}_1' y_1}\right),$$

$$\hat{u}_2 = y_2 - y_3\left(\frac{\hat{u}_3' y_2}{\hat{u}_3' y_3}\right) \tag{4.3}$$

$$= y_2 - y_3\left\{\frac{\left[y_3 - y_1 \dfrac{\hat{u}_1' y_3}{\hat{u}_1' y_1}\right]' y_2}{\left[y_3 - y_1 \dfrac{\hat{u}_1' y_3}{\hat{u}_1' y_1}\right]' y_3}\right\},$$

where $\hat{u}_1 = y_1 - \hat{\beta}^*_{12} y_2$ for some consistent estimator $\hat{\beta}^*_{12}$. Now consider two estimators for β_{12}:

$$\hat{\beta}_{12} = (\hat{u}_2' y_2)^{-1} \hat{u}_2' y_1 , \qquad \tilde{\beta}_{12} = (\hat{u}_3' y_2)^{-1} \hat{u}_3' y_1 .$$

Substituting for $\hat{u}_2$ and $\hat{u}_3$ from equation (4.3) in the above shows that: (i) $\tilde{\beta}_{12}$ reduces identically to the initial consistent estimate $\hat{\beta}^*_{12}$. Thus $\hat{u}_3$ has no effect as an instrument in the first equation. (ii) $\hat{\beta}_{12}$ reduces to a quadratic equation in $\hat{\beta}_{12}$ and moments of (y_1, y_2, y_3). Thus the FIML estimator of β_{12} can be interpreted as the IV estimator $\hat{\beta}_{12}$ which, in the final iteration, solves the quadratic equation (4.2).

In this example, there is an *assignment* of residuals to equations such that $\hat{u}_j$ is assigned to equation i and $\hat{u}_i$ is *not* assigned to equation j, for all i and j in the system. Provided such an assignment exists, the FIML solution to the first order conditions will not fail due to the rank considerations in Proposition 8. This assignment condition seems to provide a useful approach to identification in the full information setting and possibly yields more stringent necessary conditions than Fisher's generalized rank and order conditions: Fisher [**3**, Theorem 4.6.2]. Investigation of this possibility and the more difficult question of necessary conditions for identification awaits future research.

5. SUMMARY

Identifiability of the first structural equation is equivalent to the existence of a unique solution $(\mathbf{B}_1', \mathbf{\Gamma}_1')$ to the equations which link the structural parameters and the reduced form parameters

$$\mathbf{\Pi}'\mathbf{B}_i' = -\mathbf{\Gamma}_i',$$
$$\mathbf{\Omega}\mathbf{B}_i' = \mathbf{B}^{-1}\mathbf{\Sigma}_i',$$

for $(i = 1, \ldots, G)$. Coefficient or covariance restrictions are useful for identification if they restrict the set of solutions $(\mathbf{B}_1^*, \mathbf{\Gamma}_1^*)$ above. For limited information analysis, covariance restrictions are useful if and only if they identify elements (or linear combinations of elements) of the vector $\mathbf{B}^{-1}\mathbf{\Sigma}_1'$. This condition, in turn, is equivalent to the existence of linear combinations of the endogenous variables which are predetermined in the first equation. Our approach yields necessary and sufficient conditions for identification in linear models with linear coefficient and covariance restrictions under limited information (i.e., Proposition 3). Now all restrictions which are useful for identification correspond to instruments which can be used for estimation, extending—for the limited information case—the intuition developed for coefficient restrictions in Fisher [**3**, Theorem 2.7.2].

Under full information, necessary and sufficient conditions for identification have not been established. From the FIML first order conditions, it is clear that if the system of equations is identified, then the FIML estimator can be treated as an IV estimator where the instruments include residuals uncorrelated with the disturbance due to the covariance restrictions. We do obtain a sufficient condition for identification when an ordering of equations is possible so that earlier equations are identifiable without reference to later equations. Finally, an example is given in which no equation is identifiable by coefficient restrictions alone, but an assignment of residuals to equations exists so that identification and instrumental variable estimation is possible.

