Manuscript:
Debt Instruments and Markets
(MBA Course B40.3333)

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Preface

These notes are a little rough in places, but my hope is that they will allow us to make better use of class time: you'll already have most of what we cover written down in black and white, so we can spend more time thinking about what we're doing. Comments welcome on content, style, typos, etc.

I owe many of the best things in here to Ned Elton, with whom I taught the course in Fall of 1995.

Chapter 1

Debt Instruments

1.1 Disasters of '94

1994 was a landmark year for fixed income securities, with a record six Fed-instigated increases in short-term interest rates and an associated collapse in the bond market. More interesting, for students of fixed income and fans of car accidents, were a startling number of derivatives disasters, including:

- April 12: Procter & Gamble announces a $157 million pretax charge for estimated losses on derivatives, many of them with Bankers Trust Company, a leading trader of over-the-counter derivatives.

- May 20: Piper Jaffray, a respected Minneapolis broker, injects $10 million (not nearly enough, as it turned out) into its Institutional Government Income Fund as partial repayment for losses on mortgage derivatives.

- December 6: Orange County, California, files for bankruptcy following losses on leveraged fixed income positions.

These events and others triggered vigorous responses, with Congress threatening to regulate derivatives trading, the Financial Accounting Standards Board (FASB) reexaming disclosure requirements, nonfinancial companies expressing renewed interest in financial risk management and control, and regulators from the Federal Reserve to the Securities and Exchange Commission proposing new standards for derivatives activity.

This course is concerned with the instruments associated with each of these disasters. Along the way we will consider their use and regulation, but our starting point is simply to describe what these instruments are.

1.2 Overview

This is a course in what is commonly referred to as *debt instruments* or *fixed income securities*. Both of these labels should give you the idea that we're talking here about financial instruments like bonds that commit the issuer to a series of fixed payments (coupons and principal, for example). Such instruments differ only in the size and timing of these payments. In fact, both phrases now encompass an enormous range of related securities, with payments both fixed and contingent in various ways on future events. Prominent examples include treasury bills, notes, and bonds; corporate bonds; eurodollar contracts of various maturities; futures contracts on government bills, bonds, and notes, and on eurodollars; options on fixed income securities and their derivatives; interest rate swaps; and mortgage passthroughs and related derivatives. Few of these instruments have terms that are even remotely "fixed." What they have in common is that their values are sensitive to movements in interest rates, so that some now use the more informative label *interest-sensitive assets*.

One question you might naturally ask is why we need so many different instruments to manage what seems like a single risk: the risk that interest rates might change. A glib answer (but not a bad one) is that assets are no different from any other good or service in a modern economy. You might as well ask: Why do we need 47 kinds of breakfast cereal, and 5 kinds of Cheerios? Presumably the variety is demanded by the market. My 4-year-old son Paul, for example, has definite opinions about the kinds of Cheerios he'll eat, so in his mind the variety is useful. A more complex answer might touch on these issues:

- Increased interest rate volatility over the last two decades, which created a market for instruments that can be used to manage interest rate risk.

- Lower costs of financial services, apparently due to changes in regulation and improvements in technology, which triggered an increase in the volume and diversity of financial products.

- Accounting and tax rules, which treat seemingly comparable interest-rate derivatives differently.

We'll come back to each of these in the course.

1.3 The Instruments

Probably the best place to start is with an overview of the most common fixed income securities, which gives us an excuse to talk about what they are and how they're used. Our list starts with government and corporate bonds, then turns to more exotic fixed-income derivatives.

US Treasury Securities

The US Treasury regularly issues securities with maturities between three months and 30 years. Treasury *bills* are issued in maturities of 3, 6, 9, and 12 months, *notes* in maturities of 1-10 years, and *bonds* in maturities up to 30 years. Principle characteristics include:

- Large, liquid market. We see in Table 1.1, for example, that at the end of 1993 the outstanding value of marketable US treasury debt was over 3 trillion dollars.

- Riskfree, for all practical purposes.

- Generally fixed coupon "bullets," meaning the principal is not repaid until maturity.

- Free from state and local tax. This tax advantage is one of the reasons yields on treasuries are lower than those on similar corporate securities.

- Special features: callable bonds can be repurchased at par by the treasury, flower bonds are valued at par to satisfy federal tax liabilities, etc.

Corporate Bonds

Corporate bonds differ from treasuries primarily in having a greater probability of default (so-called *credit risk*, leading to a significant role for rating agencies). Special features, like call provisions, are also quite common. Principle characteristics include:

- Substantial market: about two-thirds of the treasury market, with more than 2 trillion outstanding at 1993 year-end (Table 1.1).

- Credit risk ranges from extremely unlikely (the ten-year default rate for Aaa debt is about 0.2%) to possible (about 5% for Baa debt) and worse.

- Liquidity variable. High-grade "Fortune 500" issues are easily traded on markets, junk bonds less so, and so-called private placements limited by regulation to a small market of sophisticated investors.

- Many varieties: fixed coupons and floating or adjustable rates, bullets and amortizing, and so on.

- Special features: sinking funds, callable for issuer, covertible to equity by owner, and so on.

Interest Rate Swaps

Interest rate swaps are a major component of the large and growing market for interest-rate derivatives, and are widely used to manage interest rate risk. They are arranged directly between financial institutions and end users, rather than through organized exchanges. Principle characteristics include:

- Large over-the-counter market. Bigggest players are the large money-center commercial banks, but investment banks have made considerable inroads in the 1990s. Estimates exceed $3 trillion dollars notional value worldwide for 1992.

- Risk depends on safety of counterparty, provisions for default. Credit risk managed through high-quality counterparties (like Merrill and Salomon's Aaa-rated derivatives subsidiaries) and posting of collateral.

- International Swap Dealers Assocation (ISDA) standard contract helps to establish common legal framework.

- Contracts can be custom-tailored to user's needs.

- Liquid market for standard contracts, less so for custom-made exotics.

Mortgages

Mortages are an enormous market in the US, made more attractive by government agency initiatives to support securitization. The critical feature of fixed rate mortgages is that they can be prepaid at the borrower's option, which makes them less attractive to investors (borrowers typically refinance when rates fall, leaving an investor with poorer opportunities). Various *Collateralized Mortgage Obligations* (CMOs) attempt to divide and repackage mortgages in more attractive combinations. Principal characteristics include:

- Large market: larger, in aggregate, than the treasury market (Table 1.1).

- Amortizing: unlike treasury notes and bonds, mortgages typically call for a series of constant payments that gradually reduce the principal to zero at maturity.

- Substantial liquidity for some standardized "pools" of single-family mortgages, less for others.

- Government agencies, and some private institutions, insure interest and principal payments on mortgages pools. These pools of relatively homogeneous home mortgages are the basis of a large secondary market in mortgage-related derivatives.

- Prepayments: for government-insured mortgages, the primary risk is that mortgagors will refinance when rates fall.

- Other varieties: adjustable rates tied to market interest rates, sometimes with limits; collateralized mortgage obligations (CMOs) repackage mortgages to better suit investor needs.

Agency Issues

A number of federal government and international agencies issue securities, including: the Federal National Mortgage Association (FNMA), the Government National Mortgage Association (GNMA), the Resolution Funding Corporation (RFC), and the World Bank. Of the US agencies, mortgages constitute most of the total ($1.348 trillion at the end of 1993). Characteristics and perceived credit risk vary widely.

Tax-Exempts

Outstanding issues of state and local governments in the US totaled, in 1993, more than a trillion dollars. Principle characteristics include:

- Claims to varying sources of revenue, ranging from general revenue (general obligation bonds) to revenue from specific projects (turnpikes, airports).

- Credit risk varies by issuer, structure of security.

- Exempt from federal taxation, and from state taxation in state of issue.

- Liquidity variable, generally low.

Futures

Futures contracts are traded on organized exchanges for a number of interest-rate sensitive instruments, including US treasury bonds and notes, eurorates (interbank interest rates for major currencies), and British and German government bonds; see the tables of the *Wall Street Journal* or *Financial Times*. Principle characteristics include:

- Low credit risk: contracts are generally guaranteed by the exchange. Exchanges, in turn, protect themselves by requiring investors to post margins.

- Highly liquid.

- Easy to short.

- Less variety available than in comparable over-the-counter markets.

Options

Even more than futures, options on interest-sensitive assets come in a variety of forms, ranging from traded options on (say) futures contracts, to options on interest-rate swaps (swaptions), to options imbedded in other instruments (callable bonds, mortgages). Exchange traded options exist for the major futures contracts, with similar principle characteristics.

Money Market Instruments

We will ignore, for the most part, markets for short-term fixed income securities: commercial paper, bankers acceptances, repurchase agreements, and so on. It's clear from Table 1.1, however, that these are substantial markets.

Summary

1. The phrases *debt instruments* and *fixed income securities* refer, in common usage, to financial instruments whose value is sensitive to interest rate movements.

2. Important examples include: government bonds, corporate bonds, interest rate swaps, mortgages, futures contracts, and options on all of these instruments.

3. Fixed income securities vary in the timing and contingency of cash flows, in credit risk, in liquidity of markets, and in tax and accounting treatment.

Further Reading

Most fixed income texts cover the markets reviewed in this chapter; specific sources will be provided when the time comes. For the short end of the maturity spectrum, which is ignored in this course, see Cook and Rowe (1996).

Table 1.1
Major Categories of Primary Fixed Income Securities

Entries are year-end values for 1993, billions of dollars, as reported in the Board of Governors of the Federal Reserve System, *Flow of Funds Coded Tables*, September 20, 1994. Updated data is available from the Fed: http://www.bog.frb.gov.

Category	Amount (Billions)
Treasury Securities	3309.9
Corporate Bonds	2066.7
Tax-Exempt Securities	1217.0
Mortgages	4209.9
US Agency Issues	1898.9
Commercial Paper	553.8
Federal Funds and Repurchase Agreements	457.8

Part I

Instruments with Fixed Payments

Chapter 2

Bond Arithmetic

Bonds are contracts that specify fixed payments of cash at specific future dates. If there is no risk to the payments, then bonds differ only in the timing and magnitude of the payments.

Pricing bonds, then, involves the time value of money: a dollar next year or ten years from now is not worth as much as a dollar now. This chapter is concerned with different ways of expressing this value, including prices, yields, forward rates, and discount factors. Probably the most common form of expression is the *yield curve*, a graph of yield vs maturity for bonds of the same type. The most popular example in the US is the yield curve for US treasuries, published daily in the *Wall Street Journal* and elsewhere. Similar curves are available for treasury securities in other countries, and for securities like corporate bonds and interest rate swaps that have some risk of default. We see, generally, that yields vary by maturity, most commonly with yields for long maturities greater than those for short ones.

Later in the course, we will try to explain why the yield curve typically slopes upward. For now, our objective is to be clear about what the yield curve means. We do this primarily for the US treasury market, but touch briefly on other markets along the way.

2.1 Prices and Yields in the US Treasury Market

The thing to remember, before we get bogged down in algebra, is that *price is fundamental*. Once you know the price of a bond, you can compute its yield using the appropriate formula. The *yield* — more completely, the *yield-to-maturity* — is just a convenient short-hand for expressing the price as an annualized interest rate: the price is the present value of the

bond's cash flows using the yield as the discount rate. The details reflect a combination of the theory of present values and the conventions of the market.

In markets for US treasury notes and bonds, two conventions are paramount: (i) prices are quoted for a face value or principal of $100 and (ii) yields are compounded semi-annually. The first is standard across fixed income markets. The second stems from the tradition of paying coupon interest twice a year, making six months a natural unit of time. With this in mind, consider an arbitrary instrument specifying cash payments of c_1 in six months, c_2 in twelve months, c_3 in eighteen months, and so on, for a total n six-month periods. We say that the instrument has a *maturity* of n six-month periods or $n/2$ years.

The price of this arbitrary instrument can be interpreted as the present value of its cash flows, using the yield y as the discount rate:

$$\text{Price } = \frac{c_1}{(1 + y/2)} + \frac{c_2}{(1 + y/2)^2} + \cdots + \frac{c_n}{(1 + y/2)^n}. \tag{2.1}$$

(Division by 2 in this formula converts y from a six-month yield to an annual yield.) Since six-month periods have different numbers of days, this relation is really an approximation in which we ignore the differences.

Although equation (2.1) reads naturally as telling us the price, given the yield, we will use it to define the yield: given a price, we solve the equation for the yield y. This isn't the easiest equation to solve, but financial calculators and spreadsheets do it routinely.

Discount Factors and Yields for Zero-Coupon Bonds

The easiest place to start is with zero-coupon bonds, or "zeros." These instruments actually exist, most commonly in the form of STRIPS, with prices reported under "Treasury Bonds, Notes & Bills" in Section C of the *Wall Street Journal*. But even if they didn't exist, we'd find that they were nevertheless useful as conceptual building blocks for bond pricing. By way of example, consider these price quotes, loosely adapted from the *Journal* of May 19, 1995:

Maturity (Years)	Price (Dollars)
0.5	97.09
1.0	94.22
1.5	91.39
2.0	88.60

In practice, prices are quoted in 32nds of a dollar, not cents, something we'll ignore from here on.

The prices of various zeros correspond to cash delivered at different future dates. $100 deliverable now is worth, obviously, $100. But $100 deliverable in six months can be purchased for $97.09 now. The difference is the time value of money: it's cheaper to buy money deliverable at a future date, and the farther away the delivery date, the lower the price.

One way to express the current value of money delivered at some future date is with a *discount factor*: the current price of *one* dollar delivered at the future date. These are, of course, just the prices of zeros divided by one hundred. They include, in our example,

Maturity (Years)	Price ($)	Discount Factor
0.5	97.09	0.9709
1.0	94.22	0.9422
1.5	91.39	0.9139
2.0	88.60	0.8860

For future reference, we use d_n to denote the discount factor for a maturity of n six-month periods, or $n/2$ years, so that (for example) $d_1 = 0.9709$. Figure 2.1 extends discount factors to a broader range of maturities. The figure illustrates the principle that the value of money declines as the delivery date moves farther into the future: one dollar in ten years is worth (in this figure) about 52 cents, a dollar in twenty years about 25 cents, and a dollar in thirty years about 13 cents.

The decline in the discount factor with maturity is a reflection, obviously, of the positive rate of interest required by lenders. A second way of expressing the time value of money makes this explicit: the *yield* implied by equation (2.1) using the market price of the zero and the cash flow of $100 at maturity. We refer to a graph of yield vs maturity for zero-coupon bonds as the *spot rate curve*.

The yield is particularly easy to compute for zeros, which have only a single cash payment at maturity. For a zero maturing in $n/2$ years, we apply the present value relation, equation (2.1), with $c_1 = c_2 = \cdots = c_{n-1} = 0$ (no coupons) and $c_n = 100$ (the principal):

$$\text{Price} \;=\; \frac{100}{(1 + y/2)^n}. \tag{2.2}$$

The analogous expression for discount factors is

$$d_n \;=\; \frac{1}{(1 + y_n/2)^n}, \tag{2.3}$$

since the discount factor prices a "principal" of one dollar, rather than one hundred. The subscript n in y_n here makes it clear which yield we have in mind. For the prices quoted earlier, the implied yields are

Maturity (Years)	Price ($)	Spot Rate (%)
0.5	97.09	5.991
1.0	94.22	6.045
1.5	91.39	6.096
2.0	88.60	6.146

The complete spot rate curve is pictured in Figure 2.2.

Coupon Bonds

With coupons bond pricing gets a little more complicated, since cash flows include coupons as well as principal, but the ideas are the same. The fundamental insight here is that an instrument with fixed payments, like a coupon bond, is a collection of zeros. Its price is the sum of the prices of the individual payments:

$$\text{Price} = d_1 c_1 + d_2 c_2 + \cdots + d_n c_n. \tag{2.4}$$

This relation is obvious in some respects, but it's so important that I'll repeat it with a box around it:

$$\boxed{\text{Price} = d_1 c_1 + d_2 c_2 + \cdots + d_n c_n.} \tag{2.4}$$

Equivalently, we could use equation (2.3) to replace the discount factors d with spot rates y:

$$\text{Price} = \frac{c_1}{(1 + y_1/2)} + \frac{c_2}{(1 + y_2/2)^2} + \cdots + \frac{c_n}{(1 + y_n/2)^n}. \tag{2.5}$$

We see in this version that each cash flow is discounted by a date-specific yield.

As an example, consider the "8 1/2s of May 97": a treasury note with an 8.5% coupon rate, issued on May 15, 1987, and maturing May 15, 1997. In May 1995 this is a two-year bond, with cash payments (per $100 principal or face value) of $4.25 in November 1995, $4.25 in May 1996, $4.25 in November 1996, and $104.25 (coupon plus principal) in May 1997. Its value is easily computed from the discount factors:

$$\begin{aligned}
\text{Price} &= 0.9709 \times 4.25 + 0.9422 \times 4.25 + 0.9139 \times 4.25 + 0.8860 \times 104.25 \\
&= 104.38.
\end{aligned}$$

Trading activity in the treasury market guarantees that this is, in fact, the market price: if the market price were less, investors would buy the equivalent cash flows in the STRIPS market, and if it were more, no one would buy the STRIPS at the quoted prices. The bid/ask spread gives us some margin for error, but the margin is small relative to the accuracy of these calculations.

A somewhat different way to think about a coupon bond is to associate it with its own yield. As with zeros, the price is the present value of the cash flows using the yield as the discount rate. For an arbitrary bond with n coupon payments remaining, equation (2.1) reduces to

$$\text{Price} \ = \frac{\text{Coupon}}{(1 + y/2)} + \frac{\text{Coupon}}{(1 + y/2)^2} + \cdots + \frac{\text{Coupon} + 100}{(1 + y/2)^n}. \tag{2.6}$$

For the 8 1/2s of May 97, this is

$$104.38 = \frac{4.25}{(1 + y/2)} + \frac{4.25}{(1 + y/2)^2} + \frac{4.25}{(1 + y/2)^3} + \frac{104.25}{(1 + y/2)^4},$$

which implies a yield y of 6.15 percent. This calculation involves some nasty algebra, but is easily done on a financial calculator or spreadsheet. See Appendix C.

The yield on a coupon bond is not generally the same as the yield on a zero with the same maturity, although for short maturities the differences are typically small. The reason they differ is that a coupon bond has cash flows at different dates, and each date is valued with its own discount factor and yield. If (as in this case) yields are higher for longer maturities, then the yield is lower on a coupon bond, which has coupon payments prior to maturity as well as a final payment of principal. The yield on a coupon bond is, approximately, a weighted average of the spot rates for the coupon dates and and maturity. The coupon dates get smaller weights in this average because coupons are smaller than principal.

We see the effect of coupons on yield clearly in the *par yield curve*, constructed from yields on bonds with coupon rates equal to their yields. These bonds sell at par by construction. We can derive par yields for our example from the discount factors d_n. The price of an n-period bond is related to discount factors by

$$\text{Price} \ = 100 = (d_1 + \cdots + d_n) \ \text{Coupon} \ + d_n 100, \tag{2.7}$$

a variant of our fundamental pricing equation, (2.4). If we solve for the annual coupon rate, or par yield, we get

$$\text{Par Yield} \ = 2 \times \text{Coupon} \ = \ 2 \times \frac{1 - d_n}{d_1 + \cdots + d_n} \times 100. \tag{2.8}$$

(The 2 in this formula comes from using semiannual coupons: we multiply the coupon by two to get the annual coupon rate.) The initial maturities give us par yields of

Maturity (Years)	Price ($)	Spot Rate (%)	Par Yield (%)
0.5	97.09	5.991	5.991
1.0	94.22	6.045	6.044
1.5	91.39	6.096	6.094
2.0	88.60	6.146	6.142

You'll note that these are slightly lower (by less than one basis point, or 0.01 percent) than yields for zeros, as we suggested. For longer maturities the discrepancy can be larger, as we see in Figure 2.3.

For now, simply note that yields on zeros and coupon bonds of the same maturity are not generally the same. A summary of the various yield formulas is given in Table 2.1. We return to the par yield calculation in Chapter 5, when we examine interest rate swaps.

Although yields on coupon bonds differ from spot rates, we can nevertheless compute spot rates from prices of coupon bonds — indeed, we could even do this if zeros did not exist. Suppose we had prices for coupon bonds with maturities $n = 1, 2, 3$:

Maturity (Years)	Coupon Rate	Price ($)
0.5	8.00	100.98
1.0	10.00	103.78
1.5	4.00	97.10

We find the discount factors using equation (2.4) repeatedly for bonds of increasing maturity. The first discount factor is implicit in the price of the one-period bond:

$$100.98 = d_1 \times 104,$$

implying $d_1 = 0.9709$. We find the second discount factor from the two-period bond:

$$103.78 = 0.9709 \times 5 + d_2 \times 105,$$

implying $d_2 = 0.9422$. We find the third discount factor from the three-period bond:

$$97.10 = (0.9709 + 0.9422) \times 2 + d_3 \times 102,$$

implying $d_3 = 0.9422$. (Your calculations may differ slightly: mine are based on more accurate prices than those reported here.)

In short, we can find the complete set of discount factors from prices of coupon bonds. From the discount factors, we use (2.3) to compute spot rates.

2.2   Day Counts and Accrued Interest

We have computed, thus far, prices and yields for bonds with maturities in even half-years. For zeros this presents no difficulty: we can use (2.2) for any maturity we like. For coupon bonds with fractional maturities, there are two additional conventions we need to know. The first convention is that prices are quoted net of a pro-rated share of the current coupon

payment, a share referred to as *accrued interest*. The second convention governs the use of fractional time periods in computing yields.

By longstanding convention, price quotes in bond markets are not prices at which trades are made. Trades are executed at the *invoice price*, which is related to the quoted price by

$$\text{Invoice Price} \;=\; \text{Quoted Price} \;+\; \text{Accrued Interest} \,,$$

where accrued interest is a fraction of the next coupon payment. Specifically: accrued interest is the next coupon payment multiplied by the fraction of time already passed between the previous payment and the next one. Time is measured in days, according to conventions that vary across markets. In the treasury market, we count the "actual" number of days between scheduled payments and refer to the convention as "actual/actual."

As usual, this is easier to explain with an example. Consider the May 18, 1995, price of the "8 1/2s of April 97," 7-year US Treasury notes issued April 16, 1990. This note has scheduled coupon payments on 10/15/95, 4/15/96, 10/15/96, and 4/15/97. As noted, accrued interest is based on the actual number of days between scheduled coupon payments. If these dates fall on a weekend or holiday — 10/15/95, for example, is a Sunday — the payments are made on the next business day, but we nonetheless compute accrued interest using the scheduled dates. For our example, there are a total of 183 days between the previous coupon date (4/15/95) and the next one (10/15/95), computed as follows:

Month	Day Count
April	15
May	31
June	30
July	31
August	31
September	30
October	15
Total	183

Of this 183 days, 33 have passed between the previous coupon date and the presumed *settlement date*, May 18. We compute accrued interest as the pro-rated share of the coupon since the previous coupon date:

$$\text{Accrued Interest} \quad = \quad \frac{33}{183} \times \frac{8\frac{1}{2}}{2} = 0.77.$$

Given a quoted price of 104:06 (104 and 6 32nds, approximately 104.19), the invoice price for the note is 104.95 (= 104.19 + 0.77, subject to rounding).

More generally, suppose u days have passed since the last coupon date and v days remain until the next one, as in this diagram:

Previous   Settlement   Next
Coupon   Date   Coupon

4/15/95   5/18/95   10/15/95

u=33 Days v=150 Days
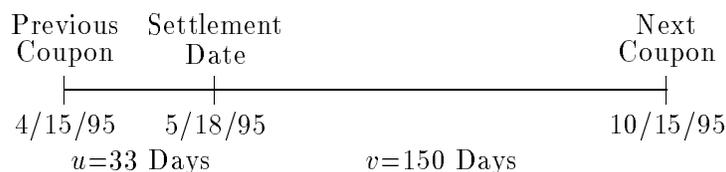
Then accrued interest is

$$\text{Accrued Interest} \quad = \quad \frac{u}{u+v} \times \text{Coupon}$$

$$= \quad \frac{u}{u+v} \times \frac{\text{Annual Coupon Rate}}{2}. \qquad (2.9)$$

You might verify for yourself that our calculation for the 8 1/2s of April 97 satisfies this relation.

The second convention concerns the relation between price and yield for fractional periods of time. For an instrument with arbitrary cash flows $(c_1, c_2, \ldots, c_n)$, the analog of equation (2.1) is

$$\text{Price} \; = \frac{c_1}{(1+y/2)^w} + \frac{c_2}{(1+y/2)^{w+1}} + \cdots + \frac{c_n}{(1+y/2)^{w+n-1}}, \qquad (2.10)$$

where $w = v/(u+v)$ is the fraction of a semiannual period remaining until the next coupon date. For a bond with n coupon payments remaining, this becomes

$$\text{Invoice Price} \; = \frac{\text{Coupon}}{(1+y/2)^w} + \frac{\text{Coupon}}{(1+y/2)^{w+1}} + \cdots + \frac{\text{Coupon} + 100}{(1+y/2)^{w+n-1}},$$

For the 8 1/2s of April 97, we have Coupon = 4.25 and $w = 150/183 = 0.82$. With an invoice price of 104.95, the yield is 6.14 percent.

2.3   Other Conventions

The yield and day count conventions used for US treasury notes and bonds are by no means the only ones used in fixed income markets. We review some of the more common alternatives below, and summarize them in Table 2.2.

US Corporate Bonds

Like US treasuries, US corporate bonds have semiannual coupons. The day counts, however, are 30/360. There is a subtle difference in the day counts between corporates and eurobonds,

which I'll mention and then ignore. In the 30/360 convention, if the next coupon is on the 31st of the month, and the settlement date is not on the 30th or 31st, then we count all 31 days in that month. (If this sounds confusing, never mind.)

By way of example, consider Citicorp's 7 1/8s, maturing March 15, 2004 with semiannual coupon payments scheduled for the 15th of September and March. Bloomberg's quoted price on June 16, 1995, for June 21 settlement, was 101.255. The 30/360 convention gives us a day count of $u = 96$ days since the previous coupon date, accrued interest of 1.900, an invoice price of 103.155, and a yield of 6.929.

Eurobonds

The term *eurobonds* refers to bonds issued in the European market, or more generally outside the issuer's country, often to avoid some of the regulations governing public issues. Typically coupon interest is paid annually, yields are annually compounded, and day counts are based on a "30E/360" (E for euro) convention. In this convention, we count days as if there were 30 days in every month and 360 days in a year. If there have been u days (by this convention) since the previous coupon, accrued interest is

$$\text{Accrued Interest} = \frac{u}{360} \times \text{Coupon}$$

and the yield y is the solution to

$$
\begin{aligned}
\text{Invoice Price} \quad &= \quad \text{Quoted Price} + \text{Accrued Interest} \\
&= \quad \frac{\text{Coupon}}{(1+y)^w} + \frac{\text{Coupon}}{(1+y)^{w+1}} + \cdots + \frac{\text{Coupon} + 100}{(1+y)^{w+n-1}},
\end{aligned}
$$

where n is the number of coupons remaining, $w = v/(u+v)$, and $v = 360 - u$.

As an example, consider the dollar-denominated 9s of August 97, issued August 12, 1987, by the International Bank for Reconstruction and Development (IBRD, the World Bank), and maturing August 12, 1997. The Bloomberg price quote on June 15, 1995, for June 20 settlement, was 106.188. We compute the yield as follows. There are $n = 3$ remaining coupon payments. The day count convention gives us $u = 308$. Accrued interest is therefore

$$\text{Accrued Interest} = \frac{308}{360} \times 9.00 = 7.700,$$

giving us an invoice price of 113.888. The yield is 5.831 percent.

A similar convention applies to the World Bank's euroyen bonds: the 5-1/8s of March 98, maturing March 17, 1998. The Bloomberg price quote on June 21, 1995, for June 27 settlement, was 109.670. You might verify that the corresponding yield is 1.472 (rates are very low now in Japan).

Foreign Government Bonds

Foreign governments use a variety of conventions: there's no substitute for checking the bond you're interested in. Most of this is available online — through Bloomberg, for example.

Examples:

- Canada. Semiannual interest and compounding, actual/actual day count.

- United Kingdom. Semiannual interest and compounding, actual/365 day count. One of the wrinkles is that "gilts" trade ex-dividend: the coupon is paid to the registered owner 21 days prior to the coupon date, not the dividend date itself.

- Germany. Annual interest, 30E/360 day count, own yield convention. Also trades ex-dividend.

US Treasury Bills

The US treasury issues *bills* in 3, 6, 9, and 12 month maturities. These instruments are zeros: they have no coupons. Yields on treasury bills are computed on what is termed a *bank discount basis*: the yield y solves

$$\text{Price} \ = 100 \times [1 - y(v/360)], \tag{2.11}$$

where v is the number of days until the bill matures. The bank discount basis has nothing to do with discounting in the sense of present value, but gives us a simple relation between price and yield. The price is 100 minus the *discount*, with

$$\text{Discount} \ = 100 \times y(v/360)$$

per 100 face value. This basis has little to do with the bond yields we quoted earlier. For comparison with bond yields, we often use the *bond equivalent yield*, the value of y that solves

$$\text{Price} \ = \frac{100}{1 + (y/2)(v/365)}.$$

This relation does two things simultaneously: it compounds the yield semi-annually, and it converts it to a 365-day year.

As an example, consider the bill due 8/10/95, which had 80 days to maturity on May 18. The quoted bank discount yield (asked) is 5.66 percent, which translates into a price of 98.74 and a discount of 1.26. The bond equivalent yield is 5.81 percent.

Eurodollar Deposits

One of the more popular markets on which to base fixed income derivatives is the eurocurrency market: short-term deposits by one bank at another. The most common location is London, although the prefix "euro" is now generally understood to include such sites as the Bahamas, the Cayman Islands, Hong Kong, and even International Banking Facilities (don't ask) in the US. If the deposits are denominated in dollars we refer to them as *eurodollars*; similarly, euroyen or euromarks. Rates quoted in London are referred to as LIBOR, the London Interbank Offer Rate (the bank is "offering" cash at this rate). As an interbank market, rates vary among banks. A common standard is the British Bankers' Association (BBA) average.

Interest rates for eurodollar deposits are quoted as "simple interest," using an "actual/360" day count convention. Consider a six-month eurodollar deposit. The interest payment is

$$\text{Interest Payment } = \text{ Principal } \times \text{LIBOR} \times \frac{\text{Days to Payment}}{360}.$$

For a one million dollar deposit made June 22, at a quoted rate of 5.93750 percent (five-digit accuracy being a hallmark of the BBA), we get

$$\text{Interest Payment } = 1,000,000 \times 0.0593750 \times \frac{183}{360} = 30,182,$$

there being 183 days between June 22 and December 22.

When we turn to floating rate notes and interest rate swaps, we'll see that it's necessary to convert LIBOR rates to the kinds of semiannual yields used for bonds. In this case, the so-called bond equivalent yield y satisfies

$$y/2 = \text{LIBOR} \times \frac{\text{Days to Payment}}{360}, \tag{2.12}$$

a minor correction for the difference in reporting conventions.

The same convention is used for rates on deposits denominated in many other currencies. The notable exception is pounds, which are quoted on an actual/365 basis.

Continuous Compounding (optional)

One of the troublesome details in computing bond yields is that they depend on how often they are compounded. Eurobonds are compounded annually, US treasuries and corporates semiannually, and mortgages (as we'll see later) monthly. Yet another convention, widely

used by academics, is continuous compounding. We'll see in a number of applications that continuous compounding gives us cleaner results in some cases.

By continous compounding, we mean the ultimate effect of compounding more and more frequently. Consider the price of a n-year bond, with interest compounded k times a year. The appropriate present value formula is

$$\text{Price} \ = \frac{100}{(1 + y/k)^{kn}}.$$

For $k = 1$ this defines the annually compounded yield, for $k = 2$ the semiannually compounded yield, and so on. As k gets large, this expression settles down. Mathematically we write

$$\lim_{k \to \infty} \frac{100}{(1 + y/k)^{kn}} = \frac{100}{e^{ny}},$$

where e is a fixed number, referred to as Euler's number, equal approximately to 2.7183. It's not apparent yet, but this will be useful later.

2.4   Implementation Issues

There are a number of practical difficulties in constructing and using yield curves. They include:

- *Interpolation.* We do not always have yields for all the relevant maturities. The standard solution is to interpolate, for which many methods exist. The details are interesting only to aficionados.

- *Smoothing.* The yields reported for zeros are extremely bumpy, as you can see in Figure 2.4, which was constructed from yields on zeros reported in the *Journal* on May 19, 1995. Most users smooth the data, as I did in Figures 2.1 to 2.3 (I used a polynomial approximation to the raw data).

- *Bid/ask spread.* There is generally a spread between the bid and ask prices of bonds, which means each observed yield is a range, not a point. Standard approaches include the average of the bid and ask, or simply the ask (on the grounds that the ask price is what an investor would have to pay to get the bond). In any case, the spread adds noise to the data.

- *Nonsynchronous price quotes.* If prices for different maturities are observed at different times, they may not be comparable: the market may have moved in the interim.

- *Day counts.* Yields on different instruments may not be comparable due to differences in day count conventions, holidays or weekends, and so on.

- *Special features.* Some bonds have call provisions, or other special features, that affect their prices. A callable bond, for example, is generally worth less than a comparable noncallable bond. This is a particular problem for maturities beyond 10 years, since there are no noncallable bonds due between Feb 2007 and November 2014. In the *Journal* yields on callable bonds are computed the standard way for bonds with prices less then 100, and are truncated at the first call date for bonds selling for more than 100 (yield to call, more later).

- *Credit quality.* When we move beyond treasuries, bonds may differ in credit quality. We may find that bonds with lower prices and higher yields have higher default probabilities.

- *Issue scarcity.* Occasionally a specific bond will become especially valuable for use in settling a futures position or some such thing, resulting in a lower yield than otherwise comparable bonds. Eg, a firm apparently cornered the market in 1993 in the issue used to settle 10-year treasury futures, raising its price about 15 cents per hundred dollars.

In short, even the treasury market has enough peculiarities in it to remind us that the frictionless world envisioned in Proposition 2.1 is at best an approximation.

2.5   Common Yield Fallacies

Yields Are Not Returns

A bond yield is not generally the return an investor would get on the bond. Yields are simply a convenient way of summarizing prices of bonds in the same units: an annual percentage rate. If you remember that, you can turn to the next section. If not, stay tuned.

Zeros are the easiest, so let's start there. The yield to maturity on a zero is, in fact, the compounded return if one held the bond to maturity. To see this, rewrite (2.2) as

$$(1 + y/2)^n = \frac{100}{\text{Purchase Price}}.$$

In this sense the yield and return are the same.

We run into trouble, though, if we compare zeros with different maturities. Suppose a two-year zero has a yield of 5% and a four-year zero has a yield of 6%. Which has the higher return? The answer depends on the time horizon of our investment and, perhaps, on future interest rates. In short, we should say that we don't know: the yield does not give us enough information to decide. If this is unclear, consider the returns on the two

instruments over two years. For the two-year zero, the return is 5%. For the four-year zero, the return h solves

$$(1 + h/2)^4 = \frac{\text{Sale Price}}{\text{Purchase Price}}.$$

If the two-year spot rate in two years is 7% or below, the four-year zero has a higher two-year return r. But for higher spot rates the two-year zero has a higher return.

Over shorter investment periods we face similar difficulties. Over six months, for example, the return h on a zero is the ratio of the sale price to the purchase price. Since the latter depends on the future values of spot rates, both bonds have uncertain returns and we can't say for sure which one will do better.

High Yield Need Not Mean High Return

With coupon bonds, even those of the same maturity, we have similar difficulties if the coupons differ. Generally speaking, bonds with higher yields need not have higher returns, even over the maturity of the bonds. Consider these two bonds:

	Bond A	Bond B
Coupon	10	3
Principal	100	100
Price	138.90	70.22
Maturity (Years)	15	15
Coupon Frequency	Annual	Annual
Yield (Annual %)	6.00	6.10

The two bonds have the same maturity and B has a higher yield. Does B also have a higher return over the full 15-year life of the bonds? The issue here is the rate at which coupons are reinvested. We can be more specific is we are willing to tolerate some algebra. Let r be the reinvestment rate and n the number of coupons and years remaining. Then the value of the investment at maturity is

$$\text{Final Value} = \left[(1+r)^{n-1} + (1+r)^{n-2} + \cdots + (1+r) + 1 \right] \times \text{Coupon} + 100.$$

The cumulative return over the n periods is

$$\text{Total Return} = (1+h)^n = \frac{\text{Final Value}}{\text{Purchase Price}}.$$

For our example, bond A has a higher return than B at reinvestment rates greater than 5.7%, since it has larger coupons. Bond B has a higher return at lower investment rates. In short, there is no reason *a priori* to suspect that Bond B is superior to Bond A.

Yields Are Not Additive

The last fallacy is that yields are additive: that the yield of a portfolio is the value-weighted average of the yields of the individual assets. In fact yields are not additive, as the next example illustrates. Consider three bonds with cash flows c_1, c_2, and c_3 over three annual periods:

Bond	Price	c_1	c_2	c_3	Yield	Yield Avg
A	100	15	15	115	15.00	
B	100	6	106		6.00	
C	92	9	9	109	12.35	
A+B	200	21	121	115	11.29	10.50
A+C	192	24	24	224	13.71	13.73
B+C	192	15	115	109	9.65	9.04

Note, for example, that an investment of 100 each in A and B has a yield of 11.29, substantially larger than the average yield of 10.50.

2.6 Forward Rates (optional)

Spot rates are, approximately, average interest rates for the period between the price quote and maturity. Forward rates decompose this average into components for individual periods. We do this for the standard treasury conventions: time units of six months and semiannual compounding.

Recall that STRIPS prices tell us the value, in dollars today, of one hundred dollars at a particular future date. If we again denote the value of an n-period (or $n/2$-year) STRIP by p_n, then the yield or spot rate y satisfies

$$p_n = \frac{100}{(1 + y_n/2)^n}.$$

This is just equation (2.2) with subscripts n added to make the maturity explicit. The spot rate y_n is the rate used to discount each of the n periods until the bond matures.

Alternatively, we might consider using different interest rates for each period: f_0 for the intial period, f_1 for the next one, f_2 for the one after that, and so on. These are the interest rates for "forward" one-period investments; f_2, for example, is the interest rate on a one-period investment made in two periods — what is referred to as a *forward contract*. Using forward rates we can write the present value of zeros as

$$p_1 = \frac{100}{(1 + f_0/2)},$$

$$p_2 = \frac{100}{(1 + f_0/2)(1 + f_1/2)},$$

and so one. For a STRIP of arbitrary (integer) maturity n, this would be

$$p_n = \frac{100}{(1 + f_0/2)(1 + f_1/2)\cdots(1 + f_{n-1}/2)},$$

which differs from the spot rate formula, equation (2.2), in using a potentially different rate for each period.

For a maturity of 3 six-month periods, we might picture yields and forward rates like this:
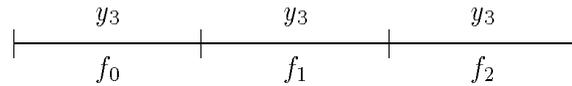
The yield y_3 applies equally to all three periods, but each period has its own specific forward rate.

Forward rates are more than a theoretical abstraction: they exist on traded forward contracts. We can also compute them from prices of zeros. The first forward rate can be derived from the price of a one-period (6-month) zero:

$$1 + f_0/2 = \frac{p_1}{100}.$$

For longer maturities we can "pick off" the forward rate from prices of consecutive zeros:

$$1 + f_n/2 = \frac{p_n}{p_{n+1}}$$

For the example of Section 2.1, forward rates are

Maturity (Years)	Price ($)	Spot Rate (%)	Forward Rate (%)
0.5	97.09	5.99	5.99
1.0	94.22	6.05	6.10
1.5	91.39	6.10	6.20
2.0	88.60	6.15	6.29

A complete forward rate curve is pictured in Figure 2.5.

