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Quantifying Interest Rate Risk

0. Overview

- Tools and Their Uses
- Price and Yield
- DV01, Duration, and Convexity
- Value-at-Risk
- Active Investment Strategies

1. Examples

- Example 1: Bell Atlantic
  - Long or short (floating rate) debt?
  - Liquidity management
  - Risk exposure
  - Accounting methods

1. Examples (continued)

- Example 2: Banc One
  - Measurement of interest-sensitivity
  - Regulatory reporting
  - Liquidity management
  - Risk management and financial engineering
  - Accounting methods

1. Examples (continued)

- Example 3: Proprietary Trading at Long-Term Capital
  - Making money
    - Spotting good deals
    - Taking calculated risks
  - Risk assessment
    - How much risk does the firm face?
    - How much risk does a specific trader or position add?
  - Risk management
    - Should some or all of the risk be hedged?
    - Silber: good traders know what they're betting on

1. Examples (continued)

- Example 4: Bond Funds
  - Mission:
    - Target specific markets?
    - Target specific maturities?
    - Index or active investing?
  - Risk reporting to customers
  - Risk management

1. Examples (continued)

- Example 5: Dedicated Portfolios
  - Purpose: fund fixed liabilities
  - Example: defined-benefit pensions
  - Objective: minimize cost
  - Approach tied to accounting of liabilities

2. Tools and Their Uses

- Toolkit 1: DV01, duration, and convexity
  - Risk management: approximation based on parallel shifts in yield curve
  - Crude but relatively easy to implement
  - Standard language among practitioners

- Toolkit 2: Statistical risk measures
  - Risk management: variance of portfolio based on variances and covariances of individual positions
  - Greater complexity leads hopefully to greater accuracy
  - Current standard in risk management

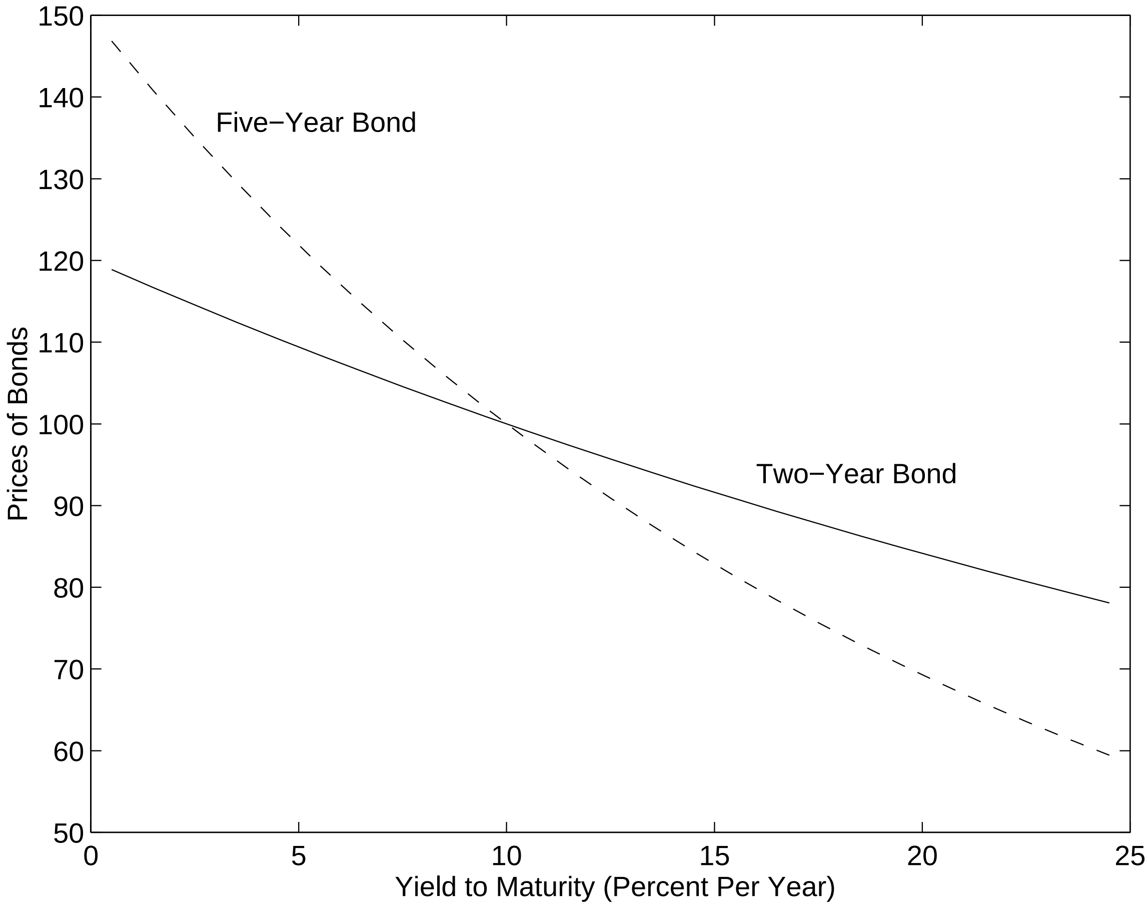
- Toolkit 3: State-contingent claims models
  - Standard tool for valuing derivatives
  - Difference from statistical models is an indication that this is an art, not a science