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APPENDIX

LEMMA: $(\mathbf{B})_{ij}^{-1} = 0$ *if and only if equations* (i, j) *are relatively triangular.*

PROOF: $(\mathbf{B})_{ij}^{-1}$ is proportional to the minor $|\mathbf{B}_{(ji)}|$ which is the determinant of the matrix $\mathbf{B}_{(ji)}$ formed by striking out the jth row and ith column of $\mathbf{B}$. $|\mathbf{B}_{(ji)}|$ is defined as the sum of the $(G-1)!$ distinct terms formed by multiplying together $G-1$ elements of $\mathbf{B}_{(ji)}$, exactly one from each row and column, with a rule for determining the sign.

Let $\mathbf{B}_{[i,j]}^{(G-p)}$ denote the set of all chain products containing exactly $(G-p)$ elements of $\mathbf{B}_{(ji)}$. In each chain product, exactly $p-1$ indices do not appear; call them $(m_1, m_2, \ldots, m_{p-1})$. Let $\mathbf{B}_{\{m_1, \ldots, m_{p-1}\}}$ represent the set of all products of $(p-1)$ elements of $\mathbf{B}_{(ji)}$, with exactly one element from each row and column indexed by $(m_1, \ldots, m_{p-1})$. Finally, multiply each chain product in $\mathbf{B}_{[i,j]}^{(G-p)}$ by one of its associated members of $\mathbf{B}_{\{m_1, \ldots, m_{p-1}\}}$ and denote the sum of all such terms by $\mathbf{\Sigma}[\mathbf{B}_{[i,j]}^{(G-p)} \otimes \mathbf{B}_{\{m_1, \ldots, m_{p-1}\}}]$.

Two characteristics of the series

$$\Sigma\left[\mathbf{B}_{[i,j]}^{(G-p)} \otimes \mathbf{B}_{\{m_1,\ldots,m_{p-1}\}}\right]$$

are important:

(i) *It contains* $(G-2)!$ *distinct terms*. An element of $\mathbf{B}_{[i,j]}^{(G-p)}$ is uniquely specified by listing the $G-p-1$ unfixed column indices of the elements, since the chain product property then determines the row indices. The first row index is i, the final column index is j, and (i, j) appear in no other index. The first column index can be chosen $(G-2)$ different ways, the second, $(G-3)$ different ways, etc., so that there are

$$(G-2)\times(G-3)\times\cdots\times[G-(G-p+1)]=\frac{(G-2)!}{(p-1)!}$$

different elements in $\mathbf{B}_{[i,j]}^{(G-p)}$. Similarly, reordering the elements of $\mathbf{B}_{\{m_1,\ldots,m_{p-1}\}}$ so that the row indices increase from left to right, the first column index can be chosen $(p-1)$ different ways, etc. The number of distinct elements of $\mathbf{B}_{\{m_1,\ldots,m_{p-1}\}}$ is $(p-1)!$, so that the series $\Sigma[\mathbf{B}_{[i,j]}^{(G-p)} \otimes \mathbf{B}_{\{m_1,\ldots,m_{p-1}\}}]$ contains $(G-2)!$ distinct terms.

(ii) *Each term is a product of* $(G-1)$ *elements of* $\mathbf{B}_{(ji)}$ *with exactly one element from each row and column*. This is derived from the definition of a chain product and the construction of the residual product $\mathbf{B}_{\{m_1,\ldots,m_{p-1}\}}$.

Now consider the following expansion for the minor $|\mathbf{B}_{(ji)}|$:

$$\text{(A.1)}\qquad \begin{aligned}|\mathbf{B}_{(ji)}| = {} & \Sigma\mathbf{B}_{[i,j]}^{(G-1)} \\ & \pm \Sigma\mathbf{B}_{[i,j]}^{(G-2)} \otimes \mathbf{B}_{\{m_1\}} \pm \cdots \\ & \pm \Sigma\mathbf{B}_{[i,j]}^{(G-p)} \otimes \mathbf{B}_{\{m_1,\ldots,m_{p-1}\}} \pm \cdots \\ & \pm \Sigma\mathbf{B}_{[i,j]}^{(1)} \otimes \mathbf{B}_{\{m_1,\ldots,m_{G-2}\}}\,.\end{aligned}$$