Summary

1. Discount factors summarize the time value of fixed payments at future dates.

2. We can replicate coupon bonds with zeros, and vice versa.

3. In a frictionless, arbitrage-free world, the same information is contained in prices of zeros, spot rates (yields on zeros), prices of coupon bonds, yields on coupon bonds, and forward rates. Given one, we can compute the others.

4. Yields are not returns: bonds with high yields need not have high returns.

5. Prices are additive, yields are not.

Practice Problems

1. Consider the following zero coupon yields, or spot rates, for US treasury STRIPS:

Maturity (Years)	Spot Rate (%)
0.5	5.00
1.0	6.00
1.5	7.00
2.0	8.00

   (a) Compute, for each of these maturities, the (i) discount factor, (ii) STRIPS price, and (iii) forward rate.

   (b) Use the discount factors to compute the price of a two-year bond with an annual coupon rate of 5.00%.

   (c) Compute the par yield curve and graph it against the spot rate and forward rate curves. Explain any differences.

2. Consider these price quotes for coupon bonds:

Maturity (Years)	Coupon Rate	Price
0.5	8.00	103.00
1.0	4.00	98.60
1.5	8.00	101.80
2.0	6.00	99.00

   The coupon rates are annual, hence twice the size of the coupon.

   (a) Compute the yield for each bond.

   (b) Compute the discount factors and spot rates implied by the coupon bond prices.

3. Consider the following US Treasury prices:

Maturity	Coupon Rate	Price
1	$9\frac{3}{8}$	101:28
2	8	102:09
3	$6\frac{1}{2}$	101:08

The maturity is listed in half-years, the coupon rate is an annual percentage, and the price is the ask, measured in 32nds.

Suppose a second three-period bond is available, with coupon $8\frac{1}{2}$ and price 103:15.

(a) Use the prices of the first three bonds to estimate the discount factors.

(b) Use these discount factors to compute the "fair value" of the second three-period bond.

(c) Compare the fair value of the bond to its price. Why might they differ?

4. Consider prices for three bonds with two semiannual periods until maturity:

Bond	Coupon Rate	Price
A	10	106.56
B	4	100.70
C	6	103.12

(a) Use any two of these bonds to compute the discount factors.

(b) Use the discount factors to value the third bond.

(c) Deduce an arbitrage strategy to exploit the inconsistency of these bond prices.

5. You are given, on August 15, 1995, the following data for various US treasury bonds maturing February 15, 1997:

Invoice Price	Coupon Rate
102.36	6.75
103.56	7.875
103.75	8.25
103.87	8.375
106.34	10.875

(a) Find an arbitrage opportunity with these bonds.

(b) If the prices above are ask prices and the bid prices are 1/8 lower, does the arbitrage opportunity still exist?

6. Consider the US treasury 6 1/2s of August 97, issued August 15, 1994, and maturing on August 15, 1997. On May 18, 1995, the quoted price was 100:21 (100 + 21/32). Compute (i) accrued interest, (ii) the invoice price, and (iii) the yield.

7. The UK issued a dollar-denominated bond on December 9, 1992, maturing December 9, 2002, with an annual coupon of 7.25 percent. Bloomberg quoted a price of 105.226 for settlement June 20, 1995. Using the 30E/360 day-count convention for eurobonds, compute (i) the number of days since the previous coupon, (ii) accrued interest, (iii) the invoice price, and (iv) the yield.

8. Pacific Bell issued a thirty-three year bond in 1993, maturing March 15, 2026, with an annual coupon rate of 7.125 and semiannual coupon payments scheduled for the 15ths of September and March. Bloomberg's quoted ask price on June 16, 1995, for June 21 settlement, was 97.134. Using the appropriate day count convention, compute (i) the number of days since the previous coupon, (ii) accrued interest, (iii) the invoice price, and (iv) the yield.

9. For May 18, 1995, the yields reported in the *Journal* on two US treasury bonds of identical maturity were substantially different, apparently because one was callable and the other was not.

   - *Bond 1.* 8s of Aug 01, issued August 16, 1976, maturing August 15, 2001. Callable at par on August 15, 1996. Quoted price: 102. Quoted yield: 6.29.
   - *Bond 2.* 13-3/8s of Aug 01, issued July 2, 1981, first coupon February 15, 1982, maturing August 15, 2001. Not callable. Quoted price: 135:04 (135 4/32). Quoted yield: 6.45.

   Our mission is to explain the difference in yield and approximate the cost to the government of the call feature on the first bond.

   (a) For each bond, compute accrued interest and the invoice price.

   (b) Use the invoice prices to compute yields for both bonds. How do these yields compare to those quoted in the *Journal*?

   (c) The government has the option, which it must announce 120 days in advance, of "calling" the bond at par: of paying 104 (100 principal and 4 coupon interest) on August 15, 1996, and retiring the bond. The *yield-to-call* is the yield computed for the cash flows received by an investor if the government does, in fact, call the bond. What is the yield-to-call for Bond 1?

   (d) Consider a noncallable bond like our first one: a straight 8 of Aug 01. If its yield were 6.45, what would its invoice price be? Comment, if you like, on this estimate of the yield.

   (e) We now have prices for 8s of August 96 with and without a call option. Use the difference to compute the cost to the government of the option.

Further Reading

Stigum (1981) is still the standard reference for US bond formulas. The essentials are also reviewed in most fixed income textbooks, including Fabozzi (1996) and Garbade (1996). Fabbozzi (1993, pp 67-74) has a nice overview of various day count conventions. Tuckman (1995, ch 4) has the best discussion I've seen on the practical aspects of estimating a yield curve.

There are a number of online sources of bond information, including Bloomberg and Reuters. Bloomberg is available in a number of places at Stern and is relatively easy to use. To get started: hit the yellow GOVT or CORP button (for government and corporate securities, respectively), then hit the large green GO button and follow directions. The examples in the text are based on information from the DES (Description) and YA (Yield Analysis) options for specific bonds.

Table 2.1
Bond Yield Formulas

Zero-Coupon Bonds

Price/Yield Relation

$$p_n = \frac{100}{(1 + y_n/2)^n}$$

$$
\begin{aligned}
p_n &= \quad \text{price of zero} \\
y_n &= \quad \text{yield or spot rate} \\
n &= \quad \text{maturity in half-years}
\end{aligned}
$$

Discount Factors

$$d_n = p_n/100$$

Forward Rates

$$1 + f_n/2 = p_n/p_{n+1}$$

Coupon Bonds

$$p = \frac{\text{Coupon}}{(1 + y/2)} + \frac{\text{Coupon}}{(1 + y/2)^2} + \cdots + \frac{\text{Coupon} + 100}{(1 + y/2)^n}.$$

$$
\begin{aligned}
p &= \quad \text{Price of Bond} \\
\text{Coupon} &= \quad \text{Annual Coupon Rate}/2 \\
n &= \quad \text{Number of Coupons Remaining}
\end{aligned}
$$

Par Yield

$$\text{Par Yield} = 2 \times \text{Coupon} = \frac{1 - d_n}{d_1 + \cdots + d_n} \times 200.$$

Bonds Between Coupon Dates

$$p = \frac{\text{Coupon}}{(1 + y/2)^w} + \frac{\text{Coupon}}{(1 + y/2)^{w+1}} + \cdots + \frac{\text{Coupon} + 100}{(1 + y/2)^{w+n-1}},$$

$$
\begin{aligned}
p &= \quad \text{Quoted Price } + \text{ Accrued Interest} \\
&= \quad \text{Quoted Price } + \frac{u}{u+v} \text{ Coupon} \\
u &= \quad \text{Days Since Last Coupon} \\
v &= \quad \text{Days Until Next Coupon} \\
w &= \quad v/(u + v)
\end{aligned}
$$

Table 2.2
Common Day Count Conventions

This table is based partly on Exhibit 5-7 of Fabozzi, *Fixed Income Mathematics (Revised Edition)*, Probus, 1993.

Type of Bond	Coupon Frequency	Day Count
US Treasury Notes/Bonds	Semiannual	Actual/Actual
UK Treasuries (Gilts)	Semiannual	Actual/365
German Government	Annual	30E/360
US Corporates	Semiannual	30/360
Eurodollar Bonds	Annual (mostly)	30E/360
Eurocurrency Deposits		
US Dollars	None	Actual/360
Deutschemarks	None	Actual/360
Pounds	None	Actual/365

Figure 2.1
Discount Factors for US Treasury STRIPS, May 1995

Figure 2.2
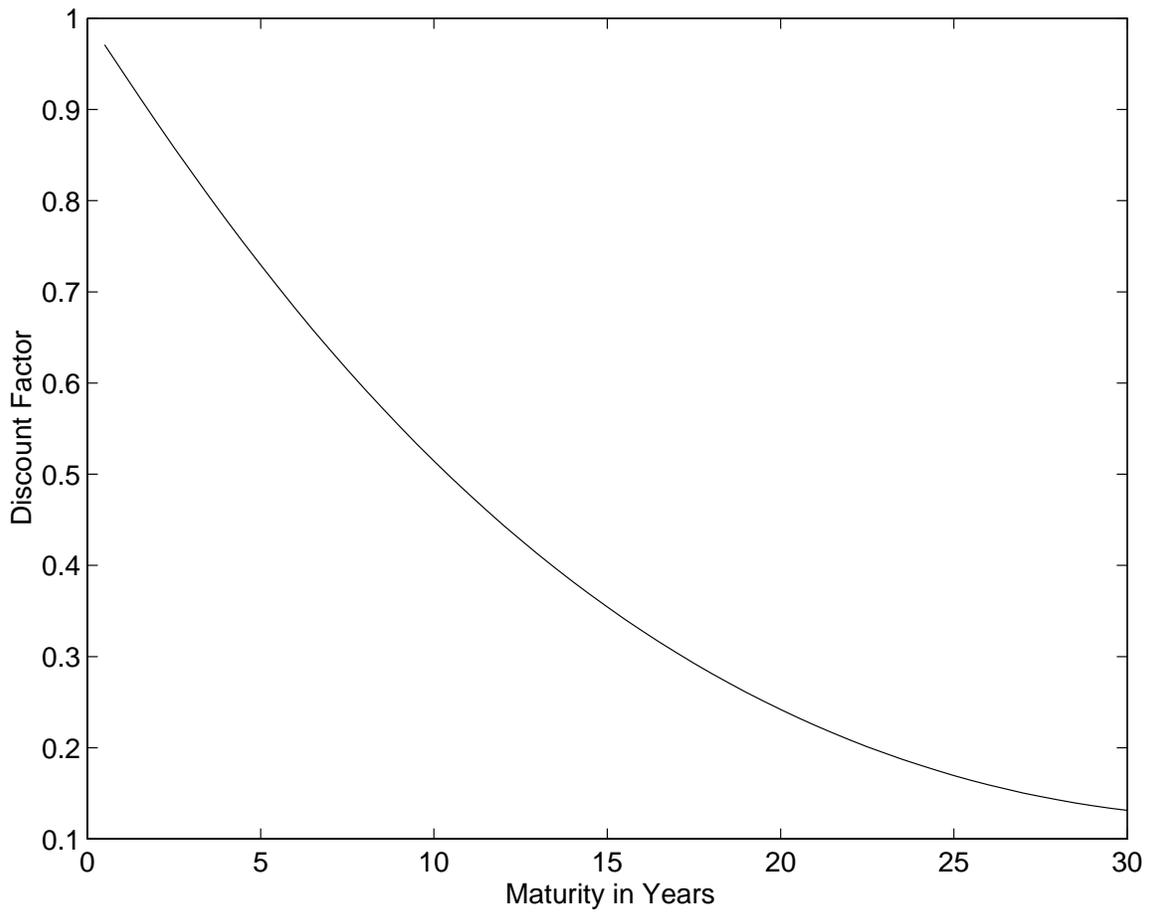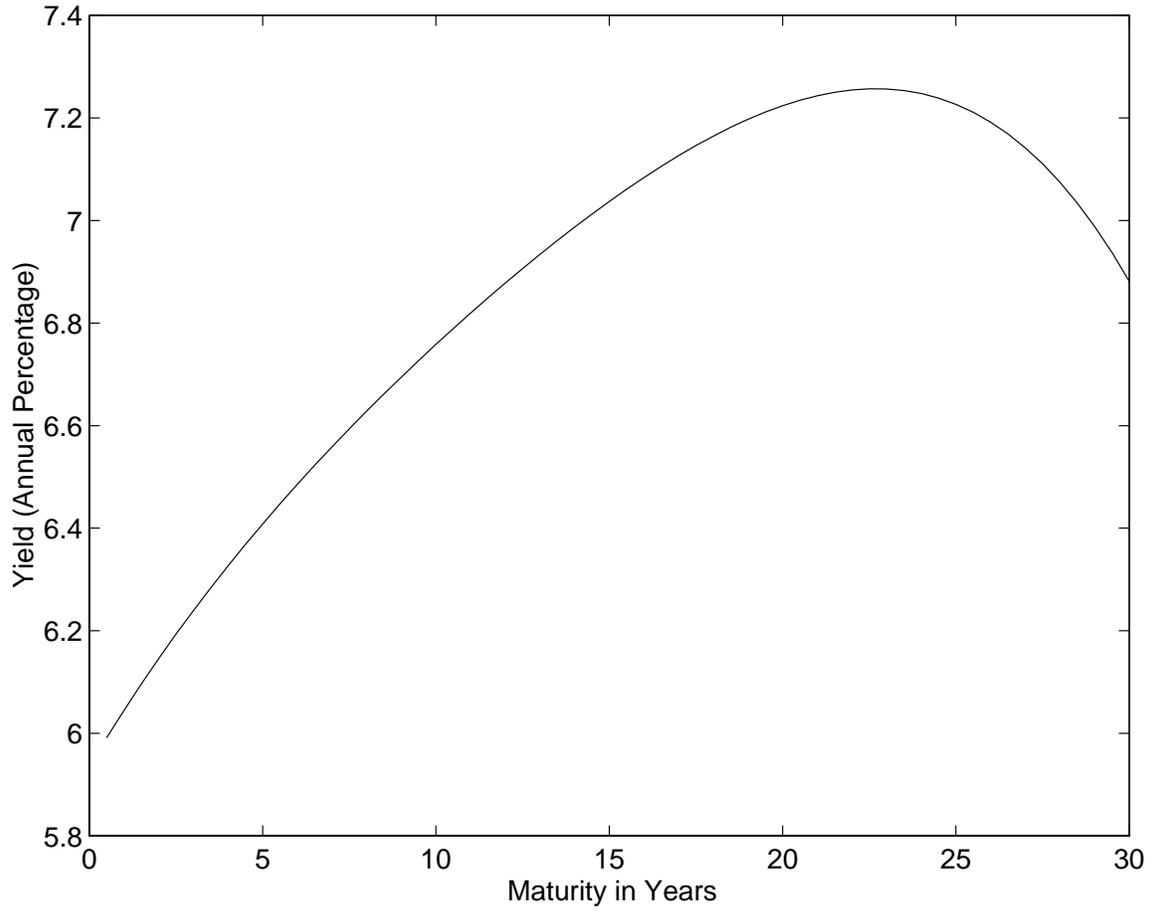**Spot Rate Curve from Treasury STRIPS, May 1995**

Figure 2.3
Par Yield Curve, May 1995
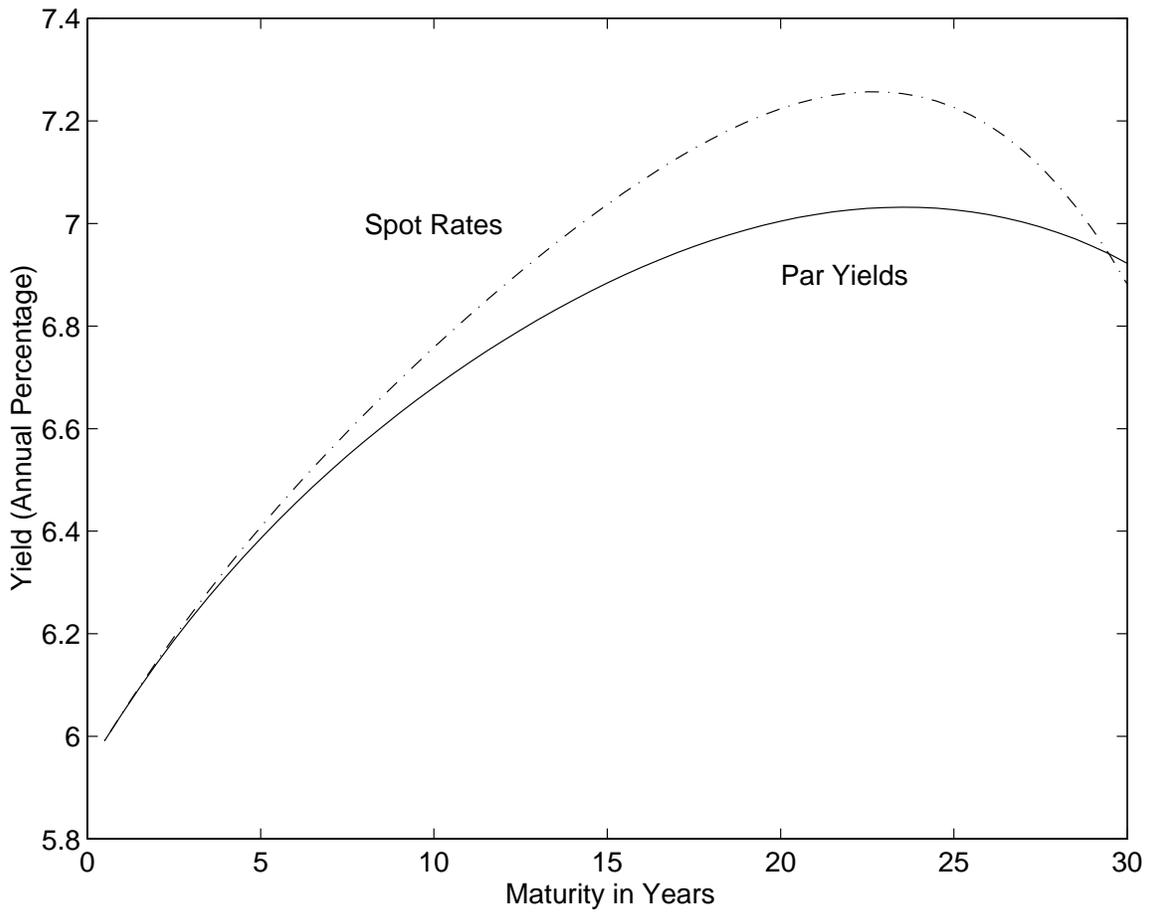
Figure 2.4
Raw and Approximate Spot Rate Curve, May 1995

Circles are raw data, the line is a smooth approximation
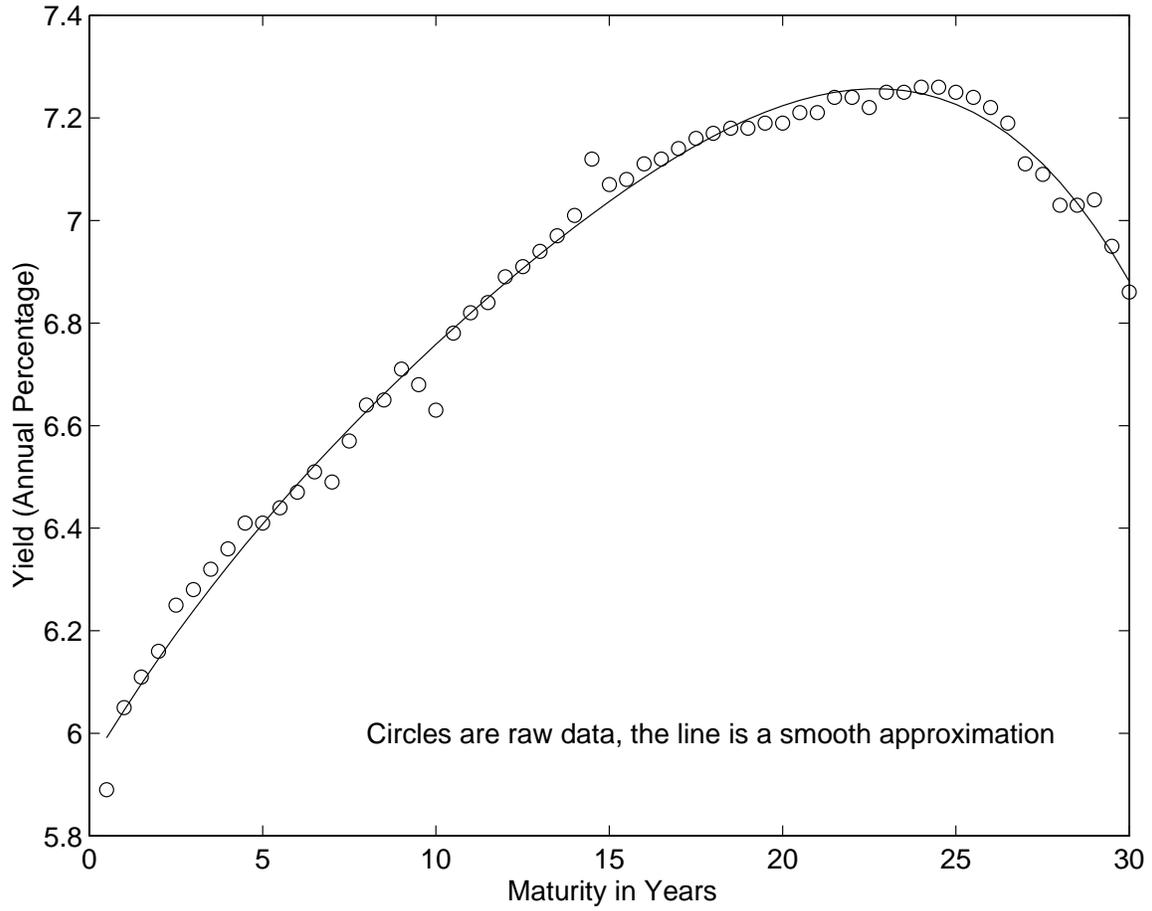
Figure 2.5
Forward Rate Curve, May 1995
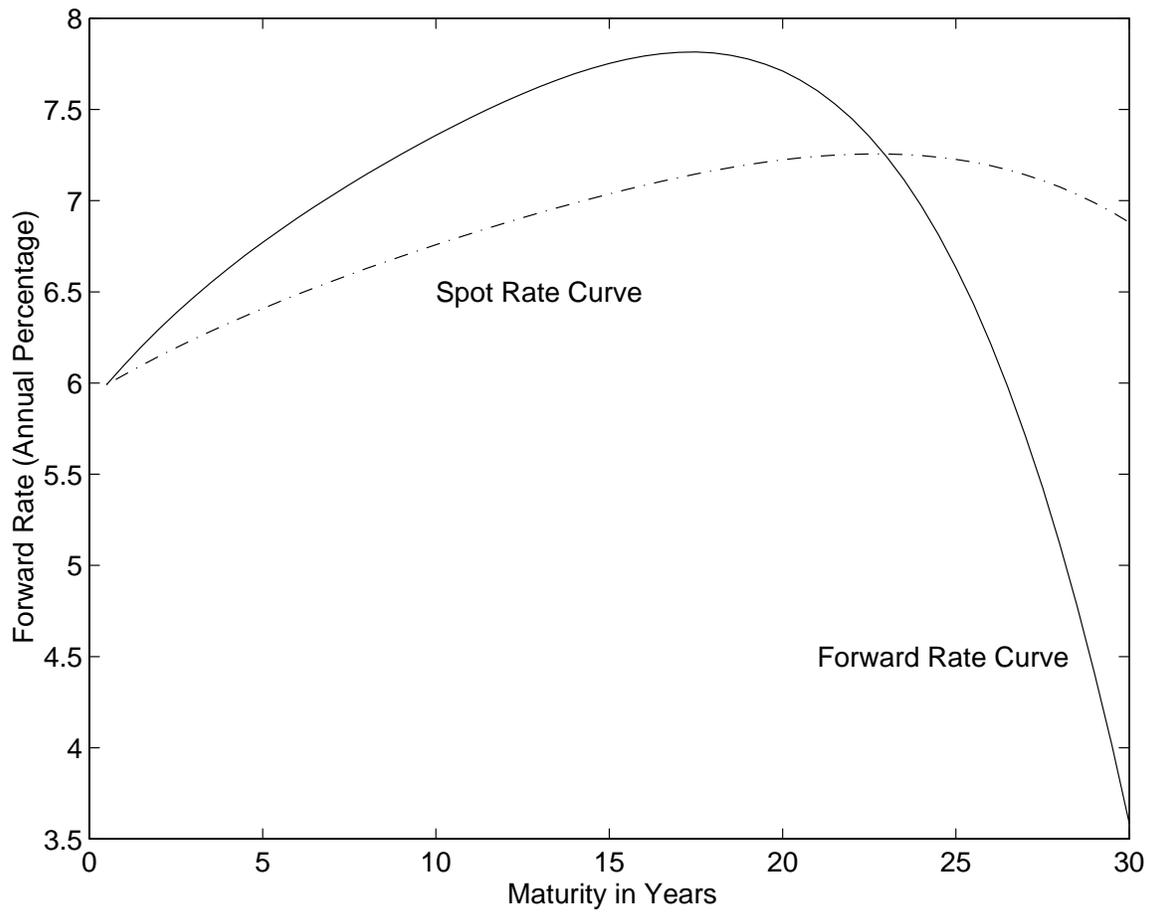
Chapter 3

Macrofoundations of Interest Rates

Maybe some other time!

Further Reading

See Chapter 2 of the Roubini-Backus macro notes:

Table 3.1
Properties of Yields and Forward Rates

Maturity (Months)	Mean	Std Deviation	Autocorrelation
		A. Yields	
1	6.683	2.703	0.959
3	7.039	2.781	0.971
6	7.297	2.774	0.971
9	7.441	2.725	0.970
12	7.544	2.672	0.970
24	7.819	2.495	0.973
36	8.009	2.373	0.976
48	8.148	2.287	0.977
60	8.253	2.224	0.978
84	8.398	2.141	0.979
120	8.529	2.073	0.981
		B. Forward Rates	
0	6.683	2.703	0.959
1	7.098	2.822	0.969
3	7.469	2.828	0.968
6	7.685	2.701	0.966
9	7.812	2.487	0.966
12	7.921	2.495	0.969
24	8.274	2.264	0.977
36	8.498	2.135	0.979
48	8.632	2.059	0.980
60	8.714	2.013	0.980
84	8.802	1.967	0.980
120	8.858	1.946	0.980

Entries are sample moments of continuously-compounded forward rates and yields constructed with data and programs supplied by Robert Bliss of the Atlanta Fed (Smoothed Fama-Bliss method). Maturity is measured in months. The data are monthly, January 1970 to December 1995 (312 observations).

Figure 3.1
Average Yield and Forward Rate Curves
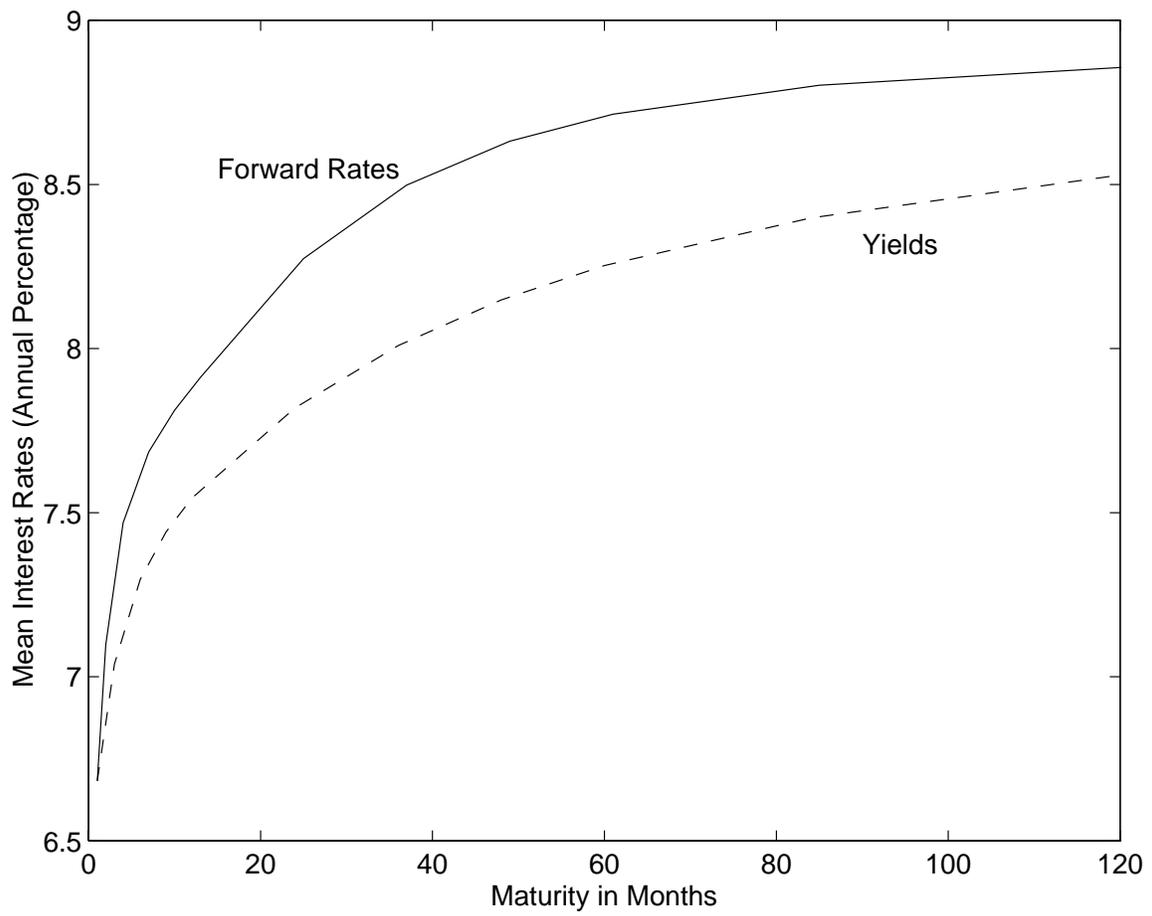
Figure 3.2
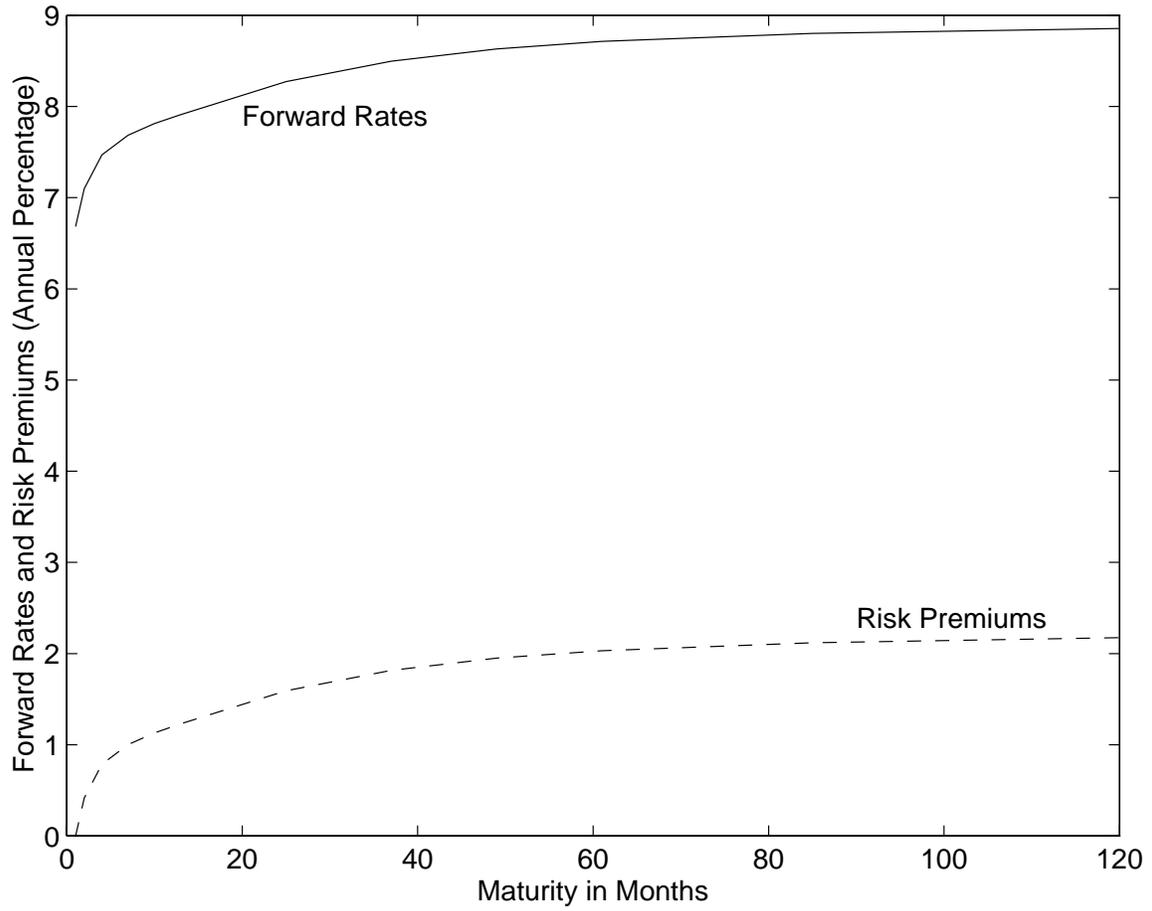**Average Forward Rates and Risk Premiums**

Figure 3.3
April 29, 1994: Forward Rates and Expected Short Rates
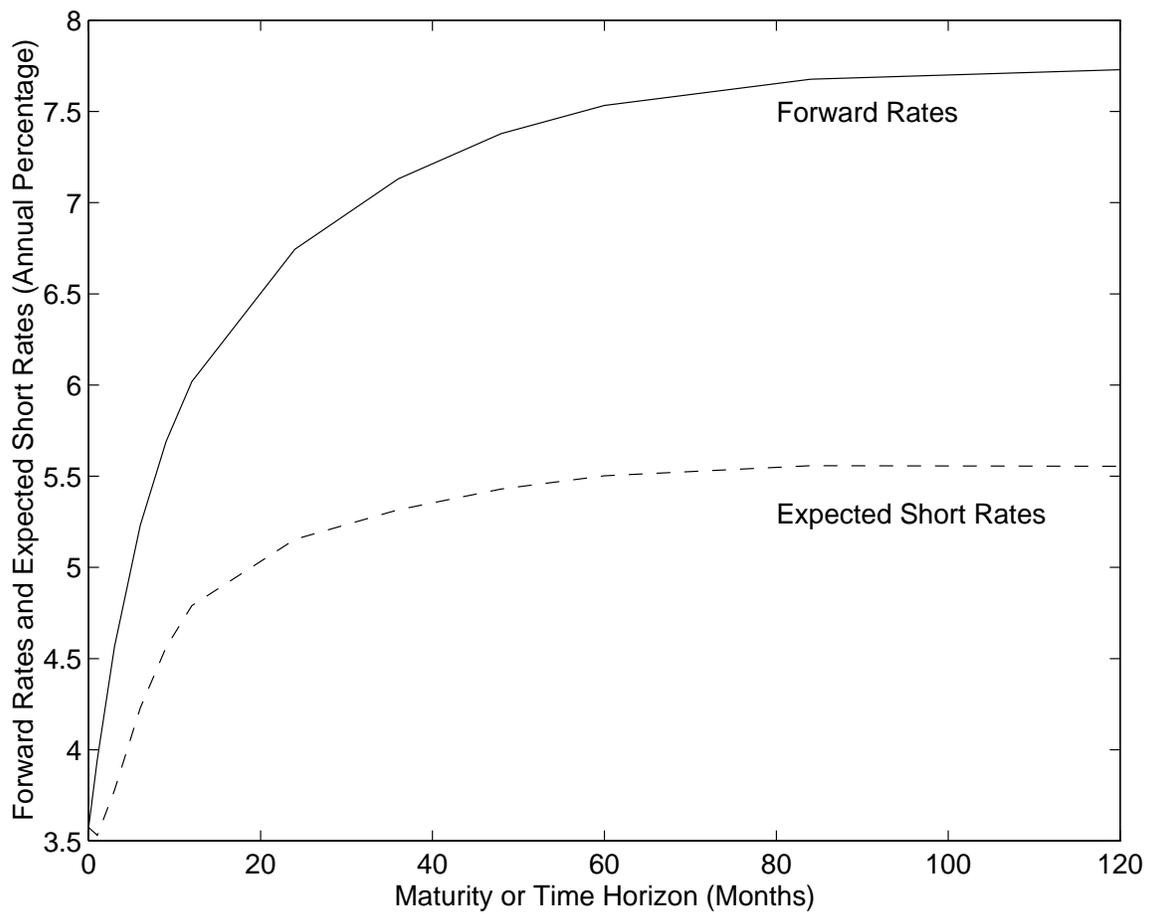
Chapter 4

Quantifying Interest Rate Risk

Prudent management and the fear induced by periodic derivatives disasters have spurred the development and widespread use of tools for managing financial risk. One of these tools is a system for quantifying risk — for answering the question: How much can we lose over the next day, week, or year?

We focus here on tools used to quantify interest rate risk. For bonds and related assets, there are two rules of thumb that cover the basics:

- When yields rise, bond prices fall.

- Prices of long bonds fall more than prices of short bonds.

The first is an obvious consequence of relations, like equations (2.1) and (2.10), connecting bond prices to yields. The second is based on long experience. This chapter is devoted to methods of translating this experience into quantifiable measures of risk.

Before we do this, it's useful to consider contexts in which such risk management tools might be used. A few that come to mind are:

- Investing pension funds. Some pension funds (this is less common than it used to be) are committed to paying benefits that are relatively easy to forecast, and that must be financed by investing current contributions. The question is how they invest these contributions to guarantee that they'll be able to satisfy claims for benefits.

- Managing interest rate risk at a financial institution. Commercial banks, for example, often borrow short and lend long. What kinds of risk does this expose them to?

- Managing a bond fund. How should a fund manager trade off the risk and return of various combinations of bonds?

- Treasury management at an industrial corporation. Should it issue long debt or short? Long debt has less uncertainty over future interest payments, but short debt is cheaper on average.

We'll come back to specific examples later in the course.

4.1   Price and Yield

Equations (2.6) and (2.10) connect bond prices to yields. We used them in the previous chapter to compute the yield associated with a particular price. Here we consider the effect on the price of a change in the yield.

In Figure 4.1 we graph the so-called *price-yield relation* for a two-year 10% bond selling at par. The graph is based on equation (2.6) for a bond with cash payments at even six month intervals. Consider, in general, a bond with n cash payments: c_1 in six months, c_2 is twelve months, and so on, through the final payment of c_n in $n \times 6$ months or $n/2$ years. Equation (2.6) — repeated here for emphasis — relates the price p to the yield y:

$$
\begin{aligned}
p &= \frac{c_1}{(1 + y/2)} + \frac{c_2}{(1 + y/2)^2} + \cdots + \frac{c_n}{(1 + y/2)^n} \\
&= \sum_{j=1}^{n} (1 + y/2)^{-j} c_j.
\end{aligned}
\tag{4.1}
$$

For the two-year bond in the figure, the price is 100 if the yield is 10%. But if the yield rises, the price falls below 100. And if the yield falls, the price rises above 100. Uncertainty over future bond yields thus translates into uncertainty over the value of the bond.

In Figure 4.2 we add a second bond: a five-year bond with the same coupon also selling at par. We see there that its price-yield relation is steeper: If two-year and five-year bond yields rise by the same amount, the price of the five-year bond falls by more than the price of the two-year bond.

Price Value of a Basis Point

One way to quantify the sensitivity of the price of a bond to changes in the yield is with the *price value of a basis point* or PVBP (sometimes called the "dollar value of an 01" or

DV01): the fall in price associated with a rise of one basis point in the yield. If the current price is p_0 and the price at the new yield is p, then

$$\text{PVBP} \quad = \quad -(p - p_0). \tag{4.2}$$

Since a rise in the yield is associated with a fall in the price, the minus sign gives us a positive PVBP for long bond positions.