- Remark:
  - We'll study toolkit 1 in depth, toolkit 2 a little, and toolkit 3 later on

2. Price and Yield

- Price and yield are inversely related:

$$p(y) \;=\; \frac{c_1}{(1+y/2)} + \frac{c_2}{(1+y/2)^2} + \cdots + \frac{c_n}{(1+y/2)^n}$$



- Long bonds are more sensitive to yield changes than short bonds

3. DV01

- Our task is to produce numbers that quantify our sense that the price-yield relation is "steeper" for long bonds

- Measure 1 is the DV01: "Dollar Value of an 01" (aka Present Value of a Basis Point or PVBP)

- Definition: DV01 is the decline in price associated with a one basis point increase in yield:

$$\begin{aligned} \text{DV01} &= -\text{ Slope of Price-Yield Relation } \times \ 0.01\% \\ &= -\frac{dp}{dy} \times \ 0.0001 \end{aligned}$$

- Calculation (direct method):

  (a) compute yield associated with (invoice) price

  (b) compute price associated with yield plus 0.01%

  (c) DV01 is difference between prices in (a) and (b)

  (d) NB: This method requires precision in (a)

- Usage:

$$\Delta p \ \cong \ -\text{DV01} \times (10000 \Delta y)$$

  (Approximation good for small changes in y)

3. DV01 (continued)

- Example 1 (2-year 10% bond, spot rates at 10%)
  - Initial values: $p = 100$, $y = 0.1000$
  - At $y = 0.1001$ (one bp higher), $p = 99.9823$
  - DV01 $= 100 - 99.9823 = 0.0177$

- Example 2 (5-year 10% bond, spot rates at 10%)
  - Initial values: $p = 100$, $y = 0.1000$
  - At $y = 0.1001$ (one bp higher), $p = 99.9614$
  - DV01 $= 100 - 99.9614 = 0.0386$
  - More sensitive than the 2-year bond

- Example 3 (2-year zero, spot rates at 10%)
  - Initial price: $p = 100/1.05^4 = 82.2702$
  - At $y = 0.1001$ (one bp higher), $p = 82.2546$
  - DV01 $= 82.2702 - 82.2546 = 0.0157$

- Example 4 (10-year zero, spot rates at 10%)
  - Initial price: $p = 100/1.05^{20} = 37.6889$
  - DV01 $= 37.6889 - 37.6531 = 0.0359$

4. DV01 Formulas

- Formulas that reproduce Bloomberg's calculation
  - Same setup as bond yield calculations:
    - Coupon C paid k times per year
    - Fraction w of a period left till next coupon
  - Given yield y, compute $d = 1/(1+y/k)$
  - Price-yield relation is

$$\text{Invoice Price} \;=\; d^w \left(\frac{1-d^n}{1-d}\right) C + d^{w+n-1} 100$$

  - Derivative formula:

$$A \;=\; \left[d^{n+1}(w+n-1) - d^n(w+n) + d(1-w) + w\right]$$

$$-\frac{dp}{dy} \;=\; \frac{d^{w+1} A}{k(1-d)^2} \times C + \frac{(w+n-1)d^{w+n}}{k} \times 100$$

(Sorry, I know how ugly this is.)

  - DV01: minus slope times 0.0001 (one bp):

$$\text{DV01} \;=\; -\frac{dp}{dy} \times 0.0001$$

4. DV01 Formulas (continued)

- Example 5 (Citicorp 7 1/8s revisited)
  - Terms:
    - Semi-annual US corporate $\Rightarrow$ $C = 7.125/2$
    - Settlement 6/16/95, matures 3/15/04
      $\Rightarrow$ $n = 18, w = 0.494$
    - Invoice price = 103.056
    - Yield = 6.929% $\Rightarrow$ $d = 0.966515$
  - DV01 (direct method):
    - Price at $y = 6.939\%$ (+1 bp) is $p = 102.991$
    - DV01 = 103.056 − 102.991 = 0.065
  - DV01 (formula): 0.065
  - Remarks
    - The two approaches give slightly different answers (the formula is an approximation based on a linear approximation to the price-yield relation)
    - Don't get bogged donw in the math — think of the formula (if you use it) as a useful shortcut.