That is, we expand $|\mathbf{B}_{(ji)}|$, organizing terms by the length of the $[i, j]$ chain product they contain. To verify that this equals $|\mathbf{B}_{(ji)}|$, observe that (i) terms in different rows in the expansion are distinct, since they contain chain products of different lengths, and (ii) there are $(g-1)$ rows, each containing $(G-2)!$ distinct terms. The expansion thus represents the sum of $(G-1)!$ distinct terms, consisting of products of $G-1$ elements of $\mathbf{B}_{(ji)}$ with one element from each row and column in each term. Up to the sign rule for each term, Equation (A.1) is thus a valid expression for the minor $|\mathbf{B}_{(ji)}|$. Since we are only interested in $|\mathbf{B}_{(ji)}|=0$ and have ruled out the possibility that $\Sigma[\mathbf{B}_{[i,j]}^{(G-p)} \otimes \mathbf{B}_{\{m_1,\ldots,m_{p-1}\}}]$ can equal zero unless each term is zero, the sign rule is irrelevant for our purposes.

The Lemma follows immediately from the expansion (A.1). Every term in the expansion contains an element of $\mathbf{B}_{[i,j]}$, so that $\mathbf{B}_{[i,j]}=\{0\}$ is obviously sufficient for $|\mathbf{B}_{(ji)}|=0$. For necessity, every distinct chain product $\mathbf{B}_{[i,j]}^{(G-p)}$ appears once in a term in (A.1) with residual elements of the form $\prod_i(\beta_{m_im_i})$, and in the usual normalization for simultaneous equations systems, $\prod_i(\beta_{m_im_i})=1$. Thus if any chain product differs from zero, at least one term in (A.1) will differ from zero and, by assumption, $|\mathbf{B}_{(ji)}| \neq 0$.

LEMMA: *If no linear function of the vector* $\mathbf{B}^{-1}\mathbf{\Sigma}_1'$ *is known a priori, then a necessary and sufficient condition for the identifiability of* $[\mathbf{B}_1, \mathbf{\Gamma}_1]$ *is given in equation* (3.3).

PROOF: Suppose that no element of $\mathbf{B}^{-1}\mathbf{\Sigma}_1' = c^*$ is known a priori, and that $[\mathbf{B}_1'^*, \mathbf{\Gamma}_1'^*]$ solves

$$\begin{bmatrix}[\mathbf{\Pi}' & I] \\ & \mathbf{\Phi}\end{bmatrix}\begin{bmatrix}\mathbf{B}_1' \\ \mathbf{\Gamma}_1'\end{bmatrix}=\begin{bmatrix}\mathbf{0} \\ \phi\end{bmatrix}.$$

We must show that $\mathbf{B}^{-1}\mathbf{\Sigma}_1' = c^*$ is consistent with any $\mathbf{B}_1$. In general,

$$\text{(A.2)}\qquad c^* = \mathbf{B}^{-1}\mathbf{\Sigma}_1' = \sum_j b^j\sigma_{1j}$$

where $\boldsymbol{b}^j$ is the jth column of $\mathbf{B}^{-1}$. We show that every element of $\boldsymbol{b}^1$ contains unrestricted elements of $\mathbf{B}$, so that the first term in the above sum is unrestricted. The ith element of $\boldsymbol{b}^1$ is proportional to the minor $|\mathbf{B}_{(1,i)}|$, which, following equation (A.1), has an expansion containing nonzero elements from the chain product $\mathbf{B}_{[i,1]}$. For $i \neq 1$, no member of $\mathbf{B}_{[i,1]}$ contains elements of $\mathbf{B}_1$. For $i = 1$,

$$\mathbf{B}_{[1,1]} = \{1, \beta_{1a}\beta_{a1}, \dots, \beta_{1a}\beta_{ab} \dots \beta_{f1}, \dots\}$$

and since the first equation has at least one nonzero β_{1j}, $\boldsymbol{b}_1^1$ will be proportional to a sum containing unrestricted coefficients from other equations. Since the first term in the sum in equation (A.2) is unrestricted in every element, $\mathbf{B}^{-1}\Sigma_1' = c^*$ is consistent with any value of $\mathbf{B}_1'^*$.

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