A few examples should make this clear. All of them are based on a flat spot rate curve at 10% per year.

Example 1 (two-year 10% par bond). At $y_0 = 0.1000$ the bond sells for $p_0 = 100$. At $y = 0.1001$ the bond sells for $p = 99.9823$. PVBP $= 100.00 - 99.9823 = 0.0177$. Thus a rise in yield of one basis point results in a loss on the bond of 1.77 cents.

Example 2 (five-year 10% par bond). At $y_0 = 0.1000$ the bond sells for $p_0 = 100$, at $y = 0.1001$ the bond sells for $p = 99.9614$. PVBP $= 100.00 - 99.9614 = 0.0386$. Note that the longer bond is more sensitive to yield changes.

Example 3 (two-year zero). At $y_0 = 0.1000$ the bond sells for $p_0 = 100/1.05^4 = 82.2702$, and at $y = 0.1001$ the bond sells for $p = 82.2546$, so PVBP $= 0.0157$.

Example 4 (ten-year zero). At $y_0 = 0.1000$ the bond sells for $p_0 = 100/1.05^{20} = 37.6889$, and at $y = 0.1001$ the bond sells for $p = 37.6531$, so PVBP $= 0.0359$.

Duration

A closely related measure of price sensitivity is *duration*, a term that alludes to the close connection between the price sensitivity and maturity of a bond. We define duration as

$$
\begin{aligned}
D \quad &= \quad - \frac{\text{Slope of Price-Yield Relation}}{\text{Price}} \\
&= \quad - \frac{dp/dy}{p}.
\end{aligned}
\tag{4.3}
$$

Since the price-yield relation in Figure 4.2 is steeper for the five-year bond, it evidently has a longer duration. Where the PVBP gives us the absolute decline in value associated with a given rise in yield, duration gives us the proportionate decline in value. Thus a bond with duration D declines in value by D percent if the yield rises by 1 percent or 100 basis points.

We can be more specific about the slope if we are more specific about the relation between price and yield. For bonds with semiannual payments we use equation (4.1). The

derivative in (4.3) is then

$$\frac{dp}{dy} = -\frac{1}{2}(1+y/2)^{-1}\sum_{j=1}^{n} j \times (1+y/2)^{-j}c_j$$

$$= -(1+y/2)^{-1}\sum_{j=1}^{n}(j/2) \times (1+y/2)^{-j}c_j.$$

(This should be clear if you know calculus, but if you don't we'll come to something more transparent shortly.) The one-half here comes from the semiannual compounding, and converts semiannual time intervals j into annual units. Duration is therefore

$$D \equiv \frac{dp/dy}{p} = -(1+y/2)^{-1}\sum_{j=1}^{n}(j/2) \times w_j, \qquad (4.4)$$

$$\text{with} \qquad w_j = \frac{(1+y/2)^{-j}c_j}{p}.$$

The weights w_j give the proportion of value due to the jth payment when the yield is used to discount future payments. From (4.1) we can show that they sum to one, so in that sense they really are weights. The summation in (4.4) is therefore a weighted average of the lives of the payments, measured in years.

This may be a little hard to swallow the first time through, so let's turn to our examples.

Example 1 (two-year 10% par bond). The timing and size of the cash flows are summarized in the first two columns:

Payment Number (j)	Cash Flow (c_j)	Present Value	Weight (w_j)
1	5	4.762	0.04762
2	5	4.535	0.04535
3	5	4.319	0.04319
4	105	86.384	0.86384

The weights are computed according to the formula in (4.4). The first one, for example, is

$$w_1 = \frac{(1+.10/2)^{-1}5}{100} = 0.04762,$$

as noted. Given the weights, we compute duration as

$$D = (1+0.10/2)^{-1}(0.5 \times 0.04762 + 1.0 \times 0.04535 + 1.5 \times 0.04319 + 2.0 \times 0.86384)$$
$$= 1.77 \text{ years}.$$

We would say that this two-year bond has a duration of 1.77 years, meaning that a rise in its yield of one percent would result in a decline in its value of 1.77 percent.

Thus a rise of 100 basis points would result in a loss of (approximately) 1.77 percent of the bond's value, or \$1.77.

Example 2 (five-year 10% par bond). Similar methods give us $D = 3.86$ years. Given a uniform change in bond yields, the price of the five-year bond would fall by more than twice that of the two-year bond.

Example 3 (two-year zero). Since zeros have only one cash flow, equation (4.4) reduces to

$$D = (1 + .10/2)^{-1}(n/2),$$

where n is the maturity measured in six-month periods. For the two-year zero and spot rates of 10%, $n = 4$ and $D = 1.90$ years. Note that the two-year zero has (slightly) longer duration than the two-year coupon bond: The coupons give the bond a shorter average life.

Example 4 (ten-year zero). Here $n = 20$ and $D = 9.51$ years.

4.2 More on Duration

We now have two measures of the sensitivity of bond prices to yield changes, the PVBP and duration. The two contain, as we will see, the same information. Duration is more common in practice, and is often the first line of attack in measuring, reporting, and even regulating interest rate risk.

This section is devoted to miscellaneous features of duration that you might run across in applications.

Duration and PVBP

PVBP and duration are two different ways of reporting essentially the same information. Since a basis point or "01" is a small change, PVBP is essntially

$$\text{PVBP} = \frac{dp}{dy} \times 0.01\% = \frac{dp}{dy} \times 0.0001.$$

If we compare this to (4.3) we see that both PVBP and duration are based on the slope of the price-yield relation.

Duration with Uneven Payment Dates

Our measure of duration was derived for an instrument with fixed payments at even six-month time intervals. For bonds between coupon dates, we use equation (2.10) relating price to yield to compute duration. The result is

$$D = \frac{dp/dy}{p} = -(1+y/2)^{-1} \sum_{j=1}^{n} [(j-1+w)/2] \times w_j, \qquad (4.5)$$

$$(4.6)$$

$$\text{with} \qquad w_j = \frac{(1+y/2)^{-j+1-w} c_j}{p},$$

where w is the fraction of a semiannual period remaining until the next coupon date and p is the invoice price. When $w = 1$ this reverts to equation (4.4). Except for the fractions, the method is the same as before. Bloomberg uses this method to compute duration for bonds with semiannual coupons, applying the relevant day-count convention for w.

Similar extensions apply to other payment patterns and compounding intervals.

This expression is most useful, in my view, for allowing us to reproduce (and therefore understand) calculations reported by Bloomberg and other financial information services. Consider:

Example 5 (five-year 5% US Treasury note due 1/31/99). On July 15, 1996, Bloomberg quoted an ask price of "96-22" for settlement the next day. With 15 days until the next coupon, the invoice price is 98.9815, $w = 15/182$, $y = 6.432\%$, and $D = 2.286$ years.

Duration of Portfolios

One of the nice features of PVBP and duration is that they are additive. The PVBP of a portfolio of bonds, for example, is the sum of the PVBPs of the individual bonds. And the duration of a portfolio of bonds is the weighted average of the durations of the individual bonds. To make this specific, consider a collection of positions in m securities in which the values of the individual positions are $(v_1, v_2, \ldots, v_m)$ and total value $v = v_1 + v_2 + \cdots + v_m$. Let us say that the jth position has duration D_j. Then the duration of the portfolio is

$$D = \sum_{j=1}^{m} D_j \times w_j, \qquad (4.7)$$

$$\text{with} \quad w_j = \frac{v_j}{v}.$$

The logic is similar to that used for duration itself: the sensitivity of the portfolio is a weighted average of the sensitivities of the components. The primary difference is that the components in (4.4) were individual payments, while the components here are securities.

We return to our examples.

Example 6 (barbell). Consider a portfolio consisting of 100 two-year and 100 five-year 10% bonds trading at par, as in examples 1 and 2 above. The portfolio is referred to as a *barbell* because a graph of the cash flows over time has two spikes (the principal payments of the two bonds). Since both bonds sell for 100, we have fifty percent of our portfolio in each bond. The duration of the portfolio is therefore

$$D = 1.77 \times 0.5 + 3.86 \times 0.5 = 2.82,$$

the average of the two.

Example 7 (another barbell). A similar example is a combination of two-year and ten-year zeros constructed to have the same duration as the five-year bond in example 2. If we invest fraction w_1 in the two-year zero and $w_2 = 1 - w_1$ in the ten-year zero, w_1 satisfies

$$1.90 \times w_1 + 9.51 \times w_2 = 3.86.$$

Thus $w_1 = 0.743$.

Alternative Duration Measures (optional)

One variant of duration eliminates the initial term in equation (4.4), which we might write more simply as

$$D = (1 + y/2)^{-1} \times \text{Weighted Average Life}.$$

In common parlance, the second term (weighted average life) is referred to as *Macaulay duration* after the person who orginated this line of analysis. D itself is then referred to as *modified duration* to distinguish it from Macauley. Since yields are typically small numbers, the difference between the two measures is typically small.

A second variant is the so-called *Fisher-Weil duration*, which uses spot rates rather than yields to compute the weights w_j in (4.4). This makes no difference in two cases: if we are looking at zeros or if the spot rate curve is flat. On the whole this seems like a sensible thing to do, but it requires more information (we need the whole spot rate curve rather than just the yield to maturity of the bond) and does not often give significantly different estimates.

Duration as a Local Approximation

Duration is a local approximation in two senses. One is that it is based on the slope of the price-yield relation *at a specific point*. If we choose a different point, we typically get a different answer. You can see this in Figures 4.1 and 4.2 if you look carefully: the lines are slightly curved. We say that duration is based on a linear or straight line approximation. To see what this approximation is, return to the definition. If we consider the slope dp/dy at the point (y_0, p_0), and let $dp = p - p_0$ and $dy = y - y_0$, equation (4.3) implies

$$p - p_0 = -D \times p_0 \times (y - y_0), \qquad (4.8)$$

a linear relation between price p and yield y. The general mathematical theory tells us that linear approximations are fairly good for small changes, but possibly less good for large ones. We see just that in Figure 4.3, where the approximation gets worse as the yield gets farther from y_0.

The other sense in which duration is a local approximation concerns time. Taken literally, our analysis ignores the passage of time: duration applies to a bond of fixed maturity. In practice, the maturity declines as time passes. For short time intervals, this doesn't make much difference, but if we are measuring risk over intervals that are close to the maturity of the instruments, duration may be misleading. An extreme case is the risk of a six-month zero over six months. Its duration is approximately six months, indicating some exposure to yield changes. But since the zero is worth exactly 100 in six months, there is no risk in this instrument over this time period.

4.3   Immunization

Now that we have established duration as a measure of exposure to interest rate risk, we can turn to ways of controlling that risk. We consider two approaches to this problem in the context of two examples. Cash flow matching serves to illustrate an alternative and highlight the imperfections in duration as a measure of interest rate risk.

Cash Flow Matching

There are cases in which an insurance company or pension fund is committed to making a series of known payments. Let us say, to make this specific, that the Balduzzi Insurance and Pension Corporation has arranged to pay the retired employees of NYU the following amounts over the next four half-year intervals:

Time (Yrs)	Payment
0.5	2000
1.0	1900
1.5	1800
2.0	1700

(This is much shorter than we would see, but makes the arithmetic easier.) The idea is that the payments decline as retirees die off. Companies generally have a good idea about mortality, so we are not violating the spirit of the problem very much by assuming that the amounts are known.

There are a number fo ways Balduzzi could arrange to make sure it had the right amount of money on hand to pay the retirees. One of the more straightforward is to set up a "dedicated" portfolio of investments designed to cover these payments — dedicated to making the required payments. In this case, Balduzzi would like to know how much money is needed now to cover these payments and how the money should be invested. The list of possible investments consists of the following bonds:

Bond	Coupon (%)	Maturity (Yrs)	Price (Ask)	Yield (%)
1	0	0.5	96.618	7.000
2	0	1.0	92.456	8.000
3	0	1.5	87.630	9.000
4	0	2.0	83.856	9.000
5	4	1.0	96.750	7.432
6	8	2.0	97.500	9.400

With four payements and six bonds, we have some room for choice.

In the simplest dedicated portfolio, Balduzzi buys combinations of bonds whose cash flows exactly match those of the required payments. This subjects Balduzzi to absolutely no interest rate risk, since the bonds are all held to maturity. If we did this with zeros, we would buy 20 six-month zeros, 19 twelve-month zeros, 18 eighteen-month zeros, and 17 twenty-four-month zeros, for a total cost of

$$\text{Cost} = 20 \times 96.618 + 19 \times 92.456 + 18 \times 87.630 + 17 \times 83.856 = 6691.91.$$

This covers all of the cash flows exactly: we simply take the principal payments as they occur and redirect them to the pensioners.

In a frictionless, arbitrage-free world there would be no difference in cost between this and any other combination of bonds. But in the real world there might be. If we consider the problem of finding the least-cost combination of investments that covers the required

payments, we have a linear program like that described in Section **??**: choose the cheapest combination of bonds that delivers the required cash flows. We can almost guess the answer in this case. Relative to the zeros, bond 5 is overpriced (we can reproduce its cash flows more cheaply with the zeros) but bond 6 is underpriced. Therefore we buy as many units of bond 6 as we need to cover the cash flows in the final period (16.35) and make up the remaining cash flows with zeros. Our investments are then

Bond	Price	Units
1	96.618	19.35
2	92.456	18.35
3	87.630	17.35
4	83.856	0.00
5	96.750	0.00
6	97.500	16.35

The total cost is 6679.19, which is slightly cheaper than the portfolio of zeros.

This approach is, in many respects, ideal. We simply buy the appropriate bonds and transfer the coupons and principal from our investments to the retirees. It requires no analysis of interest rate risk or of current bond market developments. As Ned Elton puts it: "We make the investment and head to the Bahamas to relax."

Duration Matching

The difficulty with cash flow matching is that it applies to an ideal world. We often find that we do not have all the required bond maturities, or that the number of maturities is so great that we incur enormous transactions costs to match them all exactly. An alternative is to rely on a specific measure of interest rate risk, like duration and choose a portfolio equal in value and duration to the required payments. Matching duration is less restrictive than matching cash flows, and provides some protection against adverse movements in interest rates.

To see how one might do this, consider the Foresi Bank of Mahopac (FBM). FBM has assets of $25b, liabilities of $20b, and shareholders' equity of $5b. The market values of both assets and liabilities are sensitive to movements in interest rates. Assets (largely business loans) have an overall duration of 1.0 years, largely the result of customer preferences: some customers prefer floating rate loans, others fixed rate, with a duration of 1.0 summarizing the average. Liabilities, on the other hand, can be managed by the bank. Of the $20b total, currently $10b is in six-month commercial paper and $10b in 2-year notes just issued at par. With a flat spot rate curve at 10%, the commercial paper has a duration of 0.48 and

the 2-year notes have a duration of 1.77 (see example 1). The duration of liabilities, then, is

$$D_L = 0.48 \times 0.5 + 1.77 \times 0.5 = 1.12.$$

Thus we see that the bank is exposed to increases in interest rates: a generalized increase in rates will reduce the value of both assets and liabilities, but since the former is larger and has greater duration the bank loses overall.

What should the bank do? One possibility is to convert some of the commercial paper to 2-year notes. But how much? FBM would like for a change in interest rates to change the value of assets and liabilities by the same amount. Equation (4.8) tells us that the change in value of assets is approximately

$$\text{Change in Asset Value} = -D_A \times 25b \times \Delta y,$$

where D_A is the duration of assets and Δy is the change in the yield. Similarly the change in the value of liabilities is

$$\text{Change in Liability Value} = -D_L \times 20b \times \Delta y,$$

where D_L denotes the duration of liabilities. With $D_A = 1.0$, the two changes will be equal if we set $D_L = D_A(25/20) = 1.25$. Note specifically that we do not equate the duration of assets and liabilities. The liabilities, in this case, must have greater interest rate sensitivity since they have smaller overall value.

The next step is to rearrange the liabilities to come up with an overall duration of 1.25. If we issue fraction w_1 of commercial paper and $1 - w_1$ of two-year notes, we need

$$0.48 \times w_1 + 1.77 \times (1 - w_1) = 1.25.$$

This implies that we issue $w_1 \times 20 = 0.40 \times 20 = 8b$ in paper and 12b in notes.

Thus we have given the bank a more neutral interest rate posture. There are a couple of fine points worth noting. One is that this kind of duration matching reduces the bank's exposure to equal changes in yields of all maturities. It should be apparent, for example, that we assumed that Δy was the same for both assets and liabilities, which need not generally be true if they pertain to different maturities. Second, liability management of this sort is an ongoing process. As time goes on, the durations of assets and liabilities change: Even if the bank does nothing, long maturities get shorter all the time. For this reason, most financial institutions redo the calculations frequently, at least monthly and some even daily.

4.4   Convexity (optional)

Convexity is an attempt to go beyond the linear approximation of the price-yield relation used in PVBP and duration. I don't find it particularly useful, but it provides us with a good opportunity to discuss the major weakness of duration as a measure of price sensitivity.

The idea behind convexity is that the price-yield relation isn't linear: in Figure 4.3 and more generally, the price-yield relation for a bond is convex, meaning it curves away from the origin. Convexity, as the term is used in finance, is a measure of the curvature.

Convexity Formulas

This is one of those issues that is most easily addressed mathematically. We can try to translate it into words later on. We define *convexity C* as

$$C \equiv \frac{d^2p/dy^2}{p} = (1+y/2)^{-2}\sum_{j=1}^{n}[(j-1+w)(j+w)/4] \times w_j, \qquad (4.9)$$

$$\text{with} \qquad w_j = \frac{(1+y/2)^{-j+1-w}c_j}{p}.$$

The term "d^2p/dy^2" means the second derivative of price with respect to yield. The formula applies to a semiannual bond with fraction w of a period left until the next coupon payment. If there is exactly six months to the first payment, $w = 1$ and convexity is

$$C = (1+y/2)^{-2}\sum_{j=1}^{n}[j(j+1)/4] \times w_j,$$

$$\text{with} \quad w_j = \frac{(1+y/2)^{-j}c_j}{p}.$$

Note that both imply positive values of C when cash flows c_j are positive, indicating that convexity is always positive: the curvature away from the origin of the price-yield relation is a general feature of bonds. The only issue is how much.

The convexity of a portfolio, like its duration, is a weighted average of its components. The analog of equation (4.7) is

$$C = \sum_{j=1}^{m} C_j \times w_j, \quad \text{for} \quad w_j = \frac{p_j}{p},$$

where p is the value of a portfolio consisting of investments p_j in various assets.

To see what this means, consider our examples:

Example 1 (two-year 10% par bond). Since we have even time intervals, $w = 1$. The weights w_j are the same ones we used to compute duration earlier. And since we have an even first time interval, $w = 1$. Convexity is then

$$
\begin{aligned}
C &= (1 + 0.10/2)^{-2}(0.5 \times 0.04762 + 1.5 \times 0.04535 + 3.0 \times 0.04319 + 5.0 \times 0.86384) \\
&= 4.12.
\end{aligned}
$$

Example 2 (five-year 10% par bond). Convexity is 18.74, a quantitative indication that the price-yield relation is more curved for the five-year bond than the two-year bond (see Figure 4.2).

Example 3 (two-year zero). Convexity is 4.53.

Example 4 (ten-year zero). Convexity is 95.14.

Example 5 (five-year 5% US Treasury note due 1/31/99). We need first to adapt the convexity formula to an uneven first period, as we did with duration. I leave the details to the reader. The result gives us convexity $C = 6.62$. Bloomberg divides this by 100 and reports 0.066.

Example 6 (barbell). The convexity of the portfolio of two bonds is $4.12 \times 0.5 + 18.74 \times 0.5 = 11.43$.

Example 7 (another barbell). Convexity is $4.53 \times 0.743 + 95.14 \times 0.257 = 27.81$. Although this portfolio has the same duration as the five-year bond in example 2, it has greater convexity. We see this graphically in Figure 4.4. The reason is that its cash flows are more spread out.

4.5 Fixed Income Funds

Fixed income mutual funds are, in aggregate, one of the largest purchasers of fixed income securities and a major component of most individual and institutional portfolios. Relative to equity funds, fixed income funds generally have lower expected returns. Their returns also tend to be less volatile (they have a smaller standard deviation) and are not very highly correlated with equity returns, which makes them a useful part of a diversified portfolio. Most financial experts suggest that investors with long horizons emphasize equity (since the expected return is higher), but those with short horizons should invest a substantial fraction in fixed income securities (diversification).

Fixed income funds vary considerably, however, in their composition. A typical range of possibilities is illustrated by the Dean Witter fixed income funds, a subset of which is described in Table 4.1. Federal Securities is a general bond fund investing in US government securities, probably including agency issues as well as treasuries. The duration on June 30, 1996, was 5.2 years. Many such funds vary the duration depending on the managers' view of interest rates. If they expect yields to rise, they shorten the duration by seeling long bonds and buying shorter ones. If they expect yields to fall, they do the reverse. For investors who would like less sensitivity to interest rate movements, the Intermediate- and Short-Term US Treasury funds offer durations of 4.3 and 1.6, respectively. Diversified Income invests in corporate bonds as well as governments, which generally raises the yield but exposes the fund to credit risk (the risk that the bond issuer will default). High Yield Securities focuses on corporate issues with higher yields and greater credit risk. Premier Income invests in mortgage securities; adjustable rate mortgages tend to have higher yields than treasuries, but because the rate adjusts the sensitivity to interest-rate movements is modest (duration 1.9). Finally, the World Wide Income fund invests in foreign currency denominated, as well as US dollar denominated, fixed income securities. The fund thus exploits income opportunities in other markets, but in return may be exposed to movements in currency prices. Funds differ in whether they hedge this exposure.

4.6   Statistical Measures of Price Sensitivity (optional)

Let us now take a completely different approach to measuring exposure to interest rate risk. Many of you would guess, I think, that to find the risk of a portfolio you would want to start with the variances and covariances or correlations of the returns for all the assets in which you have positions. You would then apply the principles of portfolio theory to compute (say) the standard deviation of the overall return. This standard deviation is a statistical measure of the risk you face.

What we have done so far differs substantially from this. The premise behind duration analysis is that yields of all maturities change by the same amount. In statistical language, we would say that yields (i) are perfectly correlated and (ii) have the same standard deviation. Neither is a bad approximation, but then again, neither is true either. We know, for example, that the yield curve occasionally moves in complex ways, like the sharp twist between August 1994 and August 1995 in which short rate rose and long rates fell (Figure 4.5). Examples like this tell us that yields are not perfectly linked. We see the same thing in Table 4.2, which lists standard deviations and correlations for one-month yield changes for five maturities between one and ten years. We see that the standard deviation declines with maturity, and that correlations are generally strong but less than one.

As a result, most quantitative risk analysis — for fixed income securities or otherwise — is based on the statistical properties of returns. I want to give you the feel for this without

belaboring the mathematics. This runs the risk of being imprecise, but I'd like to focus on the ideas rather than the math.

Comparison with Duration

We can see the difference between statistical methods and traditional duration with examples.

One example is a comparison of the risk in two- and ten-year zeros. With duration, the possible proportionate price change for an arbitrary bond is

$$\frac{\Delta p}{p} = -D \times \Delta y,$$

so the standard deviation is

$$\text{Std}\left(\frac{\Delta p}{p}\right) = D \times \text{Std}\left(\Delta y\right). \tag{4.10}$$

In duration analysis, we assume that changes in yields are the same for all maturities, so the only thing that differs across positions is duration. Thus if the two-year zero has a duration of 1.9 and the ten-year zero has a duration of 9.5, we would say that the latter is about 5 times more risky.

Statistical measures tell us, instead, that the ten-year zero is more risky than the two-year, but not by a factor of five. If we use equation (4.10) again to convert variation in yields to variation in price, we might compute the standard deviation of the ten-year zero from

$$\text{Std}\left(\frac{\Delta p_{10}}{p_{10}}\right) = D_{10} \times \text{Std}\left(\Delta y_{10}\right)$$

where the subscript "10" refers to the ten-year zero. The result is $9.5 \times 0.309\% = 2.8\%$.

Now consider the two-year zero. The standard deviation of the two-year zero isn't reported in Table 4.2, but we can estimate it by taking the average of the one- and three-year zeros — say 0.494%. Then the standard deviation of the relative price change is about 0.94%. Using this method, the ten-year zero is only about 3 times more risky than the two-year. The reason for this difference is that the ten-year spot rate is less variable than the two-year spot rate, something duration analysis doesn't consider.

The other difference between statistical measures and duration analysis concerns correlations. Things like the barbells of examples 6 and 7 have less volatility than duration might indicate, because their exposure to different segments of the spot rate curve results in a small amount of diversification: the standard deviation of the position is less than the the sum of the standard deviations of the individual positions. The reason is that spot rates of different maturities are imperfectly correlated.

JP Morgan's RiskMetrics

JP Morgan announced RiskMetrics, a commercial risk management product, several years ago in an apparent attempt to define the standards in a rapidly evolving field. They supply daily and monthly standard deviations and correlations for returns in more than twenty countries and covering fixed income securities, foreign exchange, and equities. Their approach shares a number of features with the statistical methods just outlined, but a number of differences are worth noting:

- Standard deviations and correlations are updated frequently to reflect changes in market volatility.

- Yield volatilities pertain to proportional changes. In Table 4.2, statistics are based on yield changes:
$$\Delta y = y_t - y_{t-1}.$$
  In RiskMetrics they are based on proportional changes,
$$\log\left(y_t/y_{t-1}\right) \approx \frac{y_t - y_{t-1}}{y_{t-1}},$$
  which makes the link between yield volatility and price volatility less clear. Perhaps for this reason, they also report price volatilities.

- Yields are reported for a more extensive set of maturities: 1 day; 1 week; 1, 3, and 6 months; and 1-5, 7, 9, 10, 15, 20, and 30 years. Other maturities are handled by a more sophisticated interpolation than the one used above.

Shifts and Twists

One last issue to tie things together. You might be asking yourself at this point: How much do I miss by using duration analysis? Duration analysis is based on the assumptions, we might recall, that yield changes have the same standard deviation and are perfectly correlated. The former is certainly wrong: standard deviations decline substantially with maturity. But the latter doesn't seem so bad: correlations are less than one, but they are generally large.

With a little more statistical technology, we can be more specific about this. The method of principal components (think of this as a name and don't worry about the details) decomposes the variance-covariance matrix for yield changes into uncorrelated factors. The two most important components are listed in Table 4.3 and graphed in Figure 4.6. The first is almost a parallel shift in the yield curve, although there is a distinct tendency for long yields to move less than short yields (a reflection of the tendency for long yields to

vary less). This component accounts for 89% of the variance of the one-year spot rate and about 82% of the variance of the ten-year spot rate. The second component is a "twist:" a decline in short rates and rise in long rates. It accounts for 10% of the one-year rate and 14% of the ten-year rate.

I summarize the behavior of yield changes this way: Yield changes are 90% parallel shifts and 9% twists, with about 1% left over.

Summary

1. Bond prices fall when yields rise.

2. Prices of long bonds react more to a given change in bond yields than prices of short bonds.

3. The present value of a basis point and duration measure the sensitivity of bond prices to yield changes. Duration is proportional to the weighted average life of the bond.

4. Cash flow matching eliminates all exposure to interest rate movements.

5. Duration matching, or immunization, eliminates exposure to parallel shifts in the yield curve. Moreover, an immunized portfolio may not remain immunized: duration changes as time goes by.

6. Risk management is moving toward statistical measures of risk: overall risk depends on the variances and covariances of the individual assets or "risk factors." For fixed income securities, the factors are related to spot rates for different maturities.

7. Statistical measures indicate that (i) long yields are less variable than short yields and (ii) the yield curve exhibit twists as well as shifts. In less technical language: the presumption of parallel shifts in duration analysis is 80-90% right.

Practice Problems

1. Suppose the spot rate curve is

Maturity (Years)	Spot Rate (%)
0.5	5.000
1.0	6.000
1.5	7.000
2.0	8.000

Compute the yield, PVBP, and duration of (a) a two-year zero, (b) a two-year 5% bond, and (c) a two-year 10% bond. Explain why they differ.

2. For a bond of your choice, reproduce Bloomberg's day count, invoice price, yield, PVBP, duration, and convexity calculations.

3. Consider ten percent bonds, all trading at par, of different maturities n measured in six-month intervals. For $n = 4, 10, 20, 30$, compute duration. How does duration vary with n? Comment.

4. Consider the spot rate curve:

Maturity (Years)	Spot Rate (%)
0.5	2.000
1.0	4.000
1.5	6.000
2.0	7.000

   (a) Use the spot rates to compute the price of a two-year 10% bond. What is its yield?

   (b) Compute the duration of the two-year bond.

   (c) Construct a portfolio of zeros that replicates the two-year bond. Use the durations of the zeros and the formula for the duration of a portfolio to compute the duration of the two-year bond.

   (d) Explain the difference between your two duration calculations.

5. We explore the difference between cash flow matching and duration matching for a stylized pension manager who must fund a series of payments. The payments and the prices of zeros associated with the same dates are:

Date (Years)	Payment	Zero Price
0.5	1000	97.50
1.0	1500	95.00
1.5	2500	92.00
2.0	1500	89.20

   (a) Describe the portfolio of zeros that delivers exactly the cash flows required by the pension fund manager.

   (b) Use this portfolio and the prices of zeros to compute the present value of the pension fund's liabilities. Use the durations of the zeros to compute its duration.

   (c) Suppose the pension manager decided to fund the payments using a combination of only 0.5- and 2.0-year zeros. What combination of these two instruments has the same value and duration as the fund's liabilities?

(d) What changes in the spot rate curve would reduce the value of the portfolio below that of the liabilities it is intended to finance? Use your answer to comment on the strengths and weaknesses of cash flow matching and duration matching ("immunization").

Further Reading

The material on duration and convexity is boilerplate. Textbook references include Fabozzi (1996), Garbade (1996), and Tuckman (1995). On RiskMetrics, visit the JP Morgan home page and follow directions: http://www.jpmorgan.com/. The documentation is first-rate, but moderately verbose and technical.

Table 4.1
Dean Witter's Fixed-Income Mutual Funds

Information reported in Dean Witter InterCapital Inc., *Fixed-Income Mutual Funds*, 1996.

Fund	Objective	Lipper Category	Duration
Federal Securities	high level of current income	general US govt	5.2
Intermediate-Term US Treasury	current income	intermediate US tsy	4.3
Short-Term US Treasury	current income, preservation of principal and liquidity	intermediate US tsy	1.6
Diversified Income	high level of current income	general bond	(na)
High Yield Securities	high level of current income; capital appreciation secondary	high current yield	(na)
Premier Income	high level of current income consistent with low volatility	adjust rate mortgage	1.9
World Wide Income	high level of current income; capital appreciation secondary	general world income	4.7

Table 4.2
Statistical Properties of Monthly Yield Changes

Statistics are based on monthly changes in spot rates for US Treasury securities computed by McCulloch and Kwon. The data cover the period January 1952 to February 1991 (470 observations).

	1-Year	3-Year	5-Year	7-Year	10-Year
A. Standard Deviations of Yield Changes (Percent)					
	0.547	0.441	0.382	0.340	0.309
B. Correlations of Yield Changes					
1-Year	1.000	0.920	0.858	0.800	0.743
3-Year		1.000	0.967	0.923	0.867
5-Year			1.000	0.980	0.930
7-Year				1.000	0.970

Table 4.3
Principal Components of Monthly Yield Changes

Statistics are based on the same data and sample period as Table 4.1.

	1-Year	3-Year	5-Year	7-Year	10-Year
	A. Contribution to Yield Changes				
Component 1	0.585	0.494	0.424	0.366	0.317
Component 2	−0.721	−0.003	0.285	0.420	0.471
	B. Percent of Variance Accounted for				
Component 1	89.1	97.2	95.3	89.8	81.6
Component 2	10.5	0.0	3.3	9.2	14.0

Figure 4.1
Price-Yield Relation for Two-Year Bond
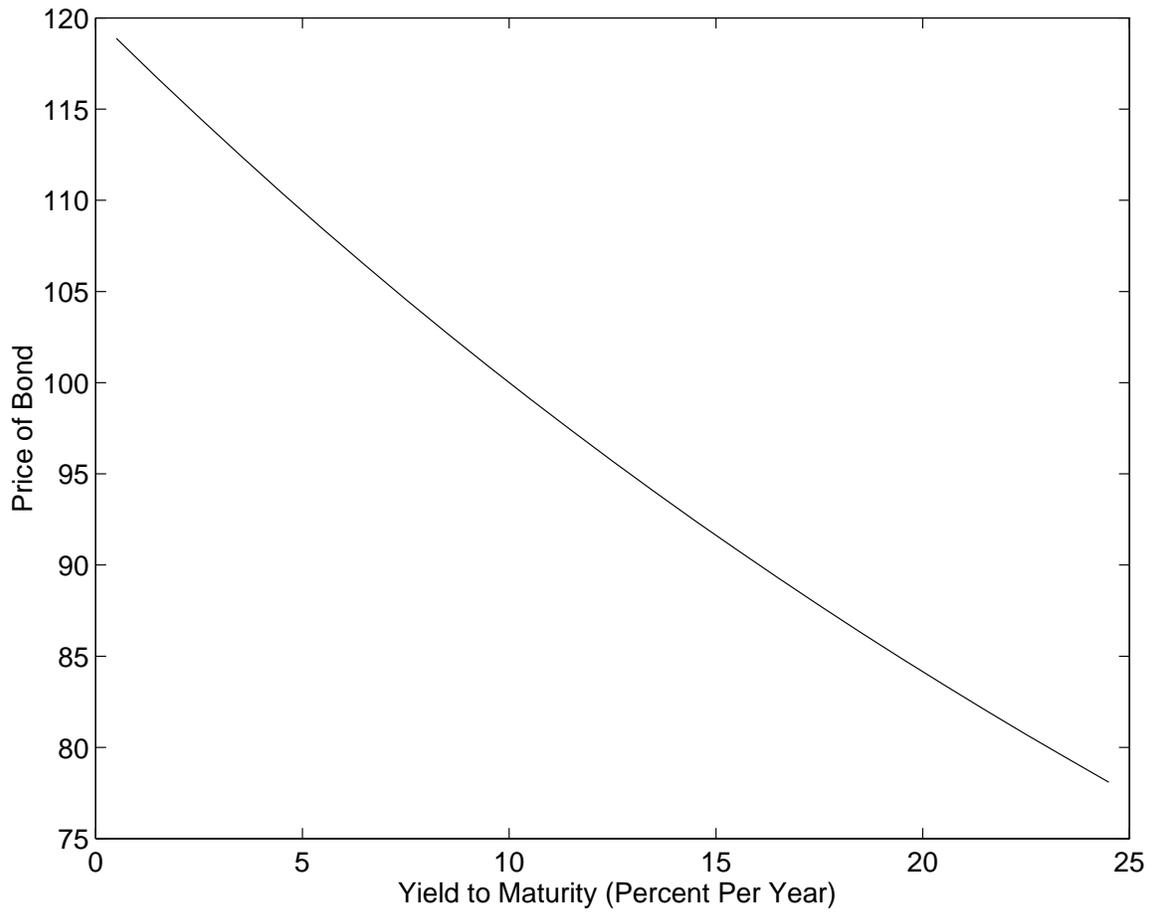
Figure 4.2
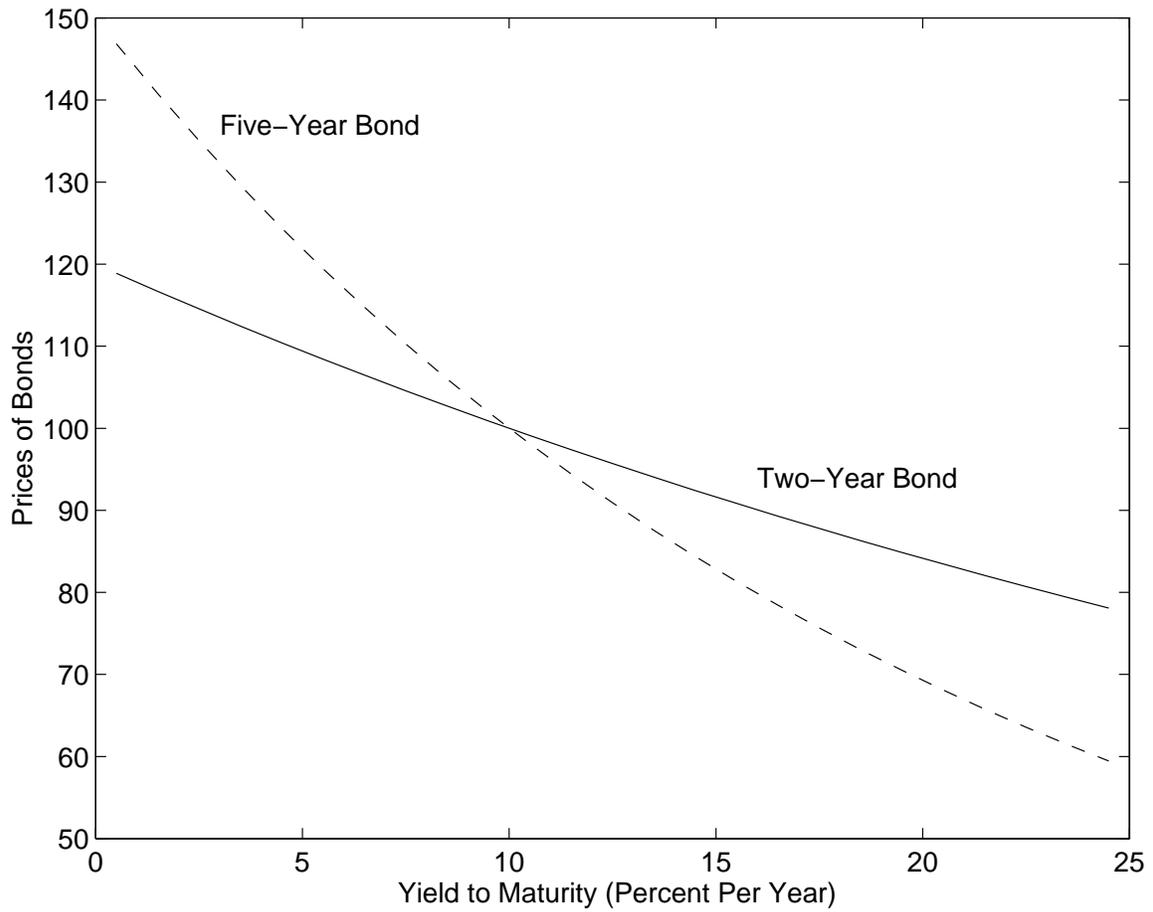**Price-Yield Relations for Two- and Five-Year Bonds**