5. DV01 for Portfolios

Similar methods work for portfolios

- Consider a position with x units of a bond:

$$\begin{aligned} \Delta v &= x\Delta p \\ &\cong -x \times \text{DV01} \times (10000\Delta y) \end{aligned}$$

- Consider a portfolio with positions in two bonds:
  - Portfolio has value

$$v = x_1 p_1 + x_2 p_2$$

  - Change in value is

$$\begin{aligned} \Delta v &= x_1 \Delta p_1 + x_2 \Delta p_2 \\ &\cong -x_1 \times \text{DV01}_1 \times (10000\Delta y_1) - x_2 \times \text{DV01}_2 \times (10000\Delta y_2) \end{aligned}$$

- Consider an arbitrary bond portfolio:
  - Portfolio has value

$$v = \sum_j x_j p_j$$

  - Change in value is

$$\begin{aligned} \Delta v &= \sum_j x_j \Delta p_j \\ &\cong -\sum_j x_j \times \text{DV01}_j \times (10000\Delta y_j) \end{aligned}$$

5. DV01 for Portfolios (continued)

- We define the DV01 of a portfolio as the change in value resulting from equal one basis point declines in all yields (ie, $\Delta y_j = -0.0001$ for all j):

$$\text{DV01} \;=\; \sum_j x_j \times \text{DV01}_j$$

- Summary for emphasis: the DV01 for a bunch of positions is the sum of the DV01's of the individual positions
  - Very helpful, since it's portfolios we care about
  - Based on equal changes in all yields
  - This is a standard risk management number: How sensitive is the portfolio to general changes in bond yields?

- Example 6: one 2-year bond and three 5-year bonds (examples 1 and 2)

$$\text{DV01} \;=\; 1 \times 0.0177 + 3 \times 0.0386 \;=\; 0.1335$$

In words: if yields rise 1 bp, we lose 13 cents.

6. Application: Yield Spread Trades

- Betting on yield spreads
  - Scenario: Spot rates flat at 10%
  - We expect the yield curve to steepen, but have no view on its level. Specifically, we expect the 10-year spot rate to rise relative to the 2-year.
  - Strategy: buy the 2-year, short the 10-year, in proportions that leave no exposure to overall yield changes
- Using DV01 to construct the trade:
  - We showed earlier that a one basis point rise in yield reduces the price of the 2-year zero by 0.0157 and the 10-year zero by 0.0359 (examples 3 and 4).
  - To eliminate exposure to equal changes in yields:

$$\begin{aligned}\Delta v &= (x_2 \times \mathrm{DV01}_2 + x_{10} \times \mathrm{DV01}_{10})\,(10000\Delta y) \\ &= 0\end{aligned}$$

  - Hence we buy more of the 2-year than we sell of the 10-year:

$$\frac{x_2}{x_{10}} = -\frac{\mathrm{DV01}_{10}}{\mathrm{DV01}_2} = -\frac{0.0359}{0.0157} = -2.29$$

    (the minus sign tells us one is a short position)
  - Remark: 2.29 is sometimes referred to as a *hedge ratio*

7. Duration

- Definition: (modified) ***duration*** is the proportional decline in price associated with a unit increase in yield:

$$D = -\frac{\text{Slope of Price-Yield Relation}}{\text{Price}}$$
$$= -\frac{dp/dy}{p}$$

- Formula with semiannual compounding and even first period:

$$D \equiv -\frac{dp/dy}{p} = (1+y/2)^{-1} \sum_{j=1}^{n} (j/2) \times w_j$$

with

$$w_j = \frac{(1+y/2)^{-j} c_j}{p}.$$

- Usage:

$$\Delta p \cong -p \times D \times \Delta y$$

(Approximation good for small changes in y)

- Remarks:
  - Weight w_j is fraction of value due to the jth payment
  - Sum is weighted average life of payments
  - Conveys same information as DV01

7. Duration (continued)

- Example 1 (2-year 10% bond, spot rates at 10%)

  Intermediate calculations:

Payment (j)	Cash Flow (c_j)	Value	Weight (w_j)
1	5	4.762	0.04762
2	5	4.535	0.04535
3	5	4.319	0.04319
4	105	86.384	0.86384

  Duration:

$$\begin{aligned} D &= (1 + .10/2)^{-1}(0.5 \times 0.04762 + 1.0 \times 0.04535 \\ &\quad +1.5 \times 0.04319 + 2.0 \times 0.86384) \\ &= 1.77 \text{ years} \end{aligned}$$

  If y rises 100 basis points, price falls 1.77%.

6. Duration (continued)

- Example 2 (5-year 10% bond, spot rates at 10%)
  $D = 3.86$.