Figure 4.3
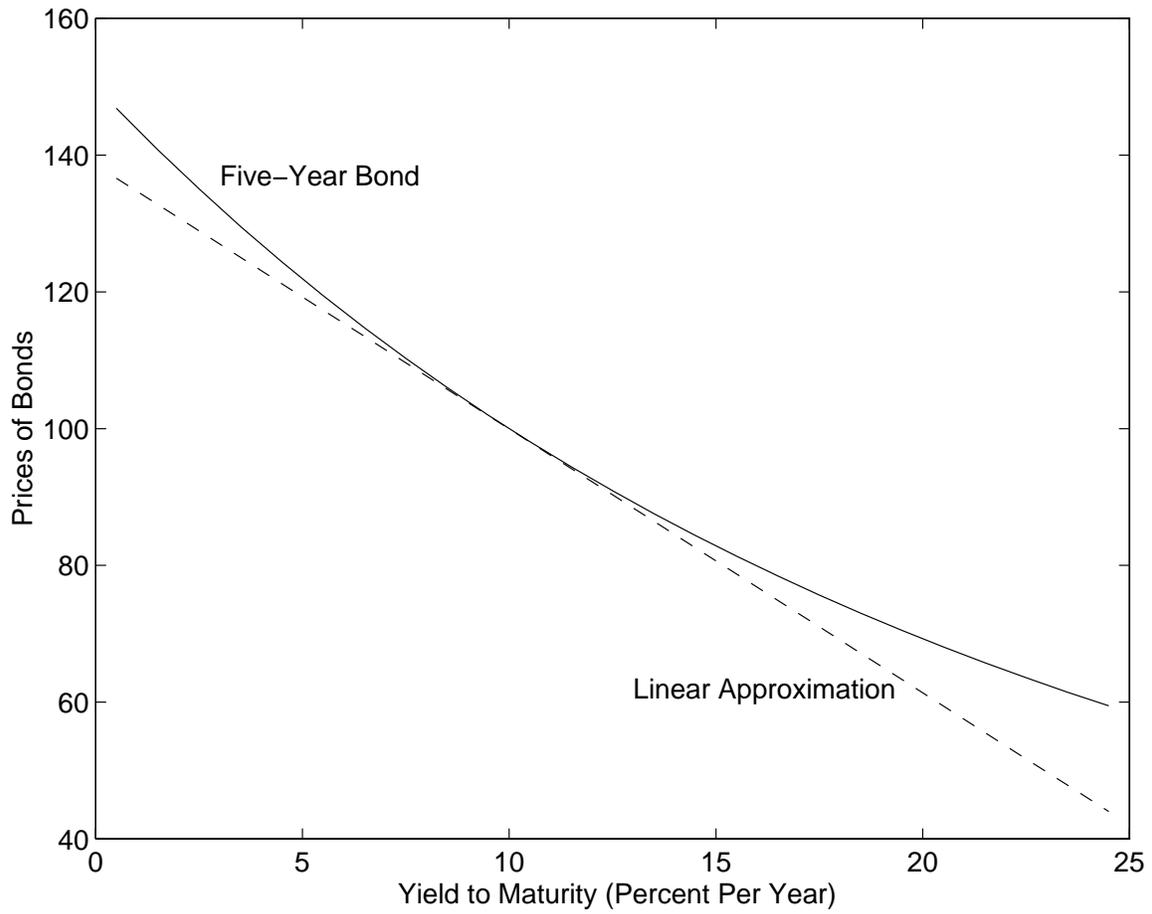**Price-Yield Relation and Linear Approximation**

Figure 4.4
Two Positions with Equal Duration but Different Convexity
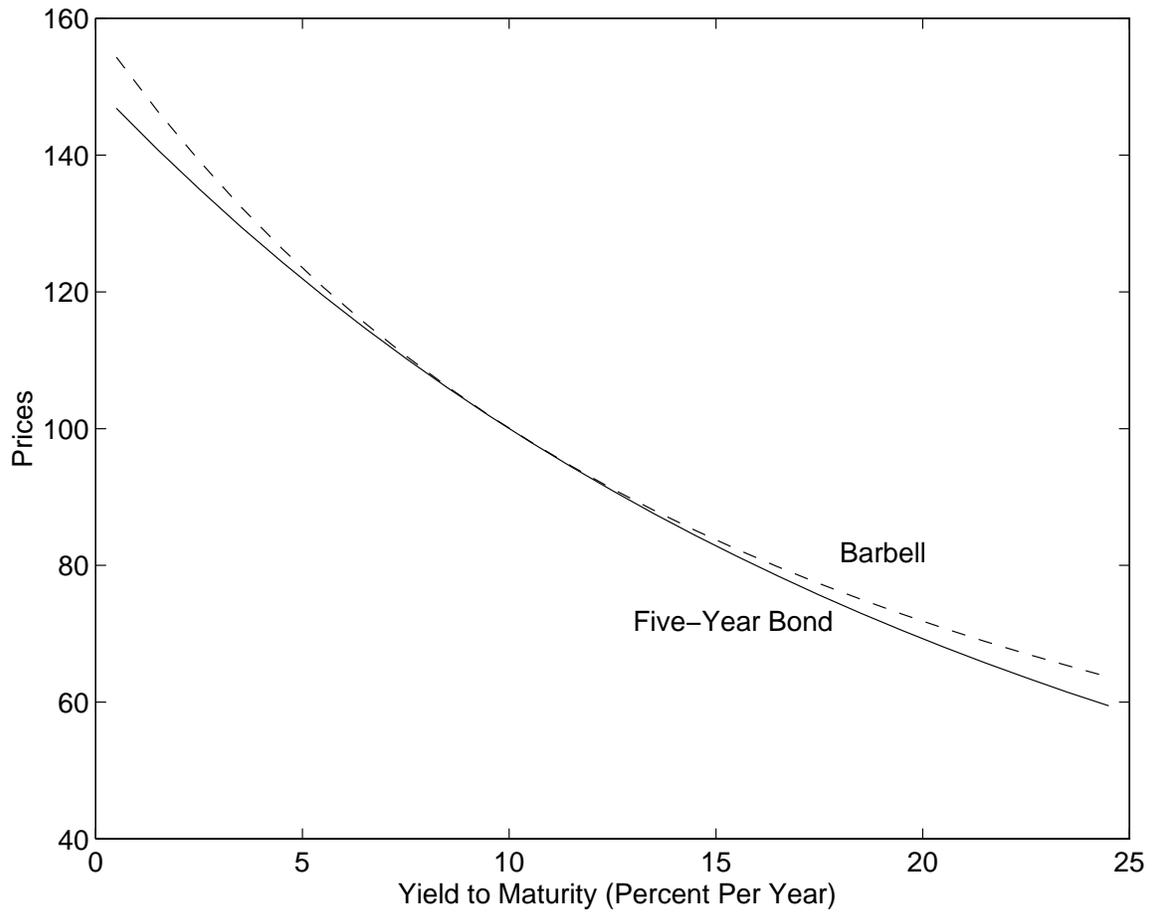
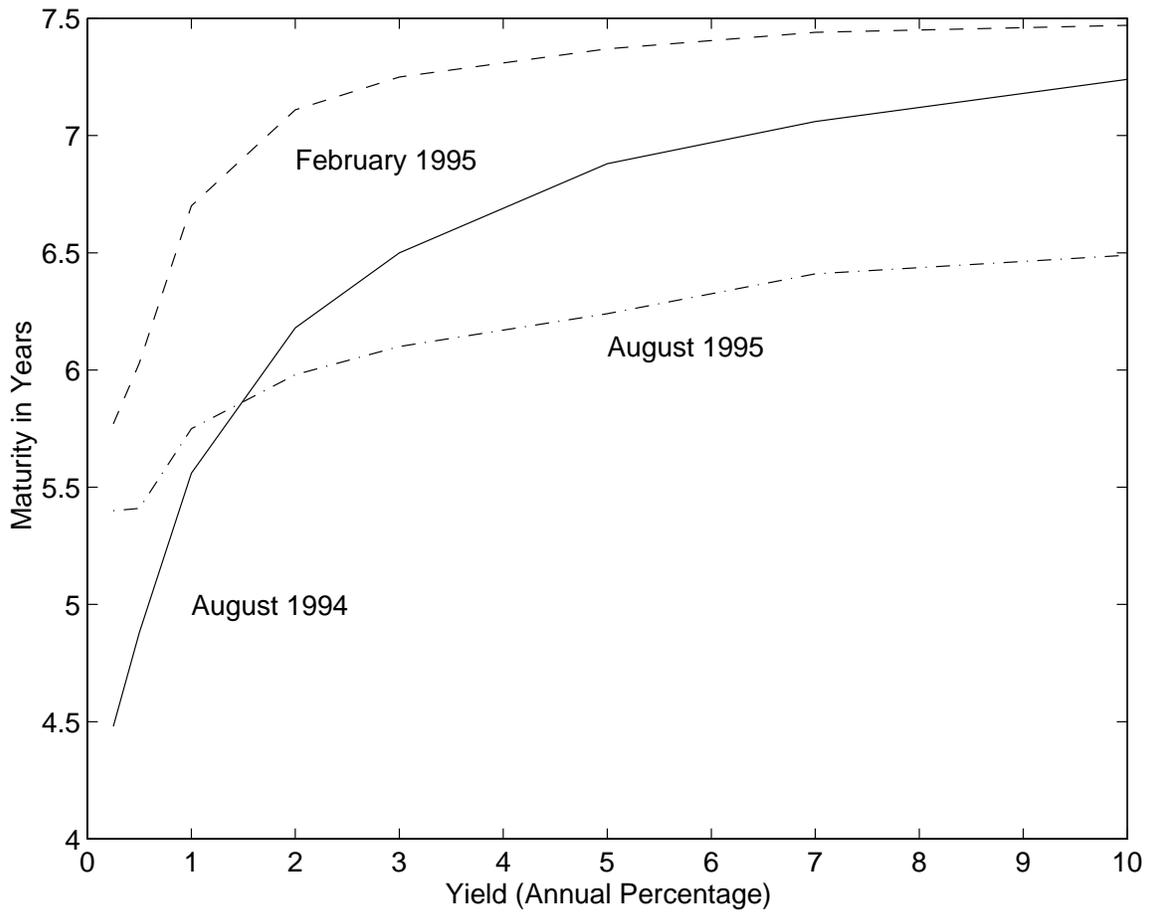**Figure 4.5**
Three Yield Curves, 1994-95

Figure 4.6
Principal Components of Yield Changes
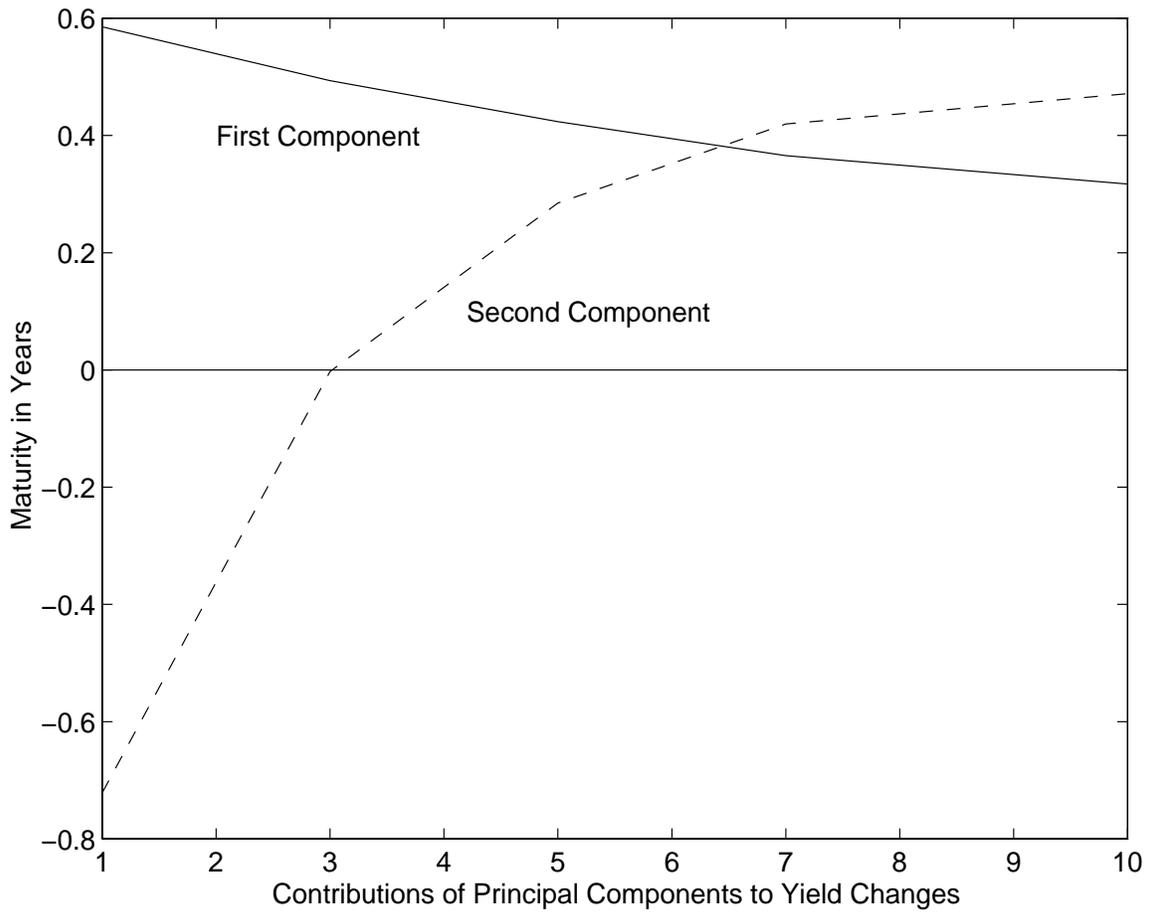
Chapter 5

Floating Rate Notes
and Interest Rate Swaps

One of the most active fixed income markets is the swap market, with outstanding notional value comparable to US treasury markets. The swap market is an over-the-counter market, with end users dealing directly with major financial institutions. It is also a global market, with trade taking place around the world, 24 hours a day, in instruments denominated in all of the major currencies — and lots of the minor ones, too. For many users, swaps are now the instrument of choice for interest rate and currency risk management.

We examine in this chapter standard forms of two of the most popular swaps: interest rate swaps and cross-currency swaps. In a standard *interest rate swap*, one party pays the other the difference between a fixed rate and a floating rate of interest for several periods (commonly between two and ten years). If you think of the fixed rate as analogous to the coupon on a long bond, this allows the buyer to transform long debt into short debt, or the reverse. In a standard *cross-currency swap*, one party pays the other the difference between interest on a loan denominated in one currency and interest on a loan denominated in a second currency. Interest payments in both cases can be either fixed or floating. In this case the swap allows the buyer to transform debt in one currency into debt in another, thus altering its exposure to movements in currency prices.

We review both of these instruments, and a few more complex variants, in the following pages. But first we need some background on floating rate notes.

5.1 Floating Rate Notes

In Chapters 2 and 4 we looked at bonds with fixed coupon payments. Here we look at analogous instruments in which the interest payment varies with current short-term interest

rates — *floating rate notes* or FRNs. These instruments are common in the eurobond market, with rates typically (but not always) tied to LIBOR, a standard eurocurrency rate (see Section 2.3).

Some examples illustrate the general structure and some of its permutations.

Examples of FRNs

Example 1. Istituto Bancario San Paolo Torino, a large private Italian bank, issued five-year ecu notes June 16, 1992, due June 18, 1997. (The ecu, or "European currency unit," is a composite of European currencies.) The notes have semi-annual payments of 6-month ecu LIBOR plus 20 basis points. The 20 BP penalty is apparently an adjustment for credit risk: the bank's credit rating on the issue date was A1, somewhat lower than the banks used to calculate LIBOR). Interest payments work as follows. The initial payment rate on 12/16/92, for example, was 10.716, 20 points above 6-month ECU LIBOR for 6/16/92. The interest payment per 100 principal is computed from the quoted rate using an actual/360 day count convention, as described in Chapter 2:

$$\text{Interest Payment } = \frac{183}{360} \times 10.716 = 5.45.$$

On each payment date, the rate for the next payment is set at 6-month ECU LIBOR + 20 BPs.

Example 2. Citicorp's DM notes, issued December 15, 1994 and maturing December 15, 1999. Interest payments are quarterly, at a rate of DM LIBOR plus 25 basis points.

Example 3. Daiwa Europe's floating rate yen notes, issued December 13, 1994, maturing March 17, 2000. Semiannual payments at the 5-year yen swap rate (coming shortly) minus 90 BPs.

Example 4. IBRD (the World Bank) issued five-year, US dollar-denominated floating rate notes on October 1, 1992, due October 1, 1997. The rate varies inversely with 6-month LIBOR:

$$\text{Rate } = 14.5\% - 2 \times \text{LIBOR}.$$

This instrument is referred to as an *inverse floater*, for what I hope are obvious reasons. For most inverse floaters, it is understood that the rate can never be negative: there is an implicit floor at zero.

Example 5. Deutsche Bank Finance NV, 10-year Canadian dollar floating rate notes issued September 1, 1992, due September 3, 2002. Guaranteed by Deutsche Bank and rated Aaa. Interest payments are quarterly, at a rate equal to that on 3-month Canadian dollar bankers acceptances minus 30 BPs, with a minimum (floor) of 5-7/8 and a maximum (cap) of 8.9.

Floating Rate Note Arithmetic

Despite the enormous variety of floating rate notes, it's useful to consider a relatively simple version with semiannual payments at 6-month LIBOR, in any currency we like, with no day count shenanigans. Such instruments are the first step in pricing more complex notes and a critical building block for interest rate swaps.

At first glance an FRN appears to be a more complex instrument than a fixed coupon bond, since its interest payments are not known when the note is issued: its cash flows are not "fixed." A closer look tells us, though, that the price should never be far from par. Take a floater with 6 months to maturity. If the bond equivalent (see below for details) of 6-month LIBOR is y, then we are promised a payment of $100(1 + y/2)$ interest and principal in six months. The value of this payment, discounted semiannually at the six-month rate, is

$$\text{Price} = \frac{100(1 + y/2)}{1 + y/2} = 100.$$

To wit: it trades at its par value of 100 on the reset date.

Now consider a floater with twelve months to maturity. We now know that in six months, when the rate is reset, it will trade at par — 100. Its price now is the discounted value of the cash flow in six months, namely its price of 100 plus interest of $100 \times y/2$. By the same reasoning we used above, this discounted value is also 100. Repeated use of this argument tells us that a floating rate note of any maturity trades at par on its coupon dates. Between dates, an FRN is essentially a short, fixed rate instrument.

Day Counts (optional)

There is one subtle feature of floating rate arithmetic that you might have guessed from Chapter 2: euromarket interest rates are quoted not as semiannual bond yields, but as simple interest. Take an FRN with six-month resets. The rate is typically tied to eurorates, like 6-month LIBOR. For the dollar, the interest rate convention (see Section 2.3) is actual/360. To convert 6-month LIBOR to a semiannually compounded zero-coupon yield y, we use

$$\frac{y}{2} = \text{LIBOR} \times \frac{\text{Days in Period}}{360}.$$

Similarly, for 12-month LIBOR we use

$$\left(1 + \frac{y}{2}\right)^2 = 1 + \text{LIBOR} \times \frac{\text{Days in Year}}{360}.$$

Similar conventions are used for other currencies; see Table 2.2.

Duration

On the reset date, a floating rate note trades at par so its duration is zero: any change in yield changes the interest payment six months hence, but has no effect on the price.

After the rate has been set, the floating rate note behaves just like a fixed rate note with less than one six-month period remaining. To be concrete, let us say that a fraction w of a six-month period remains. Then the invoice price of the note is related to its yield by

$$\text{Price}\ =\frac{100 + \text{Interest Payment}}{(1 + y/2)^w}.$$

Hence its duration, as we learned in the previous chapter, is

$$D = (1 + y/2)^{-1}w/2. \tag{5.1}$$

The bottom line of all this is that FRNs with semiannual resets have a duration between zero and six months. Their prices are not especially sensitive to movements in yields.

Inverse floaters are another matter. We typically find the duration by decomposing it into floating and fixed rate notes. To see how this works, consider a relatively simple example of a five-year floater with rate

$$\text{Rate}\ = 10\% - \text{LIBOR}$$

and suppose the spot rate curve is flat at 10%. We decompose the floater into these pieces:

1. Fixed rate note with 10% coupons. This sells for 100 and has a duration of 3.86 (see example 2 of the previous chapter).

2. Short position in a five-year floating rate note. Its price, at issue, is 100, and its duration is $0.48 = 1.05^{-1}1/2$.

3. Long position in a five-year zero. Its price is $61.39 = 100/1.05^{10}$ and its duration $4.76 = 1.05^{-1}10/2$.

This ignores the floor on the rate, but is otherwise a perfect replication of the inverse floater.

We find the price of the floater from the prices of the pieces:

$$\text{Price}\ = 100 - 100 + 61.39 = 61.39.$$

Similarly, its duration is a weighted average of the duration of the components, as in equation (4.7):

$$D = 3.86\left(\frac{100}{61.39}\right) + 0.48\left(\frac{-100}{61.39}\right) + 4.76\left(\frac{61.39}{61.39}\right) = 10.27.$$

Thus the inverse floater has greater duration than the floater itself.

5.2 The Plain Vanilla Interest Rate Swap

Interest rate swaps constitute close to half the swap market. And like FRNs, they come in many flavors. We will focus on the so-called "plain vanilla" interest rate swap, in which one counterparty pays the other the difference between fixed and floating rates. This is often represented visually as
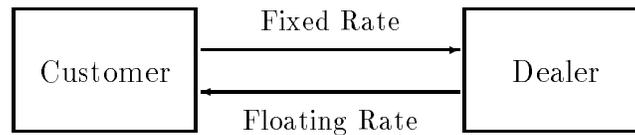
As illustrated, the customer — in this case the fixed rate payer — pays a fixed rate to the dealer, and receives a floating rate in return. In fact they would net the two payments: the counterparty with the larger payment sends the difference to the other one. Both the fixed and floating rates are applied to a *notional principal*. We refer to the principal as "notional" to emphasize that the two parties do not exchange principal payments (which are typically the same anyway).

Although the parties do not exchange principal, it's helpful to think of an interest rate swap as an exchange of a floating rate note for a fixed rate note. For notes of equal value at the settlement date, this arrangement has the same net cash flows as the interest rate swap above. The principal payments at the end, for example, net to zero, so for this purpose it's immaterial whether we include them or not. The advantage is that we can apply what we know about the pricing of bonds and FRNs. The value of a swap to a fixed rate payer is then the difference between the value of an FRN and a fixed rate note with the stated coupon.

For the rest of this section we look at the arithmetic and mechanics of a standard plain vanilla swap, in which one party pays LIBOR ("LIBOR flat," in the jargon of the trade) and the other pays a fixed rate.

Swap Arithmetic

Swap arithmetic shows up in two guises: when we compute the fixed rate appropriate to the swap, and when we value an existing swap with a specific fixed coupon rate. In both cases we need the discount factors appropriate to this market. By and large the discount factors in the swap market are somewhat lower, and the yields somewhat higher, than those in the US treasury market. Most people attribute the difference to credit risk.

Estimating Discount Factors. As with bonds, zeros are the building blocks and discount factors are a convenient way of representing their prices. In the swap market, there are several common ways of estimating discount factors. One, used primarily for US dollar swaps, is to start with US treasury discount factors. The difficulty here is that the discount factors and zero-coupon yields for US treasuries and swaps aren't the same (credit risk again), and moreover the spread between them changes through time. A second approach, which we will imitate shortly, is to use eurocurrency rates at the short end and publically quoted swap rates for standard maturities at the long end. The primary difficulty here is that these quotes cover a limited number of maturities: 1 to 12 months for eurocurrency rates, and 2, 3, 4, 5, 7, and 10 years for swap rates. A third way is to infer discount factors from eurocurrency futures, which we'll study later on. This currently limits us, however, to relatively short maturity swaps (less than four years).

I'll run through the second method to give you a rough idea what's involved. The starting point is Bloomberg's BBA LIBOR and generic swap quotes for June 22, 1995:

Instrument	Rate (Annual %)
6-Month LIBOR	5.94
12-Month LIBOR	5.81
2-Year Swap	5.84
3-Year Swap	5.98
4-Year Swap	6.10
5-Year Swap	6.19
7-Year Swap	6.34
10-Year Swap	6.54

Some quick and dirty calculations lead to these discount factors and zero yields for the short end:

Maturity (Years)	Discount Factor	Zero Yield (Annual %)
0.5	0.9707	6.036
1.0	0.9443	5.809
1.5	0.9175	5.824
2.0	0.8913	5.839
2.5	0.8644	5.914
3.0	0.8378	5.989

The mechanics of these calculations are not especially revealing. They involve (i) interpolation between maturities, (ii) converting LIBOR rates to a consistent semiannual basis, and (iii) converting swap rates to zero rates. These are issues that arise in practice, but I suggest you skip to the next paragraph unless you're really curious.

i. Interpolate. I set the 18-month swap rate to the average of 12-month LIBOR and the two-year swap rate, the 30-month swap rate to the average of the 2- and 3-year swap rates, and so on.

ii. Compute 6- and 12-month discount factors from LIBOR:

$$d_1 = \frac{1}{1 + y_1/2} = \frac{1}{1 + \text{LIBOR6} \times \frac{183}{360}}$$

$$d_2 = \frac{1}{(1 + y_2/2)^2} = \frac{1}{1 + \text{LIBOR12} \times \frac{365}{360}}$$

The subscripts for d and y refer to their maturity measured in six-month periods.

iii. Compute other discount factors from swap rates by reversing the par yield calculation of Chapter 2. Let the swap rate, expressed as an annual percentage, be C. Then you might recall that C satisfies

$$100 = (d_1 + \cdots + d_n) \times C/2 + d_n 100,$$

an application of equations (2.4) and (2.7). This implies

$$d_n = \frac{100 - (d_1 + \cdots + d_{n-1}) \times C/2}{C/2 + 100}.$$

In words: given the first $n - 1$ discount factors and the swap rate C, this equation tells us how to get the next one. We can then use the discount factors, if we like, to compute the zero yields using equation (2.3).

Valuing Swaps. Once we have the market discount factors, it is relatively simple to price a swap, even one that was issued some time ago. Consider a swap with arbitrary annual fixed rate C and n remaining payments. And to keep things simple, suppose we have just made the previous net interest payment so that we can ignore fractional time periods. The price of the swap to the fixed rate payer is the difference between the price of an n-period floating rate note and an n-period fixed rate note with annual coupon rate C:

Price of Swap = Price of Floating Rate Note − Price of Fixed Rate Note.

What we need to do, evidently, is value each of the two notes.

The floating rate note is easy: on a reset date, the price is par: 100. The fixed rate note is also relatively easy. Given the discount factors d and the fixed rate C, the price is

$$\text{Price of Fixed Rate Note} = (d_1 + \cdots + d_n) \times C/2 + d_n 100, \tag{5.2}$$

an application of equation (2.4). The price of the swap is then the difference.

By way of example, consider a five-year swap arranged June 22, 1993, with a fixed rate of 5.20 percent annually and six semiannual payments remaining. With rates rising in the meantime, we might expect the fixed rate payer to benefit. We see this here: our discount factors above imply a price of the fixed rate note of 97.88, so the swap is now worth +2.12 (=100.00−97.88) to the fixed rate payer.

Duration

We assess the risk of a swap by computing the risk of its component parts. Note, first, that we cannot compute the duration of the swap directly. Since its value is zero, the proportionate change in value is zero. (Stated more directly: if we apply equation (4.7) we find that we end up dividing by the overall value $p = 0$.) What this means is that we should be using the PVBP or something similar that doesn't divide by zero. We'll run across a similar issue when we turn to futures.

We can, however, compute the duration of the two sides of the swap separately. We compute duration for the floating leg using (5.1). The fixed leg is just a coupon bond, and we use the usual formula for duration, equation (4.4).

??

More commonly, we compute duration of swap plus other positions. Give example...

5.3   Cross-Currency Swaps

In a plain vanilla cross-currency swap, one party pays the other the difference between interest and principal in two currencies, illustrated here for the US dollar and the DM:

The interest payments can be fixed or floating in either direction. Thus we see that a cross-currency swap differs from an interest rate swap in denominating the interest and principal in different currencies and in calling for an exchange of principals at maturity. The principals generally have the same value when the swap is arranged, but can be substantially different at maturity — a factor that plays an important role in the swap's risk characteristics.

Consider, as an example, a three-year swap of fixed DM vs fixed dollar payments on June 22, 1995, on a notional value of $100m. For the sake of discussion, let us say that the spot rate on the same day was 1.400 DMs per dollar (not a bad approximation), so the equivalent DM notional principal is DM 140m. The three-year dollar swap rate on this date was 5.98% (see Section 5.2), so the dollar fixed rate payments are $2.990m every six months. DM rates on the same date were

Maturity	Discount Factor	Zero Yield	Swap Rate
0.5	0.9778	4.535	na
1.0	0.9558	4.574	na
1.5	0.9309	4.833	na
2.0	0.9041	5.105	5.09
2.5	0.8761	5.361	na
3.0	0.8468	5.621	5.58

The DM fixed rate payments are DM 3.906m every six months. At maturity the two parties swap the last interest payment plus principal: $102.990m for DM 143.906m.

As before, we can think of the swap as an exchange of notes, one denominated in each currency. For the swap in the diagram, we might write

$$\text{Price of Swap} = \text{Price of Dollar Note} - \text{Price of DM Note},$$

and for the fixed/fixed structure of the example,

$$\text{Price of Swap} = \text{Price of Fixed Rate Dollar Note}$$
$$- \text{Price of Fixed Rate DM Note}.$$

From this it's apparent that the swap of the example is exposed to several types of risk: to the risk that dollar yields will change, to the risk that DM yields will change, and to the risk that the dollar value of the DM will change. For the example, the customer benefits from a fall in US yields, a rise in DM yields, and fall in the value of the DM.

Swap Arithmetic

As before, we value the swap as the difference between two notes. To do this we generally need (i) discount factors in both currencies and (ii) the spot exchange rate between them.

Step 1. Value each note in its own currency, just as we did with interest rate swaps. For a floating rate note the price is close to par. For a fixed rate note, we use the discount factors for its currency of denomination and apply (5.2).

Step 2. Use the spot exchange rate to convert one note to the currency of the other.

Example. Consider a 3-year pay-fixed-DM/receive-fixed-dollar cross-currency swap with two years of semiannual payments remaining. The DM fixed rate is 6.40 and the dollar fixed rate is 6.54, the 3-year swap rates for June 22, 1994. The spot exchange rate on the same day was 1.58 DMs per dollar, the current rate is 1.40. The notionals are $200m and DM 316m, which were equivalent on the trade date.

We estimate the June 22, 1995, value of the swap by applying the appropriate discount factors to dollar and DM cash flows. The dollar note has a value of $202.61m, the premium over par reflecting the fall in dollar yields since the swap was executed. The calculation simply applies dollar discount factors to payments of interest and principal. The DM note has a value of DM 319.75m (DM yields have fallen, too) or, at the current rate of 1.4 DMs per dollar, $228.39m. The value of the swap is the difference: $25.78m. Almost all of this comes from the sharp rise in the DM vs the dollar between June 94 and June 95.

Risk Assessment (optional)

Risk analysis for cross-currency swaps requires a combination of the kinds of risk assessment methods we described for fixed income positions and a compatible method for currency risk. As a general rule, the currency risk is dominant.

For a standard cross-currency swap, we can express the change in the swap's value in terms of the prices of the two underlying bonds (say p for the dollar bond and p^f for the DM bond) and the spot exchange rate (s, say, for the dollar price of one DM). For small changes, the change in value is approximately

$$\Delta \text{Swap Value} = \Delta p + s\Delta p^f + p^f \Delta s.$$

We can get burned, that is, by changes in the price of the dollar bond, in the DM bond, or in the value of the currency.

We can estimate the likely magnitude of bond price changes from their durations. A one-standard deviation price change might be estimated by one standard deviation of the appropriate interest rate times duration:

$$\Delta p \quad = \quad -D\,p\,\Delta r$$
$$\Delta p^f \quad = \quad -D^f s\,p^f \Delta r^f.$$

If we start with bonds of roughly equal value, the magnitudes are governed by the durations of the bonds. For our three-year fixed/fixed swap, the durations are about 2. The standard deviation depends on the time interval. For a period of six months, the US short rate has a standard deviation of about 100 basis points, the German interest rate a little lower. These give us, for each price, a one-standard deviation change of about 2 percent.

Currency movements are considerably larger. For a period of six months, the standard deviation of dollar/DM spot rate changes is about 8 percent, which gives rise to a similar standard deviation in the dollar price of the DM bond.

A more systematic risk analysis would take into account, in addition, the correlation between interest rate and currency changes. Risk management tools like JP Morgan's RiskMetrics provide both a framework for doing this and current estimates of the required standard deviations and correlations.

5.4 Other Swaps

The examples earlier of floating rate notes should give you the idea that there is no end to the diversity of financial instruments. Swaps are no exception. A short list of common and nontrivial variations includes:

Amortizing and Accreting Swaps

There is no particular reason for the notional principal to be constant throughout the life of the swap and, in fact, in amortizing and accreting swaps they are not. In an amortizing swap the notional declines through time, in an accreting swap the reverse. This makes the arithmetic a little more complicated, but there is nothing conceptually different or difficult about such structures. These swaps illustrate one of the major advantages of swaps over more standardized instruments: they can be custom-taylored to the user's needs.

Step Up/Down Swaps

The wrinkle here is to have a coupon rate that starts low and then "steps up" (a step up swap, obviously), or the reverse (a step down swap). Most people will recognize this trick from home mortgages: start with a low "teaser" rate then adjust upwards after a year or two. For swaps this possibility gives users a great deal of flexibility over the timing of their cash flows.

Basis Swaps

Another common swap involves an exchange of two floating rates. One example is an exchange of the 3-month treasury bill rate for three-month eurodollar rates (LIBOR, in other words). The so-called TED spread (T for treasury, ED for eurodollars) is generally less than a hundred BPs but is moderately variable, and institutions operating in both markets may want to protect themselves against a change. Another example is an exchange of the prime loan rate for LIBOR, which commercial banks use to protect themselves against adverse movements in loan income (tied to the prime rate) relative to short-term liabilities (often closely related to LIBOR).

A popular variant is the constant maturity Treasury (CMT) swap, in which a long treasury rate (say, the five-year rate) is swapped for a short one (the six-month rate). Investors in mortgages find that prepayments are related most closely to such yield spreads, and that CMT swaps provide useful hedges for prepayment risk.

Diff Swaps

Finally we have what seem like relatively strange objects called diff swaps which are easier to explain than define. Party A pays interest in currency 1. Party B pays interest in currency 1 as well, but at a rate given by yields in a second currency. For example, Party A might pay 6-month US dollar LIBOR, and Party B Japanese yen LIBOR *in dollars*. These instruments give users the possibility of exploiting differences in yields across countries, without subjecting them to currency risk.

CMT Swaps

Constant Maturity Treasury swaps, in which one or more legs has a floating rate tied to a long treasury yield, have been increasingly popular in recent years. For example, one counterparty might make semiannual payments tied to six-month LIBOR and the other make payments tied to the five-year treasury yield. This floating payment at a long-term yield introduces a new difficulty, since the corresponding floating rate note no longer trades at par on reset dates.

5.5   Credit Risk

Perhaps the major factor differentiating swaps from treasury securities for major countries is credit risk: there is some chance (typically not large, but not zero either) that a swap counterparty will default. Many of the features of swaps can be thought of as ways of minimizing the damage of from such events.

One feature is that the principal payments are not exchanged. This reduces the risk of default enormously relative to a note, since the interest payments are generally worth much less than the principal.

A related feature is netting. The ISDA agreement establishes, and US law confirms, that in the event of default creditors have a claim only on the net value of the swap, not on the receive side on its own. In this sense swaps are much different from the combinations of notes that we used to price them. With two notes, the long position would go into the pool used to pay creditors, including the other side of the short position.

Another feature is the standard legal agreement used for swaps, designed by the International Swap Dealers Association (ISDA). The ISDA master agreement, as it's called, has standardized swap documentation. Given the way precedent works in the US and UK legal systems, this has helped to establish groundrules for how swaps are treated in the event of default or bankruptcy.

These are standard features of swap agreements. In addition, some counterparties add (i) collateral, (ii) letters of credit that guarantee payment by an insolvent counterparty, (iii) insurance, (iv) termination triggered by counterparty downgrades, or (v) mark-to-market requirements, analogous to futures exchanges. All of these reduce exposure to credit risk.

5.6 Why Swaps?

We come back to two versions of a question raised in Chapter 1: Why so many kinds of instrument? And why swaps, in particular? The swap market has grown enormously since its advent in 1981: what makes it so useful?

Among the reasons most commonly given are:

- Lower cost than public issues. A company wanting to retire fixed rate debt and replace it with floating rate debt might find it cheaper, and easier, to arrange an interest rate swap.

- Custom design. Unlike publically traded instruments, which come with fixed specifications, swaps can be designed to suit a particular users' needs.

- Accounting and tax treatment. there are cases in which a publically traded instrument, like an interest rate futures contract, is treated differently on financial statements than a swap. This can extend to tax liabilities. Swaps, for example, are less often marked to market, probably because valuing a swap is not as easy as looking up the closing price of a futures contract.

- Regulation. This is changing continuously, but off-balance sheet items like swaps are often treated differently by regulators of financial institutions from economically similar positions in (say) interest rate futures. In some cases, banks find that swaps have lower capital requirements.

Summary

1. Floating rate notes make "coupon" payments tied to market interest rates, most commonly LIBOR.

2. Inverse floaters, in which interest payments vary inversely with market rates, often have long durations.

3. In a plain vanilla interest rate swap, two parties exchange the difference between a floating and a fixed rate of interest applied to a fixed notional principal.

4. In a plain vanilla cross-currency swap, two parties exchange the difference between interest and principal payments denominated in two different currencies.

5. For those who dislike vanilla, there's no end to the variety of swap flavors.

Practice Problems

1. Consider an inverse floater like Example 4 of Section 5.1 but due June 22, 1997, with exactly two years to maturity. Interest payments by the issuer are

$$\text{Rate } = 14.5\% - 2 \times \text{LIBOR}.$$

We will use the discount factors of Section 5.2, and our knowledge of fixed and floating rate notes, to approximate its price.

The trick is to decompose the note into pieces that we know how to value. One approach is to think of the buyer of the note as short two standard floating rate notes (thus taking care of the $2\times$LIBOR part of the interest rate) and long one fixed rate note with coupon rate 14.5% (ditto the fixed rate). This handles the interest payments, but leaves us short one principal, and as the owner of the note we should be long. We therefore add to our collection of pieces long positions in two two-year zeros. In sum, the inverse floater is equivalent to: two short positions in standard FRNs; one long position in a fixed rate note with 14.5% coupons; and two long positions in two-year zeros.

   (a) Value the floating rate notes.
   (b) Value the fixed rate note.
   (c) Value the zeros.
   (d) Compute the value of the note.
   (e) If we assume that the note was issued at par in 1992, how do you account for its current price?
   (f) What is the duration of this note?

2. Our objective is to value an inverse floater issued by Deutsche Bank NV, maturing August 29, 2000, and paying

$$\text{Rate } = 15.125\% - \text{ 6-month DM LIBOR}$$

every 6 months. Today's date is August 29, 1996, and we would like to assess the note's value. Using euromark futures and swap rates (type EMS GO in Bloomberg), we estimate DM spot rates at

Maturity (Years)	Spot Rate
0.5	3.38
1.0	3.62
1.5	3.98
2.0	4.29
2.5	4.59
3.0	4.86
3.5	5.20
4.0	5.45

(a) Design a combination of traditional instruments that reproduces the cash flows of the inverse floater.

(b) Use the estimated spot rates to compute the values of the components and of the note itself.

(c) What is the (modified) duration of the note?

(d) This note, like others, has an implicit lower limit on the payment rate of zero: if 6-month LIBOR exceeds 15.125, the note pays nothing. Describe qualitatively how this feature might affect the interest sensitivity of the note at high interest rates. How might this change your calculation of the note's value?

3. Using the same spot rates as the previous problem:

(a) Compute swap rates for semiannual DM interest rate swaps with maturities 2, 3, and 4 years.

(b) Consider a four-year swap with a fixed rate of zero in the first year. What is a fair coupon rate for the following three years?

4. The German subsidiary of an American industrial firm issued DM250m of five-year floating rate notes one year ago, but with the German economy heating up is now concerned that interest rates might rise and is thinking of locking in a fixed rate now.

(a) Describe qualitatively how the firm might use an interest rate swap to modify the form of its interest payments.

(b) Compute, using your answer to the previous problem, the duration of the fixed rate side of swaps with maturities 2, 3, and 4 years.

(c) What notional principal is required to produce an overall duration of 2 years if the firm uses 2-year swaps? 4-year swaps? In what respects do these two strategies differ? Which would you recommend?

5. Using once more the discount factors of Section 5.2, we will consider a simple amortizing swap. The swap has a maturity of two years and involves semiannual fixed rate payments based on six-month dollar LIBOR. For the first two payments, interest payments are based on a notional principal of $100m. The last two are based on a smaller notional principal of $50m.

   (a) Decompose the swap into combinations of fixed and floating rate notes.