- Example 3 (2-year zero, spot rates at 10%)
  Duration for an n-period zero is

  $$D \;=\; (1 + .10/2)^{-1}(n/2)$$

  Note the connection between duration and maturity.
  Here $n = 4$ and $D = 1.90$.

- Example 4 (10-year zero, spot rates at 10%)
  $D = (1 + .10/2)^{-1}(20/2) = 9.51$.

8. Duration Formulas

Reproducing Bloomberg's calculation

- Duration formula becomes (semiannual compounding)

$$D \equiv -\frac{dp/dy}{p} = (1+y/2)^{-1} \sum_{j=1}^{n} [(w+j-1)/2] \times w_j$$

  with

$$w_j = \frac{(1+y/2)^{-w-j+1} c_j}{p}.$$

- Short cut for coupon bonds:

$$A = -\frac{[n(1-d)-(1-d^n)]C}{(1-d)(1-d^n)C+(1-d)^2 d^{n-1} 100}$$

$$D = \frac{d}{k}\left[(w+n-1)+A\right]$$

  where k is the number of coupons per year, C is the coupon (not the annual coupon rate!), and $d = 1/(1+y/k)$.

- Remarks
  - Don't ask, it works!
  - Role of k: duration depends on interest rate convention

8. Duration Formulas (continued)

- Example 5 (Citicorp 7 1/8s again)
  - Parameters: $n = 18$, p (invoice price) $= 103.056$, $w = 0.494$, $y = 6.929\%$
  - Answer: $D = 6.338$ years (less than maturity of 8.75 years)

9. Duration for Portfolios

Similar methods work for portfolios

- Consider an arbitrary portfolio of bonds:
  - Portfolio has value
$$v = \sum_j x_j p_j$$
  - Change in value is
$$\begin{aligned} \Delta v &= \sum_j x_j \Delta p_j \\ &\cong -\sum_j x_j p_j \times D_j \times \Delta y_j \end{aligned}$$
  - Proportional change in value is
$$\begin{aligned} \Delta v / v &\cong -\sum_j (x_j p_j / v) \times D_j \times \Delta y_j \\ &\cong -\sum_j w_j \times D_j \times \Delta y_j \end{aligned}$$
    where $w_j = x_j p_j / v$ is the fraction of value in bond j.
- We define the duration of a portfolio as the proportional change in value resulting from equal changes in all yields ($\Delta y_j = \Delta y$ all j):
$$D = -\frac{dv/dy}{v} = \sum_j w_j D_j$$

9. Duration for Portfolios (continued)

- Example 6: one 2-year bond and three 5-year bonds (examples 1 and 2)
  - Since prices are equal, value weights are 1/4 and 3/4
  - $D = 0.25 \times 1.77 + 0.75 \times 3.86 = 3.38$

- Example 7: combination of 2- and 10-year zeros with duration equal to the 5-year par bond
  - This kind of position is known as a barbell, since the cash flows have two widely spaced lumps (picture a histogram of the cash flows)
  - If we invest fraction w in the 2-year, the duration is

$$D = w \times 1.90 + (1 - w) \times 9.51 = 3.86$$

Answer: $w = 0.742$.

9. Duration for Portfolios (continued)

- Spread trade for 2- and 10-year zeros
  - Recall: exploit expected yield-curve steepening
  - Durations are 1.90 (2-year) and 9.51 (10-year)
  - Dollar sensitivity is duration times price
  - To eliminate overall sensitivity, set

$$\begin{aligned} \Delta v &= (x_2\, p_2\, D_2 + x_{10}\, p_{10}\, D_{10})\, \Delta y, \\ &= 0, \end{aligned}$$

    which implies

$$\frac{x_2}{x_{10}} = -\frac{p_{10} D_{10}}{p_2 D_2} = -\frac{37.69 \times 9.51}{82.27 \times 1.90} = -2.29$$

  - Same answer as before: DV01 and duration contain the same information (slope of price-yield relation).

10. Duration: History and Assessment

Duration comes in many flavors:

- Our definition is generally called "modified duration"