   (b) Compute the swap rate that equates the values of the fixed and floating rate legs of the swaps.

   (c) Compare the interest-rate sensitivity of the swap to that of plain vanilla one-year and two-year swaps.

6. *(This problem and the next one are adapted from one developed by Suresh Sundaresan at Columbia.)* A large, Aaa-rated industrial corporation is rethinking its debt strategy in late 1988. For several years it has exploited a steep yield curve by shortening the length of its debt. It did this in two ways: by rolling over 6-month commercial paper and by issuing $200m of a plain vanilla floating rate notes maturing November 1, 2015. In October of 1988 the company is reviewing this strategy. Between July and October of 1988 the spread between 5-year and 6-month yields fell from about 300 BPs to just over 100. The CFO wonders if this is the time to switch to longer-term debt and insulate the firm from further rate increases, but worries that it may not be feasible to retire the floating rate note.

At this point, one of the firm's bankers suggests that interest rate swaps might be an effective tool in this regard, and quotes a rate of 5-year treasury plus 100 BPs against 6-month LIBOR flat on a 5-year pay-fixed/receive-float swap. The swap would commence November 1, 1988, and have semiannual payments on a notional principal of $200m. The question is whether the quoted rate is a fair one.

   (a) Your research department has constructed discount factors from cash and futures markets for eurodollars:

May 2, 1989	0.9587
November 1, 1989	0.9190
May 1, 1990	0.8803
November 1, 1990	0.8416
May 1, 1991	0.8050
November 1, 1991	0.7683
May 1, 1992	0.7353
November 2, 1992	0.7022
May 3, 1993	0.6716
November 1, 1993	0.6410

Given these discount factors, what is the annual coupon rate for a five-year swap?

   (b) At the current 5-year treasury rate of 8.368, the quoted price implies a fixed rate of 9.368 percent. How does this rate compare to the one you computed above? Since your firm is Aaa, and the eurodollar market is approximately Aa, what rate would you suggest as a counteroffer?

   (c) Compute the duration of the combination of the swap and the floating rate note.

7. *(Continued).* Six months later, on May 2, 1989, interest rates have changed, and your research department gives you updated discount factors:

November 1, 1989	0.9508
May 1, 1990	0.9053
November 1, 1990	0.8641
May 1, 1991	0.8228
November 1, 1991	0.7862
May 1, 1992	0.7496
November 2, 1992	0.7163
May 3, 1993	0.6830
November 1, 1993	0.6542

   (a) Describe, for those of us who find discount factors hard to understand, what has happened to the zero yield curve for swaps between November 1988 and May 1989.

   (b) Given changes in yields, and your previous calculation of duration, what would you expect to have happened to the value of the swap?

   (c) Compute the current value of your interest rate swap and compare it to your duration-based estimate.

8. A US exporter to Germany is interested in hedging its exposure to the DM over the next three years. In anticipation of rising DM revenues, its bank designs an accreting cross-currency swap, in which the notional principal is DM 50m the first year, DM 100m the second, and DM 150m the third. It pays semiannual fixed rates on both sides.

   (a) Use the dollar discount factors from Section 5.2 to compute the fixed rate that prices the dollar fixed rate note at 100.

   (b) Use the DM discount factors from Section 5.3 to compute the fixed rate that prices the DM fixed rate note at 100.

9. An A-rated commercial bank would like to arrange a \$20b notional semiannual pay-float interest rate swap to reduce its sensitivity to interest rate movements, but would like to raise interest revenue in the current year. It suggests to Merrill Lynch Derivative Products (MLDP) a three-year step down swap, in which it receives a fixed rate of 8 percent in the first year and (in all likelihood) a lower rate thereafter.

   (a) Use the discount factors in Section 5.2 to compute the appropriate fixed rate for the final two years of the swap.

   (b) Describe briefly, from the perspective of MLDP, the credit risk of such a swap.

10. Effective July 14, 1993, Greyhound Lines Inc entered into a five-year semiannual interest rate swap with Bankers Trust Company (hereafter BT). A stylized version follows, modified in the interest of illustrating the mechanics of an interest-rate swap. Greyhound closed out this position in January 1994 by arranging with BT to fix the rates on both sides for the remainder of the swap.

    In the original swap agreement, Greyhound paid a floating rate of 6-month LIBOR as determined by the British Bankers Association. LIBOR on July 14, 1993, was 3.50. BT paid a rate (called the "fixed rate" in the documentation) governed by LIBOR on January 25, 1994 according to the following formula:

    - If LIBOR is between 3.25 and 3.90, the rate is 6.05.
    - If LIBOR is greater than 3.90, the rate is

    $$6.05 - 7 \times (\text{LIBOR} - 3.90).$$

    - If LIBOR is less than 3.25, the rate is

    $$6.05 - 7 \times (3.25 - \text{LIBOR}).$$

    (a) Describe the swap.
    (b) Describe the interest sensitivity of the swap under each of the three possible payment scenarios.
    (c) Suppose you are working for BT. What is your sales pitch for this swap?

Further Reading

Swaps are by now standard instruments, but textbook treatments remain poor. Sundaresan (1997, ch 14) and Tuckman (1995, ch 16) are two of the best. Galitz (1995) is a good source for both analytics and descriptions of exotic swaps. Backus, Klapper, and Telmer (1995) is concerned with Banc One's $39 billion notional in interest rate swaps in 1993.

Data: for Bloomberg swap quotes, type IRS [GO] and follow instructions.

Chapter 6

Risk Management, Accounting, and Control

Outline...

1. Accounting. The key issue is whether positions are reported at market or book value. With hedging, it's important that both the position and its hedge be treated the same way. Otherwise a hedge could appear as a gain or a loss. Other than that, the rules are complicated and change all the time.

2. Taxes. (i) By long tradition, municipals are exempt from federal tax and treasuries from state and local tax. Accountants sometimes "gross up" returns on such instruments to the taxable equivalent. (ii) Interest income and capital gains are sometimes taxed at different rates. Tax authorities have rules for computing income to avoid tricks like counting returns on zeros as capital gains.

3. Control. Most large derivatives losses seem to be the result of inadequate corporate controls. Since derivatives give people more powerful instruments than they had before, it's essential that firms have procedures for approving and monitoring derivatives activity. Examples illustrate some common pitfalls.

   - Kidder/Jett. Did supervisors fail to investigate good news that rewarded them well? Did the power of an apparently successful trader hinder controls?
   - Barings. Why was Leeson allowed to verify his own trades?
   - Lufthansa. Hedge or bad news?
   - Orange County. Why was Citron allowed to make such large interest rate gambles?
   - Proctor and Gamble. Did the CEO and CFO fail to supervise their young financial wizards?

Part II

Quantitative Modeling
of Fixed Income Securities

Chapter 7

Introduction to
State-Contingent Claims

We have looked so far at traditional fixed income securities, traditional meaning that the securities have known cash flows. The great growth in fixed income markets, however, has come largely in derivatives whose cash flows are not known in advance: futures, options, swaptions, caps and floors, and so on. Since these instruments have uncertain cash flows, we cannot simply apply the appropriate discount factors to price them. We need a new framework for thinking about claims to uncertain cash flows.

Although uncertainty is a new topic for us, our approach is similar in most respects to our earlier analysis: we will think of an asset as a collection of cash flows that can be valued separately. The plan is to list all possible future events, which we call *states*, and to associate with each one the cash flows generated by the asset in that situation. Since the cash flows depend on the situation or state that occurs, we refer to them as *state-contingent*. And since future states are uncertain, so are the cash flows that depend on them. Instruments with uncertain cash flows differ from traditional instruments in requiring prices across states (*state prices* or their equivalent) as well as time (the familiar discount factors).

This chapter is concerned with explaining and applying the ideas of states and of assets as claims to state-contingent cash flows. The ideas are more conceptual than mathematical, but no less sophisticated for that. An analogy with fruit baskets is intended to convince you that none of it is especially difficult. Following chapters review some of the common applications of this framework to fixed income securities.

7.1   Pricing Fruit

Before turning to interest rate derivatives, we're going to take a quick look at fruit baskets, with the objective of making it absolutely clear how straightforward the theory is. Consider — to be concrete, if a little artificial — a market in which people can buy and sell fruit. By convention, the fruit is sold in baskets that contain different combinations of fruit, rather than separately. The question is whether we can, nevertheless, talk about prices of individual pieces of fruit. The answer: Of course! The only thing you may not have guessed on your own is that this is possible only when prices of baskets do not allow pure arbitrage opportunities. If this sounds familiar from Section **??**, that's because it's essentially the same idea.

As a start, consider three examples:

Example 1. Suppose we have two kinds of fruit, apples and bananas, and two marketed baskets. Basket 1 consists of 25 apples and 100 bananas and sells for \$150. Basket 2 consists of 50 apples and 50 bananas and sells for \$150. It's easy to verify that this is consistent with prices q_a for apples and q_b for bananas of:

$$q_a = 2, \qquad q_b = 1.$$

With these prices the value of the first basket is 150 $[= 25 \times 2 + 100 \times 1]$ and the price of the second is 150 $[= 50 \times 2 + 50 \times 1]$, as given.

Example 2. Basket 1 consists of 50 apples and 50 bananas and sells for \$100. Basket 2 consists of 25 apples and 25 bananas and also sells for \$80.

Example 3. Basket 1 consists of 50 apples and 50 bananas. Basket 2 consists of 25 apples and 75 bananas. Basket 3 consists of 75 apples and 25 bananas. All three baskets sell for \$100.

Example 1 suggests that we should be able to deduce the prices of baskets from their content and individual fruit prices. If a basket has x_a apples and x_b bananas, then its price might be

$$\text{Basket Price} = q_a x_a + q_b x_b.$$

for some fruit prices q_a and q_b. This is a direct analog to equation (2.4) for discount factors. For similar reasons, you might guess (and you'd be right) that we can value all baskets this way when prices of baskets rule out fruit arbitrage:

Proposition 7.1 *(fruit prices in arbitrage-free settings).* Consider a market in which people can buy and sell fruit baskets in any quantities they like. Then if (and only if) the prices

of baskets do not allow arbitrage opportunities, we can derive positive prices for individual pieces of fruit that are consistent with the pricing of baskets.

We can get a feeling for what this means by going back to the examples:

Example 1 (continued). Here we guessed the fruit prices. The proposition says that if we have positive fruit prices consistent with prices of baskets, which we verified, then the basket prices rule out arbitrage.

Example 2 (continued). You might guess here that there is an arbitrage opportunity here. If we buy two units of Basket 2 for \$80 ("buy low"), and sell short one unit of Basket 1 for \$100 received ("sell high"), we have no net obligation to deliver fruit: we use the fruit from the long position to cover our obligation on the short position. That leaves us with \$20 pure profit.

Example 3 (continued). A subtlety that we will not have much reason to return to concerns situations in which the numbers of marketed baskets and types of fruit are different. If we have more fruit than baskets, the proposition still applies, but the individual fruit prices are not unique. Here we have the opposite: more baskets (3) than kinds of fruit (2). In this case, either the prices admit arbitrage or (as here) we have more assets than we really need. Note, for example, that 2 units of Basket 1 is equivalent to one each of Baskets 2 and 3. To rule out arbitrage, the prices of the equivalent positions must be the same, as in fact they are. The implied fruit prices are $q_a = q_b = 1$.

7.2 Two-Period Contingent Claims

Assets are like fruit, as you will see.

States

Uncertainty is a little more abstract than fruit, but if we look at it the right way the logic is surprisingly similar. Here's the trick. When we say that tomorrow is uncertain, we mean that we face more than one — and possibly many — possible situations tomorrow. Depending on our interests, the relevant situations might include: the Knicks win, the Fed raises short-term interest rates, the Bureau of Labor Statistics reports higher inflation, General Motors defaults on its bonds, and so on. We refer to these situations as *states*. The idea behind modern asset pricing theory is to treat different states as we treated different kinds of fruit. Just as a basket of fruit can be described by its contents, an asset can be described by its cash flows in various states.

A picture should make this clear. If there were only two possible states tomorrow, we might express them in an *event tree* like this:

$$\text{Today} \Big\langle \begin{array}{l} \text{Tomorrow: ``Up'' State} \\ \text{Tomorrow: ``Down'' State} \end{array}$$
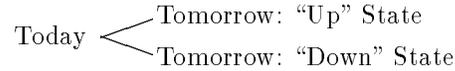
The branches illustrate how the future unfolds, moving from left (now, today) to right (later, tomorrow). We could handle more than two possibilities by including more branches in the tree, but two is sufficient to make the point that the future differs from the present in having more possibilities. We breathe some life into this framework by being specific about the states. Examples include:

Up State	Down State
Knicks Win	Knicks Lose
Yields Rise	Yields Fall
GM Defaults	GM Does Not Default

Which states are most useful depends on the application. If we're betting on basketball the Knicks' score is the most relevant, but if we're interested in fixed income securities the other two are probably more useful. Either way, you should be getting the idea that this as as much art as science.

By far the most common application in fixed income analysis concerns the behavior of the one-period "short" rate of interest. Let us say, to be concrete, that the short rate $r = y_1$ (the yield on a six-month zero) is currently 5.00 percent, and may either rise to 6.00 or fall to 4.00 in six months. We can summarize this in a *short rate tree* like this:

$$5.000 \Big\langle \begin{array}{l} 6.000 \\ 4.000 \end{array}$$

We'll use similar trees repeatedly in the rest of the course.

Cash Flows

Given a list of states, assets can be viewed as combinations or "baskets" of cash flows in different states. They are baskets in the same sense as the fruit baskets of the previous section. We refer to the assets and cash flows as *state-contingent claims*, since the payments to which the owner has claim are contingent on the state. If you bet on the Knicks, your bet only has value if they win.

Several examples of state-contingent claims fit naturally into the framework of the short rate tree we just looked at. One is a one-period riskfree bond, whose cash flows can be represented

$$0 \diagdown \genfrac{}{}{0pt}{}{100.00}{100.00}$$

This diagram is an example of a *cash flow tree*. It tells us, in this case, that the bond delivers 100 to its owner regardless of the state — in both states, that is. A second example is a two-period zero-coupon bond. Since a two-period bond is a claim to a one-period bond one period from now, its cash flows might be represented

$$0 \diagdown \genfrac{}{}{0pt}{}{97.087}{98.039}$$

The cash flows in this case are $100/(1 + r/2)$ for the appropriate values of r. For example, $97.087 = 100/(1 + 0.06/2)$.

A final example is an option on a two-period (one-year) zero: a European (more later) call option that allows the owner to buy the zero in six months for 98. The owner will only do this if the price of the zero is greater than 98, in which case he can pocket the difference. It thus generates a cash flow in each state of

$$\text{Option Cash Flow} = \max\left(0, \text{Price} - 98\right).$$

The cash flow tree is therefore

$$0 \diagdown \genfrac{}{}{0pt}{}{0.000}{0.039}$$

with the owner of the option getting 3.9 cents if the six-month yield falls, nothing if it rises.

State Prices

Our objective, if you recall, is to value fixed income derivatives. With this in mind, we now have

i. a list of states, which we label s in general, or u and d in our up/down examples, and

ii. a description of an asset as a collection of state-contingent cash flows, which we label c_s for state s.

If we had *state prices* q_s, say, giving us the value of one dollar in each state s, we could value an asset by summing the products of price and cash flow. For the up/down examples we could express this mathematically as

$$\text{Asset Price} = q_u c_u + q_d c_d, \tag{7.1}$$

and in more general settings (trees with more than two branches) we could write

$$\text{Asset Price} = \sum_s q_s c_s.$$

I refer to equation (7.1) as the *pricing relation*, since it tells us how to compute an asset's price from state prices and cash flows. The logic is just what we've done since the start of the course: we divide an asset into simpler components and find its price by summing the values of the components.

This has an abstract sound to it, but the reasoning is no different from anything we've done before. As long as asset prices are immune to arbitrage, the mysterious state prices exist whether we can find them in the *Journal* or not:

Proposition 7.2 *(state prices in arbitrage-free settings)*. Consider a market in which people can buy and sell assets in any quantities they like. Then if (and only if) the prices of assets do not allow arbitrage opportunities, we can derive positive state prices that are consistent with the prices of assets.

To see how this works, let us continue the option example from the previous subsection. With two states (up and down), we can find the state prices q_u and q_d if we know the prices of two assets. One asset is a six-month zero, which is worth 97.561 [=100/(1+.05/2)]. This is a claim to 100 in both states, so the price should satisfy

$$97.561 = q_u \times 100 + q_d \times 100.$$

A second asset is the call option. Let us say that the option's price is 0.0191 (1.91 cents). Since the option is a claim to 0.039 in the down state, we have

$$0.0191 = q_d \times 0.039.$$

The two equations together give us the state prices, $q_u = 0.4878$ and $q_d = 0.4878$, which we can express in the *state price tree*,

$$1.0000 \begin{cases} 0.4878 \\ 0.4878 \end{cases}$$

In the tree we have added the price 1.0000 at the root of the tree: a dollar today is worth one dollar.

In practice, we generally do the reverse: we start with a model of state prices and use it to find prices of fixed income securities. A popular convention, not entirely innocuous, is to divide the discount factor exactly in half, so that half of the value goes to each state. We know that the one-period discount factor d_1 is the sum of the state prices:

$$d_1 = q_u + q_d. \tag{7.2}$$

This says simply that the value of a claim to one dollar in both states is the sum of the prices in the two states. If the prices are equal, then each state price is one-half the discount factor:

$$q_u = q_d = 0.5 \times d_1 = \frac{0.5}{1 + r/2}. \tag{7.3}$$

I refer to this equation as the *fifty-fifty rule*, since it tells us to compute the state prices by dividing the one-period discount factor in two equal pieces. Once we have the state prices, it is relatively easy to find the prices of assets like call options.

Equations (7.1) and (7.3) form the basis of our theory of asset pricing. Given them, and a tree for the short rate of interest, we can compute prices for any asset we like.

7.3   Alternative Representations of State Prices (optional)

When you read books and articles on financial derivatives, you may come across two other ways of expressing state prices. The choice among them is pure convenience: all three versions contain the same information.

One common way of representing state prices is with a *pricing kernel*, a random variable whose values we denote by m_s. The pricing kernel divides the state price up into two components:

$$q_s = \pi_s m_s$$

where π_s is the probability that state s occurs. The analog to equation (7.1) in this case is

$$\text{Asset Price} = \sum_s \pi_s m_s c_s.$$

This is commonly written in more compact form as

$$\text{Asset Price} = E(mc), \tag{7.1a}$$

the expectation (using the probabilities π_s) of the product mc. The value of one dollar in each state is $d_1 = E(m)$.

The pricing kernel representation highlights the role played by probabilities in state prices: we might guess that uncommon events have low prices as a result. It also highlights the role of risk aversion. To see this, suppose m_s is the same in all states. Since the probabilities sum to one (they're probabilities, after all), equation (7.2) tells us that the constant value is just the discount factor d_1. The pricing relation then becomes:

$$\text{Asset Price} = d_1 E(c).$$

The price, in other words, is the discounted value of expected cash flows. Since only the expected cash flow affects the price we think of pricing being risk neutral, this being the way a person who didn't care about risk would value it. More generally, of course, m_s is not the same in all states, and we can think of variations in m_s across states as reflecting the attitudes toward risk of market participants.

A second popular way of representing state prices is with what are referred to, somewhat misleadingly, as *risk-neutral probabilities*. We denote them by π_s^* and define them as

$$\pi_s^* = \frac{q_s}{\sum_{s'} q_{s'}} = \frac{q_s}{d_1},$$

the second equality following from equation (7.2). The pricing relation becomes

$$\text{Asset Price} = d_1 \sum_s \pi_s^* c_s = d_1 E^*(c), \qquad (7.1b)$$

where $E^*(c)$ means the expectation of c using probabilities π^*.

The terminology deserves some explanation. People refer to the π^*'s as *probabilities* for one obvious reason: they are positive (since state prices are positive) and they sum to one. But unless m is constant, they are not the true probabilities π. The modifier *risk neutral* is added to distinguish them from the true probabilities, and because the form of (7.1b) is the same as our risk-neutral pricing with constant m. This is a little misleading: the effects of risk aversion are built in.

The option example continued. Suppose the short rate rises and falls with equal probability, so that $\pi_u = \pi_d = 0.5$. Then the pricing kernel takes on the values $m_u = 0.9756$ and $m_d = 0.9756$ and the risk neutral probabilities are $\pi_u^* = 0.500$ and $\pi_d^* = 0.500$.

The distinction between true probabilities π_s and risk-neutral probabilities π_s^* is often ignored, but it comes to the fore when we think about returns. An asset with with price p cash flows c_s has gross return

$$1 + h_s = \frac{c_s}{p} - 1.$$

If the cash flows are uncertain, then so is the return. But what about the expected return? Is there a premium for assuming risk? One view of this problem builds on equation (7.1a)

and bears a passing resemblance to the CAPM — in fact, it's a generalization of it. The expected return is

$$E(1 + h) = \frac{E(c)}{E(mc)} = \frac{E(c)}{d_1 E(c) + Cov(m, c)}.$$

(The second equality stems from the definition of the covariance of the product of two random variables, if that helps, and $d_1 = E(m)$.) The covariance plays the same role as "beta" in the CAPM: the only risk that affects the expected return is risk that's correlated with the "market" m.

Another view is based on risk-neutral probabilities. If we use equation (7.1b), the expected return is

$$E(1 + h) = \frac{\pi_u c_u + \pi_d c_d}{d_1(\pi_u^* c_u + \pi_d^* c_d)}.$$

If the $\pi_u = \pi_d^*$ (the true and risk-neutral probabilities are the same), then the expected return is $E(1 + h) = 1/d_1 = 1 + r/2$, the same as the one-period riskfree rate. The asset has a larger expected return (a positive risk premium) if the cash flows are greater in the state in which the true probability is less than the risk-neutral probability.

Suffice it to say that it's not entirely innocuous to equate, as we have done, the true probabilities and the risk-neutral probabilities. Nevertheless, that's what most people do. For most applications, I view this as a useful simplification rather than an outright mistake.

7.4   Multi-Period Contingent Claims

The bad news is that we've just begun: to make this useful, we need to extend the work of Section 7.2 to many periods. The good news is that there's not much more to it than what we've done so far. The algebra gets a little more involved, but the calculations are readily done on a spreadsheet (see, for example, Appendix C). I refer to the model in this section as the Ho and Lee model, after the people who first designed it. (Ho was a professor at Stern who now runs his own business.)

States and Interest Rate Trees

Although states can represent many kinds of events, for the rest of this chapter we consider only one: short term interest rates. We do this because the short rate is the single most important factor affecting prices of fixed income (or interest-sensitive) securities, and because we can (and will) add other factors later on when an application calls for it.

We need, then, a description of how the one-period "short" rate of interest evolves through time: what the possibilities are six months or even five years from now. If we

continue to think of a period as six months, then the short rate for our purposes is the yield on a six-month zero. The list of future short rate possibilities we get by extending the tree of the previous section.

Let us say, to be concrete, that the short rate r at any date $t+1$ is related to the short rate at t by

$$r_{t+1} = r_t + \mu_{t+1} + \varepsilon_{t+1}, \tag{7.4}$$

with

$$\varepsilon_{t+1} = \begin{cases} +\sigma & \text{with probability one-half} \\ -\sigma & \text{with probability one-half} \end{cases}$$

This is a little complicated, so let's consider it term by term. The parameter μ_t is the predictable part of the change in r. If we expect r to rise by 0.01, or 100 basis points, between dates t and $t+1$, we set $\mu_{t+1} = 0.01$. We need a value of μ for every future period we are interested in. If we want to consider instruments with cash flows over ten years, then we need values for 20 periods: μ_{t+1}, μ_{t+2}, and so on, through μ_{t+20}. The random variable ε is the unpredictable part of the change in r. It causes r either to rise or fall by an amount σ. I've assumed that each happens with the same probability, one half. As a result, σ is the standard deviation or *volatility* of a six-month changes in the short rate. You can imagine using interest rate data to get an estimate, something we'll come back to later on. For now, think of the parameters, μ and σ, as having known values.

We use (7.4) to generate a multiperiod short rate tree. To make this concrete, suppose the current short rate is $r_t = 0.05$, that $\mu_{t+n} = 0$ for all n, and that $\sigma = 0.01$. Then at date $t+1$ the short rate takes on the values 0.06 and 0.04, as in the previous section. At date $t+2$ the short rate takes on the values 0.07, 0.05, and 0.03. The short rate tree over four periods looks like this:
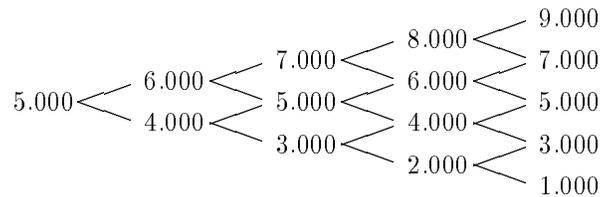
States, in this diagram, are represented by nodes of the tree: a total of fifteen if we include the "root." You'll note that states "recombine" in this tree: up/down is the same as down/up, for example — we get 5 either way. There is no particular reason for this, but one consequence is that we have fewer states to worry about than if states branched out without combining again.

Each route or path through the tree tells us a story about interest rate movements. Two examples are:

- *Path A (up,down,up,down).* The short rates are 5, 6, 5, 6, and 5.

- *Path B (down,down,down,up).* The short rates are 5, 4, 3, 2, and 3.

To fix this in your mind, you might draw the path through the tree for each one. For a tree of this size, there are 16 [$=2^4$] different paths, and for a tree of length n there are 2^n paths: the longer the tree, the more possible interest rate paths there are.

With a relatively large number of states, it's convenient to have a shorthand way of representing them. A relatively simple way is to label nodes by their location: the date, given by the horizontal position, and the total number of up moves, indicated by the vertical position. We number the dates n by 0, 1, 2, and so on. We number the up moves by i, a number between zero and n (since over n periods we can't have more than n up moves). The pair (i,n) then tells us which node we have in mind and thus uniquely identifies the state. One example is the root, which we label $(0,0)$. Another is the node with the box around it:
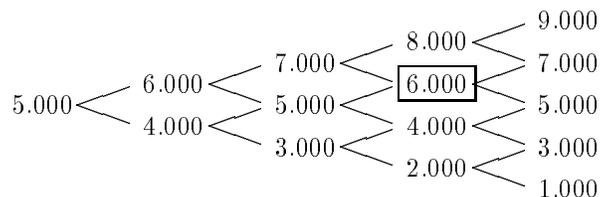
Its position is three periods to the right and is reached by two up moves and one down move, so we label it $(2,3)$.

For future reference, note that equation (7.4) can be be used to generate a formula for the short rate in each state. If the current short rate is $r(0,0)$, the short rate in state (i,n) is

$$r(i,n) = r(0,0) + \sum_{j=1}^{n} \mu_{t+j} + (2i - n)\sigma. \tag{7.5}$$

The last term is a little mysterious, so let's think through what it means. In state (i,n) we have experienced a total of i up moves over n periods. Apparently we experienced $n - i$ down moves, as well, since we move either up or down every period. The total effect of up and down moves is therefore

$$i\sigma - (n - i)\sigma = (2i - n)\sigma,$$

as written in equation (7.5).

Price Paths

The next step is to expand the rate tree to a broader range of assets — to describe, for example, the possible paths for the price of a long bond, the slope of the spot rate curve, the price of an option, or some other feature of interest. Our approach to each of these issues is to break the problem into a series of two-period problems, like those of Section 7.2. Like most things in this course, it's easier to follow by example. I use two: a three-period bond and an abstract state-contingent claim.

The principles of asset pricing are the same as Section 7.2, but for emphasis I repeat them and add boxes. Principle one: at each node, the value of an asset is sum of the current cash flow and the values of the cash flows of the two branches coming out of it:

$$\text{Asset Price} = \text{Current Cash Flow} + q_u c_u + q_d c_d. \qquad (7.1)$$

The cash flows are understood to include both explicit claims, like coupon interest and principal, and claims to the bond in the future. A two-period zero, for example, is a claim to a one-period zero one period in the future, and a two-period bond is a claim in one period to the coupon plus a one-period bond. To make the pricing relation operational we need, first, the prices for the two branches, q_u and q_d, and the associated cash flows, c_u and c_d. Principle two: we compute prices using the fifty-fifty rule,

$$q_u = q_d = 0.5 \times d_1 = \frac{0.5}{1 + r/2}. \qquad (7.3)$$

Since r varies across nodes of the tree, so too do the prices q_u and q_d.

Now let's put this to work and follow the price of a long bond through the tree. For reasons that should become clear when we look at options, we look at the ex-dividend price — the price excluding the current coupon payment. To be specific, consider the evolution of the price of a 6 percent three-period bond, three periods being enough to make the point without killing ourselves with algebra. We start at maturity, where bond's cash flows are known: 103 (interest plus principal) in each state. In the previous period, the bond has one period left, and its price is just the discounted value of 103,

$$\frac{103}{1 + r/2},$$

for the appropriate short rate r. Using the rates from the short rate tree (7, 5, and 3) we can fill in the values for the last column of our truncated tree:

This gives us the possible values of the bond at date $t + 2$, the last column in our tree.

We continue by moving backwards in time through the tree — to the left, that is. At date $t + 1$ the bond is a claim to its own value one period later, namely the values we just calculated, plus a 3 dollar coupon. For example, at node $(1,1)$ (the first up state) the bond is a claim to 99.52+3.00 and 100.49+3.00 in the up and down states, respectively. Applying the pricing formula, equation (7.1), and the fifty-fifty rule, equation (7.3), we find that the price is

$$\frac{0.5}{1 + .06/2} \times [(99.52 + 3.00) + (100.49 + 3.00)] = 100.00.$$

By similar methods we find that the price in the down state, $(1,0)$, is 101.94. Finally, the bond in the initial period is a claim to 100.00+3.00 and 101.94+3.00 in the up and down states, respectively, and thus has value

$$\frac{0.5}{1 + 0.05/2} \times [(100.00 + 3.00) + (101.94 + 3.00)] = 101.44.$$

The complete tree is

```
                               99.52
                  100.00
     101.44                    100.49
                  101.94
                               101.48
```
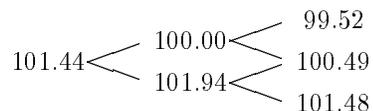
Thus we see that the bond can reach a maximum price of 101.94 (if the short rate falls to 4) or a minimum of 99.52 (if the short rate rises to 7).

A more abstract example is a claim to one dollar in state $(2,2)$, the (up,up) state two periods from now, and nothing in the other states. We refer to this asset as a pure state-contingent claim, since it has a positive cash flow in only one state. Similar methods give us a tree of

```
                               1.0000
                  .4584
     .2368                     .0000
                  .0000
                               .0000
```
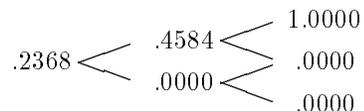
I leave the details to you. The tree tells us that the claim is worth 0.2368 now. One period from now it will be worth 0.4584 if the short rate rises to 6, or zero if the short rate falls, thereby making it impossible to get to state (2,2).

To summarize: these calculations follow from repeated application of the pricing relation, equation (7.1), using the "fifty-fifty split" of (7.3) to fix the one-period state prices.

State Prices

It's apparent from the previous subsection that we can value any asset we like using one-period state prices (the fifty-fifty split of the one-period discount factor) and the pricing relation (7.1). But it should also be apparent that it's a fair amount of work to fill out the entire tree — and, moreover, to do it again for every asset. Sometimes this is necessary, but for some purposes we can compute all the required state prices at one time and use them as needed.

To remind ourselves of notation: we labeled states by (i, n), indicating the number of periods n from the present and the number i of up moves, and state prices by $q(i, n)$. *These prices differ from q_u and q_d in the previous section in giving us the current, date-t value of one dollar in each state, not the value one period prior to the payment.*

The state prices can be computed from the following relation, which I refer to as *Duffie's formula*:

$$
q(i, n+1) = \begin{cases} \dfrac{0.5q(i, n)}{1 + r(i, n)/2} & \text{if } i = 0, n + 1 \\[3ex] \dfrac{0.5q(i, n)}{1 + r(i, n)/2} + \dfrac{0.5q(i-1, n)}{1 + r(i-1, n)/2} & \text{if } 0 < i < n + 1 \end{cases} \tag{7.6}
$$

starting with $q(0, 0) = 1$ (a dollar today is worth a dollar). With a little effort and patience, you should be able to convince yourself that this is an application of the same principles we applied earlier. Even better, Appendix C shows how it can be implemented in a spreadsheet.

By way of example, the state prices for our five-period interest rate tree are
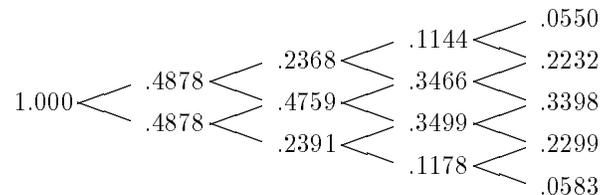
All of these numbers can be derived from the discount factors. The pair of 0.4878's at the start of the tree, for example, are half of the discount factor for the initial node of the tree:

$$
\frac{0.5}{1 + .05/2} = 0.4878.
$$

The next up node — (2,2) in our labeling system — is 0.4878 times half the discount factor from the first up node, $0.5/(1+.06/2)$. The current price of 0.2368 is the same value we computed in the previous subsection. Its different position in the tree there illustrates a change in our perspective. In the previous subsection, we followed the price of the claim through the tree. Here we are simply reporting the prices *now* of claims to different pure state-contingent claims.

Now that we have the state prices we can put them to work. One of the simplest things we can do is value known cash flows, as we did before with discount factors. Recall that the discount factor d_n is the price of one dollar n periods from now. In our current setting, we might say this more elaborately: the price of a claim to one dollar *in all states* n periods from now is

$$d_n = \sum_{i=0}^{n} q(i,n) \times 1; \tag{7.7}$$

ie, the discount factors are the sums of state prices down columns of the state-price tree. For our example, the discount factors are: $d_0 = 1.0000$, $d_1 = 0.9756$, $d_2 = 0.9518$, $d_3 = 0.9287$, and $d_4 = 0.9062$. We could go on, if we wished, to compute spot rates using equation (2.3).

I think the relation between discount factors and state prices gives us a clear idea what we've just done. In earlier sections we examined the time value of money, which we expressed in discount factors or (equivalently) spot rates. You might picture this as drawing a time line and labelling the various points with the appropriate d's or y's:

$$
\begin{array}{c c c c c c c}
 & d_1 & & d_2 & & d_3 & \\
\vdash & & \vdash & & \vdash & & \vdash \\
 & y_1 & & y_2 & & y_3 & \\
t & & t{+}1 & & t{+}2 & & t{+}3
\end{array}
$$
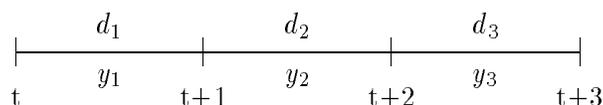
What we have done in this chapter is expand the picture vertically, in a tree, by distinguishing between different states at each date. In this sense, what we've done is natural progression from where we started.

Back to work. We use similar methods to value claims to uncertain cash flows. Consider an arbitrary claim on the state-contingent cash flows $c(n,i)$. The value of a specific cash flow is the product of price and cash flow, $q(i,n) \times c(i,n)$. The price of the asset as a whole is the sum of the prices of the individual claims:

$$\text{Asset Price} = \sum_{n} \left[\sum_{i=0}^{n} q(i,n)c(i,n) \right], \tag{7.8}$$

an extension of (7.1) to many periods.

Let me summarize briefly. We have described a method, summarized by Duffie's formula, of computing state prices: the prices of one dollar payable in specific situations, which we refer to as states. The bad news is that it's a little abstract. The good news is that the formula is basically all we need to know, for most purposes, about the theory of fixed income security pricing. The rest is judgment and arithmetic.

7.5   Choosing Parameters

What we've just done is fine, but there are a few tricks in choosing parameters that make the model more realistic. One is to think more carefully about the "50-50 split," referred to in most treatments as setting the risk neutral probabilities equal to one-half. We will follow tradition and continue to disregard this as an issue, without any particular justification. A second trick is to choose a realistic value for σ, the volatility parameter. One approach, suggested by the form of (7.4), is to set σ equal to the standard deviation of one-period interest rate changes. With a period of six months, a typical number is 1 percent, or 0.01, which is what we used above and will continue to use below. More sophisticated users might update their estimate with recent data, or infer it from option prices.

A final trick concerns the constants μ_t that govern predictable movements in the short rate. Standard practice is to choose these so that the model exactly reproduces the current spot rate curve. This is a little tedious numerically, but relatively easy if you have a good computer program. The computer chooses μ_{t+1}, μ_{t+2}, and so on, to match spot rates y_2, y_3, and so on. We could do this, if we were patient, one parameter at a time: We choose the initial short rate $r(0,0)$ equal to y_1, the first point on the spot rate curve. Then we vary μ_{t+1} until the model generates exactly the second point on the spot rate curve, y_2. This may take some time if you do it manually, but is not difficult in principle. Once we have μ_{t+1}, we vary μ_{t+2} until the model reproduces the third point on the spot rate curve, y_3. And so on for as many points on the spot rate curve as we like. The result is a model that agrees exactly with the current yield curve.

As an example, consider the LIBOR-based spot rates used in Section 5.2:

Maturity (Half-Years)	Discount Factor	Spot Rate (%)
1	0.9707	6.036
2	0.9443	5.809
3	0.9175	5.824
4	0.8931	5.839
5	0.8644	5.914
6	0.8378	5.989

To reproduce these numbers with our model, we need $\mu_{t+1} = -0.0045$, $\mu_{t+2} = 0.0028$, $\mu_{t+3} = 0.0004$, and $\mu_{t+4} = 0.0035$. These values give us a rate tree of

```
                                                      10.263
                                           8.913 <
                                 7.869 <            8.263
                      6.587 <            6.913 <
                                 5.869 <            6.263
6.036 <               4.587 <            4.913 <
                                 3.869 <            4.263
                                           2.913 <
                                                      2.263
```

and state prices

$$
\begin{array}{ccccccc}
& & & & & & .0541 \\
& & & & .1130 & < & \\
& & & .2349 & < & & .2196 \\
& & .4854 & < & .3424 & < & \\
& .4854 & < & .4722 & < & & .3342 \\
1.0000 & < & & & .3457 & < & \\
& .4854 & < & .2372 & < & & .2261 \\
& & & & .1164 & < & \\
& & & & & & .0573 \\
\end{array}
$$
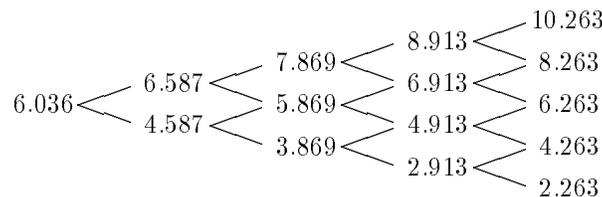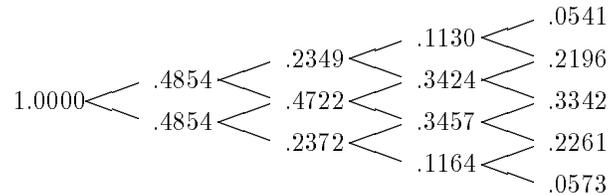
You can verify for yourself that the discount factors implied by these prices are those we started with.

Matching the spot rate curve like this requires some effort, but is absolutely necessary if we are to take the model seriously. If our model misprices assets with riskfree cash flows, how can we trust it with more complex instruments? For this reason and others, practitioners generally "tune" their models along these lines before using them.

Drift Parameters and Forward Rates (optional)

Explain connection ??

7.6 Examples

All of this is much more interesting if we have an application in mind. The following examples illustrate how we might we apply our new technology in practice. Each illustrates that something of this sort is needed: there is simply no way to apply our earlier methods for riskfree cash flows to the examples — indeed to many of the more popular fixed income instruments.

The calculations, in all cases, are based on the rate tree and state prices we calibrated in the previous subsection.