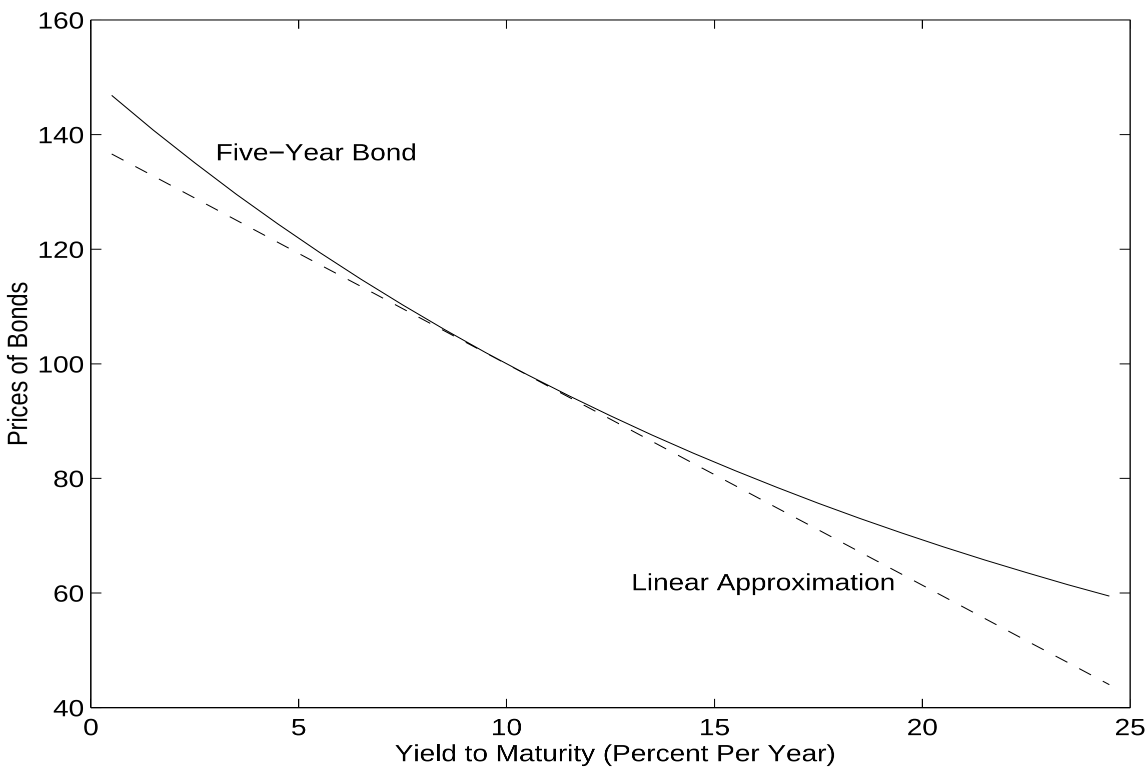
- The textbook standard is Macaulay's duration
  - Differs from ours in lacking the $(1 + y/2)^{-1}$ term:
  $$D = - \sum_{j=1}^{n} (j/2) \times w_j$$
  - Leads to a closer link between duration and maturity (for zeros, they're the same)
  - Nevertheless, duration is a measure of sensitivity; its link with maturity is interesting but incidental
  - Our approach depends on coupon frequency; since large k leads to Macaulay, hard to say the difference matters
  - Frederick Macaulay studied bonds in the 1930s

- Fisher-Weil duration: compute weights with spot rates
  - Makes a lot of sense
  - Used in many risk-management systems (RiskMetrics, for example)
  - Rarely makes much difference with bonds

10. Duration: History and Assessment (cont'd)

- Bottom line: duration is an approximation (ditto DV01)
  - Based on parallel shifts of the yield curve (presumes all yields change the same amount)
  - Holds over short time intervals (otherwise maturity and the price-yield relation change)
  - Holds for small yield changes:

$$\begin{aligned} \Delta p &= -pD \\ \Rightarrow \quad p - p_0 &\cong -p_0 \times D \times (y - y_0) \end{aligned}$$



11. Convexity

- Convexity measures curvature in the price-yield relation

- Common usage: "callable bonds have negative convexity" (the price-yield relation is concave to the origin)

- Definition (semiannual, full first period):

$$C \equiv \frac{d^2p/dy^2}{p} = (1+y/2)^{-2} \sum_{j=1}^{n} [j(j+1)/4] \times w_j$$

with

$$w_j = \frac{(1+y/2)^{-j} c_j}{p}$$

- Convexity is
  - higher for long bonds
  - higher for coupon bonds
  - higher yet for barbells (highly spread out cash flows)

11. Convexity (continued)

- Example 1 (2-year 10% bond, spot rates at 10%)
  $C = 4.12$.

- Example 3 (2-year zero, spot rates at 10%)
  Convexity for an n-period zero is
  $$C = (1 + .10/2)^{-2}[n(n+1)/4],$$
  or 4.53 when $n = 4$.

- Bloomberg calculations (fractional first period w)
  $$C = (1 + y/2)^{-2} \sum_{j=1}^{n} [(j - 1 + w)(j + w)/4] \times w_j$$
  with
  $$w_j = \frac{(1 + y/2)^{-j+1-w} c_j}{p}$$
  (Bloomberg divides this number by 100.)

- There's a formula for this, but it's *really* ugly.