Betting on Interest Rates

Consider an outright bet: we get \$10 if 6-month LIBOR is greater than 7 percent four periods (two years) from now, nothing otherwise. This is not a common instrument (bookies prefer sports), but has some similarity to the *digital options* now available. (Digital because they're on-off: the payoff depends only whether the underlying is beyond some threshold, not on its level.) If we look at the rate tree in the previous subsection, we find that the bet generates state-contingent cash flows that look like this:

```
                                                      ___ 10.000
                                            0.000 <
                                  0.000 <      ___ 10.000
                        0.000 <      ___ 0.000 <
              0.000 <      ___ 0.000 <      ___ 0.000
  0.000 <      ___ 0.000 <      ___ 0.000 <
       ___ 0.000 <      ___ 0.000 <      ___ 0.000
              ___ 0.000 <      ___ 0.000 <
                        0.000 <      ___ 0.000
                              ___ 0.000 <
                                      ___ 0.000
```
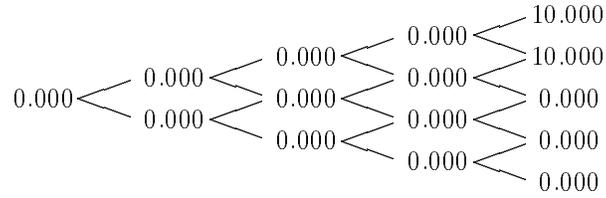
The 10's indicate the times we win. And, of course, there are no cash flows prior to 2 years from now.

What is this bet worth? The value, according to our model, is the sum of the product of state-price times cash flow:

$$(0.0541 + 0.2196) \times 10 = 2.737.$$

With nonzero cash flows in only two states, this is relatively easy.

Interest Rate Cap

A more common example of a fixed income derivative is an interest rate cap. This differs from the first example in having payoffs that depend on the amount by which the short rate exceeds its threshold or cap. Specifically, consider a two-year, semiannual 7 percent cap, on a notional principal of \$250m. The cap pays, each period for two years, the difference between 6-month LIBOR and 7 percent if positive, and nothing otherwise. As with interest payments, the cash flows occur one period after LIBOR is quoted. If we label LIBOR by r, then the payment in period $t + n + 1$ is

$$\text{Cap Cash Flow at t+n+1} = \max\left(0, r_{t+n} - 0.07\right) \times \frac{\text{Notional Principal}}{2}.$$

We divide by two because the cap is semiannual.

This way of specifying the cash flows does not fit into our framework, because the payments at date $t + n + 1$ depend on the node at $t + n$: a node at $t + n + 1$ can be associated with different cash flows depending on the path taken. This is a sign that our definition of a state, and the recombining nature of the tree in particular, is overly restrictive. In this case, however, we can get around the problem: since the cash flows are known one period in advance, we can move them back one period if we discount appropriately. The cash flows at $t + n + 1$ have value at $t + n$ of

$$\text{Cap Cash Flow at t+n} = \max\left(0, r_{t+n} - 0.07\right) \times \frac{1}{1 + r_{t+n}/2} \times \frac{\text{Notional Principal}}{2},$$

with the understanding that the rate r varies across states.

We compute the cash flows in two steps, using our LIBOR-based rate tree. Expressed as a percentage, the interest rate component, $\max(0, r - 0.07)$, is (expressed as a percentage)

```
0.000 <    0.000 <    0.869 <    1.913 <    0.000
              0.000      0.000      0.000      0.000
              0.000      0.000      0.000      0.000
                         0.000      0.000      0.000
                                    0.000      0.000
                                               0.000
```
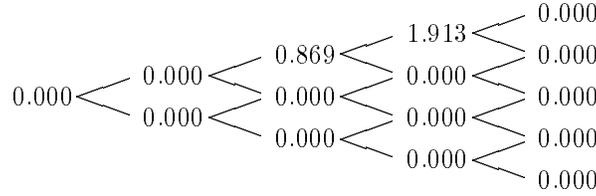
We discount this back one period and multiply by one-half the notional to get the cash flows (in millions):

```
0.000 <    0.000 <    1.045 <    2.289 <    0.000
              0.000      0.000      0.000      0.000
              0.000      0.000      0.000      0.000
                         0.000      0.000      0.000
                                    0.000      0.000
                                               0.000
```

Using the state prices, we find that the cap is worth

$$\text{Cap Price} = 0.2349 \times 1.045 + 0.1130 \times 2.289 = 0.504,$$

or 504 thousand dollars.

Price Path of a Zero

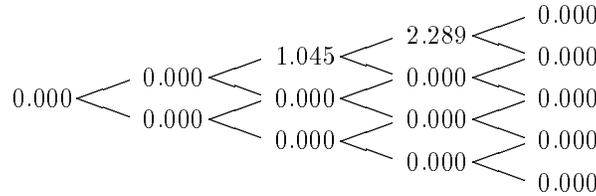A somewhat different example is a 30-month zero, which we use in the next subsection as the underlying for an option. Here we consider its possible price paths through the tree. One of the differences in our approach here is that we compute prices recursively, starting with the last period in the tree, instead of using Duffie's formula for state prices.

A five-period, 30-month zero is a claim to \$100 in 30 months. Its value at a node in the tree two years from now is 100 discounted at the appropriate rate from the tree. For example, the top-right node in the tree has a rate of 10.263%, so the value of the zero at this node is

$$\text{Price of Zero} = \frac{100}{1 + 0.10263/2} = 95.12.$$

In earlier periods, we work our way backwards through the tree using the pricing relation and the fifty-fifty rule, equations (7.1) and (7.3), respectively. The result of these calculations is

```
                                                    ╱ 95.12
                                        91.50 ─────<
                              88.89 ────<         ╲ 96.03
                    87.32 ───<        ╲ 93.27 ────< 96.96
          86.44 ───<        ╲ 91.50 ──<
                    90.78 ───<        ╱ 95.10 ────< 96.96
                              94.22 ────<         ╱ 97.91
                                        96.98 ─────<
                                                    ╲ 98.88
```
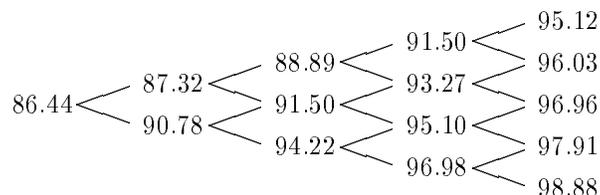
The tree summarizes the possible paths for the zero's price between now and maturity.

One feature of the price tree is that the range of variation of prices across states gets smaller as the bond approaches maturity. One way to think about this is that the price at maturity is known: 100. Another way is to think of the bond's duration declining as we move through the tree, resulting in smaller effects of yield differences as we approach maturity.
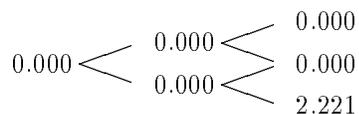
Option on a Zero

We continue the example with a call option on a zero: an option to buy the zero at a predetermined *strike price* at a future date. Specifically, consider a one-year European call option with a strike price of 92 on the 30-month zero examined in the previous subsection. The option gives its owner the right to buy the bond for 92 in exactly two periods (an "American" option can be exercised early, but a "European" option cannot).

For the option, the only relevant parts of the tree are the possible zero prices in the second period: 88.89, 91.50, and 94.22. The first two are worthless: the strike price is greater than the price in these states, so the option would not be exercised. The third, however, produces a net cash flow of 2.22: the owner exercises the option to buy the zero for 92, then sells it at its market price of 94.22. The cash flows are, in general,

$$\text{Option Cash Flow} = \max\left(0, \text{Price} - 92\right),$$

which gives us a cash flow tree of

```
                              ╱ 0.000
                    0.000 ───<
          0.000 ───<        ╲ 0.000
                    0.000 ───<
                              ╲ 2.221
```

(2.221 being a more accurate version of 2.22). The price path for the option is therefore

```
                              ╱ 0.000
                    0.000 ───<
          0.527 ───<        ╲ 0.000
                    1.086 ───<
                              ╲ 2.221
```
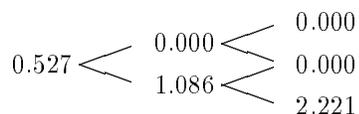
Ie, 52.7 cents per hundred dollars face value of the zero. The price may seem small, but for 52.7 cents the owner gets a chance at $2.22, more than four times the call price. This is typical of options: a small amount of money gets you a chance at much more. Of course, it also gets you a substantial chance to lose your entire investment.

Option Replication

One of the convenient features of binomial models is that there are only two possible one-period scenarios from each state (the two branches from each node). As a result, we can represent any possible one-period return with a combination of just two assets. We say that we use the two assets to *replicate* the third.

Option replication is a standard application, and gives us a new perspective on the option's price. Suppose we attempt to mimic the call option in the last subsection with a combination of the underlying zero and a one-period zero. Can we construct a combination of the two zeros that gives us exactly the payoffs of the option? If so, then the option must have the same price as the combination of the other two assets that delivers the same payoffs.

To be specific, suppose the call option at a particular node of the tree can be reproduced with x_z units of the underlying zero ("zero") and x_s units of a one-period zero ("short") that delivers 100 in both states following. The quantities x_z and x_s are chosen so that the combination delivers the same payoffs for both the up and down branches. We need, in other words,

$$\text{Call}_u = x_z \times \text{Zero}_u + x_s \times \text{Short}_u$$
$$\text{Call}_d = x_z \times \text{Zero}_d + x_s \times \text{Short}_d.$$

The notation here is a little ugly, but hopefully transparent. For the initial node, the two equations are

$$0.000 = x_z \times 87.32 + x_s \times 100$$
$$1.086 = x_z \times 90.78 + x_s \times 100,$$

which gives us $x_z = 0.314$ and $x_s = -0.274$ (the minus sign indicating a short position). With these quantities the price of the call is

$$
\begin{aligned}
\text{Price of Call Option} &= x_z \times \text{Price of Zero} + x_s \times \text{Price of Short} \\
&= 0.314 \times 86.44 - 0.274 \times \frac{100}{1 + 0.06036/2} \\
&= 0.527,
\end{aligned}
$$

as we found earlier.

Callable Zero

We continue the example through another step, and consider a "callable zero:" a 30-month zero which the issuer has the right to call in two periods. The issue is what the call option does to the duration of the zero.

Let us say, to make this concrete, that the issuer's call option is exactly as described in the previous subsection. From the perspective of the owner of the instrument, the callable zero is like owning a zero *and* being short a call option on it. Thus we might say

$$\text{Price of Callable Zero} = \text{Price of Zero} - \text{Price of Call Option}.$$

This is yet another example of dividing a complex instrument into simpler components.

With this in mind, the price of a callable zero is the difference between the prices of the zero and the call option. Over the first two periods the price path for the callable zero is therefore

$$
85.91 <
\begin{matrix}
87.32 < \begin{matrix} 88.89 \\ 91.50 \end{matrix} \\
89.69 < \begin{matrix} 91.50 \\ 92.00 \end{matrix}
\end{matrix}
$$

We might now ask ourselves how much this differs from the underlying bond. One way to address this question is to replicate the cash flows of the callable bond at each node with a combination of the underlying bond and a one-period zero, just as we did with the option. The replicating strategy consists of quantities x_z given by

$$
0.686 <
\begin{matrix}
1.000 < \begin{matrix} \text{(na)} \\ \text{(na)} \end{matrix} \\
0.183 < \begin{matrix} \text{(na)} \\ \text{(na)} \end{matrix}
\end{matrix}
$$

and quantities x_s of the one-period zero:

$$
0.274 <
\begin{matrix}
0.000 < \begin{matrix} \text{(na)} \\ \text{(na)} \end{matrix} \\
0.748 < \begin{matrix} \text{(na)} \\ \text{(na)} \end{matrix}
\end{matrix}
$$

We see that in the initial node the callable bond is equivalent to a combination of 0.686 units of the underlying bond and 0.274 units of the short bond. This gives the callable bond more of a "short flavor" than the underlying zero (which of course is 1.000 unit of itself). The replicating quantities in the second period depend on the route taken. If the short rate rises, the bond price moves away from the strike and behave more like the underlying bond. But if the short rate falls, the option behaves much more like a short bond.

This gives us an idea that call features in bonds change their interest rate sensitivity dramatically, which makes you wonder about the duration. We'll look at this in more detail when we turn to options.

7.7 Other Models

The model developed in Section 7.4 is adapted from a paper by Ho and Lee, and is widely used by practitioners. Nevertheless, it also has several features that might trouble us. One is that it gives us a pretty crude list of possibilities: two in six months, three in one year, and so on. This feature is relatively easy to correct: we simply use a finer time interval. If we use, say, a time interval of a week, then there are 26 periods in six months and thus 26 possible interest rates. The cost is small if we have good software. A second troubling feature is that interest rates can turn negative in the Ho and Lee model, and in real-life they can't (nominal interest rates on bonds are bounded below by the zero nominal return from holding cash). A third is that volatility is fixed throughout the tree: there is no possibility for changes in volatility through time.

Both of these last two problems are easily corrected with little change to our theoretical framework: we simply modify the rules for generating the short rate tree. Once we do this, the usual rules for computing state prices apply.

Multiplicative Interest Rate Model

A standard math solution to positive quantities is logarithms: if a variable x is positive, then $\log x$ can take on any real value. (The function log will be understood to mean ther natural logarithm, sometimes denoted ln.) In this spirit, suppose the short rate evolves according to

$$\log r_{t+1} = \log r_t + \mu_{t+1} + \varepsilon_{t+1}, \tag{7.9}$$

where (as before)

$$\varepsilon_{t+1} = \begin{cases} +\sigma & \text{with probability one-half} \\ -\sigma & \text{with probability one-half} \end{cases}$$

If we rewite this in terms of r, rather than $\log r$, it's apparent why this keeps r positive:

$$r_{t+1} = r_t e^{\mu_{t+1} + \varepsilon_{t+1}}.$$

The changes μ and ε are multiplicative rather than additive. Since the multiplicative factor is always positive, the short rate stays positive.

With this rule, the short rate in state (i, n) is

$$r(i, n) = r(0, 0) \times \exp\left(\mu_t^n + (2i - n)\sigma\right), \tag{7.10}$$

where $\mu_t^n \equiv \sum_{j=1}^n \mu_{t+j}$, an obvious variant of equation (7.5). Given the tree, we price assets as before. You may note, for example, that nothing in Duffie's formula depends on the way in which the short rate tree was generated.

Black, Derman, and Toy's Model

The popular Black-Derman-Toy model takes this one step further, setting short rates equal
to

$$r(n, i) = r(0, 0) \times \exp\left(\mu_t^n + (2i - n)\sigma_n\right). \tag{7.11}$$

This differs from the previous model only in the volatility, which is allowed to vary with the
time period n. In this way, the model is able to reproduce observed patterns of volatility
through time — the so-called term structure of volatility.

As with the previous model, asset pricing proceeds in the usual way from the short rate
tree.

7.8   Putting it in Perspective

We've seen how to price state-contigents claims, and can now join the "contingent claims
arms race" that is threatening to put modern finance in the hands of the nerds. But is
all this technology necessary? The answer is that we need a model to value instruments
with uncertain cash flows — there's generally no way around it. We can't apply observed
discount factors to known cash flows, as we did in earlier chapters, because the cash flows
aren't known. Sometimes we can price an asset by constructing a replicating strategy from
instruments whose prices we know, but not always. When we can't, we need the prices
of contingent claims. Period. That's why most Wall Street firms have proprietary models
(meaning they won't tell you exactly what they are) that they use to value positions that
market quotes can't handle, and why some customers might not be able to value their own
positions reliably.

The difficulty with contingent claims pricing is that the answer depends on the way in
which state prices are generated. There are good models and less good ones, and which
is which may depend on the application. Using these models in practice is a much art
as science — or maybe I should say more engineering than physics. Probably for this
reason many firms team research people with traders and salespeople, in the hope that the
combination is more powerful than either one together.

Given the elements of artistry in modeling, it's relatively easy to pick holes in any
particular model. A model is, after all, a simplification of reality, and in some cases this
results in overlooking things that we think are details but others think are critical. Among
the issues commonly raised about the binomial models we described in Sections 7.4 and 7.4
are

- *Mean reversion.* Interest rates, unlike equity prices, do not seem to be random walks, as equation (7.4) suggests. They tend, instead, to return slowly to their means. Thus when short rates hit 20 percent in the early 1980s, most observes expected them to return, eventually, to their previus levels. Moreover, mean reversion is not easily squeezed into a binomial framework.

- *Multiple factors.* We saw, when we studied duration, that we need more than one factor to account for the diverse movements in bond yields through time, yet our models focus on a single factor: the short rate. More complex models extend the binomial structure to settings with two or more factors in the hope of generating more accurate prices.

- *Calibration.* We often have several plausible ways of choosing the parameters of our models. How we do this depends on the data we have available and on the application.

In short, you can make asset pricing models as complicated as you like. Which extensions are most useful depends on the issue at hand.

Summary

1. States are examples of possible future events.

2. Pricing states is like pricing fruit.

3. Assets are claims to uncertain cash flows: the cash flows, we say, are contingent on the state.

4. Quantitative asset pricing consists of concocting a useful list of states and deducing (somehow) the prices of payments in each of them.

5. We developed a theoretical framework that identified states with the short rate, and priced state-contingent claims using the "fifty-fifty rule." The results are summarized in Duffie's formula.

Practice Problems

1. Your mission is to use the Ho and Lee model of Section 7.4 to compute state prices and discount factors.

    (a) If the current interest rate is 6 percent, $\sigma = 0.02$, and the "drift parameters" $\mu_t = 0$, what is the short rate tree over two periods (ie, for $n = 0, 1, 2$)?

(b) Consider a claim to one dollar in state $(i, n) = (1, 2)$. Use the pricing relation and fifty-fifty rule to compute the value of this claim in all states.

(c) Use Duffie's formula to compute state prices for the entire tree. Verify that your current price for state $(1, 2)$ is the same one you computed above.

(d) Use the state prices to find, for each date n, the discount factor d_n and spot rate y_n.

2. We will compute the price of an 18-month floating rate note with exaggerated upward sensitivity to interest rate movements. Calculations, in all cases, should be done with the rate tree and state prices reported in last subsection ("Calibration") of Section 7.4. The instrument calls for semiannual interest payments, over three six-month periods, of

$$\text{Interest Payment} = \frac{1}{2} \times \begin{cases} \text{LIBOR} & \text{if LIBOR} \leq 6 \\ 6 + (\text{LIBOR} - 6)^2 & \text{if LIBOR} > 6 \end{cases}$$

The one-half, as usual, converts the rate to semiannual units. After three periods, the note returns its principal of 100 as well as its final interest payment.

(a) Graph the interest payment versus LIBOR for LIBOR between 0 and 10.

(b) For the rate tree in the text (the "Calibration" section), compute the cash flows for each state and report them in a tree. Note that you will have to move them back one period, as we did with our interest rate cap example.

(c) Use the cash flows to compute the price path for this asset, indicating its value at each node in the tree. How does it differ from the price of a standard floating rate note?

3. Find the mistake, if any, in Tuckman's rate tree (Tuckman 1995, p 88). To avoid confusion, I suggest you use our notation and Tuckman's volatility, $\sigma = 0.00318198$. The problem divides into two parts:

(a) Find the μ_t's that reproduce the following discount factors and zero yields: $d_1 = 0.98044$, $d_2 = 0.95966$, $d_3 = 0.93776$, $y_1 = 0.0399$, $y_2 = 0.0416$, $y_3 = 0.0433$.

(b) What is the rate tree implied by your answer? How does it compare to Tuckman's?

4. Consider a European call option on the zero described in Section 7.6. The option expires in three periods, and has a strike price of 94.

(a) Compute the cash flow tree for this option.

(b) Use the cash flows to compute the option's price at each node of the tree.

(c) Find, for each node, the quantities of a one-period zero and the zero underlying the option that reproduce the payoffs of the option. Give an intuitive explanation for how these vary through the tree.

Further Reading

The theory in this chapter is pretty standard among academics and quant jocks on the street, but there are not many MBA level sources that approach bond pricing this way. Tuckman (1995, chs 5-8) is one of the closest. My version follows Duffie (1992), which is an extremely good book for PhD students. What I call Duffie's formula is on p 57. The model in Section 7.4 was designed by Ho and Lee (1985) when Ho was on the faculty at Stern, but is presented much differently in their paper. The Black, Derman, and Toy (1990) is a workhorse on the Street. Hull (1993) provides a relatively comprehensive review of fixed income models. Derman (1996) gives a user's perspective on how seriously to take models.

Part III

Instruments with Contingent Payments

Chapter 8

Interest Rate Futures

The leading exchange-traded fixed income derivatives are futures contracts: US treasury futures, Japanese and German government bond futures, eurodollar and euroyen futures, and so on. We review the properties of the most popular contracts and discuss how they might be used in practice.

8.1 Bond Forwards

Although futures are more common, forwards are conceptually simpler and therefore provide a useful starting point. We will see that a forward contract on a bond is equivalent to a long position in a long bond and a short position of equal value in a short bond. For this reason, forward contracts might be used to adjust the duration of a portfolio toward that of the long bond.

Cash Flows

A forward contract on a bond has three important dates — the trade date, the settlement date, and the maturity date of the underlying bond — as in this picture:
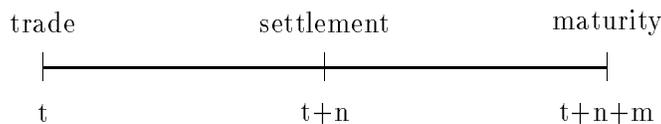
On the trade date t the buyer and seller arrange the terms of the contract, specifying payment at the settlement date $t + n$ for receipt of a bond maturing at date $t + n + m$. What makes this different from a standard purchase of a bond is that the trade date and settlement date can be separated by more than a few days — most often a few months, but possibly several years. What this does is lock in an interest rate at date t for the period between settlement and maturity. If a firm is worried that rates might rise, then it can use a forward contract to fix the rate now.

The terms of the contract (specifically, the price paid at settlement for the bond) are agreed to on the trade date. Thus a typical forward contract might specify that the purchaser agree at t to pay the seller F at date $t + n$ for the bond. We refer to the contract as a forward, because execution of the trade has been pushed forward in time.

Forward contracts sound like a relatively sophisticated instruments, but we can easily replicate them with bonds. The simplest case is a zero. Consider the forward purchase of a zero. The purchase involves payment of F at settlement in return for a zero paying 100 at maturity. Its cash flows might be listed

t	$t + n$	$t + n + m$
0	$-F$	100

The obvious question is what a reasonable price F is. To answer that, let's try to replicate the cash flows. First, we can replicate the final cash flow by buying, at date t, a zero with maturity $n + m$. But this leaves us with a payment of (say) p_{n+m} at t that the forward contract doesn't have and no payment F at settlement. To correct both, consider the simultaneous sale of x units of an n-period zero. These two trades generate cash flows

t	$t + n$	$t + n + m$
$-p_{n+m}$	0	100
$x \times p_n$	$-x \times 100$	0

If we choose x right, we can replicate the cash flows of the forward contract. To get a cash flow of zero at the trade date we need

$$x = \frac{p_{n+m}}{p_n}.$$

This tells us that the forward price must be

$$F = 100 \times \frac{p_{n+m}}{p_n}.$$

?? for n,m integers, find forward rates... Show that exposure is to the part of the forward rate curve between n and n+m.

As an example, consider a forward contract with settlement in six months on a six-month zero, meaning $n = m = 1$ measured in half-years. This is an artificial example, but bears a resemblance to a "6 by 12" *forward rate agreement*. If we use the LIBOR based interest rates from chapter 5, the prices of the two zeros are $p_1 = 97.07$ and $p_2 = 94.43$, implying a forward price of

$$F = 100 \times p_2/p_1 = 97.28.$$

This price implies, in turn, a forward *rate* of

$$97.28 = \frac{100}{1 + f_1/2},$$

or $f_1 = 5.58\%$. You can easily verify that the methods described in Section 2.6 give the same answer.

Forward contracts for coupon bonds have more cash flows — the coupons — but we approach them the same way. Let us say that the cash flows are

t	coupons	$t + n$	coupons	$t + n + m$
0	0	$-F$	C	$100 + C$

How might we replicate these cash flows? We could start, as we did before, with a long position in the bond, resulting in cash flows of

t	coupons	$t + n$	coupons	$t + n + m$
$-p_{n+m}$	C	C	C	$100 + C$

We now need to offset the purchase price p_{n+m} and the pre-settlement coupons. With par bonds and a flat yield curve this is relatively easy. The long bond sells at par. We can offset the purchase price with a short position in a short bond also selling at par:

t	coupons	$t + n$	coupons	$t + n + m$
$-p_{n+m}$	C	C	C	$100 + C$
p_n	$-C$	$-(100 + C)$	0	0

The result is a forward price of 100 (par).

If the yield curve isn't flat, we need to modify this a little. We buy the long bond, as before, and take a short position in a short bond (maturity n) with the same coupon. If the yield curve is upward sloping, then the short bond will sell for more than the long bond. We then arrange a long position in an n-period zero to make up the difference. As a result, the forward price will be less than 100. Conversely, if the yield curve is downward sloping, we need to add a short position in a short zero and the forward price will be greater than par.

Duration and Forward Contracts

Now that we've worked through the math, what have we found? Put simply, a forward contract is like a long position in a long bond and a short position of equal value in a short bond or bonds. Accordingly, the addition of a forward contract on a long bond to a fixed income portfolio is a way of would tend to add duration.

As with swaps, we cannot compute the duration of a forward contract on the trade date, since its value is zero. But we can certainly compute the duration of the bonds that replicate the forward, and of a portfolio that includes a forward contract, or compute the size of a forward position required to adjust duration to a particular value. We do this the same way we did for swaps.

By way of example, consider 100 invested in the 5-year 10% bond from Chapter 4. With a flat yield curve at 10%, the bond trades at par and has a duration of 3.86 years. Suppose we fear that yields will rise, and would like to eliminate our exposure to this possibility. One way to do this is to sell the bond forward: to arrange a price now for the bond in (say) six months. Another way is to use a popular forward contract to reduce the duration to zero. The duration strategy isn't as bullet-proof as selling the bond forward, for the usual reasons, but is not a bad first approximation.

To cast this in the terms we used earlier, suppose we have ready access to a six-month forward contract ($n = 1$) on a ten-year 10% bond ($m = 20$). What kind of forward position should we take? We are looking for a position of size x in the forward contract that gives us a duration of zero. Equation (4.7) tells us that the duration our overall position is then

$$0 = 3.86 \times 100 + D_{21} \times x - D_1 \times x.$$

Since the duration of the long bond under these circumstances is $D_{21} = 6.41(wrong??)$ and that of the short bond is $D_1 = 0.48$, the position is $x = -65$. The sign is negative since we are short.

This strategy immunizes us against general movements in bond yields, but not against "twists." If 10-year yields fall, but 5-year yields do not change, the hedged portfolio could lose money.

8.2   Bond Futures

Forwards and futures are closely related, the difference being that futures contracts are "marked to market daily," and forwards are not. It took me ten years to figure out what that means, so hold on if it sounds confusing. In principle we should value these cash flows

using our favorite pricing model. In practice, the difference between forward and futures prices is small enough that only fairly sophisticated practitioners worry about it. Suffice it to say that we will describe the issue then disregard it.

The most popular US dollar bond futures contracts are the 30- and 10-year US treasury bond futures offered by the Chicago Board of Trade. Contracts are also available on US treasury bills and 5-year treasury notes, and Japanese, German, and British government bonds. We'll focus on the 10-year treasury bond contract.

Cash Flows

A forward contract may turn out to make or lose money, depending on whether the price of the underlying bond rises above or falls below the forward price by the settlement date. In a futures contract, the long position is adjusted daily for these changes in value.

As an example, consider a contract on a 10-year US treasury note and suppose the prices on succeeding business days are

Date	Price	Cash Flow
Trade Date	104	none
Day 1	102	(2000)
Day 2	103	1000
Settlement	101	(2000)

(Like bonds, the prices are typically quoted in 32nds, but I find decimals more convenient.) The cash flow is the change in price adjusted for the size of the contract, namely,

$$\text{Cash Flow} = \frac{\text{Change in Price}}{100} \times 100,000$$

for the typical contract size of \$100,000. We refer to these daily cash flows as "marking to market."

The cash flows involved in marking to market are often done with margin accounts. The customer posts an initial margin set by the exchange Changes in the value of contract are then credited or debited to the margin account as they occur. If the account falls too low, the exchange issues a margin call for more.

Special Features of Treasury Bond Futures

There are a couple special features of treasury bond futures that deserve mention. One is the choice of bond. The Board of Trade was worried, when it designed the contract, that someone could corner the market in the bond on which the contract was written. They therefore allow sellers of futures contracts to deliver any of a number of vaguely similar bonds. But since these bonds differ in maturity and coupon, they are given conversion factors that presumably adjust them to the standard bond. In practice, some bonds are cheaper to deliver than others and sellers of futures contracts presumably deliver the cheapest one available.

A second feature is that delivery can take place on any day in the month on which the contract expires.

Benefits of Exchange-Traded Instruments

Repeat and update the list from Chapter 5, including:

1. Liquidity. The futures contracts we have in mind are actively traded and highly liquid.

2. Low credit risk. Your contract is with the exchange, whose ability to pay is backed by cash. Further, marking to market reduces exposure, since losses are paid up daily.

8.3   Eurocurrency Futures

The most actively traded interest rate futures are those on three-month eurodollars, with similar contracts on euroyen and euromarks a distant second. Eurodollar futures contracts are traded on the International Monetary Market (IMM) of the Chicago Mercantile Exchange (CME), the London International Financial Futures Exchange (LIFFE), and the Singapore International Monetary Exchange (SIMEX), and a reciprocal agreement between CME and SIMEX offers day/night trading opportunities. LIFFE contracts are not offsettable against those on the CME or SIMEX, but are otherwise similar in form.

The Contract

It's important to get the contract straight:

- The contract size is $1 million.

- Contracts are available maturing on the third Wednesday of March, June, September, and December, over forty quarters or ten years.

- The quoted index, as reported in the *Journal* and elsewhere, for the contract is

$$\text{Index} = 100 \times (1 - \text{Yield}).$$

Thus an Index of 93.89 corresponds to a Yield of 6.11 percent; more examples are reported below. At maturity/settlement, Yield is equal to 3-month dollar LIBOR expressed as an annual number (the annual percentage rate divided by 100). Prior to settlement, the Yield is referred to as the futures yield, and is similar to the forward rates mentioned in Chapters 2 and 8.

- The price of the contract differs from the Index in the scale of the contract and (this is the confusing part) in dividing the Yield by 4 to convert it to a three-month rate:

$$\text{Effective Price} = 1 \text{ million} \times (1 - \text{Yield}/4).$$

It's important to note the difference: gains and losses on the contract follow from the Effective Price, not the Index. Unlike the Index, the Effective Price results in payments that correspond exactly to interest payments on three-month loans.

- As a result of the way contracts are priced, a one basis point change in the Index results in a $25 change in the value of the contract.

Let's look at some examples. On May 19, 1995, eurodollar futures quotes reported in the *Journal* looked something like this:

	High	Low	Settle	Yield	Open Interest
June	93.92	93.88	93.89	6.11	375,294
Sept	94.01	93.95	93.97	6.03	363,938
Dec	93.99	93.90	93.92	6.08	312,979
Mr96	94.02	93.90	93.93	6.07	265,797
June	93.86	93.78	93.80	6.20	186,003
Sept	93.76	93.66	93.68	6.32	163,688

Additional contracts extend to March of 05.

Futures contracts for eurorates on other currencies are similar in form, but are available for fewer maturities. Euromark and euroyen contracts, for example, are currently available for three years. A quick look at the futures page of the *Journal* will give you the idea.

Hedging with Eurocurrency Futures

Eurodollar futures are very active, at least for short maturities. Their interest rate risk characteristics differ from bond futures in having the possibility of long contract lengths (large m) on a short instrument ($n = 1$). They offer less in the way of exposure to long yields, which is why we still see a lot of action in 30- and 10-year bond futures.

Summary

1. Forward contracts for bonds are equivalent to a combination of a long position on a long bond and a short position of equal value on a short bond.

2. Futures contracts differ from forwards in the daily mark to market.

3. The most popular fixed income futures contracts are those in eurocurrency interest rates and government bonds.

4. Forwards and futures are often used to adjust the interest sensitivity (duration, for example) of a portfolio.

Practice Problems

1. Use a recent issue of the *Wall Street Journal* or Financial Times to evaluate the use of various futures contracts. Which has greatest open interest? Volume? Should we weight the contracts by notional value, duration, or something else?

2. Conversion factors... (some examples from Bloomberg). Compute invoice prices and durations associated with each...

Further Reading

Useful references on futures contracts in general include Duffie (*Financial Futures*, Prentice-Hall, 1989), Figlewski (*Hedging with Financial Futures: From Theory to Practice*, Ballinger, 1988), and Galitz (*Financial Engineering*, Irwin, 1995). On fixed income futures, see also Tuckman (*Fixed Income Securities*, Wiley, 1995).

Many of the futures exchanges have Web sites, including

- http://www.cbot.com/

- http://www.cme.com/

- http://www.liffe.com/

Chapter 9

Options

Options are among the most widely used financial derivatives. Fixed income examples range from exchange-traded options on eurodollar futures to over-the-counter interest rate caps and floors, to options embedded in callable bonds. Basically, an option can be constructed for any instrument we like, including an option itself.

Our focus in this chapter is on both common examples of fixed income options and their general features. Unlike bonds, plain vanilla swaps, and interest rate futures, options on fixed income securities have highly nonlinear payoffs. The payoff is generally zero up to some threshold, then increases with the value of the underlying asset. As a result these securities often have, in our earlier language, variable duration and nonzero convexity, either positive or negative. The linear duration-based measures of risk, in other words, are generally inadequate.

Mortgages are an extreme but representative example of the nonlinearity induced by options, and we will study them later on in more detail. The prepayment option gives mortgages (as we will see) negative convexity: their price responds sharply to interest rate rises (thus allowing the owner to lose more money), but responds moderately to interest rate declines (thus limiting potential gains). Perhaps because this feature was not widely understood, some users expressed surprise at their losses in 1994 when short-term interest rates rose 200 basis points in a year.

Despite the complications, options can be useful instruments in the right hands. Examples include:

- Industrial firms. Many firms issue debt in variety of forms. One of the first issues a firm must address is whether to issue long or short debt. Long debt locks in a fixed coupon, and thus protects the firm from fluctuations in interest rates until the

debt matures. Short or floating rate debt is generally cheaper (since the yield curve is increasing, on average), but of course comes with the risk that rates will rise. Options provide a compromise: for a price, a firm can protect itself against extreme increases in interest rates or provide itself an opportunity to benefit from decreases in rates. Examples include: interest rate caps, floating rate notes with caps built in, and callable bonds (in which the firm issues long debt with the option to buy it back and refinance if rates fall substantially).

- Commercial banks. Banks often have substantial negative convexity in their balance sheets, from mortgages and related instruments, that they would like to manage. Banc One, for example, found that the duration of their assets increased substantially in 1994, resulting (given their overall management of interest rate risk) in substantial losses on derivatives positions. In the words of Steve Bluhm, VP of Banc One's Funds Management Company, "We should have considered the possibility of rates spiking up by as much as 200 basis points, but we didn't." (Quoted in *Derivatives Strategy*, March 27, 1995, p 6.) He noted that protection was available in the form of interest rate caps, and in early 1994 would have been very inexpensive.

- Fixed income fund managers. Fixed income funds can distinguish themselves from competitors either by finding cheaper sources of interest payments (underpriced bonds and the like) or by making better guesses about the future path of interest rates. A manager who expected rates to fall, for example, might increase the duration of her portfolio. Options can be a useful tool in this regard. Rather than change the whole portfolio, a manager can buy call options on treasury bond futures, which should rise substantially as rates fall. The latter may well be a more cost-effective strategy when transaction costs are considered. There may also be some differences in tax and accounting treatment that favor options, particularly over-the-counter products.

In short, options can be a useful tool for many users, as their enormous popularity attests.

9.1   Option Terminology

Before we start, it's useful to review some basic option terminology. The basic option contracts are *puts* and *calls*. Call options give the owner the right to buy an asset at a preset price called the *strike price*. Puts give the owner the right to sell. We say that someone who has sold a call or put is short an option, or has *written* an option.

A number of terms are used to describe the *exercise* of an option. If the owner of a call exercises the right to buy at the strike price, we say that he has *called* the underlying asset. Similar, the owner of a put option might *put* the underlying asset to the person who wrote the option.

Options generally apply to a fixed time period. An option might expire, for example, in three months, meaning that the right to buy or sell is valid for three months — after that it is worthless. *European options* can be exercised only on the expiration date. The more common, but conceptually more complex, *American options* can be exercised any time prior to the expiration date. Options imbedded in callable bonds are in between: they typically cannot be exercised for the first few years of the bond.

9.2 Optional Fundamentals

Stick to European options...

Cash flow pattern.... Options buyers have limited liability, writers usually do not

Intrinsic value...

Put-call parity...

Volatility good...

Leverage

The Black-Scholes Benchmark

We can see some of the basic features of options in the BS formula. Plus put/call parity...

1. Value increasing in underlying, decreasing in strike price.

2. Value greater than exercise (could get lucky) and increasing in volatility.

3. Observable: only volatility missing.

9.3 Options on Eurodollar Futures

By far the most actively traded interest rate options are those on three-month eurodollar futures contracts, with similar options on euroyen and euromarks a distant second. These options offer a direct bet on a short-term rate closely tied to firms' cost of funds: 3-month LIBOR in dollars or other currency.

Options on Eurodollar Futures

These contracts are a useful basis for options, and activity in them dwarfs other exchange-traded fixed income options. Specifics include:

- Options are American: they can be exercised any time up to expiration. In fact, almost all exchange-traded options are American in this sense.

- Expiration dates cover the next six futures contracts. Six options match the maturity dates of the futures contracts. Two "serial" options have monthly expiration dates that fill in the blank months between futures settlement dates in the next quarter.

- Strike prices vary by 25 basis points, generally centered around the current futures price (although this varies depending on the recent history of futures prices).

Some of this is apparent in quotes from the *Journal* of May 19, 1995:

Strike	Calls			Puts		
Price	Jun	Sep	Dec	Jun	Sep	Dec
9350	0.39	0.52	0.60	0.00	0.06	0.20
9375	0.16	0.35	0.46	0.02	0.13	0.29
9400	0.02	0.20	0.32	0.13	0.23	0.40
9425	0.00	0.10	0.23	0.36	0.38	0.55
9450	0.00	0.05	0.15	0.61	0.57	0.72
9475	0.00	0.03	0.09	0.86	0.80	...

More extensive information is available online (Bloomberg, for example).

These quotes give us an idea how the option works. Take the September call options. The first has a strike price of 93.50. We saw earlier that the current price of the September futures contract was 93.89, so this contract is (we say) *in the money*: if we exercised immediately, we gain (in terms of the Index) 0.39 [=93.89 − 95.50], or 39 basis points. In value terms, of course, this is $975: 39 basis points times $25 each. The option, though, trades at 0.52, not 0.39. The difference of 13 basis points reflects the time value of the option: we still have four months in which to get luckier, which in this case means a fall in 3-month dollar LIBOR.

Put options are similar. The put with strike price 93.50 gives its owner the opportunity to sell a futures contract for 93.50. Since the current price is 93.89, this opportunity is not one to take immediate advantage of (we say the put is *out of the money*), but it nevertheless has a traded value of 0.06. Again, the idea is that if LIBOR rises between May and September the option might be exercised profitably.

Capping Interest Rates

To see how options on eurodollar futures might be used, consider a fictitious firm that is planning to assume a three-month, \$75 million obligation in September (the third Wednesday, say), due in December, at a rate of LIBOR flat. The firm would like to explore its options (so to speak) for limiting exposure to movements in LIBOR between now and September.

One thing it could do, of course, is use a eurodollar futures contract to lock in a rate now. Another is to use eurodollar options to limit exposure to upward interest rate moves. The instrument for this is a put on a eurodollar futures, since a put gains value when LIBOR rises. Given the magnitude of the loan, the firm would need puts on 75 contracts, each contract covering interest on one million dollars. The cost depends on how much protection is desired.