- Example 5 (Citicorp 7 1/8s again)

  Answer: $C = 51.3$ (0.513 on Bloomberg)

11. Convexity (continued)

Convexity and returns

- Second-order approximation:

$$\frac{\Delta p}{p} = -D\Delta y + \frac{1}{2}C(\Delta y)^2$$

- Other things equal, high C is good
("The benter the better")

- Standard usage: "convexity added 6 bps to returns"

12. Statistical Measures of Interest Sensitivity

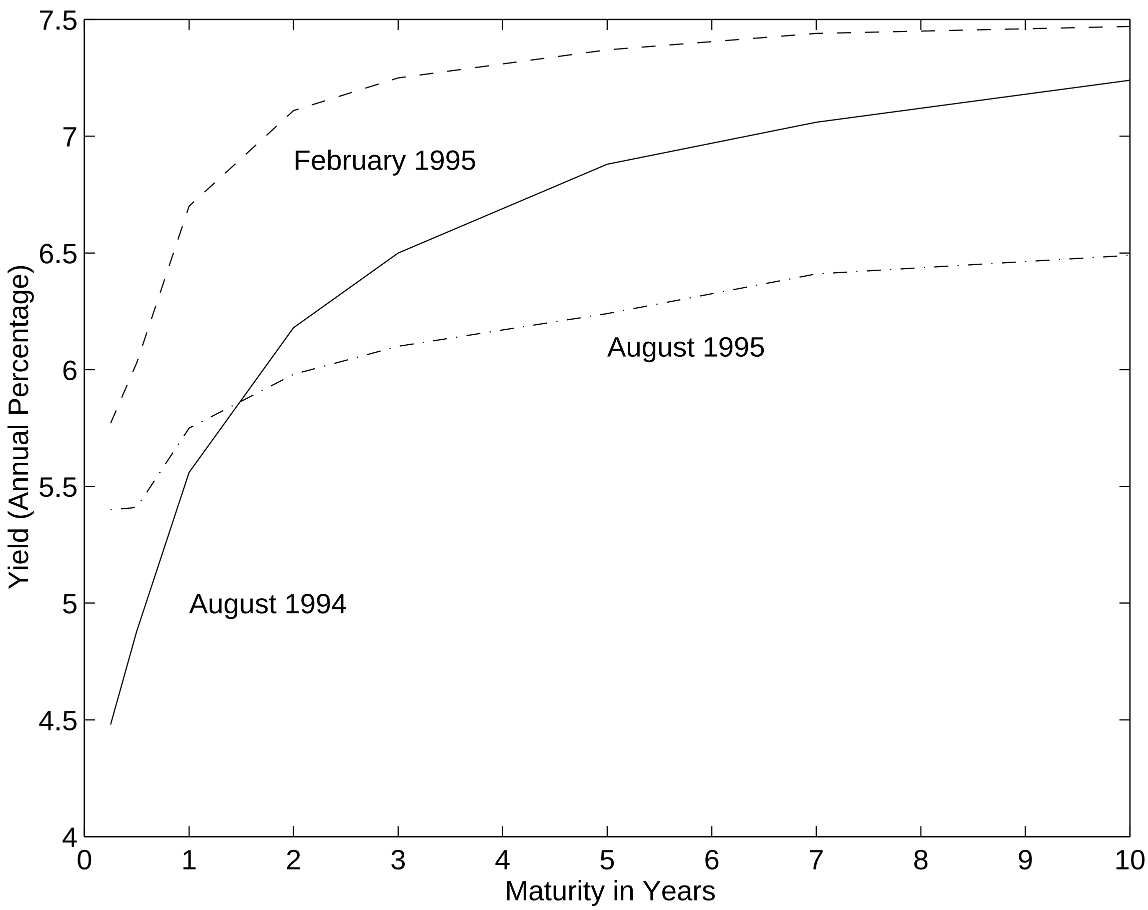
- Standard approach to measuring risk in finance: standard deviations and correlations of price changes
- Fixed income applications
  - Standard deviations and correlations of yield changes
  - Use DV01 to translate into price changes
  - Bottom line: yield changes not equal
- Statistical properties of monthly changes in spot rates

	1-Year	3-Year	5-Year	7-Year	10-Year
A. Standard Deviations of Changes (Percent)					
	0.547	0.441	0.382	0.340	0.309
B. Correlations of Changes					
1-Year	1.000	0.920	0.858	0.800	0.743
3-Year		1.000	0.967	0.923	0.867
5-Year			1.000	0.980	0.930
7-Year				1.000	0.970

Source: McCulloch and Kwon, US treasuries, 1952-91.