At current rates (our usual May 19 numbers) LIBOR is (say) 6.00 percent, so the firm would end up paying interest of

$$\left(\frac{\text{LIBOR}}{4}\right) \times 75m = \frac{0.060}{4} \times 75m = 1.125m,$$

just over a million dollars. Each rise of 25 basis points, though, costs the firm an additional \$46,875.

Put options allow us to limit this exposure. Suppose the firm buys a September put at 94.00. Then any rise in LIBOR above 6.00 percent (or decline in the futures price below 94.00) will be offset by a profit on the option. If LIBOR rises to 8.00, the interest payment rises to

$$\frac{0.080}{4} \times 75m = 1.500m,$$

an increase of 375 thousand dollars. But the put can now be exercised at 94, giving us a profit (at \$25 per basis point) of

$$200 \text{ basis points} \times 25 \times 75 \text{ contracts} = 0.375m.$$

Ie, they exactly offset. In fact the firm gets the money early: the profit is realized in September, but the interest isn't paid until December.

The protection offered by the put comes at a cost. If the firm limits its exposure to 6 percent, the cost of the put at 94.00 is another 23 basis points (the price of the option is directly comparable to the interest rate), or

$$23 \text{ basis points} \times 25 \times 75 \text{ contracts} = \$43,125.$$

Its total cost, measured as an percentage, is thus

$$\min{(\text{LIBOR}, 6.00)} + 0.23.$$

Moreover, the cost of the option must be paid now (May), while the interest is paid in until December.

We can lower the cost if we allow ourselves some exposure to interest rate moves. If we are willing to live with LIBOR up to 6.50 percent, the cost falls to 6 basis points, or

$$6 \text{ basis points} \times 25 \times 75 \text{ contracts} = \$11,250.$$

In words: it's cheaper to protect yourself against only large moves.

In short, options on eurdollar futures are useful tools for managing interest rate risk. The timing of the payments doesn't match exactly those of interest payments on floating rate debt, but the magnitudes are exactly those needed by someone dealing with LIBOR-based floating interest rates.

9.4   Interest Rate Caps and Floors

We have seen that options on eurodollar futures can be used to cap LIBOR-based interest payments, but the timing of the option's cash flows do not correspond precisely to those of the interest payments. Moreover, for a long series of interest payments we need a comparable series of options. Interest rate caps provide a more direct approach to the same problem. Like other over-the-counter products, they can be taylored directly to the needs of the user.

In this section we look at caps — and their close relatives, floors — in more detail than the example of Section 7.6, and also examine a more general property of option prices: their dependence on volatility. As a rule, a call or put option has greater value the more uncertain is the future of the underlying asset.

Caps

The example in Section 7.6 is a standard cap. They vary most commonly in maturity (like swaps, maturities out to ten years are common) and frequency of payment (annual, semiannual, quarterly, or monthly). An unlimited range of mutations starts from there. A semiannual cap generates cash flows, as we have seen, of

$$\text{Cap Cash Flow at t+n+1} = \max{(0, r_{t+n} - \text{Cap Rate})} \times \frac{\text{Notional Principal}}{2}.$$

Adjustments for other payment frequencies are relatively obvious. We say that rate is *reset* at date $t + n$ and paid at date $t + n + 1$.

In our example, the cap paid out the difference every six months for two years between 6-month dollar LIBOR and 7 percent, if positive, nothing otherwise. The owner of the cap thus gets protection against rises in LIBOR above 7 percent. The cap differs from a standard option in one respect: the exercise date is different from the payment date. If (say) LIBOR is quoted at 7.56 in May 1996, the difference of 0.56 percent is paid six months later. The convention follows that of LIBOR: today's six-month LIBOR quote is paid six months from now. As with swaps, we therefore distinguish between reset dates (when rates are quoted and set) and payment dates (when cash changes hands).

Volatility

Another general feature of call and put options is that their value is higher the greater the uncertainty about the future. We can see this easily in the interest rate cap. We computed the initial price of 0.847 million from (i) the cash flows associated with the cap at each node of the interest rate tree and (ii) the state prices. Both rates and state prices were based on an interest-rate volatility parameter σ of 0.01: the six-month standard deviation of short-term interest rate changes.

What happens to the value of the cap if we raise σ to 0.011 (a ten percent increase)? To find out, we need to redo the analysis of Section 7.6, computing a new interest rate tree, new state prices, new cash flows for the cap, and finally the value of the cap itself. The answer is 1.046, a rise of almost 200 million dollars or 23 percent.

The reasons for the increase are relatively obvious when you think about it. The increase in σ increases the spread between future interest rates in the rate tree: high interest rates are higher, and low rates are lower, than in the tree with $\sigma = 0.01$. The low rates have no effect on the price of the cap, which generates no cash flows in low interest rate states. The high interest rates, however, raise the cash flows in high interest rate states, and thus raise the value of the cap. This asymmetry is an essential feature of an option contract: by eliminating the down side risk, the contract can only gain from increases in uncertainty.

Floors

While interest rate caps can be used to put an upper bound on interest payments (the cap paying any difference above the cap rate), floors can be used to provide a lower bound. The

lower bound is typically not a objective on its own, but a floor is often used to reduce the cost of borrowing. A semiannual floor delivers cash flows of

$$\text{Floor Cash Flow at t+n+1} = \max\left(0, \text{Floor Rate} - r_{t+n}\right) \times \frac{\text{Notional Principal}}{2}.$$

Other details are analogous to those of caps.

Exchange-Traded vs OTC Products

Interest rate caps and options on eurodollar futures offer similar return profiles, but they also illustrate the some of the differences between exchange-traded and over-the-counter (OTC) products. Exchange-traded products

- come in a limited set of standardized types;
- are traded, as a result, in active, liquid markets;
- are guaranteed by the exchange.

OTC products, on the other hand,

- can be custom-taylored to the user;
- are less liquid as a result (and the more unusual the product, the less liquid it is likely to be);
- are as good as the counterparty to the trade.

Neither has an unambiguous advantage over the other. Several years ago there was concern that OTC products were excessively vulnerable to credit risk, but as we saw with swaps, the OTC market has responded by increasing the credit quality of its offerings and designing products that minimize the impact of credit risk. The choice is often more one of convenience. An interest rate cap is often more convenient than a eurodollar option for managing long streams of interest payments, but the liquidity of the option makes it more useful to active fund managers who change their positions frequently.

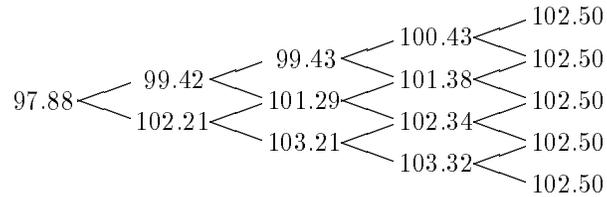## 9.5   Options on Bonds

Our next topic is options directly on bonds. These instruments are not commonly traded as such, but they are implicit in the many bonds issued with call options: options under which the issuer can call the bond in the future. We say, in this case, that the owner of the bond is short a call option.

We'll look at callable bonds shortly, but for now consider the option on its own.
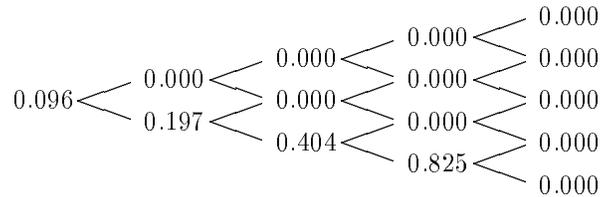
The American Call Feature

We proceed, as usual, with an example: a two-year bond with a 5 percent annual coupon rate, coupon interest paid semi-annually, callable at par plus accrued interest (102.5, in other words). With the rate tree we calibrated in Section 7.4 and have used ever since, the price of this bond follows the path

```
                                              ┌ 102.50
                                   ┌ 100.43 ┌─┤
                         ┌ 99.43 ┌─┤        └ 102.50
               ┌ 99.42 ┌─┤       └ 101.38 ──┤
               │        └ 101.29 ┌─┤        └ 102.50
     97.88 ────┤               ┌─┤ 102.34 ──┤
               │        ┌ 103.21┘ └        └ 102.50
               └ 102.21─┘               ┌───┤
                                 103.32 ┘    └ 102.50
```

This is computed by the usual methods: we work our way backwards through the tree, starting at maturity, applying as we go the state prices implied by the fifty-fifty rule.

But what is an option on this bond worth? The issue is when the option can be exercised. Clearly there is no reason to call it at maturity: the price is equal to the strike price in all states. We could, however, consider a European option expiring six months prior to maturity. We see in this case that the option generates a cash flow of 0.8248 in the low interest rate state, (0,3). The price path for the option is

```
                                              ┌ 0.000
                                   ┌ 0.000 ┌──┤
                         ┌ 0.000 ┌─┤       └ 0.000
               ┌ 0.000 ┌─┤       └ 0.000 ──┤
               │        └ 0.000 ┌─┤        └ 0.000
     0.096 ────┤              ┌─┤ 0.000 ──┤
               │        ┌ 0.404┘ └        └ 0.000
               └ 0.197 ─┘              ┌───┤
                                0.825 ─┘    └ 0.000
```

The option's current value is 0.096.

If we allow — but do not require — the option to be exercised earlier, its value can be larger. The reason is that we can always choose to exercise in period 3, as above, so the extra flexibility can't hurt us. And it may help. This is sometimes referred to as the *early exercise premium* for American options.

We can value an American option by again working our way back through the tree and asking ourself at every node: would I do better to exercise now? Consider the node (0,2). If we hold the option for another period it will be worth 0.000 if the short rate moves up, 0.825 if it moves down. The two together are worth, at node (0,2),

$$\text{Value If We Hold Option} = \frac{0.5}{1 + 0.04213/2} \times (0.000 + 0.825) = 0.404,$$

as we have seen. This is our usual application of state pricing theory, with state prices given by one-half of the one-period discount factor (the fifty-fifty rule). Alternatively, we could exercise immediately. If we exercise, we get

$$\text{Value If We Exercise Option} = 103.212 - 102.500 = 0.712.$$

The latter is clearly better in this case. That tells us that if we ever get to node (0,2) we should exercise the option, and that the cash flow at that node is 0.712.

In earlier periods we do the same thing. At node (0,1), for example, we can continue to hold the option, which generates a value of

$$\text{Value If We Hold Option} = \frac{0.5}{1 + 0.05103/2} \times (0.000 + 0.712) = 0.347.$$

If we exercise immediately, we get nothing, so holding is the right thing to do.

The result of this exercise is that the option is worth 0.169 if we take into account our ability to exercise at any date, but only 0.096 if we have to wait until period 3 to exercise. At least in this case, the difference is significant. The bad news is that it's more work.

9.6   Callable Bonds

Callable bonds are yet another fixed income instrument with option-like features. Examples range from callable US treasury notes and bonds, like the "8s of Aug 01," to callable corporate bonds like those listed below. We will see that the call provision gives these bonds negative convexity: their prices are less sensitive to yield declines than to increases.

Examples

Example 1. Apple Computer Corporation's 6-1/2s, issued February 10, 1994, due February 15, 2004, with semi-annual coupon payments. The bond is callable at "make whole." The idea behind "make whole" is that the firm compensates the owner of the bond for any difference between the bond's coupon and current interest rates (presumably lower). Details vary by bond.

Example 2. Ford Motor Company's 6.11 percent bonds, issued September 22, 1993, due January 1, 2001. Callable at "make whole under special circumstances," which circumstances I have yet to discover.

Example 3. Intel Overseas Corporation's 8-1/8s, issued April 1, 1987 (really), due March 15, 1997, callable at par. Par in this situation means par plus accrued interest: the firm

pays the relevant interest as well as the face value. The bonds were called March 15, 1994, at 100.

Example 4. Texas Instruments' 9s of July 99, issued July 18, 1989, due July 15, 1999. Callable on or after July 15, 1996, at par.

You get the idea. In practice, bonds are called for a number of reasons. We focus on refinancing, but some bonds specifically rule this out as grounds for calling (presumably to reduce the cost of the call provision). Firms also call bonds to undo covenants.

Duration of Callable Bonds

The effect of the call provision is to reduce the value of the bond, particularly in low interest rate states. The latter gives rise, as we will see, to negative convexity: duration is larger for interest rate rises than declines.

To develop this more systematically: we can separate the call option from the bond and think of the owner of a callable as having a bought a noncallable bond and sold an option on the same bond:

$$\text{Callable Bond} = \text{Noncallable Bond} - \text{Call Option}$$

The price of the bond is thus the difference between the two. We computed the price of the noncallable bond earlier: 97.880. Similarly we valued the option at 0.169. So the callable bond is worth (by our calculations) 97.711 [= 97.880 − 0.169].

More interesting than the price: the call option has a large effect on the interest-sensitivity of the bond.

We get a similar picture from the price-yield diagram for a callable bond (Figure 9.1). At high interest rates the price line looks much like that of a noncallable bond. But for low interest rates the option gains value and flattens out the price line. The option, in effect, puts an upper limit on the price of the bond, which forces the price line below that of the noncallable bond.

The unusual shape of the price-yield relation is commonly referred to as *negative convexity*. What it means in practice is that the duration of the instrument is higher for interest rate rises than falls. After the fallout of the 1994 interest rate increases, the landscape was littered with firms who claimed to have underestimated the duration of their positions, and suffered large losses as a result.

9.7   Exotics

Maybe later ...

Summary

1. Options are everywhere, explicit and otherwise. Some of the most common fixed income examples are options on eurodollar futures, interest rate caps and floors, and callable bonds.

2. Options have nonlinear payoffs, which translates into variable duration. Callable bonds, for example, have negative convexity: their prices are more responsive to interest rate rises then falls.

Practice Problems

1. Consider the callable US treasury "8s of Aug 01," callable at par every six months starting August 15, 1996. Like other callable treasuries, they are callable with 120 days notice. For further details on the bond, use Bloomberg and type: T 8 8/01 [Govt] [GO]. Your mission is to evaluate its price: given current yields, is the price too low, too high, or just about right in your judgment?

    (a) Use quoted yields for US treasury STRIPS, and any other information you like, to estimate discount factors and zero-coupon yields (spot rates) for all maturities through August 2001.
    (b) Calibrate a multiplicative interest rate model, as in equations (7.9, 7.10), to these discount factors/yields. (The reason for this model is that the Ho and Lee model tends to produce negative interest rates in some states over a time horizon of this length.) A reasonable value for σ in this case is 0.15, but you may estimate your own value if you like.
    (c) Use the model to compute the theoretical price of a callable bond of this type.
    (d) Compare the price to the market.
    (e) Using your best judgment, including an assessment of the model's strengths and weaknesses, does the market value the bond accurately?

2. A large, highly-rated manufacturing company is assessing its alternatives for debt management. Using our usual interest rate tree, compute the current interest rate for each of the following:

(a) A floating rate note.

(b) A 2.5-year fixed rate note.

(c) A 2.5-year floating rate note with a cap at the same rate as the fixed rate note.

Based on your calculations, which alternative would you recommend?

3. The Deutsche Bank Finance 10-year Canadian dollar floating rate notes described in Section 5.1 illustrate a common stragey for dealing with interest rate risk: a cap on the floating rate is used to eliminate extreme interest rate moves, but a floor is added, too, to reduce the overall cost. Your mission is to design a similar structure for an industrial client using the US dollar interest rate tree derived in Section 7.6 and applied to interest rate caps in Section 9.4. The starting point is a 2.5-year floating rate note on a principal of $100 million.

   (a) With 6-month interest rates at about 6 percent, an interest rate cap at 6.5 percent is slightly out of the money. Compute the cost of such a cap on a notional principal of $100 million. If we measure the cost of the cap — somehow — in units comparable to the interest rate, by how much does the cap raise the current cost of funds?

   (b) Your client suggests that the cost of the cap is excessive. Compare two alternatives: raising the cap to 7 percent and adding a floor at 5 percent, thereby limiting the firm's potential to benefit from declines in short-term interest rates. Which would you recommend to your client, and why?

   (c) Compute, for the floating rate note with a cap of 6.5 and a floor of 5, the theoretical duration for each node in the tree.

4. Consider a six-month option on a 2.5-year pay fixed swap on 250 million dollars — a swaption, in other words. Use the usual interest rate tree, ignoring for the moment the difference between treasury and swap rates.

   (a) Compute the fixed rate appropriate to the pay fixed swap.

   (b) Given the fixed rate, compute the price path for the swap through the tree.

   (c) Consider an option allowing you to initiate the swap in six months. What is a reasonable price for the option?

5. Suppose the Black-Scholes formula, suitably adapted, applies to the prices of eurodollar options listed in Section 9.3.

   (a) If the yield curve is flat at 6.00 percent, what is the value of the option volatility v implied by the formula?

   (b) For this value, what is the vega of the option? In words: how much does the call price change if we increase v by a small amount?

Further Reading

There are lots of good sources for option theory, but they vary widely in emphasis. Among those I've found useful: Figlewski, Silber, and Subramanyam (1990, chs 6, 8, and 9), Galitz (1995, chs 10 and 11), and Tuckman (1995, ch 17) (terrific analysis of callable bonds and sinking funds).

Data: Bloomberg has extensive option quotes.

Figure 9.1

Price-Yield Relation for Callable Bond
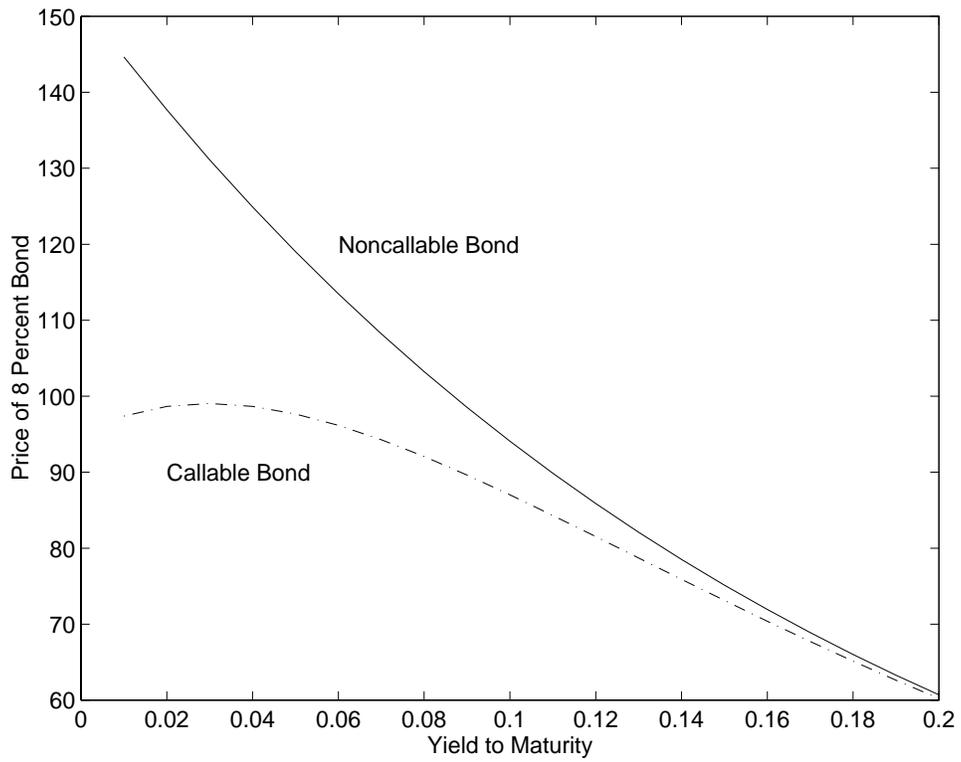
Part IV

Appendices

Appendix A

Answers to Practice Problems

Chapter 2

1. The numbers include:

Maturity	Spot Rate	Disc Factor	STRIP	Forward Rate	Par Yield
0.5	5.00	0.9756	97.56	5.000	5.000
1.0	6.00	0.9426	94.26	7.005	5.985
1.5	7.00	0.9019	90.19	9.015	6.954
2.0	8.00	0.8548	85.48	11.029	7.902

   Spot rates, forward rates, and par yields are annualized percentages. The price of the two-year 5 percent bond is 94.668.

2. The numbers are

Maturity	Coupon Rate	Price	Yield	Discount Factor	Spot Rate
0.5	8.00	103.00	1.942	0.9904	1.942
1.0	4.00	98.60	5.458	0.9472	5.494
1.5	8.00	101.80	6.718	0.9043	6.818
2.0	6.00	99.00	6.542	0.8784	6.589

3. (a) There are a variety of ways to interpret this question. One approach is to use the first three bonds to replicate the cash flows of the fourth. An equivalent approach is to use the first three bonds to estimate discount factors or spot rates, and use these to value the fourth bond. If you do this, the discount factors $d_1 = 0.9731$, $d_2 = 0.9460$, and $d_3 = 0.9202$, and the spot rates are $y_1 = 0.0552$, $y_2 = 0.0562$, and $y_3 = 0.0562$.
   (b) The discount factors imply a price for bond 4 of 104.089, which is considerably

higher than the quoted price.

(c) Among the possible rationales for such apparent mispricing are: bid/ask spreads (probably not this big), nonsynchronous price quotes, tax effects on bonds selling away from par, and call provisions on one or more of the bonds.

4. Suppose we use the first two bonds to compute the discount factors:

$$106.56 \;\; = \;\; d_1 \times 5 + d_2 \times 105$$
$$100.70 \;\; = \;\; d_1 \times 2 + d_2 \times 102.$$

The implied discount factors are $d_1 = 0.9854$ and $d_2 = 0.9679$. Using these discount factors, the implied price of Bond C is 102.65. Ie, Bond C is overly expensive relative to A and B. Consider shorting C and covering our position with a combination of A and B. The relevant strategy is to buy 1/3 of a unit of A and 2/3 of a unit of B, which replicates the cash flows of C. The net profit is $0.47 = 103.12$ (proceeds from sale of C) $- 102.65$ (cost of 1/3 A and 2/3 B).

5. There are a number of approaches you could take to this problem: you could guess which bonds are overpriced and construct a bond swap for them, you could do a linear program, or you could run a regression (with the discount factors as the parameters) and see which bonds are overpriced, and which underpriced. The guess method is the easiest. One possible guess is that the second bond is overpriced. On this premise, let us try to find a combination of bonds 1 and 3 that deliver the same cash flows and costs less. The combination includes (say) x_1 units of bond one and x_3 units of bond 3. To replicate the cash flows the quantities must satisfy

$$7.875 \;\; = \;\; x_1 6.75 + x_3 8.25$$
$$100 \;\; = \;\; x_1 100 + x_3 100$$

The answer is $x_1 = 0.75$ and $x_3 = 0.25$. The cost of this combination is

$$x_1 102.36 + x_3 103.75 = 103.40,$$

which is 16 cents less than the cost of bond 2. If we really did a bond swap, though, some of this disappears: we'd sell the bond for 103.56 minus one-eighth, or about 103.44, thus eliminating much (but not all) of our profit.

6. Accrued interest = 1.65, invoice price = 102.31, yield = 6.18 percent.

7. This is a eurobond. Numbers include: $u = 191$ days, $v = 169$, Accrued Interest = 3.847, Invoice Price = 109.073, $y = 5.340$ percent.

8. Accrued interest = 1.900, invoice price = 99.035, yield = 7.360 percent.

9. Bond 1: accrued interest = 2.03, invoice price = 104.03, yield = 7.59(!) percent. Bond 2: accrued interest = 3.40, invoice price = 138.52, yield = 6.45 percent. Yield to call = 6.29 (as reported in the Journal). For yield of 6.45, the invoice price would be 109.89. Yield likely lower than this, because coupon is smaller. The implied value of the call is $5.86 = 109.89 - 104.03$.

Chapter 4

1.

2. Answer supplied by Bloomberg (type YA for Yield Analysis).

3. Duration rises with n, but levels off. The reason is that the coupons take on a larger and larger fraction of the value of the bond as maturity increases. Even an infinite bond (what the Brits call a perpetuity or consol) has a finite duration.

4. Two aproaches to duration.
(a) The price is 105.83, the yield 6.830%.
(b) The duration is 1.804 (years).
(c) You invest these amounts in 1, 2, 3, and 4-period zeros: 4.95, 4.81, 4.58, and 91.50 (the sum equals the price). The durations of the four zeros are 0.50, 0.98, 1.46, and 1.93. The duration of the portfolio is thus

$$D = \left(\frac{4.95}{105.83}\right) \times 0.50 + \left(\frac{4.81}{105.83}\right) \times 0.98 + \left(\frac{4.58}{105.83}\right) \times 1.46 + \left(\frac{91.50}{105.83}\right) \times 1.93$$
$$= 1.801.$$

Ie, the duration of the portfolio is the value-weighted average of the durations of its components (the zeros). The answer is very close to what we found in (a).
(d) The difference is that the weights in (c) were based on spot rates (what some people refer to as Fisher-Weil duration), while in (a) they were based on the yield to maturity of the bond. We see in this case, at least, that it makes very little difference. The answer in (b) is what you'll see on a Bloomberg terminal (labelled "adj/mod duration"). Assessment: Basing duration on zeros is more accurate (RiskMetrics does something similar), but requires more information (you need all the spot rates, not just the yield on the bond).

5. Risk management: cash flow matching vs immunization.
(a) To replicate the cash flows, you buy 10 units of the 1-period zero (principal=100), 15 units of the 2-period zero, 25 units of the 3-period zero, and 15 units of the 4-period zero.
(b) Using the prices of zeros, we find that the value of the portfolio (and the value of the liabilities) is 6038. As in Problem 1, the duration of the portfolio is the value-weighted average of the durations of its components: 1.30 years.
(c) An alternative plan is to use only two instruments and match the duration. Suppose we put fractions w of our portfolio in 0.5-year zeros and $1 - w$ in 2-year zeros. The fractions are chosen to match the duration of the portfolio to that of the liabilities (namely 1.30):

$$1.30 = w \times 0.49 + (1 - w) \times 1.94,$$

so $w = 0.445$. Thus we invest 2689 in 0.5-year zeros and 3349 in 2-year zeros.
(d) The question is: what new spot rate curve would generate a smaller value of the

immunized portfolio than of the cash-matched portfolio? The guess (since duration is based on parallel shifts in the yield curve) is that some kind of twist might do this. Beyond that, it's a matter of experimenting. A relatively simple example is one in which the fourth spot rate rises, but nothing else changes. Since the immunized portfolio has more exposure to the 2-year zero, its value will fall more. Eg, suppose I change spot rates from (5.12,5.20,5.64,5.80) (that's what the problem implies) to (5.12,5.20,5.64,6.00). Then the value of the cash-matched portfolio falls to 6033, while that of the immunized portfolio falls to 6025. The point is simply that duration matching is not bulletproof.

Chapter 5

1. An inverse floater.
   (a) 100 each. (b) 116.12. (c) 89.13 each. (d) $94.38 = -2(100) + 116.12 + 2(89.13)$. (e) Guess: rates rose between 1992 and the value date. (f) The duration is a weighted average of the durations of the components (FRN, fixed rate note, zero):

$$D = 0.49 \left(\frac{-2 \times 100}{94.38} \right) + 1.75 \left(\frac{116.12}{94.38} \right) + 1.94 \left(\frac{2 \times 89.13}{94.38} \right) = 4.79 \text{ years.}$$

   Thus a two-year inverse floater has (in this case) a duration close to that of a five-year zero.

2. Another inverse floater.
   (a) One combination is (i) a long position in a 15.125% bond, (ii) a short position in a floating rate note, and (iii) a long position in a zero, with all three having a maturity of four years. This combination reproduces the interest payments and principal of the inverse floater.
   (b) The prices of the components are

$$\begin{aligned} p_{\text{bond}} &= 135.17 \\ p_{\text{frn}} &= 100.00 \\ p_{\text{zero}} &= 80.65, \end{aligned}$$

   implying a value for the inverse floater of 115.82 [= 135.17 − 100.00 + 80.65].
   (c) The duration of the inverse floater is the value-weighted average of the durations of its components:

$$\begin{aligned} D &= \left(\frac{135.17}{115.82} \right) D_{\text{bond}} + \left(\frac{-100.00}{115.82} \right) D_{\text{frn}} + \left(\frac{80.65}{115.82} \right) D_{\text{zero}} \\ &= \left(\frac{135.17}{115.82} \right) 3.19 + \left(\frac{-100.00}{115.82} \right) 0.49 + \left(\frac{80.65}{115.82} \right) 3.89 = 6.01. \end{aligned}$$

(d) When LIBOR exceeds 15.125%, the interest payment hits its lower limit of zero, at which point the interest payment is no longer sensitive to the current interest rate. This floor is not something we took into account in replicating the inverse floater, so what we've done so far has to be considered an approximation. At the floor, the inverse floater is essentially a zero, which has a substantially shorter duration. The effect of the floor, then, is to flatten out the price-yield relation at high rates of interest. With interest rates around 3%, it's hard to belive that this has much effect on its current price or duration.

3. Swap rates are essentially par yields: we use discount factors to find the coupon that makes the value of the fixed leg, including principal, equal to 100.
   (a) Swap rates are 4.273% (2-year), 4.819% (3-year), and 5.368% (4-year).
   (b) We need a high enough payment in the last three years to make up for the "teaser rate" of zero in the first year. The annualized coupon rate C is chosen to generate a total value of 100, the same as the floating rate side of the swap:

$$100 = (d_3 + d_4 + \cdots + d_8)\, C/2 + d_8\, 100,$$

so

$$C = 200 \times \frac{1 - d_8}{d_3 + \cdots + d_8} = 7.356,$$

a substantial premium over the 4-year swap rate of 5.368.

4. Using swaps to manage interest rate risk.
   (a) If the firm enters a "pay-fixed" swap, it can convert its floating rate debt into fixed rate, thereby locking in a rate now. Stated somewhat differently: the firm can use a swap to change the duration and interest-sensitivity of its debt.
   (b) The durations are computed just as they are for bonds and are 1.90 (2-year), 2.76 (3-year), and 3.56 (4-year).
   (c) Suppose the firm enters a swap in which the values of the fixed and floating legs, including principal, both have value equal to some fraction x of the debt (DM250). Then the duration of the floating rate note and swap together is (by the usual rule) the value-weighted average of the notes, the floating leg of the swap (a short position), and the fixed leg of the swap:

$$D = D_{\text{frn}} - x D_{\text{frn}} + x D_{\text{fixed}}.$$

If we set $D = 2$ and use $D_{\text{frn}} = 0.49$ and the appropriate value for the duration of the fixed rate leg (computed above), we find that $x = 0.49$ with the 4-year swap and $x = 1.07$ with the 2-year swap. You need roughly twice as much of the 2-year swap because its duration is about half.

How do these two strategies differ? Both generate the same duration. The shorter swap probably has less credit risk, since it matures earlier, but the duration reverts

to that of the FRN in two years when it expires. There is also an issue with basis risk: the value of the 2-year swap is sensitive to changes in the 2-year swap rate, the 4-year swap to the 4-year swap rate, and the two rates need not change by the same amount.

5. You can attack this a number of ways, here's one. The assumption (the question is a little vague) is that we receive a fixed rate (to be determined) and pay a floating rate. (a) The floating leg: Think of it as a combination of (i) a one-year 50m FRN and (ii) a two-year 50m FRN. The total value is 100, since each trades initially at par. The fixed leg: Think of it as a combination of (i) a one-year 50m fixed rate bond and (ii) a two-year 50m fixed rate bond, where the two have the same rate. With this decomposition the principals of the fixed and floating legs match in amount and timing.
(b) Suppose the fixed payments are at a rate $C/2$, giving us interest payments over the next four six-month periods of $100(C/2)$, $100(C/2)$, $50(C/2)$, and $50(C/2)$. The value of these interest payments, plus the two principles, is

$$(d_1 + d_2)(100)(C/2) + (d_3 + d_4)(50)(C/2) + (d_2 + d_4)50.$$

For the swap to have legs of equal value at the start, this must equal 100, implying a coupon rate of
$$C = 0.05830,$$
or 5.83 percent per year. This differs slightly from the rate on a generic two-year swap, 5.840 percent. The difference would be larger if the spot rate curve had more slope to it. This gives us, altogether, fixed payments at the ends of the first two periods of 2.915m and, after the last two periods, 1.458m.
(c) We have 50m each in one-year and two-year bonds. A good guess is that the duration is about 1.5 years. A more accurate calculation is to (i) compute separately the price and duration of the two bonds and (ii) average them using their prices. The result is a duration of 1.48 years.

6. (a) The swap rate is 9.062 percent annually. (b) You'd probably be tempted to argue for a lower swap rate, on the grounds that a Aaa-rated company should get a better rate than the (say) Aa-rated generic swap rate. (c) The swap and floating rate debt give the firm five years of essentially fixed rate debt, with an annual coupon rate equal to the swap rate. The duration of the five-year debt is 3.95 years.

7. (a) The spot rate curve has shifted up, esp at the short end. (b,c) The yield to maturity rose to 9.643, an increase of 58 basis points. Given the theoretical relation between duration and price change, we should find that the price of the fixed rate note rose by
$$\Delta p = -D \times p \times \Delta y = -3.95 \times 0.581 = -2.294,$$
a fall of $2.29 on 100 dollars. In fact the price fell by 2.263 (the fixed rate leg of the swap is now worth 97.737), which is pretty close.

8. This one is for you: it is similar to 5, but harder. The trick is coming up with a reasonable plan for adjusting the principal payments.

9. To keep things familiar, keep the notional at 100, then scale up to 20b at the end. (a) We need the value of the fixed cash flows to equal 100, the value of the floating rate cash flows (adding, as usual, the principal payments to make both sides resemble bonds). That gives us an annual fixed rate C satisfying

$$100 = (d_1 + d_2) \times (8/2) + (d_3 + d_4 + d_5 + d_6) \times (C/2) + d_6 \times 100,$$

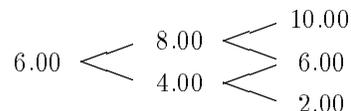implying $C = 4.879$. What we've done is collect a higher rate up front and a lower one later on.
(b) MLDP suffers a loss if (i) the bank defaults *and* (ii) the bank is in the money when default occurs. In this case the bank will tend to be slightly in the money after the first two payments, since the fixed rate is "front-end loaded." A decrease in the floating rate will operate to MLDP's advantage, thus exposing it to credit risk. Conversely, if the floating rate rises sharply, MLDP faces no credit risk, since it owes the bank money, not the reverse.

10. This is a wild one, intended to show you what a real swap can look like. (a,b) If LIBOR stays in a narrow band, BT pays a fixed rate of 6.05 and we have a vanilla swap. If LIBOR rises above 3.90, the rate floats, with BT paying a much lower rate. The swap in this case looks like an inverse floater, with the multiplicative factor giving the swap extreme sensitivity to inteerst rate movements. If LIBOR falls below 3.25, the rate also floats, but in this case the rate rises with LIBOR. In either of the latter two cases, the floating rate is very sensitive to LIBOR, but the direction varies. In both cases there is a limit (I forgot to tell you this): the rate paid by BT has a minimum of zero. (c) If there is one, I can't think of it. Greyhound was essentially making a bet on volatility: large changes in LIBOR in either direction resulted in smaller "fixed" payments by BT.

Chapter 7

1. An application of the Ho and Lee model.

   (a) The short rate tree is

$$6.00 < \begin{array}{c} 8.00 < \begin{array}{c} 10.00 \\ 6.00 \end{array} \\ 4.00 < \begin{array}{c} 6.00 \\ 2.00 \end{array} \end{array}$$

   (b) The price path for the state-contingent claim is

$$.4713 < \begin{array}{c} .4808 < \begin{array}{c} 0.0000 \\ 1.0000 \end{array} \\ .4902 < \begin{array}{c} 1.0000 \\ 0.0000 \end{array} \end{array}$$
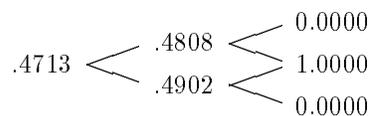
The last column is the pure state-contingent claim [1 in state (1,2), zero in the others]. The earlier columns give us the value of this claim at prior dates.

(c) Duffie's formula puts the current value of one dollar in state (1,2) in the (1,2) node of the tree, and tells us what current prices of other states are, too. The complete set of state prices is

$$
1.000 <
\begin{array}{l}
.4854 < \begin{array}{l} .2334 \\ .4713 \end{array} \\
.4854 < \begin{array}{l} .4713 \\ .2380 \end{array}
\end{array}
$$

(d) Duffie's formula is a different approach than we used in class, but is helpful in generating discount factors and spot rates (which we could use in calibrating the μ's). You could use another method if you like. Either way, the discount factors and spot rates are

Maturity (Half Years)	Discount Factor	Spot Rate
1	0.9709	6.00
2	0.9427	5.99
3	0.9155	5.97

2. This is an example of an instrument whose value is highly nonlinear in the interest rate. Similar nonlinearity is (as we will see later) a hallmark of options, and in any case is a common complication. What it means is that we have to be careful of linear risk measures, like duration. A less obvious lesson is that our methods have no trouble with something this bizarre.

(a) Interest payment is LIBOR for LIBOR up to 6, a quadratic function of LIBOR for higher values. For LIBOR greater than 7, the quadratic term makes the payment both higher than LIBOR and more sensitive to its changes.

(b) The rate tree of

$$
5.991 <
\begin{array}{l}
7.103 < \begin{array}{l} 8.214 \\ 6.214 \end{array} \\
5.103 < \begin{array}{l} 6.214 \\ 4.214 \end{array}
\end{array}
$$

implies interest payments of

$$
5.991 <
\begin{array}{l}
7.217 < \begin{array}{l} 10.900 \\ 6.046 \end{array} \\
5.103 < \begin{array}{l} 6.046 \\ 4.214 \end{array}
\end{array}
$$

As with interest rate caps, the payments are paid six months after the date they appear in the tree. If we therefore (i) divide by 2 because payments are semiannual, (ii) add the principal of 100 to the final period, and (iii) discount everything back one period, the instrument generates cash flows of
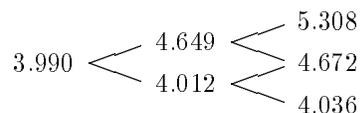
$$
2.91 <
\begin{array}{l}
3.48 < \begin{array}{l} 101.29 \\ 99.29 \end{array} \\
2.49 < \begin{array}{l} 99.29 \\ 100.00 \end{array}
\end{array}
$$

(c) The price path for this asset is

$$
100.29 <
\begin{matrix}
& & 101.29 \\
100.64 < & \\
& & 99.29 \\
& 99.96 < \\
& & 100.00
\end{matrix}
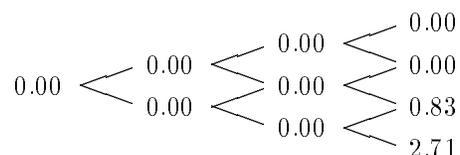$$
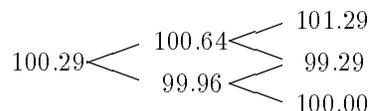
A standard FRN would be 100 at each node.

3. Tuckman's algebra mistake (see, it can happen to anyone). The μ's are $\mu_{t+1} = 0.003406$ and $\mu_{t+2} = 0.003413$. They generate a rate tree of

$$
3.990 <
\begin{matrix}
& & 5.308 \\
4.649 < & \\
& & 4.672 \\
& 4.012 < \\
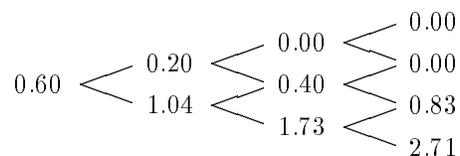& & 4.036
\end{matrix}
$$

which is different from Tuckman's.