12. Statistical Measures (continued)

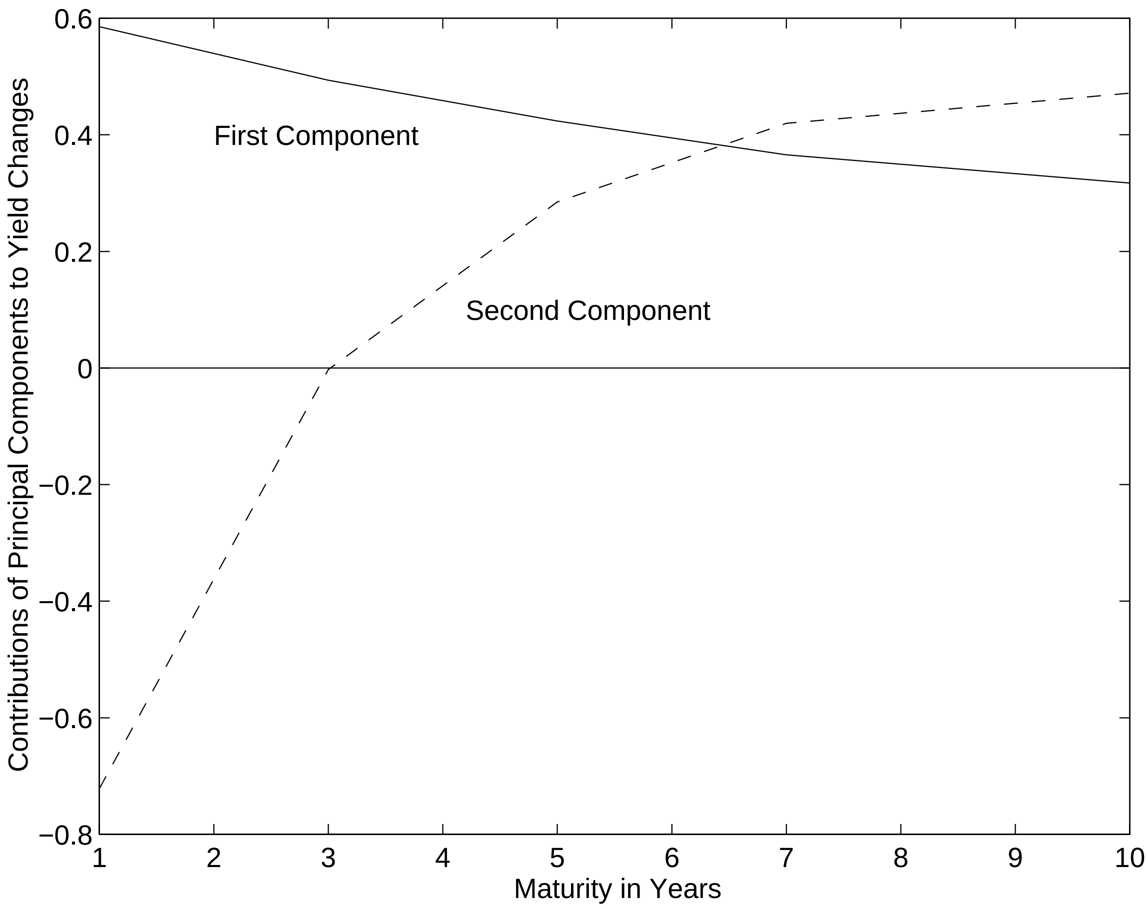
- Examples of yield curve shifts (spot rates)



- Remark: not just parallel shifts!

12. Statistical Measures (continued)

- Typical components of yield curve shifts



- Remarks:
  - Component 1: Roughly parallel, but less at long end
  - Component 2: Twist accounts for 10-15% of variance
  - Bottom line: DV01/duration only approximate

13. More on Statistical Risk Measures

Industry Practice: JP Morgan's RiskMetrics

- Daily estimates of standard deviations and correlations (This is critical: volatility varies dramatically over time)
- Twenty-plus countries, hundreds of markets
- Yield "volatilities" based on proportional changes:

$$\log(y_t/y_{t-1}) \approx \frac{y_t - y_{t-1}}{y_{t-1}}$$

- Maturities include 1 day; 1 week; 1, 3, and 6 months
  Others handled by interpolation
- Documentation available on the Web
  (useful but not simple)
- Similar methods in use at most major institutions

14. Application: Hedging

- Situation: We own x_5 units of 5-year notes.

- Problem: Short 10-years to minimize risk. How many?