4. The option gives us the right to buy the zero for 94 in 3 periods (18 months).

(a) The option generates a cash flow only in the third period:

$$
0.00 <
\begin{matrix}
& & & & 0.00 \\
& & 0.00 < & \\
& 0.00 < & & 0.00 \\
& & 0.00 < & \\
0.00 < & & & 0.83 \\
& 0.00 < & & \\
& & 0.00 < & \\
& & & & 2.71
\end{matrix}
$$

meaning (among other things) that we can buy it now for 60 cents.

(b) The price path for the option is

$$
0.60 <
\begin{matrix}
& & & & 0.00 \\
& & 0.00 < & \\
& 0.20 < & & 0.00 \\
& & 0.40 < & \\
& 1.04 < & & 0.83 \\
& & 1.73 < & \\
& & & & 2.71
\end{matrix}
$$

(c) As in the notes, we can replicate the option (or any other asset whose behavior can be put into the tree) with a combination of a one-period zero (face value 100) and the zero on which the option is writtten (the "underlying"). The quantities of the underlying are

$$
0.247 <
\begin{matrix}
& & 0.000 \\
0.156 < & \\
& & 0.458 \\
& 0.492 < \\
& & 1.000
\end{matrix}
$$
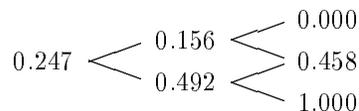
and the quantities of the one-period zero are

$$
-0.214 <
\begin{matrix}
& & 0.000 \\
-0.138 < & \\
& & -0.426 \\
& -0.445 < \\
& & -0.940
\end{matrix}
$$
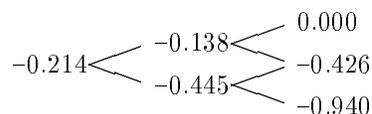
(the negative values indicating short positions). The message is similar to that from duration: we get greater sensitivity to the price of the underlying at the bottom of the tree, where the option is closer to being in the money. Again: option generate nonlinear payoffs and returns.

Appendix B

Old Exams

B.1 1995 Midterm

Midterm Examination

Monday, November 6, 1995

Answer all questions in the space provided. You may use one page of notes and a calculator. Answers with correct logic but wrong numbers will be looked at kindly. Conversely, numbers without explanation will be viewed as incomplete.

1. Banc One Corporation (ONE) has among its liabilities a corporate bond with a coupon rate of 10 maturing August 15, 2010. Bloomberg computes its (quoted) price, for November 2 settlement, at 127.730.

   (a) What is the accrued interest for this bond?

   (b) What is the invoice price?

   (c) What is the yield to maturity? (An exact description of how you would compute the yield is sufficient.)

2. Consider the following US Treasury prices from the *Journal* of October 15:

Maturity	Coupon Rate	Price
1	$9\frac{3}{8}$	101:28
2	8	102:09
3	$6\frac{1}{2}$	101:08

The maturity is listed in half-years, the coupon rate is an annual percentage, and the price is the ask, measured in 32nds.

Suppose a second three-period bond is available, with coupon $8\frac{1}{2}$ and price 103:15.

(a) Use the prices of the first three bonds to estimate the "fair value" of the second three-period bond.

(b) Compare the fair value of the bond to the price quoted in the *Journal*. Why might they differ?

3. Suppose bonds are fairly priced and are free of default risk. Explain, for each of the following pairs of bonds, which has the higher yield-to-maturity. If the answer "depends," explain what it depends on.

(a) A 6 percent bond or a 10 percent bond?

(b) A strip or a coupon bond of the same maturity?

(c) A callable bond or a comparable noncallable bond?

4. On January 1, 1993, Ford Motor Company had financial assets worth $100 million with a duration of 7 years. They also had an existing position in an interest rate swap (plain vanilla) in which they (i) received 6-month LIBOR + 200 BPs and (ii) paid a fixed rate of 8 percent, both for two more years on a notional principal of $50 million. Spot rates at the time were

Maturity (Years)	Spot Rate
0.5	4
1.0	5
1.5	6
2.0	7

(a) What was the value of the swap to Ford on January 1, 1993?

(b) Does Ford gain or lose if interest rates rise? Why?

(c) What notional principal would Ford choose on the swap if it wanted to have an overall duration (financial assets plus swap) of 6 years?

5. Fannie Mae, the US mortgage agency, issued an inverse floater maturing May 7, 1997, paying a rate set by the formula

$$\text{Floating Rate} = \text{Base Rate} - \text{6-Month LIBOR},$$

where the base rate rises by one percent a year,

$$\text{Base Rate} = \begin{cases} 12.5 & \text{for } 11/95 \text{ payments} \\ 12.5 & \text{for } 5/96 \text{ payments} \\ 13.5 & \text{for } 11/96 \text{ payments} \\ 13.5 & \text{for } 5/97 \text{ payments} \end{cases}$$

    (a) Explain how you might value this note by decomposing it into simpler components. What data would you need to estimate its price?

    (b) Estimate the duration of this note.

    (c) The floating rate also has a lower limit of zero, which we forgot to mention. Describe qualitatively the effect of this limit on the interest-rate sensitivity of the note if LIBOR rises above 12 percent.

Short Answers

1. (a) Day count is 77, accrued interest is 2.138. (b) Invoice price is 129.869. (c) The yield is the solution y to

$$\text{Invoice Price} = \frac{\text{Coupon}}{(1 + y/2)^w} + \frac{\text{Coupon}}{(1 + y/2)^{w+1}} + \cdots + \frac{\text{Coupon} + 100}{(1 + y/2)^{w+n-1}},$$

which in this case is 6.964 percent.

2. (a) There are a variety of ways to interpret this question. One approach is to use the first three bonds to replicate the cash flows of the fourth. An equivalent approach is to use the first three bonds to estimate spot rates, then use these spot rates to value the fourth bond. If you do this, the spot rates are $y_1 = 0.0552$, $y_2 = 0.0562$, and $y_3 = 0.0562$. With these spot rates, the value of bond 4 is 104.089, which is considerably higher than the quoted price.
   (b) Among the possible rationales for such apparent mispricing are: bid/ask spreads (probably not this big), nonsynchronous price quotes, tax effects on bonds selling away from par, and call provisions on one or more of the bonds.

3. These typically depend:

    (a) Depends on the slope of the yield curve: with a flat yield curve they're the same, if increasing then the 6 percent bond has the higher yield, if decreasing the reverse. If the yield curve is hump-shaped, it could go either way.

    (b) Similar thing: with a flat yield curve they're the same, with an increasing yield curve the zero has a higher yield-to-maturity (recall the par yield curve).

    (c) The callable bond is cheaper (the difference being the value of the call option) and so has a higher yield. Alternatively, you could argue that the yield for a callable bond is the yield-to-call or some other option-dependent number, in which case it could go either way.

4. (a) Use spot rates to value the two sides of the swap. The floating rate leg (a long position) has a value of (say) par if this is a reset date, namely 50. The fixed rate leg

(a short position) has a value (using the spot rates) of 51.01m. The net value of the swap, then, is −1.01m. Apparently rates fell after the swap was initiated. [Remark: the +200 is generally viewed as a penalty for risk, but we regarded any reasonable approach to it as ok.]

(b) Ford loses on its assets (they have positive duration, meaning the price falls when yields rise) and gains on the swap. In this case the former is larger, by a lot.

(c) Take the positions as is. Ford's net position in assets and the swap is

$$v = 100 + 50 - v_{\text{fixed}} = 100 + 50 - 51.01 = 98.99m.$$

Duration is the value-weighted average of the durations of the individual positions:

$$D = \frac{100}{v} \times 7 + \frac{50}{v} \times 0.5(\text{say}) - \frac{v_{\text{fixed}}}{v} \times D_{\text{fixed}}.$$

We estimate the duration of the fixed leg to be 1.89 years, giving us $D = 6.35$, a little above our target. We need to scale up the swap position, in other words. A swap with a notional principle of \$74.4mm gives us exactly $D = 6$. Details may vary depending on what duration estimates you use for the floating and fixed legs of the swap.

5. Another inverse floater.

   (a) Pieces could be (i) a long position in a bond, in which the coupon payments drift up over time as in the schedule for the Base Rate; (ii) a short position in a standard FRN; and (iii) a long position in a zero maturing 5/97. These three pieces have the same cash flows (interest and principal) as the inverse floater. To value the pieces, you need the discount factors or spot rates for maturities up to about 1.5 years.

   (b) Guesstimates: (i) a little less than 2 years, (ii) about 0.5 (we're close to the reset date), and (iii) about 2 years. The weighted average of these durations (with weights taken from their shares of the overall value of the floater) depends on the market values of the components.

   (c) If LIBOR rises above the Base Rate, the note pays nothing and we have essentially an 18-month zero. Since the duration of the zero (about 2 years in this case) is less than that of the floater (3 maybe?), we see that duration declines as rates rise. Thus in the price-yield diagram, the note has lower duration at higher yields, which we refer to as negative convexity (draw this to see why).

B.2   1996 Midterm

Midterm Examination
Monday, November 6, 1995

Answer all questions in the space provided. Each question is worth 25 points. You may use

one page of notes and a calculator. Answers with sound logic but wrong numbers will be looked at kindly. Conversely, numbers without explanation will be viewed as incomplete.

1. Deutsche Bank Finance BV issued a Swiss-franc-denominated note maturing November 28, 2000, with annual coupons of 4%. Bloomberg estimates its (quoted) price, for November 1 settlement, at 104.110.

    (a) What is the accrued interest for this bond?

    (b) What is the invoice price?

    (c) What is the yield to maturity?

2. Consider these prices and yields for US treasuries:

Maturity (Yrs)	Coupon Rate	Price (Ask)	Yield (%)
0.5	0	97.561	5.00
1.0	8	101.932	5.98
1.5	0	90.194	7.00

    (a) Use the prices of the bonds to compute the first three discount factors.

    (b) Use the discount factors to estimate the value of a 10% bond maturing in 18 months.

    (c) If the market price of the 10% bond is 103.100, how could you exploit the implied arbitrage opportunity? What practical considerations might make this difficult?

3. The IBRD (World Bank) issued a lira-denominated note maturing October 31, 2001, and making annual interest payments at a rate set by the formula

$$\text{Rate} = 18.5\% - 2 \times \text{12-Month Lira LIBOR}.$$

    (a) Explain how you might replicate the cash flows of this note with simpler components.

    (b) If the lira spot rate curve (computed with the usual semi-annual compounding) is flat at 7.5% (a good approximation at the moment), what should the note be worth?

4. A bond fund manager operates on a fiscal year ending April 30. Her current portfolio consists of $1mm (million): 20% in 12-month treasury bills (zeros, in other words) yielding 5% and 80% in three-year zeros yielding 6.5%. Thus far she has had a good year and would like to eliminate any chance that a change in interest rates could eliminate her gains over the second half of the fiscal year.

    One approach to her risk management problem is to use an interest rate swap. With this in mind:

(a) Describe qualitatively how an interest rate swap might be used to reduce the interest-sensitivity of the fund.

(b) If the swap rate for a three-year semi-annual swap is 6%, what is the duration of the fixed rate leg?

(c) If the target duration for the fund is zero, what size swap is called for? (Ie, what is the appropriate notional value?)

(d) Given the zeros in the portfolio, the fund manager would like to avoid the first two interest payments. One possibility is a forward starting swap, in which no interest payments are made — by either side — in the first two periods. If you are given the relevant discount factors, what fixed rate would equate the initial values of the fixed and floating rate legs of the forward starting swap?

Short Answers

1. It's a eurobond. (a) 333 days of accrued interest is $3.700. (b) The invoice price is 107.810. (c) The yield is 2.59%.

2. Arbitrage and discount factors.

   (a) Discount factors are 0.9756, 0.9426, and 0.9019.

   (b) The 10% bond should be worth

   $$(d_1 + d_2) \times 5 + d_3 \times 105 = 104.295.$$

   (c) The market underprices the 10% bond by about a dollar relative to the first three bonds. One way to exploit this is to find a combination of the first three that generates the same cash flows and short it. The cost of doing this is the 104.295 we just calculated. Among the practical difficulties are (i) the bid/ask spread (we can't sell at the prices quoted above) and (ii) imperfect opportunities for shorting treasuries.

3. Inverse floater.

   (a) One approach consists of (i) a long position in an 18.5% fixed rate note, (ii) two short positions in standard FRNS (each paying LIBOR), and (iii) two long positions in zeros.

   (b) Value is sum of the components:

   $$\begin{aligned} \text{Price} &= p_{\text{fixed}} - 2 \times p_{\text{frn}} + 2 \times p_{\text{zero}} \\ &= 143.77 - 2(100.00) + 2(69.20) = 82.18. \end{aligned}$$

4. Interest-rate risk management with swaps.

   (a) A pay-fixed swap tends to reduce duration by adding (effectively) a short position in a fixed-rate bond.

   (b) Swaps are generally priced at par to start, so what we want is the duration of a three-year bond selling for 100 with coupons of 3: 2.71. The duration of the floating rate side is (about) 0.49.

   (c) Start with the duration of the portfolio:

   $$D = 0.2 \times 0.98 + 0.8 \times 2.91 = 2.52.$$

   To reduce this to zero, we need a notional value fraction equal to a fraction w of the $1mm total satisfying:

   $$0 = 1 \times 2.52 - w \times 2.71 + w \times 0.49.$$

   The solution is $w = 1.14$, indicating a notional value of $1.14mm.

   (d) Warning: this is hard! With no payments the first year, the value of the floating rate leg is
   $$p_{\text{frn}} = d_2 \times 100 \text{ ("par")}.$$
   The fixed payment C satisfies

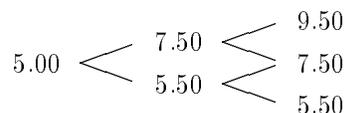 $$p_{\text{frn}} = (d_3 + d_4 + d_5 + d_6) \times C + d_6 \times 100.$$

B.3 1995 Final

Final Examination
Wednesday, December 20, 1995

Answer all questions in the space provided. You may use one page of notes and a calculator. Answers with correct logic but wrong numbers will be looked at kindly. Conversely, numbers without explanation will be viewed as incomplete.

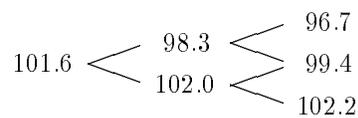1. *(20 points)* You are the manager of a fixed income portfolio and have decided to shorten its duration. List (in list format) ways in which this might be done, and discuss the advantages and disadvantages of each.

2. *(30 points)* Your mission is to use a binomial pricing model to value a one-year American call option, with a strike price of 100, on a 2.5-year bond. The interest rate tree is

$$
5.00 <
\begin{array}{c}
7.50 < \begin{array}{c} 9.50 \\ 7.50 \end{array} \\
5.50 < \begin{array}{c} 7.50 \\ 5.50 \end{array}
\end{array}
$$

As usual, the time interval is six months, so the tree describes possible paths for the six-month "short" rate between now and 12 months from now.

(a) Compute, for this tree, the 6, 12, and 18 month spot rates.

Now consider a 2.5-year, 8 percent bond, whose price path is

$$
101.6 <
\begin{array}{c}
98.3 < \begin{array}{c} 96.7 \\ 99.4 \end{array} \\
102.0 < \begin{array}{c} 99.4 \\ 102.2 \end{array}
\end{array}
$$

That is, the current price of the bond is 101.6, the price 6 months from now is either 98.3 or 102.0, and so on.

(b) Compute the price of the option noting, in particular, the times at which it is optimal to exercise the option.

(c) Comment briefly on the reliability of your valuation of the option: what factors might make you adjust your answer?

3. *(25 points)* You work for a large pension fund and your boss has given you data on 6 corporate bonds, all rated Baa and all with ten years to maturity. List (in list format) the factors you'll want to consider in choosing among the bonds. If you intend to use a complete evaluation of each bond, describe the information you will need.

4. *(25 points)* Plot the price-yield relation for each of the following bonds. Explain the reasoning for each behind your plot.

(a) A 30-year GNMA with an 8% coupon and an underlying pool of 8.5% mortgages with a normal prepayment pattern.

(b) The same as (a), but with a higher than usual expected prepayment rate (these people live in a high-turnover NJ suburb and move frequently).

(c) The same as (a), but "seasoned," with twenty years left until maturity.

Short Answers

None available.

B.4   1996 Final

Final Examination
Tuesday, December 16, 1996

Answer all questions in the space provided. Each question is worth 25 points. You may use one page of notes and a calculator. Answers with sound logic but wrong numbers will be looked at kindly. Conversely, numbers without explanation will be viewed as incomplete.

1. Short answers:

   (a) List two assets with negative convexity. Explain why.

   (b) List two reasons why the Black-Scholes option pricing model is less useful for bonds than for other assets.

   (c) List two important differences between exchange-traded and over-the-counter derivatives.
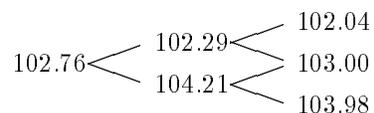
2. A bond fund manager has a \$120mm portfolio of bonds with an average duration of 2.1 years. He now suspects the US may enter a recession and that the Fed will reduce interest rates as a result. Your mission is to explain how the manager could use a forward or futures contract to raise the duration of his portfolio to 4. Assume for the purpose of your calculations that spot rates are flat at 6%.

   (a) Suppose the manager could like to use a 6-month forward contract on a 10-year zero. Describe how such an instrument would be used to increase duration. How large a contract should he enter to change the duration to 4?

   (b) Duration, we know, is a measure of sensitivity to generalized movements in interest rates. What specific interest-rate sensitivity does the forward contract add to the portfolio?

   (c) If the manager decides instead to use one of the popular bond futures contracts, what should he know about the difference between forwards and futures?

3. Consider a 3-period interest rate tree generated by the Ho and Lee binomial interest rate model:

$$
7.00 \quad < \quad {6.50 \atop 4.50} \quad {< \atop <} \quad {8.00 \atop {6.00 \atop 4.00}}
$$
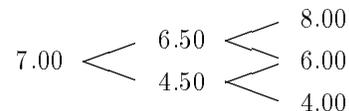
   (a) What is the value of the volatility parameter σ? What does this value mean?

   (b) What are the values of the drift parameters μ_{t+1} and μ_{t+2}? What do these values mean?

(c) Compute the first three spot rates implied by this tree. How do these reflect the values of the parameters?

4. Consider a callable 3-period 6% bond. Using the interest rate tree from the previous problem, the price path for the underlying noncallable bond is

$$
102.76 \left< \begin{array}{c} 102.29 \left< \begin{array}{c} 102.04 \\ 103.00 \end{array} \right. \\ 104.21 \left< \begin{array}{c} 103.00 \\ 103.98 \end{array} \right. \end{array} \right.
$$

The "embedded" call option has a strike price of 103 (par plus accrued interest) and can be exercised any time.

(a) Compute the cash flow at each node of the tree generated by immediate exercise of the option.

(b) Compute the value of the option. At which nodes should the option be exercised?

(c) Use your answer to compute the price of the callable bond at each node in the tree.

(d) For each of the two second-period nodes (those corresponding to interest rates of 6.50 and 4.50), find the combination of a one-period zero and the "underlying" bond that reproduces the value of the callable bond in the subsequent period. In what sense does your answer tell you that the callable bond's value is nonlinear?

Short Answers

1. Short answers:

   (a) Callable bonds: the call provision makes the price-yield relation flatten out at low yields. Mortgage passthroughs: the prepayment "option" flattens the price-yield relation at low yields (faster prepayments shorten the duration) and steepen it at high yields (slower prepayments lengthen duration).

   (b) Black-Scholes allows negative yields, and assumes volatility constant both over time and across option maturities.

   (c) Credit risk, liquidity, standard v. customized contracts.

2. Duration management with forwards and futures.

   (a) Long position in forward contract adds exposure to long zero and raises duration:

   $$4 = 2.1 + x \times D_{21} - x \times D_1,$$

   With $D_{21} = 10.194$ and $D_1 = 0.485$ we find $x = 0.196$. Meaning: the value of each side of the forward contract is $0.196 \times 120mm = 23.48mm$.

(b) Sensitivity to forward rates between 2 and 21 half-years.

(c) Futures have greater liquidity, are marked to market daily (requires cash management), and with long bond contracts it's not always clear what the bond is.

3. Rate tree:

   (a) $\sigma = 1.00\%$.

   (b) $\mu_{t+1} = -1.50\%$, $\mu_{t+2} = 0.50\%$. They indicate the predictable changes in the short rate over the next two periods.

   (c) Discount factors are $(0.9662, 0.9403, 0.9130)$ and spot rates $(7.00, 6.25, 6.16)$.

4. Callable bond:

   (a) Cash flows are

$$
0.00 \;<\; \begin{array}{c} 0.00 \\ 1.21 \end{array} \;<\!\!< \begin{array}{c} 0.00 \\ 0.00 \\ 0.98 \end{array}
$$
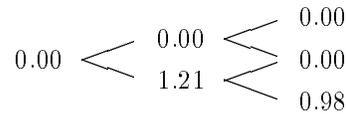
   (b) The optimal exercise plan is to call at (0,1) if you get there, leading to the option being worth

$$
0.59 \;<\; \begin{array}{c} 0.00 \\ 1.21 \end{array} \;<\!\!< \begin{array}{c} 0.00 \\ 0.00 \\ 0.98 \end{array}
$$
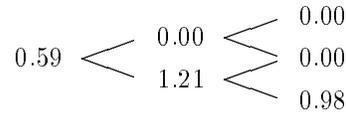
   (c) The callable bond is the difference between the noncallable bond and the call option:

$$
102.17 <\; \begin{array}{c} 102.29 \\ 103.00 \end{array} <\; \begin{array}{c} 102.04 \\ 103.00 \\ 103.00 \end{array}
$$
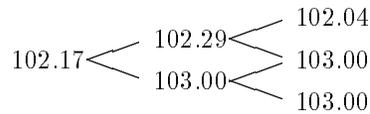
   (d) The quantities of the underlying bond and the "short" are

$$
\text{na} \;<\; \begin{array}{c} 1.00,0 \\ 0,1.03 \end{array} \;<\!\!< \begin{array}{c} \text{na} \\ \text{na} \\ \text{na} \end{array}
$$
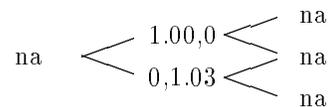
   At the up node (1,1) you buy only the noncallable bond, and at the down node (0,1) you buy only the short bond. Ie, at the up node the bond is long (the noncallable) and at the down node it's short.

Appendix C

Spreadsheet Calculations

C.1 Bond Yield Calculations

Spreadsheets can be used in a variety of ways to reduce the effort of bond yield calculations — to find the yield y satisfying (say) equation (2.10), given a quoted price. I start with the most transparent method then suggest some tricks to make this easier. For bonds other than US treasuries, adapt accordingly.

The most direct method of attack on bond yields is to use the method of repeated guesses: We guess a value of y and compute the price using the formula on the right side of (2.10). If the price from the formula is too high, we raise our guess of y, if too low we reduce y. Continue until the computed price is sufficiently close to the invoice price. A sample spreadsheet for the example of Section 2.2:

```
      A                 B            comments
 1    Price Calculations
 2    quote =           104.1875     = 104 + 6/32
 3    n =               4
 4    coupon rate =     8.5
 5    u =               33
 6    v =               150
 7    w =               0.819672     = b6/(b5+b6)
 8    accint =          0.766393     = 0.5*b4*b5/(b5+b6)
 9    invoice =         104.9539     = b2+b8
10
11    Yield Calculations: Manual Method
12    y =               6.3          <== vary until b14 equals b9
13    d =               0.969462     = 1/(1+b12/200)
14    formula =         104.6579     = 0.5*b4*b13^b7*(1+b13+b13^2)
                                       + (0.5*b4+100)*b13^(b7+b3-1)
```

You'll note that the guess of $y=6.3\%$ in B12 produces a price (B14) a little below the invoice price (B9), so we need to continue our guessing. The exact answer is 6.1376.

We can refine this method in several directions. First, we can make the formulas a little simpler with the following substitution:

$$d = \frac{1}{1 + y/200}.$$

You'll note that I did this in B13 and B14 above to save myself some typing. I've also expressed y as a percentage, requiring division by 200 rather than the usual 2. Second, with some knowledge of geometric series, namely

$$\sum_{j=0}^{n-1} d^j = \frac{1 - d^n}{1 - d}, \qquad d \neq 1,$$

we can replace (2.10) with

$$\text{Invoice Price} = \text{Coupon} \times d^w \times \frac{1 - d^n}{1 - d} + 100 \times d^{w+n-1},$$

which saves a lot of typing with bonds that have many periods remaining to maturity. Third, we can use the equation solver in our spreadsheet program to do the guessing for us. In Excel, "solver" is available in the tools menu; most other spreadsheets have a similar capability.

At some risk, you can use the bond utilities available in many programs to perform the day count, accrued interest, and yield calculations. If you use these correctly they do most of the work for you, but if not you may get garbage without knowing it.

A fine point for the aficionados: for bonds with no accured interest, d is the solution to an equation of the form

$$0 = -\text{Price} + d \times \text{Coupon} + d^2 \times \text{Coupon} + \cdots + +d^n \times \text{Coupon} + \text{Principal},$$

which is an nth degree polynomial in d and therefore has n solutions or "roots." Which one do we want? Descarte's Law of Signs tells us that the number of positive real roots is no larger than the number of sign changes in the coefficients of powers of d above. Since the price, coupon, and principal are all positive, there is only one sign change, and therefore at most one real positive root, which is the one we want. For more complex instruments, however, we could have more than one real positive solution.

C.2 Duration Calculations

Duration is easily appended to the yield calculations. The simplest way to do this is not to use one of the formulas, but to do a numerical derivative: approximate dp/dy with $\Delta p/\Delta y$

for some small Δy. The only trick is making sure we compute Δy as a number, not a percent. We add to the previous speadsheet:

```
       A                B              comments
16  Duration Calculation
17  Dy =             +0.0001    <== any small number will do
18  d01 =                       = 1/(1+b12/200+b16/2)
19  Dp =                        = 0.5*b4*b19^b7*(1+b19+b19^2)
                                   + (0.5*b4+100)*b19^(b7+b3-1)
20  Dur =                       = -(b19/b17)/b9
```

C.3   Trees for Binomial Models

We can also use spreadsheets to do the calculations of our quantititative interest rate models, like those in the examples of Section 7.4. A sample spreadsheet for this purpose is:

```
       A                      B                    C
1   Parameters
2   r(0,0) =                5
3   sigma =                 1
4   mu1 =                   0
5   mu2 =                   0
6
7   Short Rate Tree
8                                               +c9+2*b3
9                           +b10+2*b3           +c10+2*b3
10  +b2                     +a10+b4-b3          +b10+b5-b3
11
12  Discount Factor Tree
13                                              +1/(1+c8/200)
14                          +1/(1+b9/200)       +1/(1+c9/200)
15  +1/(1+a10/200)          +1/(1+b10/200)      +1/(1+c10/200)
16
17  Bond Price Path Tree (6 Percent 3-Period Bond)
18                                              +103*c13
19                          +0.5*b14*(c18+c19+6) +103*c14
20  +0.5*a15*(b19+b20+6) +0.5*b15*(c19+c20+6) +103*c15
21
22  State Price Tree (Duffie's Formula)
23                                              +b24*0.5*b14
24                          +a25*0.5*a15        +b24*0.5*b14+b25*0.5*b15
25  +1.0                    +a25*0.5*a15        +b25*0.5*b15
26
```

```
27  Discount Factors and Yields
28  +b24+b25           +c23+c24+c25          +c23*c13+c24*c14+c25*c15
29  +200*(1/a28-1)     +200*(1/b28^(1/2)-1) +200*(1/c28^(1/3)-1)
```

Some things that may not be immediately obvious:

- Unlike the text, I've computed the discount factor tree explicitly to make the state price formulas simpler. [The fifty-fifty rule applies directly to discount factors; see equation (7.3).]

- The last row can be used to adjust the drift parameters, called here mu1 and mu2, to match current spot rates: simply vary these parameters until the row matches current spot rates.

Appendix D

Caselet: Banc One 1993

Note: This example/case and those following illustrate the use of fixed income and currency derivatives. The emphasis is on derivatives debacles, on the grounds that noone can drive by an accident without looking, but they illustrate aspects of derivatives use of more general interest. They are generally very brief, and come with no guarantes of accuracy.

Banc One Corporation, the Columbus, Ohio, bank holding company, rose from roughly the 800th largest US bank in 1960 (measured by asset value) to the 7th largest at the end of 1993. Banc One was then, and remains, one of a handful of "superregional" banks that have grown rapidly by acquiring other regional banks and now challenge the money center banks for supremacy in many areas. Banc One has traditionally focused on retail banking to consumers and middle-market (ie, not Fortune 500) business customers. They also have a history of technological innovation, including both the first ATMs in the US and (currently) one of the most ambitious branch management systems anywhere.

During the summer and fall of 1993, however, Banc One came under intense scrutiny for its derivatives activity. During 1993 Banc One increased dramatically its positions in interest rate swaps, resulting in year-end positions with a total notional value of 39 billion dollars — this for a bank with total assets of $80 billion. Panic following the release of this information was apparently responsible for a sharp decline in the price-earnings ratio. The stock price decline killed off one deal, a proposed acquisition of the $20 billion FirsTier Financial of Nebraska, and in all likelihood nipped other possibilities in the bud.

D.1   Interest Rate Exposure

The swaps were part of Banc One's overall management of interest rate risk. The bank's natural positions (ie, prior to any modification with financial derivatives) left it exposed

to declines in interest rates. Banks traditionally express this sensitivity in two ways. The first, which stems from banks' use of book-value accounting, is to measure how quickly the interest payments adjust on its various assets and liabilities. Banc One reports that it is naturally *asset sensitive*, by which it means that when interest rates change, payments on assets adjust more quickly than those on liabilities. As a result, earnings tend to fall if rates fell.

The second way of expressing interest rate exposure follows from the same kinds of duration calculations we've studied. With this approach, Banc One reports that its assets have an effective duration of 1.45 years, and the liabilities 1.84, thus leaving it with a negative net duration. Banc One's net asset value, in other words, tends to decline if rates fall.

D.2   Financial Engineering

Banc One's stated interest rate policy is to stay neutral (ie, roughly equate the duration of assets and liabilities), but it is willing on occasion to depart somewhat from this goal. One way to do this is with receive fixed swaps, which raise the duration of assets and lower the duration of liabilities. At the end of 1993 it reported swap positions with a total notional value close to $37 billion dollars, as follows:

Type of Swap	Notional Principal (b$)
Receive Fixed Generic	6.7
Receive Fixed Amortizing	15.1
Pay Fixed	1.6
Forward Starting	7.5
Basis (Pay Prime)	5.6
Total	36.5

The receive fixed generic swaps are the plain vanilla swaps of Chapter 5. They transform Banc One's floating rate assets into fixed rate assets with longer duration. The amortizing swaps do the same thing in a more complex way: the notional principal falls when interest rates fall. This gives them a similar risk profile to mortgage passthroughs, with the amortization schedule mimicking prepayments. The small amount of pay fixed swaps are the result of previous asset/liability management efforts, but are held to maturity partly because of hedge accounting rules. Forward starting swaps are also receive fixed, with payments scheduled to start in 1995 and 1996. The basis swaps pay the difference between the prime loan rate and LIBOR, and thus provide a hedge for any difference between these two floating rates.

Banc One estimates that these swaps, plus smaller positions in other interest-sensitive derivatives, raise the duration of its assets to 1.73, and lower the duration of its liabilities to 1.51, giving the bank a net positive duration. Financial engineering, in other words, was used to change Banc One's net duration from negative to positive.

D.3   Credit Risk

One of the risks from swaps is that the counterparty defaults. There is not much of a track record, but credit risk is undoubtedly greater for a swap position with an A-rated bank than with (say) a futures contract on the Chicago Mercantile Exchange. Banc One reduced this risk in two ways: by requiring mutual posting of collateral and by spreading its positions across several counterparties. On October 31, 1993, it reported these positions:

Counterparty	Notional Principal
Bankers Trust	12.1
Union Bank of Switz	7.0
Goldman Sachs	6.2
Lehman Brothers	4.1
Merrill Lynch	3.3

D.4   Experience and Control Procedures

Banc One considers itself a relatively experienced user of financial derivatives. As in many of its activities, it started small. It began systematic interest rate risk (GAP) management in 1982, and arranged its first swap in 1983.

Moreover, risk management and derivatives activity are governed from the top. As a holding company, Banc One is the head of a complex group of smaller units. Banc One centralizes interest rate risk management in its corporate asset and liability committee (ALCO). The committee meets monthly, and includes: the chairman, the president, and six other officers. This structure is intended to guarantee that senior management is aware of, has responsibility for, and has control over, corporate derivatives activity.

D.5   What Happened?

As we have seen, Bank One used interest rate swaps to give it a positive net duration, thus exposing the bank to increases in interest rates. At the time, this was an understandable

gamble: with the spread between 3-month and 5-year treasury yields over 200 basis points, it was tempting to accept some risk in return for higher current income. Indeed, Banc One estimates that swaps contributed more than 400 million to 1993's net income before taxes of 1.7 billion.

The bad news is that interest rates rose sharply in 1994, with 3-month treasuries rising from 3.07percent at the end of 1993 to 5.68 percent at the end of 1994. One consequence is that Banc One wrote off 169m (after taxes) in 1994 in derivatives losses, possibly with more to come. By one report, Banc One suffered unrealized losses of 939m on derivatives and 208m on securities. Another consequence is that Banc One reduced its interest rate exposure. It did this by selling one- to three-year treasuries rather than selling the swaps, since the latter would violate the rules under which the swaps qualified for hedge accounting treatment.

Despite this, Banc One registered net earnings in excess of a billion dollars in 1994 and has a stronger capital position than ever.

Questions for Discussion

1. Was Banc One's strategy for interest rate risk management faulty, or was it simply unlucky?

2. What modifications to Banc One's risk management strategy would you suggest to avoid similar problems in the future?

3. Steven Bluhm, Banc One VP for funds management, comments that 1994 net unrealized derivatives losses of $1.3b were largely the result of non-symmetric accounting treatment. An article paraphrases his comments like this: "Because current accounting laws require Banc One to mark to market its interest rate swaps but not the underlying vehicles being hedged — credit card loans and other cash flows — they do not provide an accurate picture of Banc One's balance sheet." (*Derivatives Week*, June 5, 1995, p 7.)

4. Compare and contrast Banc One's use of interest rate swaps with other instruments for adjusting its duration:

   - Issue short debt (commercial paper or large CDs) and buy long treasuries.
   - Buy eurodollar futures.

   Include in your discussion some mention of accounting treatment, bank capital requirements, and liquidity management.

5. One unforeseen consequence of Banc One's use of derivatives was panic by some prominent bank analysts, driven as much by a general perception of "derivatives risk" as by systematic disgreement with Banc One's risk management. Similar fears have hurt other companies, as shareholders and even experienced analysts wrestle with the mind-numbing complexity of modern derivatives. Given this climate, how would you suggest Banc One deal with the problem of communicating its risk management strategy to shareholders?

Appendix E

Caselet: Orange County

Orange County, California, moved instantly from invisibility to infamy when it filed for bankruptcy on December 6, 1994, after its municipal investment fund reported losses in excess of a billion dollars. Its difficulties apparently stemmed from leveraged positions on government securities, whose value dropped suddenly when interest rates rose during 1994.

E.1   The Mission

Orange County's treasurer, Robert Citron, managed a fund of about $7b, the contributions of Orange County and a number of other California municipalities. The fund was a cash managment tool: the idea was to invest tax and bond revenue profitably before it was spent. The treasurer was told to invest only in Aaa-rated US government-backed securities, presumably to limit risk to the fund.

E.2   The Power of Leverage

Citron, however, was an aggressive fund manager who made the fund enormously profitable in the early 1990s as interest rates fell. He did this not with unusual derivatives, but with leverage.

Using repurchase agreements, he used the $7-billion fund to purchase $20b in securities. A repo is essentially a collateralized loan, and works something like this. You take the first $7b and buy securities. Then you use the securities as collateral for a loan, getting $7b in cash by pledging the securities as collateral. With the cash you buy $7b more securities, thus levering your position two-to-one. This effectively doubles the benefits of increases in the price of the securities, but it doubles the losses too. Citron did this one better: he used the second batch of securities for another repo, giving him approximately $20b worth of securities for the original $7b.

Citron used a second method to increase his exposure further. Many of the securities he bought were inverse floaters, some of them taylor-made for him by government agencies. In an inverse floater, the interest rate paid might be something like

$$\text{Interest Rate} = 17\% - 2 \text{ LIBOR}.$$

Ie, the interest rate varies inversely with LIBOR, and by a multiple (here 2). We saw a similar example in Section 5.1.

The combination of direct leverage through repos and indirect leverage through inverse floaters resulted in a portfolio with very high duration. When rates rose in 1994, the value of the portfolio dropped by an estimated 2 billion.

E.3   Legal Issues

Among the issues that come to mind:

1. Did Merrill Lynch, Citron's banker, have a legal or ethical responsibility not to sell him such products?

2. Merrill Lynch underwote a $699m bond issue by Orange County in July 1994, well after interest rate increases started to eat away at the fund. The prospectus noted that the fund was not marked to market, but did not mention the losses. Did Merrill have a legal, ethical, or professional obligation to investors in the bonds to warn them that the fund was worth less than its book value? Was there a conflict of interest between Merrill's role in advising Citron in his investment and underwriting Orange County's bonds?

Questions for Discussion

1. Describe, as specifically as possible, the effects on duration of inverse floaters and leverage.

2. Citron was a public employee, and got no performance bonus for large returns on the investment fund. What was his motivation for taking such large risks?

3. How was Citron able to do all this on his own? Ie, where was the oversight and control?

4. Some of the inverse floaters were issued by Fannie Mae and Sallie Mae, and designed specifically with Orange County in mind. Should agencies of the federal government be allowed to do this kind of thing?

Appendix F

Caselet: Intel

As usual, no guarantees of accuracy or sense.

Many industrial firms use fixed income derivatives to balance the cost and uncertainty of their liabilities. They face, in this regard, a choice between fixed and floating rate debt, with unlimited variations in between. Intel in 1994 faced a somewhat different problem: its enormously successful microprocessor business was throwing off cash faster than it could be used. With the cost of new plants in the billions, Intel certainly had plans for the money. The question was what to do with it in the meantime.

F.1   Objectives

From the 1994 10K:

> The Company's policy is to protect the value of the investment portfolio by minimizing principal risk and earning returns based on current interest rates. All hedged equity and a majority of investments in long-term fixed rate debt securities are swapped to U.S. dollar LIBOR-based returns. The currency risks of investments denominated in foreign currencies are hedged with foreign currency borrowings, currency forward contracts or currency interest rate swaps (see "Derivative financial instruments"). Investments with maturities of greater than one year are classified as long term. There were no material proceeds, gross realized gains or gross realized losses from sales of securities during the year.
>
> Investments with maturities of greater than six months consist primarily of A/A2 or better rated financial instruments and counterparties. Investments

with maturities of up to six months consist primarily of A1/P1 or better rated financial instruments and counterparties. Foreign government regulations imposed upon investment alternatives of foreign subsidiaries or the absence of A/A2 rated counterparties in certain countries result in some minor exceptions. Intel's practice is to obtain and secure collateral from counterparties against obligations whenever deemed appropriate. At December 31, 1994, investments were placed with approximately 100 different counterparties, and no individual security, financial institution or issuer exceeded 10

F.2   Derivatives Positions

Also from the 10K:

As part of its ongoing asset and liability management activities, the Company enters into derivative financial instruments to reduce financial market risks. These instruments are used to hedge foreign currency, equity market and interest rate exposures of underlying assets, liabilities and other obligations. These instruments involve elements of market risk which offset the market risk of the underlying assets and liabilities they hedge. The Company does not enter into derivative financial instruments for trading purposes. Notional amounts for derivatives at fiscal year-ends are as follows:

(In millions)	1994	1993
Swaps hedging investments in debt securities	1,080	809
Swaps hedging investments in equity securities	567	260
Swaps hedging debt	155	110
Currency forward contracts	784	620
Currency options	10	28

Questions for Discussion

1. Evaluate Intel's stated objective of converting interest payments on assets and liabilities into floating rates.