- Conventional approach: use DV01's
  - Assume equal yield changes for 5s and 10s
  - Zero change in value:

$$\begin{aligned} \Delta v &= (x_5 \times \text{DV01}_5 + x_{10} \times \text{DV01}_{10})\,\Delta y \times 10000 \\ &= 0 \end{aligned}$$

  - Hedge ratio is

$$\frac{x_{10}}{x_5} = -\frac{\text{DV01}_5}{\text{DV01}_{10}}$$

14. Application: Hedging (continued)

- Statistical approach (don't sweat the details)
  - Notation shortcut: use δ for DV01
  - Variance of change in value:

    $$\begin{aligned}\sigma^2 &= Var(\Delta v)\\ &= (x_5\delta_5)^2\sigma_5^2 + (x_{10}\delta_{10})^2\sigma_{10}^2 + 2\rho(x_5\delta_5)(x_{10}\delta_{10})\sigma_5\sigma_{10}\end{aligned}$$

    (This is the variance of a sum: σ's are standard deviations in bps and ρ is the correlation between the two yields.)

- Choose x_{10} to minimize the variance:

  $$\frac{\partial\sigma^2}{\partial x_{10}} = 2x_{10}\delta_{10}^2\sigma_{10}^2 + 2\rho x_5\delta_5\delta_{10}\sigma_5\sigma_{10} = 0.$$

- Hedge ratio:

  $$\frac{x_{10}}{x_5} = -\rho\left(\frac{\sigma_5}{\sigma_{10}}\right)\left(\frac{\mathrm{DV01}_5}{\mathrm{DV01}_{10}}\right)$$

- Remarks:
  - Last term: the conventional ratio of DV01's
  - First term: if correlation is low, do less hedging
  - Middle term: correction for different yield volatilities
  - Bottom line: conventional approach different unless $\sigma_5 = \sigma_{10}$ and $\rho = 1$

15. Value-at-Risk

- Compute and report risk to management and shareholders
- Statistical approach (continued)
- Value-at-Risk (VAR) generally defined as $k \times \sigma$
  ($k = 1$, $k = 2.336$, etc, based on level of confidence)
- Example: portfolio with 1- and 10-year zeros (5 each)
  - The usual 10% flat spot rate curve
  - DV01's are 0.0086 and 0.0359
  - Std deviations are 54.7 and 30.9 (monthly in bps)
  - Correlation is 0.748
  - Variance of change in value:

$$\begin{aligned}\sigma^2 &= Var(\Delta v) \\ &= (x_1\delta_1)^2\sigma_1^2 + (x_{10}\delta_{10})^2\sigma_{10}^2 + 2\rho(x_1\delta_1)(x_{10}\delta_{10})\sigma_1\sigma_{10}\end{aligned}$$

  - Answer: $\sigma = 7.46$.
  - Translation:
    * Portfolio is worth 641.96
    * One standard deviation is 6.78 (about 1%)

15. Value-at-Risk

- Goldman Sachs: Daily VAR ($m), May 1998

Risk category	VAR
Interest rates	33
Currencies	11
Equities	22
Commodities	7
Diversification	(26)
Total	47

- Similar table for our example:

Asset	VAR
1-Years	2.35
10-Years	5.55
Diversification	(1.12)
Total	6.78

- Remarks:
  - Individual assets are $|x_j| \times \text{DV01}_j \times \sigma_j$
  - Total is less than sum (equal if correlations are one)
  - Diversification is the difference

16. Active Investment Strategies

- Basic investment strategies
  - Indexing: make the market return
  - Exploit arbitrage opportunities
  - Bet on the level of yields
  - Bet on the shape of the yield curve (yield spreads)
  - Bet on credit spreads
- Betting on yields
  - If you expect yields to fall, lengthen duration
  - If you expect yields to rise, shorten duration
  - Modifications typically made with treasuries or futures (lower transactions costs than, say, corporates)
  - Forecasting requires a combination of bond analytics, macroeconomics, and psychology
  - Henry Kaufman made two great calls in the 1980s
- Mutual Funds
  - Prospectuses and reporting standards vague
  - Often benchmarked to indexes (which?)
  - Cross-over investors muddy the water further

17. Interview with Henry Kaufman

- Biographical sketch
  - Started Salomon's bond research group
  - Legendary interest rate forecaster
  - Picked rate rise in 1981, when most thought the "Volcker shock" of 1979 was over
  - Now runs boutique

- Investment strategy
  - Study macroeconomic fundamentals (chart room)
  - Adjust duration depending on view
  - Adjustment through sale and purchase of treasuries
  - Some foreign bonds
  - No derivatives (except occasional currency hedge)

- Thoughts on modern risk analytics
  - What's "market value"?
  - RiskMetrics-like methods look backwards, but history can be a poor guide to the future in financial markets

Summary

- Bond prices fall when yields rise

- Prices of long bonds fall more

- DV01 and duration measure sensitivity to generalized changes in yields — parallel shifts

- Statistical measures are based on historical standard deviations and correlations

- Statistical measures allow different yield changes at different maturities

- Active investment strategies include bets on the level and shape of the yield curve

- Risk measurement and management remains as much art as science